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Saving Rate Dynamics in the Neoclassical Growth Model – Hyperbolic Discounting and Observational Equivalence[1]

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Abstract. The standard neoclassical growth model with Cobb-Douglas production predicts a monotonically declining saving rate, when reasonably calibrated. Ample empirical evidence, however, shows that the transition path of a country's saving rate exhibits a rising or non-monotonic pattern. In important cases, hyperbolic discounting, which is empirically strongly supported, implies transitional dynamics of the saving rate that accords well with empirical evidence. This holds true even in a growth model with Cobb-Douglas production technology. We also identify those cases in which hyperbolic discounting is observationally equivalent to exponential discounting. In those cases, hyperbolic discounting does not affect the saving rate dynamics. Numerical simulations employing a generalized class of hyperbolic discounting functions that we term *regular* discounting functions support the results.

Keywords and Phrases: Saving rate, non-monotonic transition path, hyperbolic discounting, regular discounting, commitment, short planning horizon, neoclassical growth model

JEL Classification Numbers: D91, E21, O40

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1. Introduction

This paper considers the question of whether or not hyperbolic discounting adds enough flexibility to an otherwise standard growth model for the saving rate to exhibit non-monotonic dynamics. It is well known that the standard neoclassical growth model *with Cobb-Douglas technology* and isoelastic preferences – one of the most frequently used frameworks in macroeconomics – exhibits a monotone transition path of the saving rate. For a reasonable calibration, it exhibits a monotonously declining transition path as an economy develops. This property, however, is counterfactual. As discussed in Section 2, ample empirical evidence suggests two regularities: an increase in the saving rate as an economy experiences growth of per capita income; a non-monotone transition path, typically featuring a hump.

The standard growth model's counterfactual prediction of a monotone declining transitional path of the saving rate is unappealing. Often, analyses of the effects of tax shocks are concerned with effects on transitional paths. Or to think of another example, the analysis of development policy typically focuses on transitional dynamics. Therefore, a growth framework should be flexible enough to allow for a non-monotonic saving rate dynamics.

The problem of the standard growth model's counterfactual prediction of transitional dynamics has been addressed in the literature. Gómez (2008), among others, provides a solution by allowing for a more flexible CES technology. In this paper, we provide a different solution in that we allow preferences to exhibit hyperbolic discounting. Both approaches add enough flexibility to the otherwise standard growth model for the saving rate to exhibit non-monotonic dynamics.

In the standard neoclassical growth model, preferences are independent of time. Empirically, however, there is abundant evidence for the pure rate of time preference to decline over time, i.e., for hyperbolic discounting (cf., e.g., Ainslie 1992, and Laibson 1997). Discount rates are time sensitive, exhibiting a "present bias": people tend to put especially high weight on a given gain/loss delayed in the near future as opposed to the same gain/loss delayed in the more distant future. In this paper, we investigate the effects of hyperbolic discounting on transitional paths of the saving rate in three frameworks. The first one is a "standard" framework in which sophisticated households fully commit to their initial intertemporal consumption plans in spite of hyperbolic discounting. In the second framework, along the lines of Caliendo and Aadland (2007) as well as Findley and Caliendo (2011), naïve households, who are not aware of their future impatience, are revising their initial intertemporal consumption plans at every instant in time. In the third framework, we reconsider Barro's (1999) Cournot-Nash equilibrium without commitment. Employing these frameworks, our analysis gives rise to the following results.

First, in most cases, hyperbolic discounting adds enough flexibility to the otherwise standard growth model (with Cobb-Douglas technology) for the saving rate to exhibit non-monotonic dynamics. In some cases, however, hyperbolic discounting is observationally equivalent to exponential discounting, so that the saving rate dynamics is monotone. Second, observational equivalence occurs in two cases: in the framework with naïve households when utility is log-linear and the discounting function belongs to the class of regular discounting functions (see below); in the Cournot-Nash framework when utility is log-linear *and* the rate of interest is constant over time. Third, we introduce the class of regular discounting functions. This class captures cases in which the second order growth rate of the discount rate is a *constant* multiple of the first-order growth rate. Most discounting specifications employed in the prior literature are special cases of the regular discount function, notably exponential discounting (i.e., the discount rate is constant), less-than-exponential discounting, classical hyperbolic discounting (Ainslie 1992), or zero discounting.

This paper is related to several previous studies on saving rate dynamics. Gómez (2008) and Smetters (2003) introduce a CES production technology with elasticities of substitution differing from one. They show that a CES between capital and labor below (above) unity might imply a hump shaped (inverse-hump shaped) transitional path of the saving rate. Gómez (2008) provides a more general analysis than Smetters (2003) in the presence of CES technology. Litina and Palivos (2010) introduce endogenous technical progress. Both Gómez (2008) and Litina and Palivos (2010) show conditions under which there is overshooting (undershooting) behavior of the transition paths of the saving rate. Antràs (2001) shows that the introduction of a minimum consumption level (Stone-Geary preferences) may also imply a hump shaped savings profile. In his model, the intertemporal elasticity of substitution (IES) rises over time, which first weakens the substitution effect and later on, the substitution effect dominates the income effect, thereby generating a hump shaped transitional path. He also provides econometric evidence in support of the non-monotonic transitional path of the saving rate both in OECD countries and in a larger cross-section of countries.

The previous literature demonstrates that the saving rate may exhibit a non-monotonic transition path in a neoclassical growth model with CES technology. In our paper, we maintain Cobb-Douglas technology. However, in contrast to the prior literature, we allow preferences to exhibit hyperbolic discounting. We contribute to the existing literature by showing that the introduction of hyperbolic discounting is an alternative explanation for an increasing or non-monotonic transition path of the saving rate. The main mechanism works via the Euler equation. Hyperbolic discounting adds a *discounting effect* to the substitution and income effects. As the pure rate of time preference declines over time, the difference between the rate of interest and the rate of time preference increases which, ceteris paribus, raises the return on savings. Unless observational equivalence occurs, the discount rate effect gives rise to an increasing saving rate or to non-monotonic dynamics of the saving rate – even with Cobb-Douglas production technology.

Section 2 provides empirical evidence supporting two stylized facts: as an economy grows, its saving rate tends to rise; the transitional path of a country's saving rate behaves non-monotonically over time. In addition, Section 2 briefly discusses the theoretical argument behind the non-monotonic dynamics of the saving rate in the presence of a discounting effect. Section 3 presents the benchmark model with hyperbolic discounting under full commitment. Transitional paths of the saving-rate are shown to be non-monotonic, even in the case of logarithmic utility. In addition, we introduce a generalized class of hyperbolic discounting functions that we term *regular* discounting functions. In Section 4, we focus on a model with naïve consumers having a short planning horizon – in the absence of commitment. We also briefly review Barro's (1999) Cournot-Nash equilibrium. In both frameworks the saving rate may exhibit non-monotonic transition paths, but we also identify cases in which hyperbolic discounting is observationally equivalent to exponential discounting. Section 5 concludes, and the Appendix contains a number of derivations and proofs of propositions.

2. Empirical evidence of the behavior of the saving rate, and the theoretical argument

2.1 Empirical evidence

Data on gross national saving rates suggest two regularities: as a country develops, its saving rate tends to increase, at least over some range; and, over time, saving rates may behave non-monotonically (hump-shaped). Neither of these regularities can be explained by a (reasonably calibrated) *standard* neoclassical growth model with Cobb-Douglas production, as shown by Barro and Sala-i-Martin (2004, p.135 ff.).

Stylized Fact 1. *As a country develops, its saving rate tends to increase.*

Maddison (1992) provides evidence for 11 countries whose savings account for about half of world savings. He finds that over the last hundred-twenty years, the saving rates of all but one country (U.S.A.) increased substantially over time. Table 1, which is based on Barro and Sala-i-Martin (2004), provides empirical evidence for national saving rates.

Table 1. Gross national saving rates (percent)

Period	Australia	Canada	France	India	Japan	Korea	U.K.	U.S.A.
1870-89	11.2	9.1	12.8	-	-	-	13.9	19.1
1890-09	12.2	11.5	14.9	-	12.0	-	13.1	18.4
1910-29	13.6	16.0	-	6.4	17.1	2.4	9.6	18.9
1930-49	13.0	15.6	-	7.7	19.8	-	4.8	14.1
1950-69	24.0	22.3	22.8	12.2	32.1	5.9	17.7	19.6
1970-89	22.9	22.1	23.4	19.4	33.7	26.2	19.4	18.5

Source: Barro, Sala-i-Martin (2004, p.15)

In all countries, except for the United States, present saving rates are significantly above their levels in late nineteenth century. Similar evidence is seen in East Asia for the last half century. With the exception of the Philippines, gross national saving rates have increased in the Asian "Tiger-countries" over the last fifty years, as shown in Table 2.

Table 2. Gross national saving rates in East Asian countries (percent)

Period	Hong Kong	Taipei	Singapore	Malaysia	Thailand	Indonesia	Philippines
1960's	31	14	8	25	22	7	17
1970's	32	27	35	29	26	19	21
1980's	34	31	42	33	26	33	20
1993	37	28	50	41	35	34	14

Source: Leipziger and Thomas (1997)

Along the same lines, Loayza et al. (2000) show for 98 countries that private saving rates rise with the level of real per capita income. We now turn to:

Stylized Fact 2. *The transitional path of a country's saving rate behaves non-monotonically. For most countries, the respective transitional path exhibits a marked hump.*

That the gross saving rates are lower in the eighties than earlier is a well documented regularity (cf. Shafer et al. 1992). Schmidt-Hebbel and Servén (1999) as well as Antràs (2001) demonstrate that for most of 24 OECD countries, as well as for the OECD as a whole, the transitional paths of the saving rates exhibit a hump when considering the last half century. Maddison (1992) shows that in many countries, after World War II, the saving rate exhibits overshooting. Similar trends are reported by Bosworth et al. (1991), Christiano (1989), Chari et al. (1996), and Tease et al. (1991).

Below, we show that when preferences exhibit hyperbolic discounting, a neoclassical growth model with Cobb-Douglas technology is, in many cases, consistent with those stylized facts.

2.2 The theoretical argument, in brief

As a country develops, the real rate of interest declines, giving rise to both a substitution and an income effect. As the return on saving declines, ceteris paribus households tend to lower the saving rate over time, $\dot{s} < 0$ (substitution effect). On the other hand, the desire for consumption smoothing requires a household in an economy distant from the steady state to consume more relative to actual income. As the economy develops, however, consumption relative to income declines. As a consequence, this income effect tends to raise the saving rate over time, $\dot{s} > 0$. In general, these two effects may give rise to a complicated dynamics of the saving rate. With Cobb-Douglas production, however, it has been demonstrated by Barro and Sala-i-Martin (2004) that the dynamics of the saving rate is *always* monotonic – a counterfactual prediction, as shown in Section 2.

