Market-based incentives

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Abstract

We study optimal incentives in a principal-agent problem in which the agent's outside option is determined endogenously in a competitive labor market. In equilibrium, strong performance increases the agent's market value. When this value becomes sufficiently high, the threat of the agent's quitting forces the principal to give the agent a raise. The prospect of obtaining this raise gives the agent an incentive to exert effort, which reduces the need for standard incentives, like bonuses. In fact, whenever the agent's option to quit is close to being "in the money," the market-induced incentive completely eliminates the need for standard incentives.

1 Introduction

The amount of short-term incentives (e.g., bonuses) in compensation packages of many workers and, especially, executives has attracted a lot of attention and scrutiny in recent years.[1] Traditional principal-agent theory provides a rationale for the presence of short-term incentives in compensation packages: because workers' actual effort is too costly to monitor and reward directly, observed performance must be rewarded in order to elicit effort; short-term incentives

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[1]For example, the Federal Reserve (2011) states that "Risk-taking incentives provided by incentive compensation arrangements in the financial services industry were a contributing factor to the financial crisis that began in 2007."

are an efficient means of delivering these rewards. Our main result in this paper is that this rationale does not quite hold once theory recognizes that good to-date performance can boost a worker's market value—the value she commands in the labor market if she quits her job. We introduce a performance-dependent market value to a principal-agent model with limited commitment and show that short-term compensation incentives are usually not needed. Workers' desire to improve their market value already gives them an incentive, a market-based incentive, which in a wide range of circumstances is sufficient to elicit effort.

In practice, workers' consideration for the quality of their future labor market options is an important source of incentives in numerous occupations ranging from an intern to a tenured professor. Interns and apprentices work for little or no pay but gain useful skills and experience that increase the quality of the job they can obtain later. In academia, strong performance in research or teaching typically is not rewarded with bonuses paid for each publication or for a high teaching evaluation. Yet, academics work hard to produce strong records of research and teaching in order to improve their value in the academic labor market. Higher market value brings quality outside offers that give professors promotions and salary increases in the long run. In this paper, we capture these forward-looking, market-based incentives in a tractable model that allows for fully flexible, long-term employment contracts with performance-dependent compensation. In the optimal contract, compensation is downward-rigid and often completely free of performance-dependent incentives like piece-rate pay or year-end bonuses. Our model delivers testable predictions on how likely performance-dependent incentives should be observed in compensation packages of different types of workers.

Market-based incentives arise in our model out of two necessary ingredients. First, workers have the right to quit at any time. Second, if a worker quits, the value she obtains in the labor market is higher the stronger her record of to-date performance. The worker's right to quit implies that if the market option becomes more attractive to her than her current job, her employer will have to increase her compensation to match her market value, or she quits. In either case, the worker benefits when her market value increases. Naturally, this motivates the worker to boost her market value, which she can do by showing strong performance. Thus, the worker has an incentive to perform on her current job even if strong performance is not immediately rewarded in terms of her current pay. Since this incentive is driven by the worker's market value considerations, we will refer to it as a market-based incentive.

Why should a worker's outside market value increase with stronger on-the-job performance? In our model, it increases because the worker's productivity is assumed to be persistent over time and equally as useful to a potential new employer as it is to her current employer. By putting in (unobservable) effort on her current job, the worker improves her current productivity, which benefits her current employer in terms of the increased quantity of output she produces now. But, because her productivity is persistent, it also makes her more valuable to a potential next employer. Competition among employers in the labor market translates the next employer's higher valuation of the worker into a higher value the worker can obtain by quitting and going to the market.

It is intuitive that when current effort enhances the worker's future productivity, fewer short-term incentives should be necessary because the worker already has some "skin in the game" in that she benefits when her productivity grows and improves her market value. Since workers' productivity is persistent in our model, it can be interpreted as human capital. If working hard on the current job is not only an input into current production but also an investment in the worker's (inalienable and transferable) human capital, then it is intuitive that the objectives of the firm and the worker become better aligned and the need for short-term compensation incentives decreases.

In our analysis, we make this intuition precise. Formally, we consider a principal-agent contracting problem in which a risk-neutral firm hires a risk-averse agent/worker whose productivity is observable and persistent over time. The evolution of the worker's productivity depends on her effort and exogenous, idiosyncratic shocks, both of which are unobservable. We embed this contracting problem in a simple model of the labor market, where firms match with workers frictionlessly. The contract between the firm and the worker specifies compensation and an effort recommendation for any realization of idiosyncratic shocks to the worker's productivity. The worker can quit at any time and go back to the market. We show that this right to quit and the persistent impact of the worker's effort on her productivity (and hence on her value in the labor market) give rise to a forward-looking, market-based incentive that encourages effort.

Market-based incentives are stronger the closer the worker's option to quit is to being "in the money." This is because the firm can provide very limited insurance to the worker if the worker is about to quit. Limited insurance means the worker's continuation value is highly sensitive to the worker's performance, which gives the worker a strong incentive to exert effort.

The link between the strength of market-based incentives and the worker's "distance to default" can be easily seen in the following simple example. Suppose the firm pays the worker constant compensation for as long as the worker chooses to stay with the firm. How well this simple contract insures the worker depends on how long the worker stays, which in turn depends on how good the worker's market option is. The worse her outside option, i.e., the further away she is from quitting, the longer the expected duration of this simple contract, and, in effect, the more insurance this contract provides to the worker. A well-insured worker has little incentive to put in effort, i.e., the market-based incentive is weak. In particular, if the worker's market option is valueless or non-existent, she will never quit, so the simple contract will last forever, effectively giving the worker full insurance. With full insurance, the worker has absolutely no incentive to put in effort. In contrast, if the worker's market option is almost "in the money," a very small positive shock to her productivity is enough to elevate her market value above the value of the simple contract. Since such small shocks happen often, the expected duration of the simple contract is short and so the contract provides very little insurance to the worker. With little insurance, the worker's incentive to put in effort is strong.[2]

In our model, the worker's option to quit is formally captured by a participation, or quitting, constraint. This constraint requires that the worker's value from the contract with the firm be at all times at least as large as her market value. How close the worker is to quitting at any given time (i.e., the worker's "distance to default") is measured by how slack the quitting constraint is. We show that, in line with the intuition from the above simple example, market-based incentives are strong and standard, contract-based incentives are absent whenever slackness in the quitting constraint is lower than a threshold. Below this threshold, compensation is constant whenever the quitting constraint does not bind, and, to keep the worker from leaving, it increases monotonically whenever the quitting constraint binds.[3] Above this threshold, market-based

[2]Note that it is the upside, not downside, risk that is uninsurable when the insured agent lacks commitment.

[3]Because there is no economic role for job transitions in our stylized model of the labor market with homogeneous firms, we derive the optimal long-term contract under the assumption that workers do not quit if indifferent. The alternative assumption leads to the exact same equilibrium processes for effort and compensation.

incentives are not strong enough to elicit effort and firms supplement them with some contract-based incentives, so compensation is not completely independent of current performance. When slackness in the quitting constraint goes to infinity, the strength of market-based incentives goes to zero and the optimal contract converges to the solution of the standard principal-agent model in which there are no market-based incentives.

How frequently market-based incentives are strong in equilibrium depends on how close on average the quitting constraint is to binding. One important factor determining the average "distance to default" is the expected change in worker productivity. If productivity tends to grow over time, the worker's market value tends to increase, so the quitting constraint binds often. This makes market-based incentives strong frequently and contract-based incentives needed rarely. In particular, with a sufficiently large positive trend in worker productivity, the probability that contract-based incentives are ever used can be arbitrarily small.

We also present an extension of our model in which not only workers but also firms lack commitment. In particular, firms can fire workers upon incurring a deadweight firing cost. In this extension, thus, in addition to the worker's quitting constraint, we have a firm's participation, or firing, constraint. We show that if the firing cost is not too large, the worker is always exposed to risk and, thus, market-based incentives are always strong. If slackness in the quitting constraint is low, then, as in our basic model, market-based incentives arise because the upside risk to the worker's productivity is uninsurable. If slackness in the worker's quitting constraint becomes large, the firm's firing constraint becomes tight and market-based incentives arise because the downside risk to the worker's productivity is not fully insured.

In order to characterize the solution to our model analytically, we make several assumptions widely used in the dynamic contracting literature. Constant absolute risk aversions (CARA) preferences and Gaussian shocks let us reduce to one the dimension of the state space sufficient for a recursive representation of our contracting problem. The optimal contract is then characterized by solving an ordinary differential equation. Although needed for analytical tractability, these assumptions are not necessary for the existence of market-based incentives. We briefly consider a version of our model with log preferences and log-normal shocks and show that there, too, market-based incentives are strong when slackness in the workers' quitting constraint is not too large.

