Frequency of trade and the determinacy of equilibrium in economies of overlapping generations

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20 November 2012

Frequency of trade and the determinacy of equilibrium in economies of overlapping generations[1]

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November 20, 2012

[1]We thank a referee of this Journal for stimulating suggestions and comments. Financial support from the European Research Council (ERC Stg Grant 283953) is gratefully acknowledged.

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Abstract

Demichelis and Polemarchalis (2007) highlighted the role played by the frequency of trade on the degree of indeterminacy of equilibrium in economies of overlapping generations. Assuming that time has a finite starting point and extends into the infinite future, they prove that the degree of indeterminacy increases with the number of periods in the life-span of individuals, which is assumed to be deterministic. We show that this result does not hold when individual longevity is represented by an exponential survival function: the degree of indeterminacy depends on individual preferences and monetary policy but is independent of the frequency of trade.
Keywords: Overlapping generations · Perpetual youth model · Determinacy · Continuous time · Discrete time

JEL Classification Numbers D50 · D90

1 Introduction

The indeterminacy of the competitive equilibrium in economies of overlapping generations is a crucial issue for the design of monetary and fiscal policies (Bénassy, 2009). In an economy populated with agents living for two periods and trading one commodity, it is well known that indeterminacy is of degree one: only the level of prices is indeterminate while the inflation rate is determinate. It has been argued that increasing the number of traded goods increases the degree of indeterminacy in a truncated economy (Geanakoplos and Polemarchakis, 1991), but this property does not generalize to a model with many agents and general preferences (Kehoe and Levine, 1984). Most notably, Kehoe et al. (1991) prove the local uniqueness of the equilibrium if dated consumption goods are gross substitutes at all price ratios in the neighborhood of the steady state. Uniqueness means that the number of stable eigenvalues exactly equals the number of initial conditions describing the distribution of financial assets among generations at the initial date of the economy.

In recent articles, Demichelis (2002) and Demichelis and Polemarchalis (2007) reassessed the role played by the number of generations in the determinacy issue by focussing on the number of periods in the life-span, or equivalently on the frequency of trade among generations. In the most interesting case, such that time has a finite starting point and extends into the infinite future, they prove its influence on the degree of indeterminacy. Indeed, the algebra shows that the number of eigenvalues whose modulus is lower than one monotonously increases with the frequency of trade. However, according to the authors, despite the fact that the equilibrium equation in prices is linear, this counting of equations and unknowns does not necessarily imply an increase in the degree of indeterminacy. Using the convergence property of their discrete time model to its continuous time counterpart, they claim that the indeterminacies that appear in discrete time vanish in continuous time as solutions are time-shifts of a single path. Indeterminacies are hence a by-product of the discretization of the economy and is not comparable to those occurring with stochastic models (as e.g. in Nakajima and Polemarchakis, 2005).

In this paper, we study the same economy as the one considered by Demichelis and Polemarchakis (2007) except for the assumption made on individual longevity that is not anymore deterministic but characterized by an exponential survival function. We build a discrete time model similar to Farmer et al. (2011) and prove that it converges, when the frequency of trade is infinite, to the pure exchange economies' extensions of the Blanchard (1985) continuous time model. It is indeed known since Weil (1989) that the degree of indeterminacy is one in such a framework. The derivation of the equilibrium prices in the discrete time model allows us to claim that the degree of indeterminacy is independent of the frequency of trade. Moreover, the system is invariant by time-shift. The difference between Demichelis and Polemarchakis results and ours can be explained by the fact that the difference equation that characterize the equilibrium price dynamics remains of order 1. We are aware (d'Albis and Augeraud-Véron, 2007, 2009, 2011) that using a Poisson process to describe the survival function eases a lot the computation of the model and think that generalizing our results to realistic mortality patterns is a promising avenue of research (Azomahou et al., 2009). This simplicity, nevertheless, allow us to compute the closed form solutions of the dynamics and to extend Demichelis and Polemarchakis (2007) study to monetary equilibrium and CRRA preferences. We show that issuing money may reduce the degree of indeterminacy while CRRA preferences may increase it.

The paper is organized as follows: in Section 2, the discrete time model is presented and the exact solution for the price dynamics is given. The degree of indeterminacy is computed by distinguishing classical and monetary equilibria. Extensions of our results are given in Section 3. In Section 3.1, we show that the equilibrium of the discrete time model uniformly converges to its continuous time counterpart, while we consider, in Section 3.2, the extension to CRRA preferences. Concuding remarks are proposed in Section 4 and some proofs are gathered in Section 5.

2 A discrete time framework

The model closely follows Demichelis and Polemarchakis (2007) except for individual longevity, which is uncertain and characterized by an exponential survival function, and for the introduction of an unbacked asset. The economy is stationary, the distribution of the fundamentals being invariant with calendar time, and there is one commodity available at each date, which can not be stored or produced. Overlapping generations may trade using consumption-loans and fiat money.

The time is discrete, has a finite starting point and extends into the infinite future: $0, (1/n), ..., (t/n), ...$ where $t \in \mathbb{N}$ and $n \in [1, +\infty)$. The unit of time is given by $1/n$: the standard discrete-time framework is given for $n = 1$, while shorter time-paths are obtained by increasing n.

2.1 Agents' optimal behavior

Let us consider at date t/n an agent who was born at date τ/n, where $t \geq \max\{\tau, 0\}$. The duration of life is uncertain and, as in Blanchard (1985), the age at death is supposed to follow a Poisson process: at date t/n, the agent has a probability to be alive at date $(t+s)/n$ equals $(1-\lambda/n)^s$, with $\lambda \in (0,n)$. The hazard rate of death, λ/n, depends on the length of a period but, the life expectancy at any given age, which is written:

$$\sum_{s=1}^{\infty} \frac{s}{n}\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)^{s-1} = \frac{1}{\lambda}, \tag{1}$$

is a constant independent of n. Increasing n permits to increase the number of periods an agent may expect to live without modifying the life expectancy. Demichelis and Polemarchakis (2007) use a similar trick for a deterministic life-span.