The consideration of hyperbolic discounting in the standard framework adds a third effect that we term *discounting effect*. Over time, as the pure rate of time preference declines, the difference between the rate of interest and the rate of time preference increases, ceteris paribus. This causes the "return on savings" to increase, which *lowers* the substitution effect and tends to increase the saving rate. Taking the discounting effect into account, in addition to the substitution and income effects, may give rise to an increasing saving rate or to non-monotonic dynamics of the saving rate – even with Cobb-Douglas production technology.

This argument, while reasonable, holds true only in the absence of observational equivalence. In some cases, as analyzed below, a growth model with hyperbolic discounting is observationally equivalent to the corresponding standard growth model with a constant rate of time preference. That is, for every pattern of the hyperbolic discount function, there exists a *constant* rate of time preference that gives rise to exactly the same transitional dynamics of the saving rate (and those of the other variables of the model). In these cases, hyperbolic discounting does not affect the saving rate dynamics – specifically, hyperbolic discounting does not imply a non-monotonic saving rate dynamics.

3. The neoclassical growth model with hyperbolic discounting

We modify the standard neoclassical growth model in that we allow the pure rate of time preference to depend on time. Time is considered a continuous variable in our model.[2] The most prominent example of a time-dependent rate of time preference occurs with hyperbolic discounting. Psychologists and behavioral economists argue that an individual discounts the near future at a greater rate than the distant future (cf. Ainslie 1992 or Laibson 1997). We argue below that, in the presence of commitment technologies, the resulting model is not

[2] For simplicity of notation, we let *subscript* t denote time.

observationally equivalent to the standard neoclassical growth model (see also Gong et al., 2007). In the subsequent section, we extend the analysis to a framework without commitment.

3.1 The benchmark model

3.1.1 Production

Let the aggregate production function be

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \ 0 < \alpha < 1, \tag{1}$$

where Y_t is (date t-) output, L_t labor input, K_t capital input, and A_t an index of labor-augmenting productivity that evolves through exogenous disembodied technical change:

$$A_t = e^{\gamma t}, \ \gamma \geq 0. \tag{2}$$

We consider a closed economy so that national income accounting implies

$$Y_t = C_t + I_t, \tag{3}$$

where C_t is aggregate consumption. The capital stock develops according to

$$\dot{K}_t = Y_t - C_t - \delta K_t, \ \delta > 0, \tag{4}$$

where δ is the rate of depreciation of capital.

We now embed the described technology into a market economy with perfect competition. The representative firm chooses inputs so as to maximize the profit for a given real wage, w_t, and capital rental rate, R_t. Given equilibrium in the factor markets, the rental rate must satisfy $R_t = \alpha Y_t / K_t$, and the following no-arbitrage condition holds: $r_t = R_t - \delta$, where r_t is the rate of return on the market for loans.

The dynamics of the production sector is best described by the ratios of output to capital and consumption to capital. We denote the transformed variables by $z_t \equiv Y_t / K_t$ and $x_t \equiv C_t / K_t$.[3] In the following, dating of variables is suppressed unless when needed for clarity. Let g_y

[3] The transformation of variables allows for expressing the dynamic system as well as the phase diagrams in the Appendix in a simple way: growth rates become linear functions of (x, z). However, we continue to work with untransformed variables in the further frameworks below.

denote the growth rate of some variable y. Then the capital accumulation equation becomes $g_K = z - x - \delta$. Furthermore, the growth rate of output equals $g_Y = \alpha g_K + (1-\alpha)(\gamma + n)$, with $n \geq 0$ being the population growth rate. Combining both growth expressions, the dynamics of the production sector is given by

$$g_z = g_Y - g_K = (\alpha - 1)(z - x - \delta) + (1-\alpha)(\gamma + n). \tag{5}$$

3.1.2 A representative household

The representative household has $L_t = e^{nt}$ members, each inelastically supplying one unit of labor per unit of time. We allow the pure rate of time preference, ρ_t, to depend on time. Function ρ_t has the following properties. At $t = 0$, $\rho_0 = \overline{\rho}$. Following the literature on hyperbolic discounting, we allow ρ_t to decline over time: $\dot{\rho}_t \leq 0$, and $\lim_{t\to\infty} \rho_t = \rho$, where $\rho \geq 0$ represents a lower bound on the instantaneous discount rate. Specifically, $\overline{\rho} \geq \rho_t \geq \rho$.

We define a household's discount factor by $D_t \equiv e^{-\int_0^t \rho_s ds}$, implying that the absolute instantaneous rate of time preference at date *t* is given by $\rho_t = -\frac{\dot{D}_t}{D_t}$.

A household's preferences are described by an instantaneous CRRA utility function with absolute elasticity of marginal utility of consumption equal to θ. Facing given market prices and equipped with perfect foresight the sophisticated household chooses a consumption plan $\{c_t\}_{t=0}^{\infty}$ so as to

$$\max_{\{c_t\}} U_0 = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} L_t D_t \, dt$$

$$\text{s.t. } \dot{K}_t = r_t K_t + w_t L_t - c_t L_t, \; K_0 \text{ given,}$$

$$\lim_{t\to\infty} K_t e^{-\int_0^t r_s ds} \geq 0,$$

where c_t is per capita consumption, and the inequality is the No-Ponzi-Game condition. To ensure boundedness of the utility integral, we impose the following parameter restriction:

$$(1-\theta)\gamma + n - \lim_{t\to\infty} \rho_t = (1-\theta)\gamma + n - \rho < 0. \tag{6}$$

Define the Hamiltonian by

$$H(c_t, K_t, \mu_t, t) = \frac{c_t^{1-\theta}}{1-\theta} L_t D_t + \mu_t (r_t K_t + w_t L_t - c_t L_t) .$$

Households are impatient but not shortsighted. They are aware of the fact that they are more impatient in the near than in the distant future (sophisticated households). In this section, households are considered to be able to fully commit to their optimal consumption plans over time. Below, we discuss commitment and analyze different frameworks without commitment. An interior solution satisfies the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho_t}{\theta} = \frac{\alpha z_t - \delta - \rho_t}{\theta} \tag{7}$$

and the transversality condition $\lim_{t\to\infty} K_t e^{-\int_0^t r_s ds} = 0$.

3.1.3 Dynamics of the economy

Notice that $g_x = g_c + n - g_K$. We can therefore describe the dynamics of the economy by two differential equations in the endogenous variables x and z :

$$\begin{aligned} \dot{x}_t &= \left[\frac{1}{\theta}(\alpha z_t - \delta - \rho_t) - (z_t - x_t - \delta - n) \right] x_t \\ \dot{z}_t &= \left[(\alpha - 1)(z_t - x_t - \delta) + (1-\alpha)(\gamma + n) \right] z_t \end{aligned} , \tag{8}$$

where x_t is a jump variable, and z_t is a predetermined variable.

Figure 1 shows the $\dot{x} = 0$ - and $\dot{z} = 0$ lines in (z, x) space. The decline of the instantaneous discount rate makes the $\dot{x} = 0$ line shift downwards over time. The figure also shows a steady state. A (non-trivial) steady state (SS) of the system is a point (z^*, x^*) , with both coordinates strictly positive, such that $\dot{x}(z^*, x^*) = \dot{z}(z^*, x^*) = 0$, where an asterisk marks steady state values of variables. If parameter restriction (6) holds, there exists a nontrivial steady state with:

$$\left(z^*, x^*\right) = \left(\frac{\delta + \gamma\theta + \rho}{\alpha}, \frac{\delta + \gamma\theta + \rho}{\alpha} - (n + \gamma + \delta)\right). \quad (9)$$

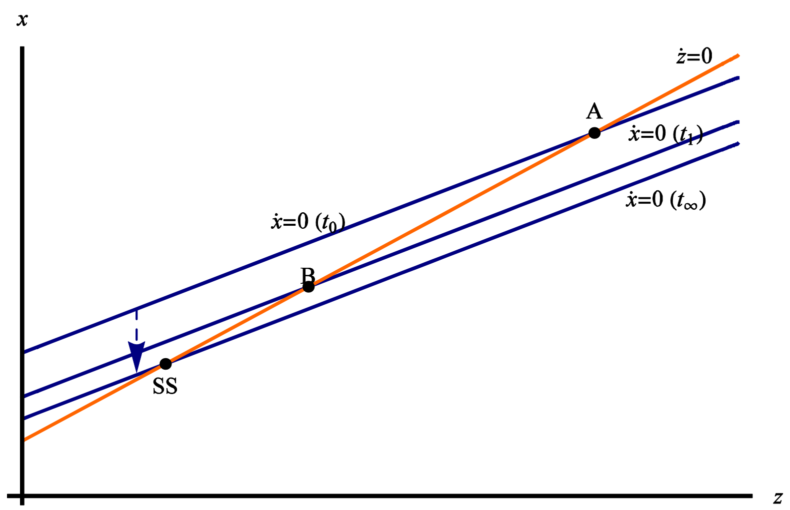

Figure 1. $\dot{x} = 0$ - and $\dot{z} = 0$ lines in (z, x) space, $t_0 < t_1 < t_\infty$

In Figure 1, points A and B do not represent steady state equilibria. As the $\dot{x} = 0$ line shifts over time (and becomes stationary only asymptotically), the dynamical system exhibits an *asymptotic* steady state, SS, which is a saddle point and is saddle point stable by the fact that $\alpha < 1$.[4] Figure 1 also shows that, both x_t and z_t decline along the transition paths, as a *growing* economy develops.

3.2. Behavior of the saving rate under commitment

At date *t*, the (gross) saving rate equals $s_t = 1 - x_t / z_t$. As an economy develops, whether the saving rate increases or decreases (possibly non-monotonically) along the transition path depends on whether z_t declines by more or by less than x_t.[5] Generally, the behavior of the saving rate is complicated along the transition path as a substitution effect opposes an income

[4] The determinant of the Jacobian of the dynamical system at the steady state equals $-\alpha(1-\alpha)x^* z^* / \theta < 0$.

[5] Due to strict concavity of the production function, as K_t increases, $z_t = Y_t / K_t$ *de*creases.

effect. As an economy develops, z_t declines and so does the rate of interest. This substitution effect lowers the return on savings and tends to lower the saving rate. At the same time, as z_t declines, the difference between current and permanent income decreases. That is, relative to income, consumption declines. This income effect tends to raise the saving rate.