Essential for the existence of market-based incentives are workers' inability to commit to staying on the job forever and a positive impact of workers' on-the-job effort on their market value. These conditions seem very plausible. The latter condition, in particular, is similar to learning-by-doing. It will be satisfied whenever putting in effort on the job helps a worker acquire any kind of skill or experience that is valued in the labor market.

Our characterization of the equilibrium contract gives the following testable predictions of our model. Performance-based incentives should be more frequently observed (a) in occupations in which workers do not acquire much general, transferable human capital, (b) when the growth of a worker's general productivity is slower, e.g., later in the life-cycle, (c) when firing workers is costly, and (d) when workers past performance is harder to observe to outsiders. Gibbons and Murphy (1992), Loveman and O'Connell (1996), and Lazear (2000) provide evidence consistent with these predictions.

Related literature Market-based incentives are similar to career concerns in that both give workers a forward-looking motivation for effort. But there are important differences in how they arise and how they affect workers' incentives. In the career concerns model of Holmstrom (1999), workers are risk-neutral, so there is no need for consumption smoothing or insurance. Workers sell labor services in spot markets every period. Because performance is assumed to be observable but not contractible, spot wages cannot be made contingent on current performance. Future spot wages can depend on today's performance, as the history of performance is available for each worker. Each period, wages reflect the market's belief about the worker's hidden productivity type. Stronger observed performance improves the market's expectation of the worker's type leading to higher wages in the future. Workers are motivated by career concerns: they choose effort to manage the market's assessment of their productivity.

Market-based incentives, by contrast, arise in our model in an environment with risk-averse workers and risk-neutral firms entering into long-term employment contracts in which compensation can be contingent on current performance. In the optimal long-term contract, compensation is often insensitive to current performance not because performance is not contractible but because this way the contract provides maximum feasible consumption smoothing and insurance to the worker. Firms provide incentives mostly through permanent compensation raises (promotions) that are necessary in order to retain workers whose strong performance bids up their market value. Since a worker's productivity is common knowledge at all times, workers cannot manage market beliefs in our model.

Our paper builds upon two strands of the literature on long-term principal-agent relationships with risk-averse agents: the studies in which the provision of insurance is impeded by moral hazard, and the studies in which insurance is limited by the lack of commitment. In the first group, the paper that we are closest to is Sannikov (2008), whom we follow in studying dynamic moral hazard in continuous time.[4] Sannikov (2008), however, does not capture market-based incentives because in his model shocks and actions only affect current output, and the agent's outside option is fixed. In our model, shocks and actions have a persistent effect and, crucially, the agent's outside option is endogenous and performance-dependent. In order to obtain a meaningful outside option function, we do not study the optimal contracting problem in isolation but rather embed it in an simple equilibrium model of the labor market. A general lesson from the dynamic moral hazard literature is that it is efficient for compensation to contemporaneously respond to observed performance. With the new elements that we add to the model, we show the existence of an incentive for effort driven by the agent's value of the outside option. This market-based incentive changes the structure of the optimal contract: compensation responds to performance to a much smaller extent than previous results suggest; often, it does not respond at all.

Among the numerous studies of optimal contracting subject to limited commitment, our paper is closely related to Harris and Holmstrom (1982) and Krueger and Uhlig (2006). As in these studies, we have in our model persistent idiosyncratic shocks, firms/principals that can commit to long-term contracts, and workers/agents who cannot. This one-sided commitment friction leads to a downward rigidity in compensation and to limited insurance of the upside risk in workers' productivity. While in Harris and Holmstrom (1982) the workers' outside option is autarky (spot markets), Krueger and Uhlig (2006) endogenize the outside option by allowing agents to enter a new long-term contract with another firm after a separation. We follow the latter approach to modeling the outside option in this paper. Grochulski and Zhang (2011) study a one-sided limited commitment contracting problem in continuous time and show that the

[4]Early contributions to the dynamic moral hazard literature include Spear and Srivastava (1987) and Phelan and Townsend (1991).

agent's continuation value is sensitive to shocks at all times, even when her current consumption is not. In the present paper, we re-examine these insights in a model that combines the one-sided commitment friction with moral hazard. We find that, with some qualifications, the results from the limited-commitment models continue to hold in our more general environment: wages are downward rigid, the upside risk is not fully insured, and the agent's continuation value is sensitive to shocks even if compensation is not. Limited commitment therefore appears to trump moral hazard considerations in our model: the optimal contract most of the time looks exactly as if moral hazard were completely absent from the model environment.

There exist a small number of studies that, like we do here, examine optimal contracts under the two frictions of private information and limited commitment.[5] Two studies closely related to our paper are Thomas and Worrall (1990, Section 8) and Phelan (1995). These papers, however, do not capture market-based incentives because the agent's outside option does not depend on her past performance in the models studied there. In Atkeson (1991), the outside option of the agent (a borrowing country) does depend on her actions (investment). For this reason, although that paper asks a different question, we expect that market-based incentives exist in that environment. However, market-based incentives are probably weak there because persistence in the impact of the private action (investment) on the value of the outside option (autarky) is low in that model. In our model, effort has a permanent effect on the worker's outside option, which makes market-based incentives much stronger and easier to identify.

Modeling compensation as part of a long-term employment contract has a long tradition in the economic theory of employment and wage determination that dates back to Baily (1974), Azariadis (1975), and Holmstrom (1983). Although in this theory, as in our model, employment contracts provide insurance to workers, shocks considered there are aggregate or industry-wide, while we consider worker-specific shocks to individual productivity. Also, that literature abstracts from incentive problems, which are the primary focus of this paper. Our main interest is in showing the effect of market-based incentives on the structure of the optimal compensation contract under moral hazard. To this end, we keep the model of the labor market simple. By assuming frictionless matching between firms and workers, we abstract from search costs and exogenous separations. All workers in our model economy are employed at all times.

[5] We will refer to moral hazard as a special case of private information.

Organization The model environment is formally defined in Section 2. Sections 3 and 4 study single-friction versions of our model, with full commitment in Section 3 and full information in Section 4. Optimal contracts from these models serve as benchmarks that we use to solve the full model in Section 5. In particular, the minimum cost functions from these models provide useful lower bounds on the cost function in the full model. Section 6 considers robustness of our results with respect to the functional forms we use as well as with respect to the assumption of full commitment on the firm side. Proofs of all results formally stated in the text are relegated to Appendix A.

2 A labor market with long-term contracts

We consider a labor market populated with a large number of agents/workers and a potentially larger number of firms operating under free entry. For concreteness, we will assume that one firm hires one worker.[6] Matching between workers and firms is frictionless: an unmatched worker can instantaneously find a match with a new firm entering the market. In a newly formed match, the firm offers to the worker a long-term employment contract. Competition among firms, those in the market and the potential new entrants, drives all firms' expected profits to zero.[7]

Workers are heterogeneous in their productivity y_t, which changes stochastically over time following a Brownian motion with drift. Let $\boldsymbol{w}$ be a standard Brownian motion $\boldsymbol{w} = \{w_t, \mathcal{F}_t; t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A worker's productivity process $\boldsymbol{y} = \{y_t; t \geq 0\}$ is $y_0 \in \mathbb{R}$ at $t = 0$ and evolves according to

$$dy_t = a_t dt + \sigma dw_t. \tag{1}$$

The drift in a worker's productivity at t, a_t, is privately controlled by the worker via a costly action she takes. Specifically, $a_t \in \{a_l, a_h\}$ with $a_l < a_h$. The volatility of y_t is fixed: $\sigma > 0$ at all t. Workers are heterogeneous in the initial level of their productivity y_0, in the realized paths of their productivity shocks $\{w_t; t > 0\}$, and, potentially, in the action path $\{a_t; t \geq 0\}$ they choose. The path of actions $\{a_t; t \geq 0\}$ taken by each worker is her private information.

[6] As long as each worker's performance is observable, our results would be unchanged if firms in the model hired multiple workers.

[7] Our results do not depend on frictionless matching or on firms making zero profit in a match.

The structure of the productivity process and each worker's productivity level y_t are public information at all times.