The agent consumes a quantity of goods $c(\tau, t; n)$ and derives some utility from the discounted flow of future consumption. The intertemporal utility at date t/n is written as:

$$\sum_{s=0}^{\infty} \left(1-\frac{\delta}{n}\right)^{s} \left(1-\frac{\lambda}{n}\right)^{s} \ln c(\tau, t+s; n), \tag{2}$$

where δ/n stands for the discount factor, which is assumed to be such that $(1-\delta/n)(1-\lambda/n) < 1$ in order to keep the objective function finite.

At date t/n, the agent receives a positive endowment of goods, denoted $e(t;n)$, that is age-independent. It is assumed that the expected flow of future endowments is normalized:

$$\sum_{s=0}^{\infty} \left(1-\frac{\lambda}{n}\right)^{s} e(\tau+s; n) = 1, \tag{3}$$

in order to keep the endowment distribution independent of changes in the unit of time.

The agent has access to competitive asset markets, where consumption loans and fiat money may be exchanged (Samuelson, 1958), and to competitive annuity markets (Yaari, 1965). The intertemporal budget constraint at date t/n is written as:

$$\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s} p\left(t+s;n\right) c\left(\tau,t+s;n\right) \leq p\left(t;n\right) a\left(\tau,t;n\right)+w\left(t;n\right), \quad (4)$$

where $p\left(t;n\right)$ is the level of prices, $a\left(\tau,t;n\right)$ are the assets holdings, and $w\left(t;n\right)$ is the human wealth defined as follows:

$$w\left(t;n\right)=\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s} p\left(t+s;n\right) e\left(t+s;n\right). \quad (5)$$

To compute the optimal choices, it is important to distinguish agents born before the initial date of the economy from those born after. At the initial date of the economy, asset holdings, $a\left(\tau,0;n\right)$ for $\tau \in \mathbb{Z}_{-}$, are given, while initial holdings are supposed to be zero after the initial date: $a\left(\tau,\tau;n\right)=0$ for $\tau \in \mathbb{N}_{+}$.

Lemma 1. *Let*

$$\phi=\left[1-\left(1-\frac{\lambda}{n}\right)\left(1-\frac{\delta}{n}\right)\right]^{-1}. \quad (6)$$

For $t \in \mathbb{N}$ and $\tau \in \mathbb{Z}_{-}$, the optimal consumption and asset holdings are:

$$\begin{aligned} p\left(t;n\right) c\left(\tau,t;n\right) &= \left(1-\tfrac{\delta}{n}\right)^{t} \phi^{-1}\left[p\left(0;n\right) a\left(\tau,0;n\right)+w\left(0;n\right)\right], \\ p\left(t;n\right) a\left(\tau,t;n\right) &= \left(1-\tfrac{\delta}{n}\right)^{t}\left[p\left(0;n\right) a\left(\tau,0;n\right)+w\left(0;n\right)\right]-w\left(t;n\right), \end{aligned} \quad (7)$$

while for $t \in \mathbb{N}$ and $\tau \in \mathbb{N}_{+}$, they worth:

$$\begin{aligned} p\left(t;n\right) c\left(\tau,t;n\right) &= \left(1-\tfrac{\delta}{n}\right)^{t-\tau} \phi^{-1} w\left(\tau;n\right), \\ p\left(t;n\right) a\left(\tau,t;n\right) &= \left(1-\tfrac{\delta}{n}\right)^{t-\tau} w\left(\tau;n\right)-w\left(t;n\right). \end{aligned} \quad (8)$$

Proof. See Section 5. □

We see that the initial distribution of assets holdings, $a\left(\tau,0;n\right)$ influences the optimal behavior of agents born before the initial date of the economy. Optimal paths (7) and (8) can be computed for any sequence of prices $\left\{p\left(t;n\right)\right\}_{t\in\mathbb{N}}$, perfectly anticipated by the agents. The equilibrium conditions presented below aim at finding the sequences that prevail at the equilibrium.

2.2 Aggregate variables and equilibrium

Each agent belongs to a large cohort of identical agents. Therefore even though longevity is stochastic, there is no aggregate uncertainty. The law of large numbers applies, and thus, the size of each cohort is decreasing at rate λ/n. Then, at date t/n, the size of the cohort born at date τ/n is $(\gamma/n)\, N(\tau; n)\,(1-\lambda/n)^{(t-\tau)}$ where $\gamma/n > 0$ is the birth rate for the unit of time $1/n$ and $N(\tau; n)$ is the size of the population at date τ/n. Since a new cohort is born at each date, the latter is obtained by summing over birth dates the size that each cohort reaches at date t/n:

$$N(t; n) = \frac{\gamma}{n} \sum_{\tau=-\infty}^{t} N(\tau; n)\left(1-\frac{\lambda}{n}\right)^{t-\tau}. \tag{9}$$

Let us assume that the population is stationary and normalized to 1. Using the previous equation, one obtains that the birth rate equals the hazard rate of death. We hence assume $\gamma = \lambda$ hereafter.