For a model with Cobb-Douglas production and without hyperbolic discounting ($\rho_t = \overline{\rho}$), it is well known that, as an economy develops, the saving rate *monotonically* decreases (increases) if $\theta^{-1} > s^* = 1 - x^* / z^*$ (if $\theta^{-1} < s^* = 1 - x^* / z^*$) (Barro, Sala-i-Martin 2004, p.135 ff.). For a reasonable calibration,[6] if $\theta < 17$ – which is considered plausible[7] – the saving rate monotonically declines as an economy develops. This implication, however, is counterfactual in the sense that more developed economies often exhibit a higher saving rate than less developed economies, as shown in Section 2.

Moreover, the discounting effect opposes the substitution effect, giving rise to non-monotonic behavior of the saving rate.

Proposition 1. *Consider the neoclassical growth model with Cobb-Douglas production and exponential discounting ($\dot{\rho}_t = 0$). Then the transition path of the saving rate is monotone.*
In the case of hyperbolic discounting ($\dot{\rho}_t < 0$) with full commitment, however, the transition path of the saving rate can also exhibit non-monotone transition paths. Specifically, along the transition path, the saving rate may overshoot or undershoot towards its steady state level.

Proof. See Appendix A.

As shown in Appendix A, the sign of $\dot{s}_t$ depends on the sign of $\psi_t \equiv (\delta + \rho_t + \gamma\theta) / (\alpha\theta) - (n + \gamma + \delta)$. Specifically,

$$\operatorname{sgn} \dot{s}_t = \operatorname{sgn}(-\psi_t) . \tag{10}$$

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[6] Barro and Sala-i-Martin (2004) suggest $\alpha = 0.3, \gamma = 0.02, \delta = 0.05, \rho = 0.02, n = 0.01$.

[7] Hall (1988) favors a value of $\theta = 5$.

Notice that $\dot{\psi}_t = \dot{\rho}_t / (\alpha\theta)$. Consider the case of exponential discounting: $\dot{\psi}_t = \dot{\rho}_t / (\alpha\theta) = 0$. In this case, the sign of ψ_t does not change over time, and the saving rate is either monotonically decreasing or monotonically increasing over time.

In case of hyperbolic discounting, $\dot{\psi}_t = \dot{\rho}_t / (\alpha\theta) < 0$, several possibilities emerge. First, $\rho \geq 0$ is large enough so that ψ_t is positive for all *t*. In this case, the saving rate monotonically declines, as was the case without hyperbolic discounting.[8]

Second, $\psi_t > 0$ initially (for large ρ_0), and $\psi_t < 0$ as time proceeds and ρ_t declines. In this case, the saving rate initially declines but then increases towards its steady state level (see Appendix A, Figure A1). Intuitively, while the lower interest rate provides the household with an incentive to reduce its saving rate, this incentive is outweighed by a larger incentive to save as the pure time preference rate declines, thus resulting in a higher saving rate ceteris paribus. Thus, households reduce savings by less as compared to the situation with a constant discount rate. Over time, the weight of the substitution effect declines, and the income effect takes over, eventually. At this point, the saving rate starts to increase towards its steady state value.

Third, if the stable arm, in a phase diagram, shifts downward over time, but still has a positive slope in steady state, the saving rate first increases but starts to decrease as of a specific date (see Appendix A, Figure A2).

In this framework, hyperbolic discounting is *never* observationally equivalent to exponential discounting. That is, given a hyperbolic discount function, there does not exist a constant rate of time preference that gives rise to exactly the same transitional dynamics of the saving rate. To see this, we employ standard methods to derive per capita consumption.[9] To simplify the exposition, we assume $\gamma = n = 0$, $L_t = 1$, $\theta = 1$. Let $R(t,0) \equiv \int_0^t r_s \, ds$. Then, per capita consumption becomes:

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[8] Similarly, if ρ_o is so small that ψ_t is negative from the beginning, the saving rate monotonically rises over time. This possibility, however, requires an unrealistically high value of θ.

[9] See Appendix B for a similar derivation.

$$c_t = \frac{\left[k_0 + \int_0^\infty w_\tau \, e^{-R(\tau,0)} \, d\tau \right] e^{R(t,0)}}{\int_0^\infty D_\tau \, d\tau} D_t \, . \tag{11}$$

The denominator of (11) equals some constant in both cases, exponential- and hyperbolic discounting. Observational equivalence requires the factor D_t in case of exponential discounting to equal a constant (ζ) times the discount factor in case of hyperbolic discounting for all $t \geq 0$:

$$e^{-\bar{\rho} t} = \zeta \, e^{-\int_0^t \rho_s \, ds}, \quad t \geq 0 \, . \tag{12}$$

Requirement (12), however, is satisfied if and only if $\rho_t = \bar{\rho}$ for all $t \geq 0$.[10] That is, under hyperbolic discounting, when ρ_t declines over time, the condition for observational equivalence, (12), is never satisfied. As a consequence, the saving rate may exhibit non-monotonic transition paths.

Observe that the results of Proposition 1 presume that the representative agent has access to commitment technologies and fully commits to his decisions. In Section 4 below, we discuss the significance of this assumption, and we consider a framework without commitment.

3.3 Regular discounting

In the following, we specify a rather general class of discounting functions that encompasses many special cases employed in the previous literature. Following the concepts employed by Groth et al. (2010), we call this class the class of *regular* discounting functions.

The *first-order* growth rate of the discount factor is given by $g_D = \dot{D}_t / D_t = -\rho_t < 0$. The second-order growth rate of the discount factor is given by $g_{2,D} = \dot{g}_D / g_D = \dot{\rho}_t / \rho_t$. Following Groth et al. (2010), we call discount functions regular, if

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[10] Take the natural logarithm on both sides: $-\bar{\rho} t + \int_0^t \rho_s ds = \ln \zeta$ = constant. Taking the derivative with respect to time yields: $\bar{\rho} = \rho_t$.

$$g_{2,D} = \beta\, g_D, \ \ \beta \geq 0, \tag{13}$$

where the *constant* β is called the dampening coefficient. Given $D_0 = 1$ and $\rho_0 = \bar{\rho} > 0$, the second order differential equation (13) has the unique solution

$$D_t = (1+\bar{\rho}\beta t)^{-1/\beta}, \ \ \rho_t = \frac{\bar{\rho}}{1+\bar{\rho}\beta t}. \tag{14}$$

The regular discount functions (14) encompass a number of special cases, depending on the specific value of the dampening parameter. First, if $\beta = 0$, $\rho_t = \bar{\rho}$. This is the case of conventional *exponential* discounting. Second, if $\beta > 0$, the discount rate declines in *t*. This is the case of hyperbolic discounting. If $\beta = 1$, $D_t = (1+\bar{\rho}t)^{-1}$. This is the case of *classical hyperbolic* discounting.[11] As the dampening parameter rises, the rate of decline of the discount rate becomes larger, and as the dampening parameter approaches infinity, the discount rate declines to zero instantly. Table 3 summarizes regular discount functions.

Table 3. Regular discount functions

	β	ρ_t	D_t
Regular discounting (general)		$\bar{\rho}/(1+\bar{\rho}\beta t)$	$(1+\bar{\rho}\beta t)^{-1/\beta}$
Exponential discounting	$\beta = 0$	$\bar{\rho}$	$e^{-\bar{\rho}t}$
Classical hyperbolic discounting	$\beta = 1$	$\bar{\rho}/(1+\bar{\rho}t)$	$1/(1+\bar{\rho}t)$
No-discounting	$\beta \to \infty$	0	1

Figure 2 shows time paths of the discount rate for various values of the dampening coefficient. The figure illustrates that regular discount functions capture *the whole spectrum*

[11] In the original, *classical* psychological literature, hyperbolic discount functions like $1/t$ or $(1+\bar{\rho}t)^{-1}$ were used (Ainslie, 1992).

of discount functions between exponential discounting, less-than-exponential (that is, hyperbolic) discounting, and no discounting at all.[12]

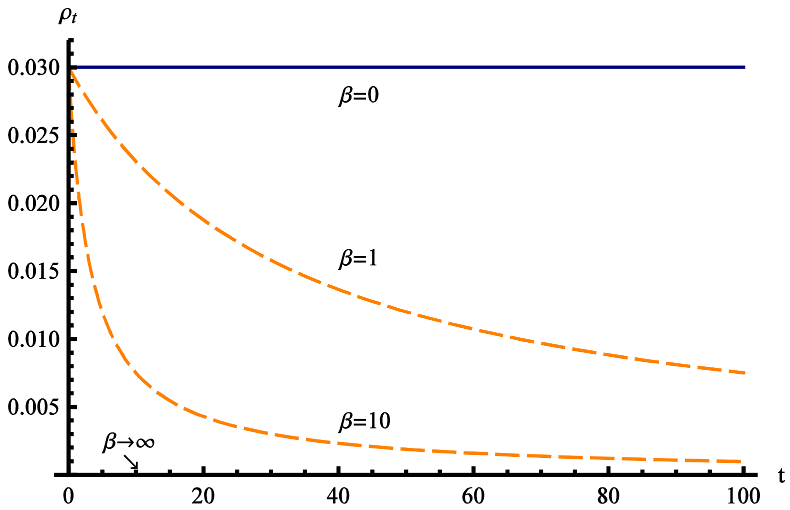

Figure 2. Time paths of the discount rate with $\rho_0 = \bar{\rho} = 0.03$.

With regular discounting, the growth rate of the saving rate becomes:

$$g_s = \alpha\left(z_t - x_t - \delta\right) + (1-\alpha)\gamma - \alpha n - \frac{\alpha z_t - \delta - \bar{\rho}/\left(1+\bar{\rho}\beta t\right)}{\theta}. \tag{15}$$

With this notation at hand, we are now prepared to study numerical simulations for transition paths of the saving rate.[13]

3.4 Numerical simulations of the saving rate for regular discounting functions

To assess the impact of hyperbolic discounting on the transition path of savings, we consider an adverse shock on the predetermined state variable *z*. At time zero, starting from an initial

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[12] If $\beta > 0$, $\lim_{t\to\infty} \rho_t = 0$ for regular discounting functions. For the more general framework discussed above, the limit, denoted by ρ, was allowed to take on a positive value as well.

[13] Regular discounting satisfies Farzin's (2006) condition for Weitzman's "*stationary* equivalence" property to hold (cf. Farzin 2006, p.528). Thus, there exists a permanently sustaining constant consumption (utility) path. This does not, however, imply *observational* equivalence, as discussed for (11) above.