We adopt a simple production function in which the revenue the worker generates for the firm equals the worker's productivity y_t at all times during her employment with the firm. In a long-term employment contract, the firm collects revenue $\{y_t; t \geq 0\}$ and pays compensation $\{c_t; t \geq 0\}$ to the worker. We will identify compensation c_t with the worker's consumption at all $t \geq 0$.[8]

Formally, a long-term contract a firm and a worker enter at $t = 0$ specifies an action process $\boldsymbol{a} = \{a_t; t \geq 0\}$ for the worker to take, and a compensation/consumption process $\boldsymbol{c} = \{c_t; t \geq 0\}$ the worker receives. Processes $\boldsymbol{a}$ and $\boldsymbol{c}$ must be adapted to the information available to the firm.

We assume that firms and workers discount future payoffs at a common rate r. In a match, the firm's expected profit from a contract $(\boldsymbol{a},\boldsymbol{c})$ is given by

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(y_t - c_t)dt\right],$$

where $\mathbb{E}^{\boldsymbol{a}}$ is the expectation operator under the action plan $\boldsymbol{a}$.

Action a_t represents the worker's effort at time t. If the worker takes the high-effort action a_h, she improves her current productivity and, hence, the revenue she generates for the firm. Because y_t is persistent, high effort a_h also increases the worker's expected productivity in the future. Action a_h, however, is costly to the worker in terms of current disutility of effort.

All workers have identical preferences over compensation/consumption processes $\boldsymbol{c}$ and action processes $\boldsymbol{a}$. These preferences are represented by the expected utility function

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t, a_t)dt\right].$$

To make our model tractable analytically, we will abstract in this paper from wealth effects in the provision of incentives. That is, we will assume constant absolute risk aversion (CARA) with respect to consumption by taking

$$U(c_t, a_t) = u(c_t)\phi^{1_{a_t = a_l}},$$

[8]We can think of the worker's savings or financial wealth as being observable and thus contractually controlled by the firm.

where $u(c_t) = -\exp(-c_t) < 0$, $0 < \phi < 1$, and $1_{a_t = a_l}$ is the indicator of the low-effort action a_l at time t. High effort a_h is costly to the agent because $U(c, a_h) = u(c) < u(c)\phi = U(c, a_l)$ for all c.[9] In Section 6, we discuss the extent to which our results depend on this form of the utility function.

Firms can commit to long-term contracts, but workers cannot. A worker has the right to quit and rejoin the labor market at any point during her employment with a firm. In the market, the worker is free to enter another long-term contract with a new firm. Any contractual promise by the worker to not use her market option would not be enforceable. The presence of this inalienable right to quit restricts firms' ability to insure workers against the upside risk to their productivity. In particular, contracts will be restricted by workers' participation (or quitting) constraints defined as follows. Denote by $V(y_t)$ the value a worker with productivity y_t can obtain if she quits and rejoins the labor market. This market value will be determined in equilibrium. We show later (in Proposition 1) that V is strictly increasing. For a worker with initial productivity $y_0 \in \mathbb{R}$, a contract $(\boldsymbol{a}, \boldsymbol{c})$ induces a continuation value process $\boldsymbol{W} = \{W_t; t \geq 0\}$ given by

$$W_t = \mathbb{E}^{\boldsymbol{a}} \left[\int_0^\infty re^{-rs} U(c_{t+s}, a_{t+s}) ds \,|\mathcal{F}_t \right]. \tag{2}$$

Contract $(\boldsymbol{a}, \boldsymbol{c})$ satisfies the worker's quitting constraints if at all dates and states

$$W_t \geq V(y_t). \tag{3}$$

This constraint is standard in models of optimal contracts with limited commitment (e.g., Thomas and Worrall (1988)). It also resembles the lower-bound constraint on the continuation value W_t used in many principal-agent models with private information (e.g., Atkeson and Lucas (1995) and Sannikov (2008)), but is in an important way different because the lower bound in these models is given by some fixed value, whereas in (3) the lower bound $V(y_t)$ changes with the worker's productivity. Later in the paper we will see that this difference has important implications for the provision of incentives to the worker at the lower bound.

In this paper, we adopt the convention that when the quitting constraint (3) binds, i.e., when the worker is indifferent to quitting, the worker stays. In our model, as in Krueger

[9] We can equivalently write $U(c_t, a_t)$ as $u(c_t + 1_{a_t = a_l} \log(\phi^{-1}))$ and interpret $\log(\phi^{-1}) > 0$ as the consumption equivalent of the utility the agent gets from leisure associated with exerting low effort.

and Uhlig (2006), there are no efficiency gains from separations. Since switching employers would not make the worker more productive, the best continuation contract that the worker's current employer can provide is as good as the best contract that the worker can get in the market. Adopting the convention that workers do not quit when (3) binds is thus without loss of generality, but lets us avoid additional notation that would be needed to describe job transitions.[10]

Because action a_t is not observable, contracts will also have to satisfy incentive compatibility (IC) constraints. A contract is incentive compatible if no deviation from the recommended action process $\boldsymbol{a}$ can make the worker better off. We will express IC constraints using the following results of Sannikov (2008).

Let $(\boldsymbol{a}, \boldsymbol{c})$ be a contract and $\boldsymbol{W}$ the associated continuation utility process as defined in (2). There exists a (progressively measurable) process $\boldsymbol{Y} = \{Y_t; t \geq 0\}$ such that the continuation utility process $\boldsymbol{W}$ can be represented as

$$dW_t = r(W_t - U(c_t, a_t))dt + Y_t dw_t^a, \tag{4}$$

where

$$w_t^a = \sigma^{-1}\left(y_t - y_0 - \int_0^t a_s ds\right). \tag{5}$$

Contract $(\boldsymbol{a}, \boldsymbol{c})$ is IC if and only if for all t and $\tilde{a} \in \{a_h, a_l\}$,

$$r\left(U(c_t, \tilde{a}) - U(c_t, a_t)\right) + \sigma^{-1}(\tilde{a} - a_t)Y_t \leq 0. \tag{6}$$

For proof of these results see Sannikov (2008).

In (4), $dw_t^a = \sigma^{-1}\left(dy_t - a_t dt\right)$ represents the worker's current on-the-job performance. Performance at t is measured by the change in the worker's productivity, dy_t, relative to what this change is expected to be at t under the recommended action plan, $a_t dt$, and normalized by σ. Note that as long as the worker follows the recommended action a_t, her (observable) performance dw_t^a will be the same as the (unobservable) innovation term dw_t in her productivity process given in (1).

[10] If we follow the alternative convention and suppose that the worker quits when (3) binds, the optimal contract is the same except it ends when (3) binds for the first time and is replaced with a new contract identical to the continuation of the original contract. This interpretation of long-term contracts is equivalent to the no-separation convention we adopt in that it leads to identical production, consumption, and welfare.

Also in (4), Y_t represents the sensitivity of the worker's continuation value to current performance. Clearly, larger Y_t will imply a stronger response of W_t to any given observed performance dw_t^a. The IC constraint (6) requires that the total gain the worker can obtain by deviating from the recommended action a_t to the alternative action $\tilde{a}$ be nonpositive. The first component of this gain shows the direct impact of the deviation on the worker's current utility. The second component shows the indirect impact of the deviation on the continuation utility expressed as the product of the action's impact on the worker's performance and the sensitivity of the continuation value to performance.

If the recommended action at time t is to exert effort, i.e., if $a_t = a_h$, then the IC condition (6) reduces to $ru(c_t)(\phi - 1) \leq \sigma^{-1}(a_h - a_l) Y_t$, or

$$\frac{Y_t}{-u(c_t)} \geq \beta, \tag{7}$$

where $\beta = r\sigma \frac{1-\phi}{a_h - a_l} > 0$. Analogously, the low-effort action a_l is IC at t if and only if

$$\frac{Y_t}{-u(c_t)} \leq \beta.$$

Written in this form, the IC constraints make it clear that the ratio $Y_t/(-u(c_t))$ measures the strength of effort incentives that contract $(\boldsymbol{a}, \boldsymbol{c})$ provides to the worker at time t. The high-effort action a_h is incentive compatible at t if and only if this ratio is greater than the constant β. Low effort is incentive compatible if and only if this ratio is smaller than β. As in Sannikov (2008), higher sensitivity of the worker's continuation value to her current on-the-job performance, Y_t, makes effort incentives stronger. Due to non-separability of workers' preferences between consumption and leisure, the level of consumption c_t also affects the strength of effort incentives in our model.[11] In particular, if the contract recommends high effort, the gain in the flow utility the worker can obtain by shirking is in our model smaller at higher consumption levels.[12] For a given level of sensitivity Y_t, thus, higher current consumption c_t makes effort incentives stronger.