The aggregate counterpart of any individual variable is obtained by summing over cohorts. Hence, the aggregate assets at date t/n, denoted $A(t; n)$, are given by:

$$A(t; n) = \frac{\lambda}{n} \sum_{\tau=-\infty}^{t} \left(1-\frac{\lambda}{n}\right)^{t-\tau} a(\tau, t; n). \tag{10}$$

Similarly, the aggregate endowment at date t/n is given by:

$$\frac{\lambda}{n} \sum_{\tau=-\infty}^{t} e(t; n)\left(1-\frac{\lambda}{n}\right)^{t-\tau} = e(t; n). \tag{11}$$

Assuming that $e(t; n)$ is constant over time, we use (3) to conclude that $e(t; n) = \lambda/n$ and, consequently, that the human wealth write:

$$w(t; n) = \frac{\lambda}{n} \sum_{s=0}^{\infty} \left(1-\frac{\lambda}{n}\right)^{s} p(t+s; n). \tag{12}$$

The goods are perishable but may be traded across cohorts. It is nevertheless necessary that aggregate consumption equals aggregate endowment at each date t/n. This condition is satisfied if and only if:

$$\sum_{\tau=-\infty}^{t}\left(1-\frac{\lambda}{n}\right)^{t-\tau}c\left(\tau,t;n\right)=1. \tag{13}$$

By replacing equations (7) and (12), the condition (13) is rewritten:

$$p\left(t;n\right)=\frac{p\left(0;n\right)\frac{A(0;n)}{\frac{\lambda}{n}}+\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s}p\left(s;n\right)+\frac{\lambda}{n}\sum_{\tau=1}^{t}\frac{\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s}p(\tau+s;n)}{\left[\left(1-\frac{\delta}{n}\right)\left(1-\frac{\lambda}{n}\right)\right]^{\tau}}}{\phi\left[\left(1-\frac{\delta}{n}\right)\left(1-\frac{\lambda}{n}\right)\right]^{-t}}. \tag{14}$$

where ϕ is given by (6). Equivalently, by changing the order of summation, (14) writes:

$$p\left(t;n\right)=\frac{\left[p\left(0;n\right)\left(1+\frac{A(0;n)}{\frac{\lambda}{n}}\right)+\sum_{i=1}^{\infty}x\left(i;t\right)p\left(i;n\right)\right]}{\phi\left[\left(1-\frac{\delta}{n}\right)\left(1-\frac{\lambda}{n}\right)\right]^{-t}}, \tag{15}$$

where:

$$x\left(i,t\right)=\left(1-\frac{\lambda}{n}\right)^{i}\left[1+\frac{\lambda}{n}\sum_{\tau=1}^{\min\{i,t\}}\left(1-\frac{\lambda}{n}\right)^{-2\tau}\left(1-\frac{\delta}{n}\right)^{-\tau}\right]. \tag{16}$$

This latter equation is similar to the one studied by Demichelis and Polemarchakis (2007) except that we allow aggregate assets to be different from zero. However, the main difference lies in the fact that equation (15) can be rewritten as the following difference equation:

$$p\left(t+2;n\right)=\left(2-\frac{\delta}{n}\right)p\left(t+1;n\right)-\left(1-\frac{\delta}{n}\right)p\left(t;n\right). \tag{17}$$

Equation (17) is simple to study (there are two stationary inflation rates: 0 and $-\delta/n$) but, as it will be shown below, it may be misleading to consider it alone. To avoid confusions, one should also use the equilibrium condition on the assets market.

On the asset markets, it is supposed that a Central Bank issues a non negative quantity M of fiat money at date 0 and nothing afterwards. The money is distributed to agents that are alive at date $t/n = 0$. The equilibrium condition on the assets market is:

$$p(t;n)\,A(t;n) = M \text{ for all } t \in \mathbb{N}, \tag{18}$$

where the aggregate assets are obtained by replacing equation (8) in (10) and rearranging using the condition on the goods market (15), such that:

$$p(t;n)\,A(t;n) = \frac{\lambda}{n}\left[\phi p(t;n) - \sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s} p(t+s;n)\right]. \tag{19}$$

The equilibrium condition on the asset market characterizes current prices $p(t;n)$ as a function of future prices. By construction, any sequence of prices satisfying it also satisfy the equilibrium condition on the goods market. It rewrites more simply as follows:

$$p(t+1;n) = \left(1-\frac{\delta}{n}\right)p(t;n) - \frac{M}{\left(1-\frac{\lambda}{n}\right)\phi}. \tag{20}$$

Let us finally notice that neither $p(0;n)$ nor $A(0;n)$ are given but, through the equilibrium condition on the assets market, their product is known.

We define an intertemporal equilibrium as a sequence of positive prices $\{p(t;n)\}_{t\in\mathbb{N}}$ that satisfies equation (20). According to Polemarchakis (1988), an equilibrium has K degree of indeterminacy if the set of distinct equilibria constains a K dimensional open set[1]. Using these two definitions, we may summarize what we obtained above in the following proposition.

Proposition 1. *At the intertemporal equilibrium, prices satisfy for* $t \in \mathbb{N}$:

$$p(t;n) = \left(1-\frac{\delta}{n}\right)^{t}\left(p(0;n) - \frac{M}{\phi\frac{\lambda}{n}-1}\right) + \frac{M}{\phi\frac{\lambda}{n}-1}. \tag{21}$$

[1] A K dimentional open set is the image of a continuously one to one function with domain an open neighborhood in $K-$dimensional Euclidian space.

The degree of indeterminacy is thus at most one and is independent of a change in the frequency of trade.

Proof. Equation (21) is the exact solution of the linear equation (20). Prices at date t/n are a function of time and $p(0;n)$. This implies that the degree of indeterminacy is one if $p(0;n)$ is unknown and zero otherwise. Moreover, a change in n has no impact on the degree of indeterminacy. □

Let us point out that all sequences given by (21) do no constitute an equilibrium as prices should be positive. This constraint is important to assess whether the equilibrium exists and is unique. Indeterminacy can be discussed by considering four cases depending on the availability of fiat money in the economy and on the various parameters values. We present these cases below in four corollaries. But whatever the case considered, we see that the frequency of trade plays no role as it does not affect the order of the difference equation (20).