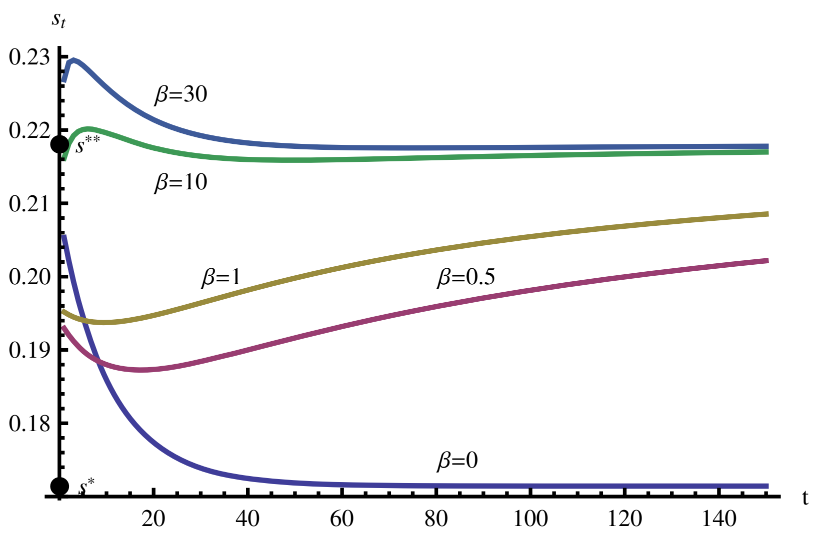
steady state, we increase the value of the predetermined variable, z, by 50%.[14] The resulting time paths show the *non-linearized* transitions of the saving rate (and other variables of interest) from far away from the steady state to the steady state equilibrium. These transition paths are interpreted as showing the development of the saving rate (and other variables) as a country develops, i.e., as its stock of capital increases (z decreases).[15]

Table 4. Baseline values of background parameters

Preference parameters	$\bar{\rho} = 0.03$,	$\theta = 3$	
Production parameters	$\alpha = 0.3$,	$\gamma = 0.02$,	$\delta = 0.05$
Population growth	$n = 0.01$		

Note. The time unit is one year; the dampening coefficient, β, varies across simulations.

Figure 3, presents transition paths of the saving rates for the baseline values of background parameters and for various values of β. The calculations of the transition paths are based on the Relaxation Algorithm (Trimborn et al., 2008).



[14] With $\alpha = 0.3$, this shock corresponds to a *decline* in the capital stock by 71%. As the capital stock *in*creases, z *de*creases because of the concavity of the production function.

[15] We employ the Mathematica implementation of the Relaxation Algorithm (Trimborn et al., 2008) to produce the numerical results documented in this paper. The code is available from the authors upon request. Notice that the shock is introduced on the state variable, not on a specific parameter. All parameters take on the same values before and after the shock. That is, the shock is introduced only to allow the Relaxation Algorithm to calculate transition paths.

Figure 3. Time paths of the saving rates with different values of the dampening factor.

The steady state value of the saving rate depends on the value of the dampening factor. If $\beta = 0$, $\lim_{t\to\infty} \rho_t = \bar{\rho}$, and the associated steady state saving rate is denoted by $s^* \ (= 0.17)$. If, however, $\beta > 0$, $\lim_{t\to\infty} \rho_t = 0$, and the associated (higher) level of the steady state saving rate is denoted by $s^{**} \ (= 0.22)$. The baseline value of the elasticity of marginal utility is $\theta = 3$. Hence, with exponential discounting ($\beta = \dot{\rho}_t = 0$), the saving rate monotonically decreases along the transition path towards s^*.

With hyperbolic discounting, however, transitional paths exhibit a non-monotonic pattern. Whether the saving rate over- or undershoots depends on the value of the dampening coefficient. Intuitively, if the dampening coefficient is "low," the rate of interest declines at a higher rate than the discount rate. In this case, the optimal consumption growth rate, as given by (7), declines over time. That is, the household shifts consumption from the future to the present, thereby lowering the saving rate initially. Over time however, as the consumption growth rate declines, the saving rate increases towards its steady state level. In contrast, if the dampening coefficient is "high," the rate of discount declines at a higher rate than the interest rate, i.e., the optimal consumption growth rate, rises over time, and the household shifts consumption from the present to the future. So, initially, the saving rate is increased. Over time however, as the consumption growth rate rises, the saving rate decreases towards its steady state level.

The key feature of this model with full commitment – allowing for nonmonotonic saving behavior – consists in the fact that the (effective) discount rate, ρ_t, declines over time. As discussed below, this feature may also occur in frameworks without commitment.

Two more points are worth emphasizing. First, hyperbolic (regular) discounting generally implies a non-monotonic transition path of the saving rate, as observed empirically. Second, with a "low" value of the dampening coefficient, the saving rate increases over (most of the) time as per capita income grows. This is true for realistic parameter values, as given in Table 2 (specifically $\theta = 3$). However, with a "high" value of the dampening coefficient, the saving rate exhibits a hump, as is consistent with Stylized Fact 2.

Corollary 1. *Consider the neoclassical growth model with Cobb-Douglas production and logarithmic utility with hyperbolic discounting ($\dot{\rho}_t < 0$) under full commitment. The saving rate regularly exhibits non-monotonic transition paths.*

It is important to emphasize that the result of non-monotonic transition paths of the saving rate is not due to the fact that $\theta = 3$. As long as the representative agent commits to her decisions, non-monotonicity of the transitional paths of savings is also present for a log-linear utility function. Figure 3A displays the transitional paths of the saving rate – parallel to those of Figure 3 – but with $\theta = 1$ rather than $\theta = 3$.

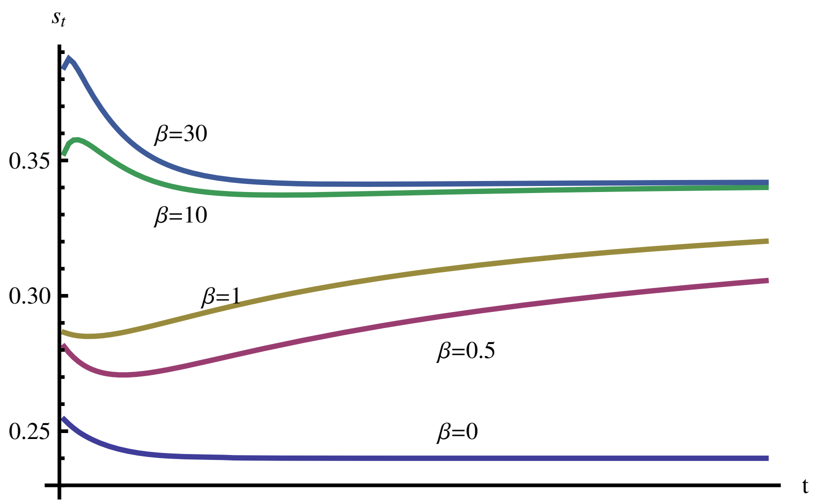


Figure 3A. Time paths of the saving rate with different values of the dampening factor under log-linear utility.

4. Behavior of the saving rate under hyperbolic discounting without commitment

In the previous section, households are assumed to commit to their decisions. Partial or full commitment is a more convincing case than one is inclined to think at first consideration. There is an abundance of commitment technologies. These include all illiquid assets. "All of the illiquid assets … have the same property as the goose that laid golden eggs. The asset promises to generate substantial benefits in the long run, but these benefits are difficult, if not impossible, to realize immediately." (Laibson 1997, p.445) Specific illiquid assets include retirement plans, or assets that are associated with a steady stream of benefits but are hard to sell, like houses. As emphasized by Laibson (1997, p.445), in the FED publication *Balance Sheets for the U.S. Economy 1945-1995*, two thirds of domestic household assets are considered illiquid – not even taking into account social security wealth or human capital.

Notwithstanding these arguments, we consider the case of no commitment in the following. The proceeding sections are concerned, respectively, with sequential planning of households with short planning horizons, and with Cournot-Nash equilibria, both in the absence of commitment.

4.1 No commitment and short-term planning

In the previous section, we argue that in a model with hyperbolic discounting and full commitment, it is the time-dependency of the discount rate that yields non-monotonic saving paths. In the prior literature, it is argued that under hyperbolic discounting – in the absence of commitment – one ends up with a constant effective discount rate so that the model is observationally equivalent to the respective model without hyperbolic discounting (Findley and Caliendo 2011). Here, we argue that this, while quite possible, is for the most part not the case.

In this subsection, we consider a framework with a naïve household who is not aware of its time-inconsistent preferences, that is, of its future impatience. The household, endowed with a short planning horizon, h, re-optimizes at *all* $t > t_0$, thereby altering its original (time-$t = t_0$) intertemporal consumption plan. Will we still encounter non-monotonic transition paths of the saving rate? The answer depends on whether or not the discounting function is regular and utility is log-linear. As shown by Findley and Caliendo (2011), short-term planning perfectly offsets hyperbolic discounting in case of log-linear utility.

In the following, the exponential discount function is characterized by[16]

[16] In the previous section, we denoted the discount factor by $D_\tau = e^{-\int_0^\tau \rho_s ds}$. Here we generalize this notation to allow for any initial t : $D(\tau, t)$.

$$\rho_s = \rho \,,\; D(\tau,t) = e^{-\int_t^\tau \rho ds} = e^{-\rho(\tau - t)} \;,\; \int_t^{t+h} D(\tau,t)\, d\tau = \frac{1-e^{-h\rho}}{\rho} \,, \tag{16}$$

while regular hyperbolic discount functions are characterized by

$$\begin{aligned} &\rho_s = \frac{\bar{\rho}}{1+\bar{\rho}\,\beta\,(s-t)},\; D(\tau,t) = e^{-\int_t^\tau \rho_s ds} = \left[1+\beta\bar{\rho}(\tau-t)\right]^{-1/\beta}, \\ &\int_t^{t+h} D(\tau,t)\, d\tau = \frac{-1+(1+h\,\beta\,\bar{\rho})^{(-1+\beta)/\beta}}{(-1+\beta)\bar{\rho}}. \end{aligned} \tag{17}$$

$D(\tau,t)$ represents the discount factor at date τ as seen from date t.

Every household only plans for some period of finite length, *h*, and we allow a household to re-optimize at every date *t*. The procedure applied follows Caliendo and Aadland (2007), and Findley and Caliendo (2011). In order to simplify notation, we consider $n = 0 = \gamma$.