[11] Compare our IC constraint (6) with the IC constraint (21) on page 976 of Sannikov (2008). Consumption c_t does not show up in the IC constraint of that model because preferences considered there are additively separable between consumption and effort.

[12] This property is particularly easy to see if we interpret $\log(\phi^{-1}) > 0$ as the consumption equivalent of the utility the agent gets from shirking. Since shirking at t is equivalent to consuming $c_t + \log(\phi^{-1})$ instead of c_t, decreasing marginal utility of consumption implies that the gain from shirking is lower when c_t is higher.

We are now ready to define the contract design problem faced by a firm matched with a worker. We will define this problem generally as a cost minimization problem in which the firm needs to deliver some present discounted utility value $W \in [V(y_0), 0)$ to a worker whose initial productivity is y_0. Let $\Sigma(y_0)$ denote the set of all contracts $(\boldsymbol{a}, \boldsymbol{c})$ that at all t satisfy quitting constraints (3) and IC constraints (6). The firm's minimum cost function $C(W, y_0)$ is defined as

$$C(W, y_0) = \min_{(\boldsymbol{a},\boldsymbol{c})\in\Sigma(y_0)} \quad \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t - y_t)dt\right] \tag{8}$$

$$\text{subject to} \quad W_0 = W. \tag{9}$$

The constraint (9) is known as the promise-keeping constraint: the contract must deliver to the worker the initial value W. In the special case of $W = V(y_0)$, the value $-C(V(y_0), y_0)$ represents the profit the firm attains in a match with a worker of type y_0 when the worker's outside value function is V.

Next, we define competitive equilibrium in the labor market with long-term contracts.

Definition 1 *Competitive equilibrium consists of the workers' market value function* $V : \mathbb{R} \to \mathbb{R}_-$ *and a collection of contracts* $(\boldsymbol{a}^{y_0}, \boldsymbol{c}^{y_0})_{y_0\in\mathbb{R}}$ *such that, for all* $y_0 \in \mathbb{R}$,

(i) $(\boldsymbol{a}^{y_0}, \boldsymbol{c}^{y_0})$ *attains the minimum cost* $C(V(y_0), y_0)$ *in the firm's problem* (8)–(9),

(ii) $C(V(y_0), y_0) = 0$ *and* $C(W, y_0) > 0$ *for any* $W > V(y_0)$.

The first equilibrium condition requires that when firms assume (correctly) that the workers' outside value is their equilibrium market value, then the equilibrium contracts are cost-minimizing (i.e., efficient) and in fact deliver to workers their market value. The second condition comes from perfect competition under free entry: profits attained by firms must be zero in equilibrium and no firm can deliver to a worker a larger value than her market value without incurring a loss.

2.1 Level-independence of incentives

The following proposition shows a simple relationship between optimal contracts offered to workers with different productivity levels. This relationship implies a particularly simple functional form for the equilibrium value function V and gives us a partial characterization of the cost function C.

Proposition 1 *If $(\boldsymbol{a}^0, \boldsymbol{c}^0)$ is the optimal contract for $y_0 = 0$, then, for any $y_0 \in \mathbb{R}$, the optimal contract $(\boldsymbol{a}^{y_0}, \boldsymbol{c}^{y_0})$ is given by*

$$\boldsymbol{a}^{y_0} = \boldsymbol{a}^0, \tag{10}$$

$$\boldsymbol{c}^{y_0} = \boldsymbol{c}^0 + y_0. \tag{11}$$

The equilibrium value function V satisfies

$$V(y) = e^{-y}V(0) \quad \forall y \in \mathbb{R}. \tag{12}$$

The minimum cost function C satisfies

$$C(W, y) = C(We^y, 0) \quad \forall y \in \mathbb{R}, W < 0. \tag{13}$$

The independence of the optimal action recommendation from y_0, shown in (10), and the additivity of the optimal compensation plan with respect to y_0, shown in (11), follow from the independence of future productivity changes dy_t from the initial condition y_0 and from the absence of wealth effects in CARA preferences. With no wealth effects, incentives needed to induce high or low effort are the same for workers of all productivity levels. The contribution of changes in a worker's productivity to a firm's revenue is also the same for all workers. Thus, the same effort process is optimally recommended to workers of all productivity levels, and output produced by a worker with initial productivity $y_0 = y > 0$ is path-by-path larger by exactly y than output produced by a worker with initial productivity $y_0 = 0$. Competition among firms implies then that in equilibrium the worker with $y_0 = y$ will obtain the same compensation process as the worker with $y_0 = 0$ plus the constant amount y at all t.

This structure of the compensation plan allows us to pin down the functional form of the workers' market value function $V(y_0)$, as given in (12). Intuitively, if a worker with $y_0 = 0$ obtains $V(0)$ in market equilibrium, then a worker with $y_0 = y$ will obtain $e^{-y}V(0)$ because her consumption is larger by y at all t and the utility function is exponential, so $u(c_t + y) = e^{-y}u(c_t)$ at all t.

In addition, this structure of optimal contracts implies a particular form of homogeneity for a firm's minimum cost function $C(W, y)$, as shown in (13). Suppose some contract efficiently delivers some value $W < 0$ to a worker whose initial productivity $y_0 = y > 0$ (i.e., this contract attains $C(W, y)$). Then a modified contract with compensation uniformly decreased by y will efficiently deliver value $e^y W < W$ to a worker whose initial productivity $y_0 = 0$ (i.e., the modified contract will attain $C(e^y W, 0)$). But these two contracts generate the same cost/profit for the firm, as in the second case the worker produces less output (uniformly less by y) and receives less compensation (also less by y).[13]

The scalability of the contracting problem and the implied homogeneity of the minimum cost function greatly simplify our analysis in this paper. In order to solve for the equilibrium, it is sufficient to find one value, $V(0)$, and one contract that supports it, $(\boldsymbol{a}^0, \boldsymbol{c}^0)$.

2.2 Optimality of high effort

In our analysis, we will focus on the case in which the recommendation of the high-effort action a_h is optimal and therefore always used by firms in equilibrium. We will verify in Section 5 that the following assumption is sufficient for high effort to be optimal.

Assumption 1 *Let $\kappa = \sigma^{-2}\left(\sqrt{a_h^2 + 2r\sigma^2} - a_h\right)$. We assume that*

$$\frac{\kappa}{1+\kappa}(a_h - a_l) \geq r \log\left(\phi^{-1}\right) + \frac{1}{2}\beta\sigma. \tag{14}$$

The set of parameter values satisfying this assumption is nonempty.[14] We will maintain Assumption 1 throughout the paper.

2.3 Recursive formulation

In order to find the cost function $C(W_t, y_t)$, we will use the methods of Sannikov (2008) to study a recursive minimization problem with control variables a_t, $u_t \equiv u(c_t)$, and Y_t. Scalability and homogeneity properties of Proposition 1 let us reduce the dimension of the state space in

[13] Similarly, a worker with initial $y_0 = -y < 0$ will produce and receive y units less than a worker with $y_0 = 0$.

[14] Take an arbitrary point in the parameter space and consider decreasing the value of a_l. Assumption 1 will eventually hold because lower a_l makes a) high effort relatively more desirable, so the left-hand side of (14) grows without a bound, and b) shirking easier to detect (β becomes smaller), so the right-hand side of (14) decreases.

this recursive problem. Instead of studying this problem in the two-dimensional state vector (W_t, y_t), we can reduce the state space to a single dimension as follows. Using (13) and (12) we have

$$C(W_t, y_t) = C(W_t e^{y_t}, 0) = C\left(\frac{W_t}{e^{-y_t}V(0)}V(0), 0\right) = C\left(\frac{W_t}{V(y_t)}V(0), 0\right). \tag{15}$$

This shows that the minimum cost $C(W_t, y_t)$ is the same for all pairs (W_t, y_t) for which the ratio $\frac{W_t}{V(y_t)}$ is the same. We will find it convenient to transform this ratio further and define a single state variable as

$$S_t \equiv \log\left(\frac{V(y_t)}{W_t}\right). \tag{16}$$

Using S_t, we can express the worker's quitting constraint (3) as

$$S_t \geq 0, \tag{17}$$

and the firm's cost function as

$$C(W_t, y_t) = C\left(\frac{W_t}{V(y_t)}V(0), 0\right) = C\left(e^{-S_t}V(0), 0\right) = C\left(V(S_t), 0\right),$$

where the first equality uses (15), the second uses (16), and the third uses (12).[15] We will denote $C\left(V(\cdot), 0\right)$ by $J(\cdot)$ and solve for this function in the state variable S_t.