Corollary 1. *Suppose that $M = 0$. Prices, which initial level $p(0;n)$ is not determined, decrease at the constant rate $-\delta/n$.*

Proof. Set $M = 0$ in equation (21). □

Using Gale (1973)' typology, this case corresponds to a classical equilibrium with zero aggregate asset holdings. Individual consumption is constant over the life cycle but, contrarily to the two-period life-span model, there exists some trade on the asset market. Using equations (7), (8), and (12) we see that cohorts who were already born at the initial date of the economy consume at each period t/n a quantity $\lambda/n + a(\tau, 0; n)/\phi$, which represents their endowment and the net present value of their asset holdings. Those being initially in debt consume less than their endowment and finance the over consumption of those who initially had positive assets holdings. Conversely, the cohorts who were born after the initial date $t = 0$ consume the endowment they receive at each date and do not hold any assets over their life. The initial price $p(0; n)$ is not determined at equilibrium and there is a one parameter family of solutions for $p(t; n)$.

Let us now consider the cases where fiat money was issued at date 0.

Corollary 2. *Suppose that $M > 0$ and $\delta > 0$. Prices, which initial level $p(0; n)$ is infinite, decrease at the constant rate $-\delta/n$ but remain infinite for all t.*

Proof. Let us proceed by contradiction: if $p(0; n)$ is finite, $\lim_{t \to +\infty} p(t; n) = M/(\phi\lambda/n - 1)$, which, using the definition of ϕ given in (6), is negative. Hence, there exists $t_0 \geq 0$ such that $p(t; n) < 0$ for all $t \geq t_0$. Consequently, any sequence starting with $p(0; n) < \infty$ cannot be a equilibrium. In order to deal with infinite $p(0; n)$, let us introduce the change in variable: $z(t; n) = [p(t; n) - M/(\phi\lambda/n - 1)]^{-1}$, which permits to rewrite (21) as follows:

$$z(t; n) = \left(1 - \frac{\delta}{n}\right)^{-1} z(0; n). \tag{22}$$

If $p(0; n)$ is infinite, one has $z(0; n) = 0$ and, using (22), $z(t; n) = 0$ for all t. We conclude that $p(t; n)$ is infinite for all t. Moreover, the inflation rate computed using (21) writes:

$$\frac{p(t+1; n)}{p(t; n)} - 1 = \frac{-\frac{\delta}{n}}{1 + \frac{M}{\left(1 - \frac{\delta}{n}\right)^t \left[\left(\phi\frac{\lambda}{n} - 1\right) p(0; n) - M\right]}}. \tag{23}$$

Using the fact that $p(0; n)$ is infinite permits to conclude. □

The considered case is a classical equilibrium such that the pure discount rate, δ/n, is greater than the population growth rate, which is here set to zero. By comparing it to the one described in Corollary 1, we see that the introduction of fiat money permits to eliminate the indeterminacy in prices. One may think it is odd to consider that infinite prices are determinate and may prefer to rewrite the equilibrium dynamics as an equation in real aggregate assets (as e.g. in Woodford, 1984), which would worth zero at each date t. In real terms the money is never valued: $M/p\,(t;n) = 0$ for all t. The prices growth rate remains the same as in the case without money, which implies that agents behaviors are the same as above.

Corollary 3. *Suppose that* $M > 0$ *and* $\delta < 0$. *Prices, which initial level* $p\,(0;n)$ *is not determinate, either converge to* $+\infty$, *or are constant.*

Proof. There exists $\bar{p}\,(0,n) \equiv M/\,(\phi\lambda/n - 1) > 0$, such that the equilibrium exists for all $p\,(0;n) \geq \bar{p}\,(0;n)$, and does not exist otherwise (as the prices would be negative). For all $p\,(0;n) > \bar{p}\,(0;n)$, the inflation rate, given by (23), converge to $-\delta/n$. For $p\,(0;n) = \bar{p}\,(0;n)$, we see with (21) that $p\,(t;n) = \bar{p}\,(0;n)$. $\square$

In this case, which is such that the pure discount rate is lower than the population growth rate, the introduction of money does not eliminates price indeterminacy. There exists a family of paths that are such that prices converge to infinity and whose growth rate is also indeterminate as it depends on $p\,(0;n)$. Conversely, aggregate assets converge to zero, which means that money is not used in the long run. However, there also exists a particular value for $p\,(0;n)$ such that prices, and aggregate asset holdings, are constant. It defines a steady-state that is locally determinate as a unique sequence of prices reaches it. The inflation rate being zero, this path corresponds to the Golden Rule, where individual consumptions and asset holdings are increasing with age.

Corollary 4. *Suppose that* $M > 0$ *and* $\delta = 0$. *Prices, which initial level*

$p(0;n)$ is infinite, are constant.

Proof. Using l'Hospital' Rule, one obtains that for $\delta = 0$, equation (21) rewrites as follows:

$$p(t;n) = p(0;n) - t\frac{\frac{\lambda}{n}M}{\left(1-\frac{\lambda}{n}\right)}. \tag{24}$$

If $p(0;n)$ is finite, one has: $\lim_{t\to+\infty} p(t;n) = -\infty$. Consequently, any sequence starting with $p(0;n) < \infty$ cannot be a equilibrium. In order to deal with infinite $p(0;n)$, let us introduce the change in variable: $z(t;n) = 1/p(t;n)$, which permits to rewrite (24) as follows:

$$z(t;n) = \frac{1}{p(0;n) - t\frac{\frac{\lambda}{n}M}{\left(1-\frac{\lambda}{n}\right)}}. \tag{25}$$

We conclude that $z(t;n) = 0$ for all t and that $p(t;n)$ are infinite for all t. Finally, we use (23) to conclude that prices are constant. $\square$

This last case, which is sometime named the coincidental equilibrium in the literature, features a Golden Rule equilibrium where the aggregate real assets are zero. As in Corollary 2, prices are determined even though they are infinite for all t.