At every point t, a household solves a short-horizon (fixed-endpoint) control problem:

$$\begin{aligned} &\max_{\{c_\tau\}} \int_{t_0}^{t_0+h} \frac{c_\tau^{1-\theta}-1}{1-\theta} D(\tau,t_0)\, d\tau,\;\; \tau \in [t_0, t_0+h] \\ &\text{s.t. } \dot{k}_\tau = r_\tau k_\tau + w_\tau - c_\tau,\;\; k_{t_0} \text{ given},\;\; \tau \in [t_0, t_0+h] \\ &\quad k_{t_0+h} = 0. \end{aligned} \tag{18}$$

In (18), $D(\tau,t_0)$ is a general discounting function, where discounting is pursued from the viewpoint of t_0. The solution to (18) is *planned* consumption from the perspective of t_0. The (fixed-endpoint) terminal condition $k_{t_0+h} = 0$ indicates that the household is concerned only with the "next" *h* periods. It does *not* imply that wealth (capital) is actually equal to zero at $t_0 + h$, as the household's planning horizon is continuously sliding forward. As the planning horizon is sliding forward, previous consumption plans are invalidated, and the household re-optimizes and updates its consumption plan at every *t*. That is, although a household plans to exhaust its resources within *h* periods, it never actually exhausts its resources in finite time, as it keeps re-planning its consumption plans.[17]

As demonstrated in the Appendix, application of the Maximum principle to (18) yields

$$c_t = \frac{k_t + \int_t^{t+h} w_\tau \, e^{-R(\tau,t)} \, d\tau}{\int_t^{t+h} D(\tau,t)^{\frac{1}{\theta}} \, e^{\frac{1-\theta}{\theta} R(\tau,t)} \, d\tau}, \qquad (19)$$
$$\dot{k}_t = r_t k_t + w_t - c_t, \; k_{t_0} \text{ given}.$$

Equation (19) presents optimal consumption of a short-sighted household with hyperbolic discounting, as captured by the denominator. In (19), c_t is derived as the envelope of infinitely many initial values from a continuum of planned time paths (cf. Appendix).

The important insight from (19) consists in the fact that the denominator accounts for the propensity to consume out of total wealth. The denominator, however, does not necessarily depend on the shape of the discounting function. Consider $\theta = 1$. Then, the propensity to consume depends on the integral, that is, on the area below the discounting function. In other words, if the integrals of different discounting functions – for example an exponential- and a hyperbolic discounting function – yield the same values then these discounting functions are observationally equivalent.

Proposition 2. *Consider a naïve household with a short planning horizon, $h \in \mathbb{R}_{++}$, and with logarithmic utility, $\theta = 1$. Then, every framework with regular hyperbolic discounting, under $D(\tau, t \mid \bar{\rho})$, is observationally equivalent to a corresponding framework with exponential discounting, under $E(\tau, t \mid \rho)$,* for some $\rho \in [0, \rho_{\max}]$, *where $\rho < \bar{\rho}$. As a consequence, the saving rate follows a monotone transition path in spite of hyperbolic discounting.*

[17] Suppose the discount rate is constant in time. Then, only in case *h* approaches infinity, the transversality condition induces the household to pick the *optimal* consumption path. As *h* approaches infinity, the short-term planning model approaches the neoclassical standard model. Once *h* is finite, however, the short-term consumption plans differ from the standard optimal neoclassical consumption plan.

Proof. Let $E(\tau,t\,|\,\rho)$ denote the exponential discount function, and let $D(\tau,t\,|\,\bar{\rho})$ be the regular hyperbolic discount function with $\beta>0$. The upper bound $\rho_{\max}$ is implicitly defined by $\int_t^{t+h} D(\tau,t\,|\,1)\,d\tau=\int_t^{t+h} E(\tau,t\,|\,\rho_{\max})\,d\tau$. Figure 4 provides intuition for the proof that is analytically given in the Appendix. ||

The idea of the proof is depicted in Figure 4. For $\rho=0$, $\int_t^{t+h} D(\tau,t)\,d\tau=\int_t^{t+h} E(\tau,t)\,d\tau=h$. For $\rho\in(0,1]$ and $\beta>0$, $\int_t^{t+h} D(\tau,t\,|\,\rho)\,d\tau>\int_t^{t+h} E(\tau,t\,|\,\rho)\,d\tau$. In Figure 4, it can easily be seen that for all $\bar{\rho}\in[0,1]$, there exists $\rho\in[0,\rho_{\max}]$ for which the condition for observational equivalence holds. Specifically, for $\theta=1$,

$$\int_t^{t+h} D(\tau,t\,|\,\bar{\rho})\,d\tau=\int_t^{t+h} E(\tau,t\,|\,\rho)\,d\tau,\ t\geq 0. \tag{20}$$

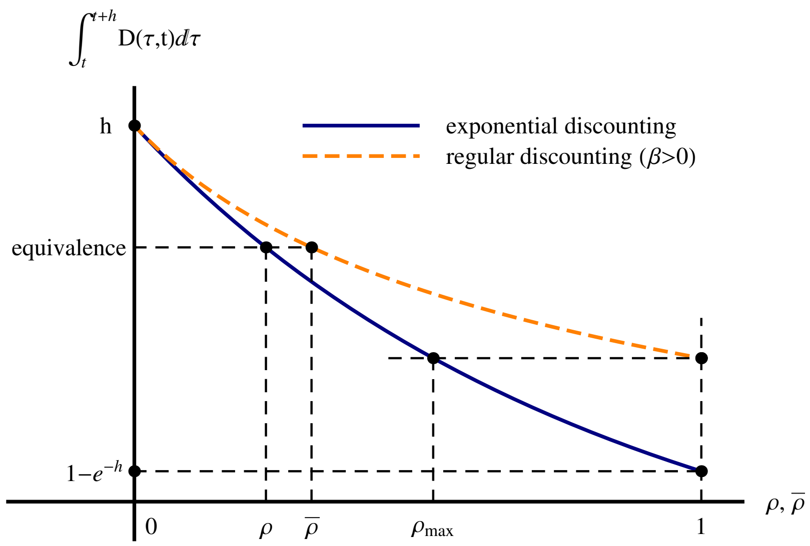


Figure 4. Observational equivalence of exponential and regular hyperbolic discounting under log-linear utility.

It follows from Proposition 2 that under hyperbolic discounting, there always exists a *constant* discount rate – independent of calendar time – for which consumption and saving rate dynamics are equal to the ones in a model with exponential discounting. As a consequence, under the conditions of Proposition 2, a model with regular hyperbolic discounting does not exhibit a non-monotonic saving rate dynamics. This result is in stark contrast to Proposition 1 and Corollary 1, where it is shown that under full commitment and an infinite planning horizon, the saving rate dynamics may be non-monotonic – even with logarithmic utility.

A special case of Proposition 2 refers to classical hyperbolic discounting ($\beta = 1$) and was previously discussed already by Findley and Caliendo (2011).

Corollary 2. (Classical hyperbolic discounting)
Consider a naïve household with a short planning horizon, $h \in \mathbb{R}_{++}$*, and with logarithmic utility,* $\theta = 1$*. Then, classical hyperbolic discounting is observationally equivalent to exponential discounting. As a consequence, the saving rate follows a monotone transition path.*

As argued by Findley and Caliendo (2011), under the conditions of Corollary 2, short period planning perfectly offsets the effects of classical hyperbolic discounting.

Proposition 2 does not capture that case of non-regular hyperbolic discount functions. Remember, following Groth et al. (2010), we call a hyperbolic discount function *regular*, if the second order growth rate of the discount rate is a *constant* multiple of the first-order growth rate. In contrast, we consider a hyperbolic discount function to be *non-regular*, if the second order growth rate of the discount function is a *time-dependent* multiple of the first-order growth rate. That is,

$$\frac{\dot{\rho}_t}{\rho_t} = -\beta_t \, \rho_t \,, \;\; \beta_t \neq \beta_{t'} \text{ for some } t, t' > 0 \,. \tag{21}$$

Corollary 3. *Consider a naïve household with a short planning horizon,* $h \in \mathbb{R}_{++}$*, and with logarithmic utility,* $\theta = 1$*. Then, non-regular hyperbolic discounting is not observationally equivalent to exponential discounting. As a consequence, the saving rate may follow a non-monotonic transition path.*

With a general path of β_t, (20) is not satisfied, as $\int_t^{t+h} D(\tau, t \,|\, \overline{\rho})\, d\tau$ depends on calendar time, while $\int_t^{t+h} E(\tau, t \,|\, \rho)\, d\tau$ does not. Therefore, with non-regular hyperbolic discounting, according to (19), the propensity to consume out of total wealth is time-varying, and so is the saving rate.

It is not that clear, though, whether or not the case of non-regular hyperbolic discounting – in a short planning horizon framework – is a natural case. In a model with cohorts, however, one may argue that β_t differs among, say, younger and older households.

Proposition 3. *Consider a naïve household with a short planning horizon,* $h \in \mathbb{R}_{++}$ *and with* $\theta \neq 1$. *If the rate of interest is not constant over time, no regular or non-regular hyperbolic discounting function allows for observational equivalence with exponential discounting. As a consequence, the saving rate may*[18] *follow a non-monotonic transition path.*

Proof. With a constant rate of interest, considering (19), an equivalence condition similar to (20) can be formulated, namely:

$$\int_t^{t+h} D(\tau,t)^{1/\theta}\, d\tau = \int_t^{t+h} E(\tau,t)^{1/\theta}\, d\tau,\ t \geq 0 .$$

This condition can be satisfied for regular discount functions. Once the rate of interest becomes time-dependent, however, the corresponding equivalence,

$$\int_t^{t+h} D(\tau,t)^{\frac{1}{\theta}}\, e^{\frac{1-\theta}{\theta}R(\tau,t)}\, d\tau = \int_t^{t+h} E(\tau,t)^{\frac{1}{\theta}}\, e^{\frac{1-\theta}{\theta}R(\tau,t)}\, d\tau \ , \tag{22}$$

is not generally satisfied for all $t \geq 0$. Both the left hand side and the right hand side integrals become time-dependent. As, over the planning horizon, different profiles $D(\tau,t)^{1/\theta}$ and $E(\tau,t)^{1/\theta}$ are multiplied by the same time-dependent factor $e^{\frac{1-\theta}{\theta}R(\tau,t)}$, (22) is generally not satisfied for all t. ||

[18] For specific parameter constellations, the saving rate may still follow a monotone transition path. Therefore we are careful to state "…the saving rate *may* follow…".

To summarize, with full commitment, hyperbolic discounting always leads to the possibility of a non-monotonic saving rate dynamics. This result does not generally carry over to a model of naïve consumers with a short planning horizon. In the latter case, regular discounting with logarithmic utility or with a constant rate of interest rules out the case of a non-monotonic saving rate dynamics – due to observational equivalence.