It is useful to note that $S_t = u^{-1}(W_t) - u^{-1}(V(y_t))$, i.e., S_t represents the difference between the worker's continuation value inside the contract and her outside option value when both these values are converted to permanent consumption equivalents. Indeed, if $S_t = S$, the worker is indifferent between giving up S units of her compensation forever and separating from the firm.[16] Because S_t shows by how much the worker prefers her current contract over the market option, it represents slackness in the worker's quitting constraint at time t. Larger S_t represents larger slackness. In particular, the quitting constraint binds at t if and only if $S_t = 0$.

With the worker equilibrium value function (12) substituted into (16), we can write the state variable S_t as

$$S_t = -y_t - \log(-W_t) + \log(-V(0)). \tag{18}$$

[15] The IC constraint (7) is not affected by the change of the state variable, as it depends on the control variables only.

[16] To see this, note that if $S_t = S$ and $\{c_{t+s}; s \geq 0\}$ is a compensation process that gives the worker the continuation value W_t, then the compensation process $\{c_{t+s} - S; s \geq 0\}$ gives the worker the continuation value exactly equal to the value of her outside option, $V(y_t)$.

Using Ito's lemma, the law of motion for y_t given in (1), and the law of motion for W_t given in (4), we obtain the law of motion for the state variable S_t under high effort as

$$dS_t = \left(r \left(-1 - \frac{u_t}{-W_t} \right) + \frac{1}{2} \left(\frac{Y_t}{-W_t} \right)^2 - a_h \right) dt + \left(\frac{Y_t}{-W_t} - \sigma \right) dw_t^a. \tag{19}$$

We will find it useful to normalize the control variables u_t and Y_t by the absolute value of the worker's continuation utility. Introducing $\hat{u}_t \equiv \frac{u_t}{-W_t}$ and $\hat{Y}_t \equiv \frac{Y_t}{-W_t}$, we express (19) as

$$dS_t = \left(r \left(-1 - \hat{u}_t \right) + \frac{1}{2} \hat{Y}_t^2 - a_h \right) dt + \left(\hat{Y}_t - \sigma \right) dw_t^a. \tag{20}$$

The *Hamilton-Jacobi-Bellman (HJB)* equation for the firm's cost function J is

$$\begin{aligned} rJ(S_t) = \quad & rS_t - r \log(-V(0)) + \min_{\hat{u}_t, \hat{Y}_t} \Bigg\{ r(-\log(-\hat{u}_t)) + \\ & J'(S_t) \left(r \left(-1 - \hat{u}_t \right) + \frac{1}{2} \hat{Y}_t^2 - a_h \right) + \frac{1}{2} J''(S_t) \left(\hat{Y}_t - \sigma \right)^2 \Bigg\}, \end{aligned} \tag{21}$$

where control variables must satisfy $\hat{Y}_t \geq -\hat{u}_t \beta$ to ensure incentive compatibility of the recommended high-effort action a_h.

The meaning of the terms in the HJB equation is standard. It may be helpful to write the HJB equation informally as

$$rJ(S_t) = \min \left\{ r(c_t - y_t) + J'(S_t) \left(\text{drift of } S_t \right) + \frac{1}{2} J''(S_t) \left(\text{volatility of } S_t \right)^2 \right\}. \tag{22}$$

Intuitively, the first derivative J' represents the firm's aversion to the drift of S_t because, as we see in (22), the total cost $rJ(S_t)$ increases by $J'(S_t)$ when the drift of S_t increases by one unit. Similarly, the second derivative J'' shows how strongly the cost function will respond to an increase in the volatility of S_t, so in this sense it represents the firm's volatility aversion. Also, using definitions of S_t and $\hat{u}_t$, it is easy to verify that the first three terms on the right-hand side of (21) represent the firm's flow cost $r(c_t - y_t)$.

In Section 5, we will characterize optimal long-term contracts by finding a unique solution to the HJB equation subject to appropriate boundary and asymptotic conditions. In the next two sections, we provide two important benchmarks by finding optimal contracts in simplified versions of our general environment in which one of the two frictions is absent.

3 Pay-for-performance incentives in equilibrium with private information and full commitment

In this section, we will assume full commitment: not only firms but also workers have the power to commit to never breaking the contract. As in our general model presented in the previous section, firms match with workers and offer them long-term contracts at $t = 0$. At this time, the worker can reject the offer and move to another match instantaneously. Upon accepting a contract at $t = 0$, however, the worker commits to not quitting at any $t > 0$. This commitment maximizes the match's surplus as it allows firms to provide better insurance against fluctuations in workers' productivity relative to the case in which the workers would not commit. In particular, it lets firms insure the upside risk to workers' productivity. We solve this version of our model in closed form. In equilibrium, firms provide incentives to workers by making compensation sensitive to current on-the-job performance.

Let $\Sigma_{FC}(y_0)$ denote the set of all contracts $(\boldsymbol{a}, \boldsymbol{c})$ that at all t satisfy the IC constraint (6). The contracting problem we study in this section is identical to the cost-minimization problem in (8) but with the quitting constraint (3) removed, i.e., with the set of feasible contracts expanded from $\Sigma(y_0)$ to $\Sigma_{FC}(y_0)$. We will use $C_{FC}(W, y_0)$ to denote the minimum cost function in this problem. The reduced-form cost function $J_{FC}(S)$ is defined analogously. Note that $J_{FC}(S)$ is defined for any S, even negative. Market equilibrium is defined as in the general case but using the cost function $C_{FC}(W, y_0)$ instead of $C(W, y_0)$.

The following proposition gives a continuous-time version of standard characterization results for optimal contracts with private information and full commitment.[17]

Proposition 2 *In the model with full commitment, workers' equilibrium compensation is given by*

$$c_t = y_0 + \frac{\mu + a_h}{r} - \mu t + \rho\beta w_t^a, \tag{23}$$

[17] Spear and Srivastava (1987), Thomas and Worrall (1990), Atkeson and Lucas (1992), and Phelan (1998) provide characterization results for optimal contracts in discrete-time models with private information and full commitment, similar to the moral hazard model with full commitment we study in this section in continuous time. Atkeson and Lucas (1995) and Sannikov (2008) characterize optimal contracts with private information assuming an exogenous lower bound on the agent's continuation utility in, respectively, discrete- and continuous-time models.

where $0 < \rho = (\sqrt{1 + 4r^{-1}\beta^2} - 1)/(2r^{-1}\beta^2) < 1$ *and* $\mu = r\,(1-\rho) - \frac{1}{2}\rho^2\beta^2 > 0$*. The sensitivity of the equilibrium continuation value* W_t *with respect to observed performance* dw_t^a *is*

$$Y_t = -u(c_t)\beta \quad \textit{at all } t. \tag{24}$$

Proposition 2 shows two main features of optimal compensation schemes in the model with private information and full commitment: contemporaneous sensitivity of compensation to performance, represented in (23) by $\rho\beta > 0$, and a negative time trend in compensation, represented in (23) by $-\mu < 0$. The positive contemporaneous sensitivity of compensation with respect to the worker's observed performance represents the standard, short-term, "pay-for-performance" incentive for workers to exert effort. The negative trend in compensation does not provide effort incentives by itself, but it improves the effectiveness of the pay-for-performance incentive.

Sensitivity Y_t in (24) shows that the IC constraint in (7) binds at all t. This means that incentives, as measured by the ratio $Y_t/(-u(c_t))$, are in equilibrium strong enough to make the recommended high-effort action a_h incentive compatible but not any stronger. Incentives are costly because they reduce insurance. The equilibrium contract is efficient in holding incentives down to a necessary minimum at all times. Because this minimum does not change over time, the strength of incentives provided to the worker is always the same in this model.

This section shows that private information requires positive sensitivity Y_t. The next section shows that positive sensitivity Y_t can arise completely independently of private information: if workers lack commitment, their productivity shocks cannot be fully insured and, therefore, their continuation values must remain sensitive to realizations of these shocks. Thus, in an environment in which private information and limited commitment coexist, limited commitment potentially could deliver the positive sensitivity Y_t that private information requires. Our main results in this paper, which we give in Section 5, consider precisely this possibility.