In the above four cases, we have seen that the price indeterminacy is at most of degree 1, which is due to the fact that the equilibrium can be characterized by a difference equation of order one. Moreover, all paths can be obtained from another one by a time-shift. This latter property, which is also found by Demichelis and Polemarchakis (2002) in the case without money, can be formally defined as follows. Let $p(t) = f(p_0, t)$ be a one-parameter family system with $p_0 = p(0)$. Trajectories are invariant by time shift if for all $t_1 \in \mathbb{N}$ one has: $f(t, f(t_1, p_0)) = f(t + t_1, p_0)$, for all $t \in \mathbb{N}$.

Lemma 2. *The equilibrium is invariant by time shift.*

Proof. Using (21), one may compute:

$$f\left(t+t_1;p_0\right)=\left(1-\frac{\delta}{n}\right)^{t+t_1}\left(p\left(0;n\right)-\frac{M}{\phi\frac{\lambda}{n}-1}\right)+\frac{M}{\phi\frac{\lambda}{n}-1}, \tag{26}$$

and:

$$f\left(t,f\left(t_1,p_0\right)\right)=\left(1-\frac{\delta}{n}\right)^{t+t_1}\left(p\left(0;n\right)-\frac{M}{\phi\frac{\lambda}{n}-1}\right)+\frac{M}{\phi\frac{\lambda}{n}-1}, \tag{27}$$

which permit to conclude. □

3 Extensions

We assess the robustness of our results by considering the extensions to the continuous time limit and to a higher degree system. The latter being obtained by considering a more general utility function.

3.1 The continuous time limit

The continuous time version of the model presented above is similar to Blanchard (1985) and, except that we do not allow for population growth, to Buiter (1988) and Weil (1989). This extension derives the equilibrium prices and demonstrates the uniform convergence of the equilibrium computed in discrete time to its continuous time counterpart.

Time has a finite starting point: $0 \leq t < +\infty$. An agent born at date τ has a probability to be alive at date t equals $e^{-\lambda(t-\tau)}$, where $\lambda > 0$ is the hazard rate. Consequently, life expectancy is $1/\lambda$ whatever the age of the agent. Consumption at age $t-\tau$ is denoted $c\left(\tau,t\right)$ and the endowment received at date t is $e\left(t\right)$.

An agent born at date $\tau > 0$ maximizes at date $t \geq \tau$:

$$\int_t^{\infty} e^{-(\lambda+\delta)(s-t)} \ln c\left(\tau,s\right) ds, \tag{28}$$

subject to:

$$\int_t^{\infty} e^{-\lambda(s-t)} p(s)\, c(\tau, s)\, ds \leq p(t)\, a(\tau, t) + w(t), \tag{29}$$

where $p(t)$ and $w(t)$ respectively denote the price and the human wealth at date t, $a(\tau, t)$ denotes the assets accumulated at age $t - \tau$ and $\delta > -\lambda$ denotes the discount rate. The human wealth satisfies:

$$w(t) = \int_t^{\infty} e^{-\lambda(s-t)} p(s)\, e(s)\, ds. \tag{30}$$

Initial and terminal conditions write: $a(\tau, \tau) = 0$ and $\lim_{t \to +\infty} p(t)\, a(\tau, t) \geq 0$. The optimal consumption path satisfies:

$$p(t)\, c(\tau, t) = (\delta + \lambda)\, w(\tau)\, e^{-\delta(t-\tau)}, \tag{31}$$

while the optimal assets holding are given by:

$$p(t)\, a(\tau, t) = w(\tau)\, e^{-\delta(t-\tau)} - w(t). \tag{32}$$

Conversely, an agent born at date $\tau \leq 0$ and still alive at date 0, maximizes at date $t \geq 0$ the objective function (28) subject to (29), $a(\tau, 0)$ given, and $\lim_{t \to +\infty} p(t)\, a(\tau, t) \geq 0$. The optimal consumption path satisfies:

$$p(t)\, c(\tau, t) = (\lambda + \delta)\, [p(0)\, a(\tau, 0) + w(0)]\, e^{-\delta t}, \tag{33}$$

while the optimal assets holding are given by:

$$p(t)\, a(\tau, t) = [p(0)\, a(\tau, 0) + w(0)]\, e^{-\delta t} - w(t). \tag{34}$$

The population is stationary and the birth rate is λ. The aggregate endowment is constant, and satisfies: $e(t) = \lambda$. The equality between aggregate demand and aggregate endowment now writes:

$$\int_{-\infty}^{t} e^{-\lambda(t-\tau)} c(\tau, t)\, d\tau = 1, \tag{35}$$

while the equilibrium on the asset market writes:

$$p(t)\int_{-\infty}^{t} e^{-\lambda(t-\tau)}a(\tau,t)\,d\tau = M. \tag{36}$$

By replacing (31) and (33) in (35) and using (30) and (36), we obtain the equilibrium equation for prices:

$$\begin{aligned} p(t) &= (\lambda+\delta)\left[M+\int_{0}^{\infty} e^{-\lambda s}p(s)\,ds\right]e^{-(\delta+\lambda)t} && (37)\\ &\quad +(\delta+\lambda)\lambda\int_{0}^{t} e^{-(\delta+\lambda)(t-\tau)}\left(\int_{\tau}^{\infty} e^{-\lambda(s-\tau)}p(s)\,ds\right)d\tau, && (38)\end{aligned}$$

which is the continuous time counterpart of (14). Similarly, replacing (32) and (34) in condition (36) and using (38), one obtains:

$$\frac{p(t)}{(\lambda+\delta)}-\int_{t}^{\infty} e^{-\lambda(s-t)}p(s)\,ds = M, \tag{39}$$

from which we deduce the equilibrium prices.