4.2 No commitment and Nash equilibrium

In this section, we focus on a Nash equilibrium involving a sophisticated representative consumer. Each date-t-self decides how much to consume and how much to save so that neither the present nor any future self will have an incentive to deviate from the equilibrium path. We employ the perturbation method developed in Barro (1999). We consider the same setup and utility integral as with full commitment, above. For the sake of simplicity, we assume a constant population, $n=0$, $L_t=1$. In order to simplify notation, we employ three discount factors: $\mathrm{P}(t,\tau)\equiv\int_\tau^t \rho(s)\,ds$; $R(t,\tau)\equiv\int_\tau^t r(s)\,ds$; $\Lambda(t,\tau)\equiv\int_\tau^t \lambda(s)\,ds$. For regular discounting, $\mathrm{P}(t,\tau)=\beta^{-1}\ln\left[1+\beta\,\bar{\rho}\,(t-\tau)\right]$, with $\lim_{\beta\to 0}\mathrm{P}(t,\tau)=\bar{\rho}(t-\tau)$, and $D(t,\tau)=e^{-\mathrm{P}(t,\tau)}$.

$$U_\tau=\int_\tau^\infty \frac{c_t^{1-\theta}-1}{1-\theta}e^{-\mathrm{P}(t,\tau)}dt=\int_\tau^{\tau+\varepsilon}\frac{c_t^{1-\theta}-1}{1-\theta}e^{-\mathrm{P}(t,\tau)}dt+\int_{\tau+\varepsilon}^\infty \frac{c_t^{1-\theta}-1}{1-\theta}e^{-\mathrm{P}(t,\tau)}dt\,. \tag{23}$$

For a small ε, we approximate[19]

$$U_\tau\approx\varepsilon\frac{c_\tau^{1-\theta}-1}{1-\theta}+\int_{\tau+\varepsilon}^\infty \frac{c_t^{1-\theta}-1}{1-\theta}e^{-\mathrm{P}(t,\tau)}dt\,. \tag{24}$$

Next, we consider the growth rate of (per capita) consumption:

[19] Between τ and $\tau+\varepsilon$, we consider consumption constant, and the discount factor equal to one. Below, we are interested in the limit, as ε approaches zero.

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \lambda_t}{\theta}, \tag{25}$$

where we call λ_t the *effective* discount rate (to be calculated). Due to hyperbolic discounting, λ_t is generally different from the discount rate ρ_t. If λ_t varies over time, the transition path of the saving rate may exhibit a non-monotonic pattern. In contrast, if it turns out that λ_t is constant over time, then the transitional path of the saving rate follows a monotone pattern, and the model with hyperbolic discounting becomes observationally equivalent to a model without hyperbolic discounting. In the proceeding analysis, we follow Barro (1999) to derive λ_t in a Cournot-Nash equilibrium.

Taking into account (25), we re-write (24) as:

$$U_\tau \approx \varepsilon \frac{c_\tau^{1-\theta} - 1}{1-\theta} + \int_{\tau+\varepsilon}^{\infty} \frac{c_{\tau+\varepsilon}^{1-\theta} e^{\frac{1-\theta}{\theta}R(t,\tau+\varepsilon) - \frac{1-\theta}{\theta}\Lambda(t,\tau+\varepsilon)} - 1}{1-\theta} e^{-\mathrm{P}(t,\tau)} dt \ . \tag{26}$$

At any τ, for a given path of λ_t, consumption is chosen so as to maximize (26). The resulting optimality condition becomes

$$\int_\tau^\infty e^{\frac{1-\theta}{\theta}R(t,\tau) - \frac{1}{\theta}\Lambda(t,\tau)} \left[e^{\Lambda(t,\tau) - \mathrm{P}(t,\tau)} - 1 \right] dt = 0 \ , \tag{27}$$

where the derivation is given in the Appendix. We employ (27) to derive a more instructive (but still implicit) path of λ_t from the optimality condition (27). A quick look at the optimality condition shows that without hyperbolic discounting – $\mathrm{P}(t,\tau) = \rho(t-\tau)$ – for (27) to hold, the term in square brackets must be equal to zero, that is, $\lambda_t = \lambda = \rho$. This is the conventional case of the neoclassical standard model with exponential discounting.

Differentiating (27) with respect to τ (see Appendix) yields:

$$\lambda_\tau = \frac{\int_\tau^\infty \omega(t,\tau)\, \mathrm{P}'(t,\tau)\, dt}{\int_\tau^\infty \omega(t,\tau)\, dt}, \qquad \omega(t,\tau) \equiv e^{\frac{1-\theta}{\theta}[R(t,\tau) - \Lambda(t,\tau)] - \mathrm{P}(t,\tau)} \ . \tag{28}$$

The effective discount rate turns out to be a weighted average of all future discount rates, $\mathrm{P}'(t,\tau)$. As demonstrated in Barro (1999), the weights $\omega(t,\tau)$ reflect the sensitivity of c_τ with respect to a marginal increase in $k_{\tau+\varepsilon}$. This result leads directly to

Proposition 4. *Consider the neoclassical growth model with Cobb-Douglas production and hyperbolic discounting. Under no commitment, for the Cournot-Nash equilibrium,*
(i) observational equivalence occurs if and only if $\theta = 1$ *or* $r_t = r$*;*
(ii) the transition path of the saving rate may exhibit a non-monotonic pattern if $\theta \neq 1$ *and* $r_t \neq r$.

Proof. Statement (i) of Proposition 4 states the conditions for observational equivalence. In this framework, by observational equivalence we mean a situation with a constant effective discount rate: $\lambda_t = \lambda$. In this case, the standard neoclassical growth model – with a constant discount rate equal to ρ – can perfectly mimic the model with hyperbolic discounting, with a constant λ (probably different from ρ). Under conditions (i), λ_τ is independent of time, as $\mathrm{P}(t,\tau)$ only depends on the difference $(t-\tau)$, but not on calendar time. Similarly, if $\theta \neq 1$ and $r_\tau = r$, all discount factors in (28) depend on the difference $(t-\tau)$, but no discount factor depends on calendar time. Therefore, under conditions (i), λ_τ is independent of time, and observational equivalence occurs.
Statement (ii) presents necessary requirements for the saving rate to exhibit non-monotonic transition paths. The weights $\omega(t,\tau)$ depend on calendar time (not only on the difference $(t-\tau)$) if and only if both conditions in (ii) are met, in which case, λ_τ is time-dependent. ||

Proposition 4 suggests several results. In the absence of commitment technologies, hyperbolic discounting does not necessarily lead to non-monotone transition paths of the saving rate. For example, if either the rate of interest is constant or the IES equals one, the Cournot-Nash effective discount rate is time-invariant. In this case, the transition path of the saving rate is monotone, in spite of hyperbolic discounting.

For the saving rate to exhibit a non-monotonic transition pattern, in addition to hyperbolic discounting, an IES different from one *and* a time-varying rate of interest are needed. Suppose $\theta > 1$ (as is empirically supported, cf. Footnote 7), then $\omega(t,\tau)$ declines with r_t. As $\mathrm{P}'(t,\tau)$ also declines over time, the effective rate of time preference, λ_t, declines over time. The fact that households effectively become more patient over time (discounting effect) gives rise to a non-monotonic transition path of the saving rate.

To summarize, with full commitment, the effective discount rate is time dependent, and so the transition path of the saving rate may be non-monotonic. With no commitment, a Cournot-Nash equilibrium implies a time-dependent effective discount rate only if utility is not log-linear *and* the rate of interest is not stationary.

5. Conclusions

The standard neoclassical growth model with Cobb-Douglas technology exhibits – for a reasonable calibration – a monotonously declining transition path of the saving rate, as an economy develops. This property is counterfactual and therefore unappealing for the analysis of policy shocks on transitional dynamics of an economy.

In this paper, we consider the question whether or not hyperbolic discounting adds enough flexibility to the otherwise standard growth model for the saving rate to exhibit non-monotonic dynamics. The answer depends on the specific framework used as well as on whether or not commitment technologies are available. The answer is “yes” for the standard framework under full commitment. For the other two investigated frameworks – naïve consumers with short planning horizons, and the Cournot-Nash equilibrium – the answer is “yes”, unless utility is log-linear *and* the rate of interest is constant. In the latter case, hyperbolic discounting is observationally equivalent to exponential discounting and does not affect the transitional dynamics of the saving rate.

We also present a functionally specified generalized discounting function (regular discounting) that nests many cases employed in the prior literature as special cases. By varying a single parameter, the uncountable collection of resulting discounting functions includes the cases of no discounting, exponential discounting, and classical hyperbolic discounting.

The prior literature shows that the saving rate exhibits a non-monotonic dynamics in a neoclassical growth model with CES technology. In our paper, we show that the introduction of hyperbolic discounting – by adding a discounting effect to the substitution- and income effects – is also able to explain a non-monotonic transition path of the saving rate, even in a framework with Cobb-Douglas production technology.

Several questions are open for future research. The propositions provide necessary, not sufficient conditions for the saving rate to exhibit non-monotonic dynamics. As seen in the figures depicting the numerical simulations, there exist parameter constellations for which transitional paths are monotone, in spite of the absence of observational equivalence. One research task then is the derivation of necessary and sufficient conditions for the saving rate to exhibit non-monotonic transition paths. Another open research question refers to partial commitment. If we allow for the more appealing case of partial rather than full commitment in the standard framework of Section 3, will the saving rate still exhibit non-monotonic transitional behavior? Notwithstanding those open questions, we still hope to have shed some light on the impact of hyperbolic discounting on saving rate dynamics in the neoclassical growth model.