4 Market-based incentives in equilibrium with limited commitment and full information

In this section, we discuss the full-information version of our model. As in the general model outlined in Section 2, firms match with workers and offer them long-term contracts at $t = 0$. A worker who has accepted a contract retains the option to quit and go back to the labor market, where she can find a new match instantaneously. Unlike in the general model, however, we will assume in this section that workers' actions on the job are observable, and that workers can contractually commit to a prescribed course of action.[18] The model we study in this section is essentially a continuous-time version of the Krueger and Uhlig (2006) model with CARA preferences. This section also generalizes the optimal insurance model studied in Grochulski and Zhang (2011), where the outside option is assumed to be autarky.

Let $\Sigma_{FI}(y_0)$ denote the set of all contracts $(\boldsymbol{a}, \boldsymbol{c})$ that at all t satisfy the quitting constraint (3). The contracting problem we study in this section is identical to the cost-minimization problem in (8) but with the IC constraint (7) removed, i.e., with the set of feasible contracts expanded from $\Sigma(y_0)$ to $\Sigma_{FI}(y_0)$. We will use $C_{FI}(W, y_0)$ to denote the minimum cost function in this problem. The reduced-form cost function $J_{FI}(S)$ is defined analogously. Market equilibrium is defined as in the general case but using the cost function $C_{FI}(W, y_0)$ instead of $C(W, y_0)$.

Proposition 3 *In the model with full information, workers' equilibrium compensation is given by*

$$c_t = m_t - \psi, \tag{25}$$

where $m_t = \max_{0 \le s \le t} y_s$ *and* $\psi = \frac{\kappa\sigma^2}{2r} > 0$. *The sensitivity of the continuation value* W_t *with respect to observed performance* dw_t^a *is*

$$Y_t = -u(c_t)\frac{\kappa}{\kappa+1}e^{-\kappa(m_t - y_t)}\sigma > 0. \tag{26}$$

As in Grochulski and Zhang (2011), the maximum level of productivity attained to date, m_t, is a state variable keeping track of the quitting constraint in this model. The quitting constraint binds whenever productivity attains a new to-date maximum, i.e., when $y_t = m_t$, and is slack whenever productivity is below its to-date maximum, i.e., when $y_t < m_t$. Since there is no private information and firms and workers discount future payoffs at the same rate, optimal contracts provide constant compensation (full insurance) to workers whenever the quitting constraint is slack. When the quitting constraint binds, i.e., when a new to-date

[18] In short, workers cannot be punished for quitting but can be punished for shirking on the job.

maximum m_t is attained, compensation c_t increases, as shown in (25). Thus, compensation never decreases in the full-information model, and it increases faster the faster new maximal levels of a worker's productivity are realized.

Since sample paths of the productivity process are continuous, a worker has a better chance of attaining a new to-date maximum of her productivity—and thus obtaining a permanent increase in her compensation—the closer her current productivity level y_t is to the current to-date maximum level m_t. The worker's continuation value in the contract, W_t, increases whenever the chance for the next permanent increase in compensation improves. This means that W_t increases when current productivity y_t increases, even during time intervals in which y_t remains strictly below m_t, i.e., when current consumption c_t does not at all respond to changes in y_t. This everywhere-positive sensitivity of the continuation value to current performance is shown in (26). Moreover, (26) shows that the continuation value's performance sensitivity Y_t increases as the distance between y_t and m_t decreases. Like S_t, the distance $m_t - y_t$ is a measure of slackness of the quitting constraint (3).[19] Thus, sensitivity Y_t is larger the closer the quitting constraint is to binding.

Positive sensitivity $Y_t > 0$ arises in this section for reasons completely distinct from those that give rise to positive sensitivity in the private-information model discussed in the previous section. There, firms pay for performance in order to elicit high effort. Here, firms can directly control workers' effort, but face the possibility of workers quitting. When the quitting constraint becomes binding, the firm must give the worker a raise in order to retain her. This raise is the source of positive sensitivity of the continuation value to current performance at all times, even when the quitting constraint is slack. Because the market option is the source of sensitivity Y_t in the model we consider in this section, we will call this Y_t market-induced sensitivity.

As we see in the IC constraint (7), incentives are measured in our model by the ratio of Y_t to $-u(c_t)$. Sensitivity Y_t is therefore closely related to the notion of incentives. Despite there being no need for effort incentives in this section, as we assume here that effort is observable (and thus contractually controllable), we should note that the contract in Proposition 3 does give the worker an incentive to exert effort because the ratio $Y_t/(-u(c_t))$ is nonzero under this

[19] In fact, S_t and $m_t - y_t$ are related by $S_t = m_t - y_t - \log\left(\kappa + 1 - e^{-\kappa(m_t - y_t)}\right) + \log(\kappa)$. Thus, S_t is strictly increasing in $m_t - y_t$, and $S_t = 0$ if and only if $m_t - y_t = 0$.

contract. Indeed, if the firm for some reason neglects to observe and control the worker's effort at some instant t, the worker would still choose to supply effort at t if the ratio $Y_t/(-u(c_t))$ she faces under her contract is larger than β, regardless of what makes this ratio large. Thus, an effort incentive can exist without private information. Since sensitivity Y_t is market-induced, we will call the effort incentive created by Y_t the market-based incentive. The next result shows that the market-based incentive can be strong in the full-information model.

Corollary 1 *The ratio* $\frac{Y_t}{-u(c_t)}$ *is strictly decreasing in* $m_t - y_t$*. In particular,* $\frac{Y_t}{-u(c_t)} \geq \beta$ *if and only if* $m_t - y_t \leq \delta$*, where* $\delta = \kappa^{-1} \log\left(\frac{\kappa}{\kappa+1}\frac{\sigma}{\beta}\right) > 0$.

This corollary shows that the equilibrium contract obtained in the full-information model formally satisfies the IC constraint (7) whenever slackness in the quitting constraint (3), as measured by $m_t - y_t$, is small. That means that the market-based incentive is strong in this region.[20] The corollary also shows that the full-information contract is not overall incentive compatible because it fails to satisfy the IC constraint (7) when the quitting constraint is sufficiently slack. Monotonicity of $Y_t/(-u(c_t))$ means that the market-based incentive is stronger when the quitting constraint is tighter (less slack).

In this section, there is no need for incentives. Yet, they exist in equilibrium as a by-product of limited commitment. In the next section, we consider the general version of our model with both moral hazard and limited commitment, where incentives are needed. There, as here, the market option improves with the worker's performance, which generates a market-based incentive. Similar to Corollary 1, the market-based incentive will be strong (sufficient to induce high effort) when slackness in the quitting constraint is smaller than a threshold. In that region of the state space, therefore, the equilibrium contract will rely completely on market-based incentives and will not use pay-for-performance incentives at all.

4.1 Further properties of equilibrium with full information

Proposition 3 describes equilibrium compensation contracts in the full-information model using two state variables: m_t and y_t. In Appendix B, we describe the equilibrium of this model

[20] In particular, the full-information equilibrium contract does satisfy the IC constraint at the onset of every employment relationship because $m_0 - y_0 = 0 < \delta$.

in terms of the single state variable S_t, and characterize the cost function $J_{FI}(S_t)$. In particular, Appendix B discusses the following properties of the equilibrium expressed in terms of S_t. The drift and the volatility of S_t are strictly decreasing in S_t. The possibility of violating the quitting constraint makes the firm infinitely averse to volatility in S_t at $S_t = 0$, hence $J''_{FI}(0) = \infty$ and the volatility of S_t at $S_t = 0$ is zero in equilibrium. Equilibrium drift in S_t at $S_t = 0$ is strictly positive (i.e., zero is a reflective barrier for S_t). In the next section we show that all these properties continue to hold when both private information and limited commitment are present in the model.

5 Market-based and pay-for-performance incentives in equilibrium with both frictions

In this section, we characterize the optimal contract in our general model, where firms face both the incentive problem and the quitting constraint.

5.1 Solving the optimal contracting problem

Standard methods for solving second-order differential equations like our HJB equation (21) require two boundary conditions. Our problem is nonstandard. It has a semi-unbounded domain (the positive half-line) with only one boundary condition: the second derivative of J at the boundary $S_t = 0$ must be infinite because otherwise the quitting constraint would be violated immediately after S_t becomes zero. Despite the lack of a second condition on J at the boundary, our analysis of the full-information model suggests an asymptotic condition that can be used to pin down the solution: the cost that the quitting constraint imposes on the firm must become negligible when S_t goes to infinity because the (time-discounted) chance of the constraint binding in the future becomes negligible when S_t is large. When S_t goes to infinity, therefore, the cost function in the model with two frictions, J, must converge to the cost function from the model with private information and full commitment, J_{FC}. In particular, first derivatives of these functions, $J'(S_t)$ and $J'_{FC}(S_t)$, must become close at large values of S_t.