Proposition 2. *At the intertemporal equilibrium, prices satisfy for* $t \in \mathbb{R}_+$:

$$p(t) = \left[p(0)+\frac{\lambda(\lambda+\delta)}{\delta}M\right]e^{-\delta t}-\frac{\lambda(\lambda+\delta)}{\delta}M. \tag{40}$$

Proof. Differentiating (39) with respect to time gives:

$$p'(t) = -\delta p(t)-\lambda(\lambda+\delta)M, \tag{41}$$

whose solution is (40). □

Lemma 3. *As* $n\to+\infty$, $p(t;n)$ *converges uniformly to* $p(t)$.

Proof. The proof proceed showing that (21) converges to (40) when $n\to+\infty$. We use $\left(1-\frac{\delta}{n}\right)^{t}\sim e^{-\frac{\delta}{n}t}$ and $\lim_{n\to+\infty}1/(\phi\lambda/n-1) = -(\delta+\lambda)/\delta$. Let us denote $t' = t/n$ and $M' = M/\lambda$, we obtain that:

$$\lim_{n\to+\infty}p(t;n) = e^{-\frac{\delta}{n}t'}\left(p(0)+\lambda\left(\frac{\delta+\lambda}{\delta}\right)M'\right)-\lambda\left(\frac{\delta+\lambda}{\delta}\right)M'. \ \square \tag{42}$$

As Demichelis and Polemarchakis (2007), we proved that the discrete time model converges to its continuous time counterpart when the frequency of trade is infinite. The degree of indeterminacy is unchanged.

3.2 CRRA utility function

Let us now consider that the intertemporal utility at date t/n writes:

$$\sum_{s=0}^{\infty}\left(1-\frac{\delta}{n}\right)^{s}\left(1-\frac{\lambda}{n}\right)^{s}\frac{c\left(\tau,t+s;n\right)^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}, \tag{43}$$

where $\sigma \in (0,1)\times(1,+\infty)$ stands for the elasticity of intertemporal substitution.

By computing the individual consumptions and aggregating them over cohorts, one obtains the following price dynamics that satisfy the equilibrium condition on the goods market:

$$p\left(t;n\right)^{\sigma}=\frac{\frac{p(0;n)A(0;n)}{\frac{\lambda}{n}}+\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s}p\left(s;n\right)}{\sum_{j=0}^{\infty}\Phi^{(j-t)}p\left(j;n\right)^{1-\sigma}}+\sum_{\tau=1}^{t}\frac{\frac{\lambda}{n}\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s}p\left(\tau+s;n\right)}{\Phi^{(\tau-t)}\sum_{j=0}^{\infty}\Phi^{j}p\left(\tau+j;n\right)^{1-\sigma}}, \tag{44}$$

where $\Phi=\left(1-\frac{\delta}{n}\right)^{\sigma}\left(1-\frac{\lambda}{n}\right)$, which satisfies $\lim_{\sigma\to1}\sum_{j=0}^{\infty}\Phi^{j}=\phi$. This equation generalizes (14) for any $\sigma\in(0,1)\times(1,+\infty)$. Aggregating the individual asset holdings, and replacing equation (44), one obtains the following equilibrium condition on the asset market:

$$M=\frac{\lambda}{n}\left[p\left(t;n\right)^{\sigma}\sum_{s=0}^{\infty}\Phi^{s}p\left(t+s;n\right)^{1-\sigma}-\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s}p\left(t+s;n\right)\right], \tag{45}$$

which generalize (19). This equation does not rewrite as a tractable difference equation but the dynamics can be studied using the following change in variables:

$$y(t;n) = p(t;n)^{\sigma}, \tag{46}$$
$$x(t;n) = \sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s} p(t+s;n)^{1-\sigma}\left(1-\frac{\delta}{n}\right)^{\sigma s}, \tag{47}$$

which permit to write a two-dimensional dynamic system:

Lemma 4. *The sequence of prices that satisfies (45) is the solution of:*

$$\begin{cases} y(t+1;n) = \left(1-\frac{\delta}{n}\right)^{\sigma}\left(y(t;n) - \frac{M}{x(t;n)-y(t;n)^{\frac{1-\sigma}{\sigma}}}\right) \\ x(t+1;n) = \Phi^{-1}\left[x(t;n) - y(t;n)^{\frac{1-\sigma}{\sigma}}\right] \end{cases} \tag{48}$$

Proof. See Section 5. □

In system (48), neither $y(0;n)$ nor $x(0;n)$ are given, except in the case $\sigma = 1$ where the latter is a constant. The system is then one dimensional and is the same as (20). We also immediately see that in the case such that aggregate assets holdings worth zero, i.e. for $M = 0$, that the price dynamics are the same as those described in corollary 1. For $M > 0$, we deduce from Lemma 4 that the degree of indeterminacy is still independent of a change in the frequency of trade but is now at most 2. The knowledge of $p(0;n)$, which gives $y(0;n)$, is not sufficient as $x(0;n)$ will not be known. We now propose a phase diagram analysis[2] that permit to perform a global analysis of the dynamics.

Let us first suppose that the pure discount rate is greater than the population growth rate: $\delta > 0$. It can be shown that there is no steady-state and that the isocline I_y, which represents the locus such that $y(t+1;n) = y(t;n)$, is always lower than the isocline I_x, which represents the locus such that

[2]The technical derivation of the diagrams can be found in Section 5.