6. Appendix

A. Proposition 1.

In the following, time indexes are suppressed, unless needed for clarity. Define $\hat{k} = K/(AL)$, $\hat{c} = (cL)/(AL) = C/(AL)$, $\hat{y} = Y/(AL)$. Clearly, $x/z = C/Y = \hat{c}/\hat{y}$. Considering the production function, $\hat{y} = \hat{k}^{\alpha}$. Finally, $s = 1 - x/z = 1 - \hat{c}/\hat{y}$. We express the dynamic system in the two variables $\left(\hat{k}^{1-\alpha}, x/z\right)$. Development is considered as the case in which $\hat{k}$ (thereby $\hat{k}^{1-\alpha}$) increases over time. Consequently, we are interested in whether the saddle path $\frac{x}{z}\left(\hat{k}^{1-\alpha}\right)$ increases or decreases in $\hat{k}^{1-\alpha}$. In the former (latter) case, the saving rate decreases (increases) as an economy develops. In order to consider this relationship, we express the dynamical system in the variables $\left(\hat{k}^{1-\alpha}, x/z\right)$ in a phase diagram. Consider first the developments of $\left(x/z\right)$ and $\hat{k}$ over time:

$$\frac{d\left(x/z\right)}{dt}=\left[\theta^{-1}(\alpha\hat{k}^{\alpha-1}-\delta-\rho_t)+n-\alpha(\hat{k}^{\alpha-1}-(x/z)\hat{k}^{\alpha-1}-\delta)-(1-\alpha)(\gamma+n)\right](x/z)\,, \quad (29)$$

$$\frac{d\hat{k}}{dt}=\hat{k}^{\alpha}-(x/z)\hat{k}^{\alpha}-\hat{k}(n+\gamma+\delta)\,. \quad (30)$$

Solve both differential equations at $d\left(x/z\right)/dt=0$, and $d\hat{k}/dt=0$ for (x/z):

$$\left.\frac{x}{z}\right|_{d(x/z)/dt=0}=(1-1/\theta)+\psi_t\hat{k}^{1-\alpha}\,, \quad (31)$$

$$\left.\frac{x}{z}\right|_{d(\hat{k})/dt=0}=1-\hat{k}^{1-\alpha}(n+\gamma+\delta)\,, \quad (32)$$

where $\psi_t=\left[\frac{\delta+\rho_t+\gamma\theta}{\alpha\theta}-(n+\gamma+\delta)\right]$.

In $(\hat{k}^{1-\alpha}, x/z)$ space, $\left.\frac{x}{z}\right|_{d(\hat{k})/dt=0}$ is downward sloping, as $(n+\gamma+\delta)>0$. At the same time, $\left.\frac{x}{z}\right|_{d(x/z)/dt=0}$ is upward (downward) sloping if $\psi_t>0$ (if $\psi_t<0$). It can easily be verified that in $(\hat{k}^{1-\alpha}, x/z)$ space, the stable arm has a positive (a negative) slope if $\psi_t>0$ (if $\psi_t<0$).[20]

Consider the case of exponential discounting ($\rho_t=\overline{\rho}$). If $\psi_t>0$ (if $\psi_t<0$), as an economy develops, (x/z) monotonically increases (decreases), and the saving rate monotonically decreases (increases). Now, consider the case of hyperbolic discounting that introduces two complexities. First, ψ_t is not constant over time, and its sign may switch from positive to negative. For this reason, (x/z) may rise for some period, followed by a decline, that is, the saving rate declines for some period and then increases towards its steady state (see Figure A1).

[20] See Barro, Sala-i-Martin (2004), p.108.

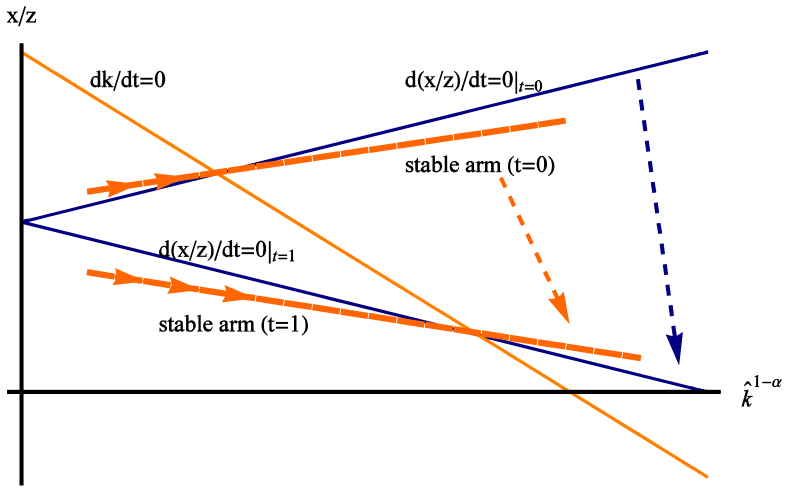


Figure A1. A decline followed by an increase of the saving rate

Second, with $\dot{\psi}_t < 0$, the stable arm changes its location from one date to another. That is, the movement of (x/z) in time not only depends on the slope of the stable arms, but also on their respective shifts over time. This presents a second reason for non-monotone behavior of the saving rate over time. Suppose, $\psi_t > 0$ – i.e., the slope of the stable time-*t*-arm is increasing – but the next date's stable arm (with lower slope) intersects with the time-*t*-arm at a lower value of $\hat{k}^{1-\alpha}$ (see Figure A2). Then, although the slope of the stable arms is positive, (x/z) declines and the saving rates increase over time. This situation is depicted as a move from point A to point B in Figure A2. Furthermore, if $\psi_t > 0$ still holds in the steady state, then (x/z) eventually increases towards its steady state. In such a situation, the saving rate first increases but starts to decrease towards its steady state level as of a specific date. ‖

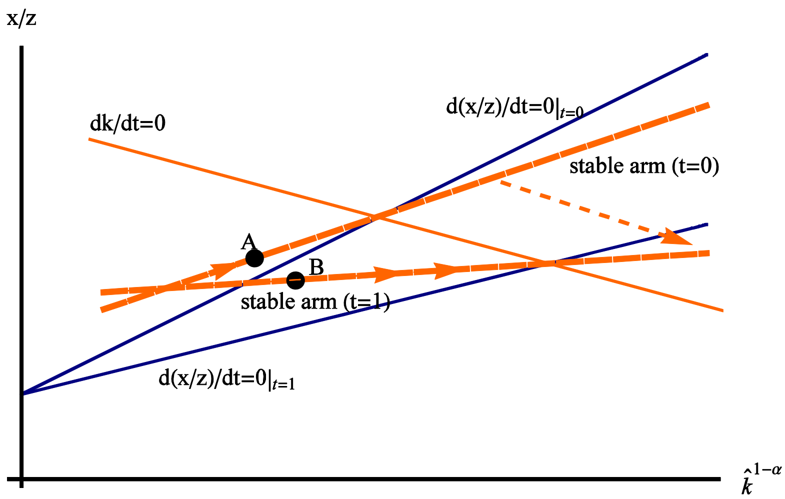


Figure A2. An increase followed by a decline of the saving rate

B. Short-term planning

B.1 Derivation of (19).

The procedure follows Findley and Caliendo (2011). Considering (18), we set up the Hamiltonian:

$$H = \frac{c_t^{1-\theta} - 1}{1-\theta} D(t,t_0) + \mu_t \left(r_t k_t + w_t - c_t \right), \; t \in [t_0, t_0 + h] \; .$$

The first order condition with respect to c_t yields: $c_t = \mu_t^{-1/\theta} D(t,t_0)^{1/\theta}$. As $\dot{\mu}_t / \mu_t = -r_t$,

$$c_t = \mu_{t_0}^{-\frac{1}{\theta}} e^{\frac{1}{\theta} R(t,t_0)} D(t,t_0)^{1/\theta} \; . \quad (33)$$

Considering (33) in the equation of motion of k_t yields

$$\dot{k}_t - r_t k_t = w_t - \mu_{t_0}^{-\frac{1}{\theta}} e^{\frac{1}{\theta} R(t,t_0)} D(t,t_0)^{1/\theta} ,$$

which can be solved as:

$$k_t = k_{t_0} e^{R(t,t_0)} + \int_{t_0}^{t} \left[w_\tau - \mu_{t_0}^{-\frac{1}{\theta}} e^{\frac{1}{\theta} R(\tau,t_0)} D(\tau,t_0)^{1/\theta} \right] e^{R(t,\tau)} d\tau \; .$$

Considering the above equation at $t = t_0 + h$ together with the terminal condition $k_{t_0+h} = 0$, and solving for the costate variable yields:

$$\mu_{t_0}^{-1/\theta} = \frac{k_{t_0} + \int_{t_0}^{t_0+h} w_\tau \, e^{-R(\tau,t_0)} \, d\tau}{\int_{t_0}^{t_0+h} D(\tau,t_0)^{1/\theta} e^{\frac{1-\theta}{\theta} R(\tau,t_0)} d\tau} \; . \quad (34)$$

Considering (34) in (33) yields an expression for planned consumption as seen from t_0:

$$c(t\,|\,t_0) = \frac{k_{t_0} + \int_{t_0}^{t_0+h} w_\tau \, e^{-R(\tau,t_0)} \, d\tau}{\int_{t_0}^{t_0+h} D(\tau,t_0)^{1/\theta} e^{\frac{1-\theta}{\theta}R(\tau,t_0)} d\tau} e^{\frac{1}{\theta}R(t,t_0)} D(t,t_0)^{1/\theta} \quad .$$

The household follows this consumption plan *only* at $t = t_0$. So, we consider the envelope, by setting all $t = t_0$. In the resulting expression, we then replace t_0 by t, which directly yields (19):

$$c_t = \frac{k_t + \int_t^{t+h} w_\tau \, e^{-R(\tau,t)} \, d\tau}{\int_t^{t+h} D(\tau,t)^{\frac{1}{\theta}} \, e^{\frac{1-\theta}{\theta}R(\tau,t)} \, d\tau} \quad .$$

B.2 Proof of Proposition 2.

(i) In the proof, we employ the following lemma.

Lemma. Let $a \in \mathbb{R}_{++}$. Then $e^a > 1 + a(1 + a/2)$.

To show the Lemma, notice that both functions, e^a and $1 + a(1 + a/2)$ are strictly monotonically increasing in a. For $a = 0$, $e^0 = 1 + 0(1 + 0/2) = 1$. To prove the lemma, we need to show that the slope of e^a exceeds the slope of $1 + a(1 + a/2)$ for all $a \in \mathbb{R}_{++}$. That is, $e^a > 1 + a$ for all $a \in \mathbb{R}_{++}$. Consider the difference $e^a - (1 + a)$. As $\partial\left[e^a - (1 + a)\right]/\partial a = e^a - 1$, the difference is strictly increasing in a. That is, e^a increases more strongly in a than $1 + a(1 + a/2)$ does, which proves the lemma.

(ii) With exponential discounting, $\int_t^{t+h} E(\tau,t)\, d\tau = (1 - e^{-\rho h})/\rho$. Regular hyperbolic discounting, according to (17), leads to $\int_t^{t+h} D(\tau\,|\,t)\, d\tau = \frac{-1 + (1 + h\beta\bar{\rho})^{(-1+\beta)/\beta}}{(-1+\beta)\bar{\rho}}$. Both integrals only depend on the planning horizon, but they are independent of calendar time t.

(iii) Both integrals have the same limit, as ρ approaches zero: $\lim_{\rho\to 0} \int_t^{t+h} D(\tau,t)\, d\tau = \lim_{\rho\to 0} \int_t^{t+h} E(\tau,t)\, d\tau = h$. Both integrals decline in ρ.