We will use this asymptotic convergence condition to pin down the solution.[21] [22]

Our analysis of the HJB equation gives the following theorem.

Theorem 1 *There exists a unique solution to the HJB equation* (21) *satisfying the boundary condition* $J''(0) = \infty$ *and the convergence condition* $\lim_{S_t \to \infty}(J'(S_t) - J'_{FC}(S_t)) = 0$. *This solution represents the true minimum cost function for the firm.*

The method of proof given in Appendix A is similar to that in Sannikov (2008) with two technical difficulties stemming from the specific boundary and convergence conditions we have. First, our HJB equation does not satisfy the Lipschitz condition at $S_t = 0$ because $J''(0) = \infty$. We overcome this difficulty by using a change of variable technique. Second, the asymptotic condition requiring convergence of $J'(S_t)$ to $J'_{FC}(S_t)$ does not provide an actual restriction on the boundary of the state space. We overcome this difficulty as follows. We determine a range of possible values for the first derivative of J at $S_t = 0$ and consider a family of candidate solutions to the HJB equation, one for each possible value of $J'(0)$ in this range. We show that the asymptotic condition requiring that $J'(S_t)$ converge to $J'_{FC}(S_t)$ as $S_t \to \infty$ is violated by all but one candidate solution. We then confirm that the one candidate solution that satisfies this asymptotic condition indeed represents the true minimum cost function J.

Lastly, we verify that the recommendation of high effort is optimal at all t. Lemma A.11 in the Appendix A shows that this conclusion follows from our Assumption 1.

5.2 Optimality of constant compensation

We now provide the main result of our paper.

[21] In order to use the cost function J_{FC} from the one-friction model with full commitment as a benchmark (lower bound) for J in the two-friction model, one must shift J_{FC} downward by a constant to account for the fact that a lower level of utility is provided to the worker in equilibrium in the model with two frictions (the value $V(0)$ is lower in this model). It is thus more convenient to express the asymptotic convergence condition in terms of first derivatives rather than levels because a uniform vertical shift of J_{FC} does not affect its derivative.

[22] Appendix B discusses the cost the quitting constraint imposes on the firm in the full-information model relative to the environment with no frictions (the first best). In that model, this cost does go to zero when slackness in the quitting constraint goes to infinity: the cost function J_{FC} and its derivatives converge to the first-best cost function and its derivatives, respectively.

Proposition 4 *In the model with two frictions, there exists a unique $S^* > 0$ such that in each time interval in which S_t remains strictly above 0 and below S^*, equilibrium compensation c_t is constant. In the equilibrium contract, the IC constraint (7) binds whenever $S_t \geq S^*$, but is slack whenever $S_t < S^*$.*

It is a standard result in the literature on optimal compensation that making pay contingent on current performance is an efficient way of providing effort incentives when the actual effort of a worker cannot be observed by the firm. That literature, however, assumes that the worker's outside option value does not depend on her performance. Our main result shows that when the worker's outside option value does depend on her performance, making current pay contingent on performance may no longer be efficient. In particular, whenever the worker's market value is close to the value she obtains by continuing to work for the current employer, optimal compensation is constant, i.e., completely unresponsive to current performance of the worker, and the worker chooses to supply effort nevertheless.

Key to this result are two facts. First, as we have seen in Section 4, when the quitting constraint binds, the firm must increase the worker's compensation in order to retain her. Second, when the quitting constraint does not quite bind but is close to binding, the worker's effort has a strong impact on the chance that the quitting constraint becomes binding. These two facts imply that when the worker is close to quitting, she will exert effort in order to actually hit the quitting constraint and obtain a raise. Knowing this, the firm does not need to provide an additional incentive via performance-dependent compensation; constant compensation is efficient.

Proposition 4 also shows that the IC constraint (7) is slack when the quitting constraint (17) is close to binding. Constant compensation is optimal when S_t remains in the interval $(0, S^*)$ precisely because both the quitting and the IC constraint are slack in that region. A slack IC constraint means that the worker's incentives are "too strong" (i.e., more than necessary to induce effort). In fact, when $S_t < S^*$ it would be efficient if the firm could provide more insurance to the worker, thus weakening her incentives, but doing so is impossible due to the worker's right to quit. The firm already insures all downside risk to the worker's future productivity, but the upside risk is not fully uninsurable. Full insurance is not possible because the threat of the worker's quitting will force the firm to adjust the worker's compensation (give her a raise) if her productivity becomes sufficiently high.

When the quitting constraint is relatively slack, $S_t > S^*$, the IC constraint binds. This is because the impact of the worker's effort on her chance of hitting the quitting constraint is smaller when the quitting constraint is more distant. The market option still gives the worker an incentive to supply effort, but this incentive is weak (i.e., not sufficient to induce effort). The firm must in this case supplement the market-based incentive with a contract-based pay-for-performance incentive. We study the optimal mix of these incentives in the next section. In the limiting case with $S_t \to \infty$, the chance of S_t ever returning to zero becomes negligible and the strength of market-based incentives goes to zero.

In sum, when S_t remains below S^* the optimal contract looks exactly like the optimal contract from the model with limited commitment and full information in Section 4.[23] When S_t goes to infinity, in contrast, the optimal contract looks like the optimal contract from the model with private information and full commitment in Section 3.

5.3 Strength of market-based incentives

In our model, the strength of effort incentives provided to a worker at time t is measured by the ratio of Y_t to $-u(c_t)$. Workers will supply effort if and only if this ratio is larger than β. Proposition 4 shows that in equilibrium, the strength of incentives is only just sufficient to induce effort when the quitting constraint is relatively slack ($S_t \geq S^*$), but is more than sufficient when the quitting constraint is relatively tight ($S_t < S^*$).

We will now decompose incentives into two parts: forward-looking market-based incentives and short-term, contract-based incentives. Market-based incentives will be those induced by the worker's outside option (as in Section 4). Contract-based incentives will be those not induced by the market option (as in Section 3). To measure the strength of market-based incentives at t, we need to compute the ratio $Y_t/(-u(c_t))$ that the firm would optimally choose at t if limited commitment were the only friction, i.e., as if the worker's effort were observable (and hence controllable) by the firm locally at t. We compute this ratio as follows. Given the optimal cost

[23] Since workers are hired in our model at market value, i.e., without any slack in the quitting constraint, optimal compensation for newly hired workers is always free of pay-for-performance incentives. New workers, however, are likely to receive a raise shortly after they are hired.

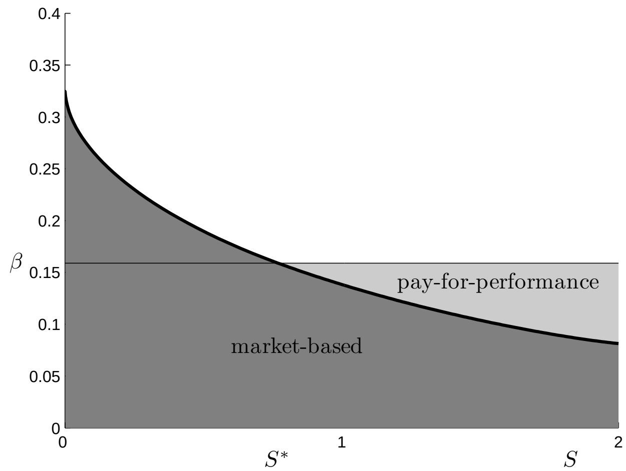


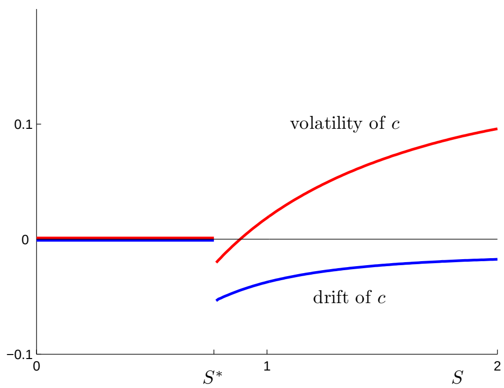
Figure 1: Composition of incentives

function J, we disregard the IC constraint at t in the HJB equation (21) and use first-order conditions to obtain current utility $u(c_t)$ and sensitivity Y_t that the firm would choose in such a relaxed problem. Denoting the ratio of Y_t to $-u(c_t)$ from this locally relaxed problem by $\tilde{Y}_t/(-\tilde{u}(c_t))$, we have

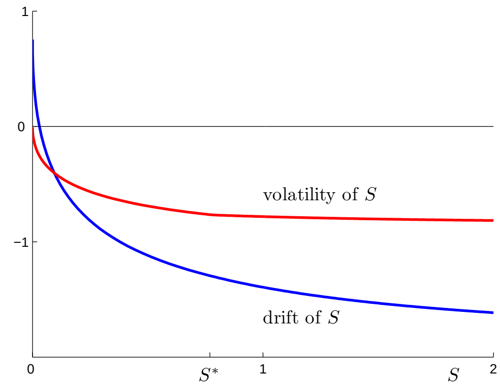
$$\frac{\tilde{Y}_t}{-\tilde{u}(c_t)} = \frac{\sigma J'(S_t)J''(S_t)}{J'(S_t) + J''(S_t)}.$$

This ratio gives the portion of the actual $Y_t/(-u(c_t))$ that is induced by the presence of the worker's market option. Thus, it represents the strength of market-based incentives at t in our model. The remainder of the actual $Y_t/(-u(c_t))$ represents contract-based incentives that the firm must inject in order to ensure incentive compatibility of high effort at t.