$x(t+1;n) = x(t;n)$. The Figures 1 represent the phase diagrams for different values of σ, which influence the shape of I_x.

Figures 1. Phase diagrams for $\delta > 0$.
about here

We see from Figures 1 that, for all $\sigma \neq 1$, there exist an infinity of admissible pairs $(y(0;n), x(0;n))$ that initiate a trajectory of positive prices satisfying (48). Asymptotically, prices are infinite. Contrarily to the case $\sigma = 1$, described in Corollary 2, the emission of money does not eliminate indeterminacy, which is of degree two.

We now consider the case such that the pure discount rate is greater than the population growth rate: $\delta < 0$. It can be shown that there exists a unique steady-state, which is an unstable focus for $\sigma < 1$ and an unstable node for $\sigma > 1$. The Figures 2 represent the phase diagrams for different values of σ.

Figures 2. Phase diagrams for $\delta < 0$.
about here

We conclude that for $\sigma < 1$, the equilibrium trajectory exists and is unique: the initial pair $(y(0;n), x(0;n))$ jumping to its steady-state value, while all other configurations lead to negative prices. Conversely, for $\sigma > 1$, in addition to the steady-state solution, there are an infinity of possible trajectories that features positive prices converging to the infinity. This latter case is thus similar to the case $\sigma = 1$, described in Corollary 3 except that the degree of indeterminacy is two, while the former implies that the equilibrium is determinate.

4 Conclusion

In this article, we developed a simple overlapping generations model that permits to easily compute the degree of indeterminacy of equilibrium paths.

We showed that this degree may increase with the dimension of the dynamic system. In Demichelis and Polemarchakis (2007), the assumption of bounded lifespans implies that the dimension of the system increases with the frequency of trade, while it is not the case with the survival function we consider. However, considering a more general utility function, namely a CRRA one rather than a logarithmic one, has been shown to increase the degree of indeterminacy. This increase is not an artifact of the discretization and does not vanish in continuous time. However, the degree of indeterminacy in large dimension dynamical systems is still an issue: with logarithmic preferences, Demichelis and Polemarchakis (2007) were able to evaluate the modulus of the eigenvalues and to conclude at the continuous time limit when the number of eigenvalues becomes infinite. This method cannot be applied to any mixed-type functional differential equation and further researches are needed in order to get general results.

5 Proofs

Proof of Lemma 1. The proof is standard. For $\tau \in \mathbb{Z}_{-}$ and $t \in \mathbb{N}$, the agent maximizes (2) subject to (4) and $a(\tau, 0; n)$ given. The Lagrangian writes:

$$\begin{aligned} L(\tau, t; n) &= \sum_{s=0}^{\infty} \left(1 - \frac{\delta}{n}\right)^{s} \left(1 - \frac{\lambda}{n}\right)^{s} \ln c(\tau, t+s; n) + \mu p(t; n) a(\tau, t; n) \\ &\quad + \mu \left[w(t; n) - \sum_{s=0}^{\infty} \left(1 - \frac{\lambda}{n}\right)^{s} p(t+s; n) c(\tau, t+s; n) \right]. \end{aligned} \tag{49}$$

The first order conditions on $c(\tau, t+s; n)$ write:

$$\frac{\left(1 - \frac{\delta}{n}\right)^{s}}{c(\tau, t+s; n)} - \mu p(t+s; n) = 0, \text{ for all } s = 0, 1, ... \tag{50}$$

The "Keynes-Ramsey" equation can thus be easily derived:

$$\frac{c(\tau, t+s+1; n)}{c(\tau, t+s; n)} = \left(1 - \frac{\delta}{n}\right) \frac{p(t+s; n)}{p(t+s+1; n)}. \tag{51}$$

Replace (50) in (4) to compute the Lagrangian multiplier:

$$\mu = \frac{1}{1-\left(1-\frac{\lambda}{n}\right)\left(1-\frac{\delta}{n}\right)} \frac{1}{p\left(t;n\right)a\left(\tau,t;n\right)+w\left(t;n\right)}, \tag{52}$$

and replace it in (50) to obtain:

$$c\left(\tau,t+s;n\right) = \frac{\left(1-\frac{\delta}{n}\right)^{s}\left[1-\left(1-\frac{\lambda}{n}\right)\left(1-\frac{\delta}{n}\right)\right]\left[p\left(t;n\right)a\left(\tau,t;n\right)+w\left(t;n\right)\right]}{p\left(t+s;n\right)}. \tag{53}$$

At date $t/n=0$, the optimal consumption is:

$$c\left(\tau,0;n\right) = \frac{\left[1-\left(1-\frac{\lambda}{n}\right)\left(1-\frac{\delta}{n}\right)\right]\left[p\left(0;n\right)a\left(\tau,0;n\right)+w\left(0;n\right)\right]}{p\left(0;n\right)}. \tag{54}$$

The first equation in (7) is obtained using (51) and (54). Replacing (7) in (53) gives the first equation in (8).