(iv) For any given $\rho > 0$, with $\beta > 0$, $\int_t^{t+h} D(\tau, t\,|\,\rho)\, d\tau > \int_t^{t+h} E(\tau, t\,|\,\rho)\, d\tau$. To show this inequality, first, we notice that $\int_t^{t+h} D(\tau,t)\, d\tau$ is strictly monotonically increasing in β.

Graphically, this is seen as the area *above* the discounting curves in Figure 2. We take a first-order approximation about $\beta = 0$ to evaluate

$$\int_t^{t+h} D(\tau,t\,|\,\rho)\,d\tau - \int_t^{t+h} E(\tau,t\,|\,\rho)\,d\tau \approx \frac{\beta\, e^{-h\rho}\left[e^{h\rho} - \left(1 + h\rho(1 + h\rho/2)\right)\right]}{\rho} \ .$$

By the Lemma, taking $a = h\rho$, the right hand side is strictly positive for any given $\rho > 0$, showing that $\int_t^{t+h} D(\tau,t\,|\,\rho)\,d\tau > \int_t^{t+h} E(\tau,t\,|\,\rho)\,d\tau$ in fact holds. Therefore the discounting curves can be drawn as depicted in Figure 4.

(v) As a consequence, there exists $\rho < \bar{\rho}$ for which $\int_t^{t+h} D(\tau,t\,|\,\bar{\rho})\,d\tau = \int_t^{t+h} E(\tau,t\,|\,\rho)\,d\tau$. Observational equivalence follows immediately from considering (19). In spite of hyperbolic discounting, consumption and saving follow the same path as under exponential discounting. As a consequence, Barro and Sala-i-Martin's (2004, p.135 ff.) result applies, i.e., the dynamics of the saving rate always is monotonic. ||

C. Cournot-Nash: Derivation of conditions (27) and (28).

Following Barro (1999), we first approximate

$$k_{\tau+\varepsilon} \approx k_\tau(1 + \varepsilon r_\tau) + \varepsilon w_\tau - \varepsilon c_\tau\,, \tag{35}$$

implying that $\frac{\partial k_{\tau+\varepsilon}}{\partial c_\tau} \approx -\varepsilon$. Next, we employ the intertemporal budget constraint

$$\begin{aligned} &\int_{\tau+\varepsilon}^{\infty} c_t e^{-R(t,\tau+\varepsilon)} dt = k_{\tau+\varepsilon} + h_{\tau+\varepsilon}\,, \\ &h_{\tau+\varepsilon} \equiv \int_{\tau+\varepsilon}^{\infty} w_t e^{-R(t,\tau+\varepsilon)} dt\,, \end{aligned} \tag{36}$$

where h_τ denotes the present (date-τ) value of human capital. Taking the consumption growth rate (25) into account, we are able to re-express (36) as:

$$c_{\tau+\varepsilon} \underbrace{\int_{\tau+\varepsilon}^{\infty} e^{\frac{1-\theta}{\theta} R(t,\tau+\varepsilon) - \frac{1}{\theta}\Lambda(t,\tau+\varepsilon)} dt}_{\equiv \Delta(\tau+\varepsilon)} = k_{\tau+\varepsilon} + h_{\tau+\varepsilon} \ . \tag{37}$$

From (37), we infer that $\frac{\partial c_{\tau+\varepsilon}}{\partial k_{\tau+\varepsilon}} = \Delta(\tau+\varepsilon)^{-1}$. Next, consider $\frac{dc_{\tau+\varepsilon}}{dc_{\tau}} = \frac{dc_{\tau+\varepsilon}}{dk_{\tau+\varepsilon}} \frac{dk_{\tau+\varepsilon}}{dc_{\tau}} \approx -\varepsilon\, \Delta(\tau+\varepsilon)^{-1}$. The first-order condition of (26) with respect to c_τ yields:

$$-c_{\tau}^{-\theta} \Delta(\tau+\varepsilon) + c_{\tau+\varepsilon}^{-\theta} \int_{\tau+\varepsilon}^{\infty} e^{\frac{1-\theta}{\theta} R(t,\tau+\varepsilon) - \frac{1-\theta}{\theta}\Lambda(t,\tau+\varepsilon)} e^{-\mathrm{P}(t,\tau)} dt = 0 \, ,$$

where we divided by ε and multiplied by $\Delta(\tau+\varepsilon)$. We next take the limit as ε approaches zero, and divide by $c_{\tau}^{-\theta}$:

$$\int_{\tau}^{\infty} e^{\frac{1-\theta}{\theta} R(t,\tau) - \frac{1}{\theta}\Lambda(t,\tau)} \left[e^{\Lambda(t,\tau) - \mathrm{P}(t,\tau)} - 1 \right] dt = 0 \, ,$$

which corresponds to (27). ||

In order to derive (28), we note that (27) is identically true. Define $\upsilon(t,\tau) \equiv e^{\frac{1-\theta}{\theta} R(t,\tau) - \frac{1}{\theta}\Lambda(t,\tau)}$, and $\omega(t,\tau) \equiv e^{\frac{1-\theta}{\theta} R(t,\tau) - \frac{1-\theta}{\theta}\Lambda(t,\tau) - \mathrm{P}(t,\tau)}$. Then (27) becomes $\int_{\tau}^{\infty} \omega(t,\tau) dt = \int_{\tau}^{\infty} \upsilon(t,\tau) dt$. Differentiating with respect to τ yields:

$$\begin{aligned} &\int_{\tau}^{\infty} \left[-\frac{1-\theta}{\theta} r_\tau + \frac{1-\theta}{\theta} \lambda_\tau + \mathrm{P}'(t,\tau) \right] \omega(t,\tau) dt = \left[-\frac{1-\theta}{\theta} r_\tau + \frac{1}{\theta} \lambda_\tau \right] \int_{\tau}^{\infty} \upsilon(t,\tau) dt \\ &= \left[-\frac{1-\theta}{\theta} r_\tau + \frac{1}{\theta} \lambda_\tau \right] \int_{\tau}^{\infty} \omega(t,\tau) dt \, , \end{aligned}$$

which, upon simplifying, becomes:

$$\int_{\tau}^{\infty}\left[\frac{1-\theta}{\theta}\lambda_{\tau}+\mathrm{P}'(t,\tau)\right]\omega(t,\tau)dt=\left[\frac{1}{\theta}\lambda_{\tau}\right]\int_{\tau}^{\infty}\omega(t,\tau)dt \iff$$

$$\int_{\tau}^{\infty}\left[-\lambda_{\tau}+\mathrm{P}'(t,\tau)\right]\omega(t,\tau)dt=0 \iff$$

$$\lambda_{\tau}=\frac{\int_{\tau}^{\infty}\mathrm{P}'(t,\tau)\,\omega(t,\tau)dt}{\int_{\tau}^{\infty}\omega(t,\tau)dt}.\ \|$$

References

Ainslie, G. (1992), Picoeconomics, Cambridge: Cambridge University Press.

Antràs, P. (2001), Transitional Dynamics of the Savings Rate in the Neoclassical Growth Model, Mimeo, Department of Economics, Harvard University.

Barro, R.J. (1999), Laibson Meets Neoclassical in the Neoclassical Growth Model, Quarterly Journal of Economics 114, 1125—1152.

Barro, R.J., X. Sala-i-Martin (2004), Economic Growth, Cambridge MA, MIT Press.

Bosworth, B., G. Burtless, J. Sabelhaus (1991), The Decline in Saving: Evidence from Household Surveys, Brookings Papers on Economic Activity 1, 183—256.

Caliendo, F.N., D. Aadland (2007), Short-Term Planning and the Life-Cycle Consumption Puzzle, Journal of Economic Dynamics and Control 31, 1392—1415.

Chari, V.V., P.J. Kehoe, E.R. McGrattan (1996), The Poverty of Nations: A Quantitative Exploration, NBER Working Paper 5414.

Christiano, L.J. (1989), Understanding Japan's Saving Rate: The Reconstruction Hypothesis, Federal Reserve Bank of Minneapolis Quarterly Review 13, 10—25.

Farzin, Y.H. (2006), Conditions for Optimal Sustainable Development, Review of Development Economics 10, 518—534.

Findley, T.S., F.N. Caliendo (2011), Interacting Mechanisms of Dynamic Inconsistency, Working Paper, Department of Economics, Utah State University, Logan, Utah.

Gómez, M.A. (2008), Dynamics of the Saving Rate in the Neoclassical Growth Model with CES Production, Macroeconomic Dynamics 12, 195—210.

Gong, L., W. Smith, H. Zou (2007), Consumption and Risk with Hyperbolic Discounting, Economics Letters 96, 153—160.

Groth C., K.J. Koch, T.M. Steger (2010), When Growth is Less Than Exponential, Economic Theory 44, 213—242.

Hall, R.E. (1988), Intertemporal Substitution in Consumption, Journal of Political Economy 96, 339—357.

Laibson, D. (1997), Golden Eggs and Hyperbolic Discounting, Quarterly Journal of Economics 62, 443—477.

Leipziger, D., V. Thomas (1997), An Overview of East Asian Experience, in: D. Leipziger (ed.), Lessons from East Asia, Ann Arbor: University of Michigan Press.

Litina, A., T. Palivos (2010), The Behavior of the Saving Rate in the Neoclassical Optimal Growth Model, Macroeconomic Dynamics 14, 482—500.

Loayza, N., K. Schmidt-Hebbel, L. Servén (2000), What Drives Private Saving around the World?, Policy Research Working Paper 2309, Washington D.C.: The World Bank.

Maddison, A. (1992), A Long-Run Perspective on Saving, Scandinavian Journal of Economics 94, 181—196.

Merton, R.C. (1971), Optimum Consumption and Portfolio Rules in a Continuous-Time Model, Journal of Economic Theory 3, 373-413.

Schmidt-Hebbel, K., L. Servén (1999), The Economics of Saving and Growth: Theory, Evidence, and Implications for Policy, Cambridge: Cambridge University Press.

Shafer, J.R., J. Elmeskov, W. Tease (1992), Saving Trends and Measurement Issues, Scandinavian Journal of Economics 94, 155—175.

Smetters, K. (2003), The (Interesting) Dynamic Properties of the Neoclassical Growth Model with CES Production, Review of Economic Dynamics 6, 697—707.

Tease W., A. Dean, J. Elmeskov, P. Hoeller (1991), Real Interest Rate Trends: The Influence of Saving, Investment and other Factors, OECD Economic Studies 17, 107—144.

Trimborn, T., K.J. Koch, T.M. Steger (2008), Multidimensional Transitional Dynamics: A Simple Numerical Procedure, Macroeconomic Dynamics 12, 301—319.