For $\tau \in \mathbb{N}_{+}$ and $t \in \mathbb{N}_{+}$, the agent maximizes (2) subject to (4) and $a\left(\tau,\tau;n\right)=0$. Equation (53) still holds, but the initial consumption writes:

$$c\left(\tau,\tau;n\right) = \frac{\left[1-\left(1-\frac{\lambda}{n}\right)\left(1-\frac{\delta}{n}\right)\right]w\left(\tau;n\right)}{p\left(\tau;n\right)}, \tag{55}$$

from which the second equation in (7) is deduced. Replacing the latter in (53) gives the second equation in (8). □

Proof of Lemma 4. This proof presents simple computations made to transform (45) in a two dimensional dynamic system. Equation (45) can be rewritten as:

$$\begin{aligned} M &= \frac{\lambda}{n}\left[\frac{p\left(t+1;n\right)^{\sigma}}{p\left(t;n\right)^{\sigma}} \frac{p\left(t;n\right)^{\sigma}\sum_{s=0}^{\infty}\Phi^{s}p\left(t+s;n\right)^{1-\sigma}-p\left(t;n\right)}{\Phi}\right. \\ &\qquad \left. -\frac{\sum_{s=0}^{\infty}\left(1-\frac{\lambda}{n}\right)^{s}p\left(t+s;n\right)-p\left(t;n\right)}{\left(1-\frac{\lambda}{n}\right)}\right], \end{aligned} \tag{56}$$

or, equivalently:

$$M = \left[z\left(t;n\right)-1\right]p\left(t;n\right)\left(1-\frac{p\left(t+1;n\right)^{\sigma}}{p\left(t;n\right)^{\sigma}\left(1-\frac{\delta}{n}\right)^{\sigma}}\right), \tag{57}$$

where

$$z(t;n) = p(t;n)^{\sigma-1} \sum_{s=0}^{\infty} \Phi^s p(t+s;n)^{1-\sigma}. \tag{58}$$

Thus:

$$z(t+1;n) = \frac{p(t+1;n)^{\sigma-1}}{p(t;n)^{\sigma-1}} \frac{z(t;n)-1}{\left(1-\frac{\lambda}{n}\right)\left(1-\frac{\delta}{n}\right)}. \tag{59}$$

Using the changes in variables, (46) and (47), one obtains (48). □

<u>Phase diagrams of Section 3.</u>

The isocline I_y, which represents the locus such that $y(t+1;n) = y(t;n)$, is described by the equation $x = \eta(y)$, where η is given by:

$$\eta(y) = -\frac{M}{y} \frac{\left(1-\frac{\delta}{n}\right)^{\sigma}}{1-\left(1-\frac{\delta}{n}\right)^{\sigma}} + y^{\frac{1-\sigma}{\sigma}}. \tag{60}$$

As $\eta(y)$ has the same sign of $\eta'(y)\,y^2$, we obtain the following configuration. For $\delta > 0$, $\eta'(y) > 0$, for $\delta < 0$ and $\sigma > 1$, $\eta'(y) < 0$ and for $\delta < 0$ and $\sigma < 1$, there exists a unique minimum to $\eta(y)$.

The isocline I_x represents the locus such that $x(t+1;n) = x(t;n)$. It is given by:

$$x = \frac{y^{\frac{1-\sigma}{\sigma}}}{(1-\Phi)}. \tag{61}$$

For $\sigma < 1$, I_x is the graph of an increasing function, which is convex for $0 < \sigma < 1/2$ and concave for $1/2 < \sigma < 1$. For $\sigma > 1$, I_x is the graph of a decreasing function.

Figures 1 and 2 represent the isocline for the various combinations of parameters. Let us now turn to the analysis of the existence and local stability of steady-states. they solve:

$$\begin{cases} y^{\frac{1}{\sigma}} = -M\frac{(1-\Phi)}{\left(1-\frac{\lambda}{n}\right)-\Phi} \\ x = -\frac{M}{y}\frac{1}{\left(1-\frac{\lambda}{n}\right)-\Phi} \end{cases} \tag{62}$$

There is no positive (x, y) if $\delta > 0$, and there is a unique positive (x, y) if $\delta < 0$. This steady state is an unstable focus if $\sigma < 1$ and an unstable node if

$\sigma > 1$. The latter case can be seen from the phase diagram, while the former can be shown by computing the Jacobian matrix:

$$J = \begin{bmatrix} \left(\left(1-\frac{\delta}{n}\right)^{\sigma} + \frac{\frac{1-\sigma}{\sigma}(1-\Phi)\left(1-\left(1-\frac{\delta}{n}\right)^{\sigma}\right)}{\Phi}\right) & \frac{y^2\left(1-\left(1-\frac{\delta}{n}\right)^{\sigma}\right)^2}{\left(1-\frac{\delta}{n}\right)^{\sigma} M} \\ -\frac{1-\sigma}{\sigma}\frac{y^{\frac{1-2\sigma}{\sigma}}}{\Phi} & \frac{1}{\Phi} \end{bmatrix}$$

and its determinant, which worth: $D = 1/\left(1-\lambda/n\right) > 1$ and implies that the modulus of eigenvalues are larger than 1.

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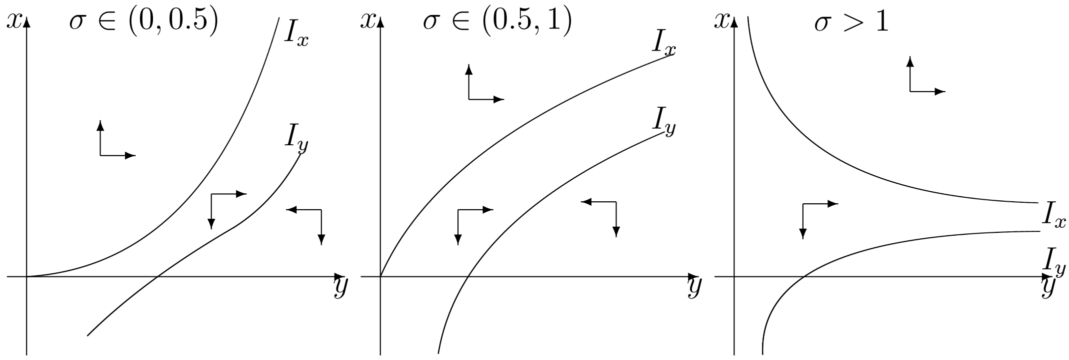
Figures 1



Figures 2