Figure 1 plots the ratio $\tilde{Y}_t/(-\tilde{u}(c_t))$ against S_t in a typically parameterized numerical example. The strength of market-based incentives decreases as the quitting constraint becomes more distant. Below S^*, market-based incentives are strong, meaning they are sufficient to induce effort, i.e., $\tilde{Y}_t/(-\tilde{u}(c_t)) \geq \beta$, and contract-based pay-for-performance incentives are zero. An implication of strong market-based incentives when $S_t < S^*$, as we have seen in Proposition 4, is that compensation is flat and workers provide effort without being compensated for current performance. Above S^*, market-based incentives are weak, i.e., not strong enough to induce worker effort, and the optimal contract supplements them with pay-for-performance incentives. This means that compensation does depend on current performance above S^*. Pay-



(a) Dynamics of compensation.



(b) Dynamics of S_t.

Figure 2: Example with $a_h > 0$. Threshold $S^* = 0.76$. Stationary point for S_t is 0.05.

for-performance incentives become stronger as market-based incentives become weaker when the quitting constraint becomes more slack.

5.4 Dynamics of the equilibrium contract

Unlike the two single-friction models studied in Sections 3 and 4, the model with both frictions does not admit a closed-form solution. In this section, therefore, we describe the dynamics of the equilibrium contract by characterizing the drift and the volatility of compensation c_t and the state variable S_t. We provide a mix of analytical and numerical results in this section.

We start out by presenting in Figure 2 the drift and the volatility of c_t and S_t computed numerically under the parametrization of our model used earlier to produce Figure 1. In panel (a), we can identify the region of strong market-based incentives by noting that for all S_t above zero and below S^* the drift and the volatility of compensation are both zero, which means that $dc_t = 0$, i.e., compensation remains constant in this region, as predicted earlier in Proposition 4. When S_t goes to infinity, the impact of the quitting constraint vanishes and optimal compensation converges to the optimal compensation from the full-commitment model, where, by Proposition 2, the drift of c_t is $-\mu < 0$ and the volatility of c_t is $\rho\beta > 0$.

In addition to these properties of compensation at low and high values of the state variable, where market-based incentives are respectively strong and negligible, numerical analysis lets us characterize the dynamics of c_t in the intermediate region of the state space, where market-based incentives are not strong but are not negligible either. As we see in panel (a), at all S_t greater than S^* the volatility of compensation is increasing in S_t but remains smaller than its asymptotic value of $\rho\beta$. The intuition for this follows from the monotonicity of the strength of market-based incentives in S_t shown earlier in Figure 1. If at some $S_t > S^*$ the worker's observed performance is positive, $dw_t^a = dy_t - a_h dt > 0$, then both the worker's continuation value inside the contract and her outside market value increase. Because the contract provides some insurance to the worker, the outside market value increases by more than the continuation value inside the contract does. This means that the quitting constraint becomes less slack (S_t decreases) and, thus, the chance of entering the area of constant compensation (below S^*) and eventually hitting the quitting constraint (when the worker receives a raise) improves. This improvement provides some incentive for the worker to supply effort. Therefore, even in the region of weak market-based incentives, compensation can be less sensitive to contemporaneous performance than what it must be in the standard principal-agent model, or in our model at S_t approaching infinity, where market-based incentives are absent. Because, as shown in Figure 1, the market-based incentive is stronger at smaller S_t, the sensitivity of c_t to performance decreases when S_t decreases at all $S_t > S^*$.

Panel (a) of Figure 2 shows that at $S_t = S^*$ (and, by continuity, also right above S^*), the sensitivity of compensation to observed performance is actually negative. This feature of the optimal contract is due to the non-separability in the worker's preferences between consumption and leisure.[24] The intuition for this is as follows. As we see in (7), higher current compensation c_t relaxes the IC constraint in our model. When the IC constraint binds, the firm saves on incentive costs by paying higher compensation now. If the IC constraint does not bind, this effect is absent. At the threshold point $S_t = S^*$, positive and negative worker productivity shocks dw_t have an asymmetric effect on the incentive benefit of high current compensation: positive shocks decrease S_t and take it into the region in which the IC constraint does not bind, where high current compensation is not needed, while negative shocks increase S_t and take it into the region where the IC constraint binds, where high current compensation does have a

[24]In numerical examples with separable preferences we computed, the volatility of consumption is everywhere weakly positive. It is zero at all S_t below S^* and positive at all S_t above S^*.

benefit. This produces negative sensitivity of compensation c_t to innovations in w_t at $S_t = S^*$: a positive shock $dw_t > 0$ will not affect c_t and a negative shock $dw_t < 0$ will increase c_t.

In addition, panel (a) of Figure 2 shows that the drift of compensation is lower than its asymptotic value of $-\mu$ at all S_t above S^*, and is monotonic in S_t in this region. Similar to the negative volatility of compensation, these properties of its drift are due to the fact that compensation increases when the state variable crosses the S^* threshold and enters the region of weak market-based incentives. A more strongly negative drift in c_t right above S^* helps average out the monotonic increase in c_t occurring at S^* as the state variable fluctuates around this threshold level. When S_t grows and moves away from S^*, this need for a more strongly negative drift vanishes and the drift in c_t approaches $-\mu$.

These dynamic properties of compensation are robust in the numerical experiments with the model we conducted. The discontinuity in the drift and the volatility of c_t at S^* can be shown analytically, but we do not have analytical results for the monotonicity of the drift and the volatility of c_t above S^*.

Moving on to the dynamics of the state variable, we note in panel (b) of Figure 2 that the volatility of S_t is everywhere negative and monotonically increasing toward zero as S_t decreases toward the boundary $S_t = 0$. The intuition for this, which we already mentioned earlier, follows from the fact that the optimal contract provides more insurance to a worker who is further away from quitting. At the boundary itself, the contract cannot provide any insurance, i.e., the volatility of the continuation value inside the contract has to match the volatility of the worker's outside option to ensure that the quitting constraint is not violated immediately after the state variable hits its lower bound. The further away S_t is from zero, the less likely it is that the quitting constraint becomes binding, the more the firm can insure the worker against her productivity shocks, and, in effect, the more negative the volatility of S_t becomes. Asymptotically, the volatility of S_t converges to its value from the model without quitting constraints.

The negative volatility of slackness S_t in the quitting constraint (3) means that this constraint can become binding only after the worker's good performance, which is exactly opposite to the pure moral hazard model of Sannikov (2008). In both models, poor performance decreases the worker's continuation value W_t. In Sannikov (2008), the lower bound on W_t is fixed, so when W_t decreases, the distance between W_t and its lower bound decreases. In our model, the lower bound on W_t, $V(y_t)$, is not fixed: it is strictly increasing in y_t. In fact, $V(y_t)$ responds to the worker's performance more strongly than W_t. When performance is poor, thus, $V(y_t)$ decreases faster than W_t, so the distance between W_t and its lower bound increases. When performance is strong, $V(y_t)$ increases faster than W_t, i.e., the lower bound "catches up" to the continuation value W_t. The closer $V(y_t)$ approaches W_t the slower this catching up becomes. When slackness S_t in the quitting constraint is zero, W_t and $V(y_t)$ respond to good performance exactly the same (S_t has zero volatility), so $V(y_t)$ "pushes up" W_t but never exceeds it.

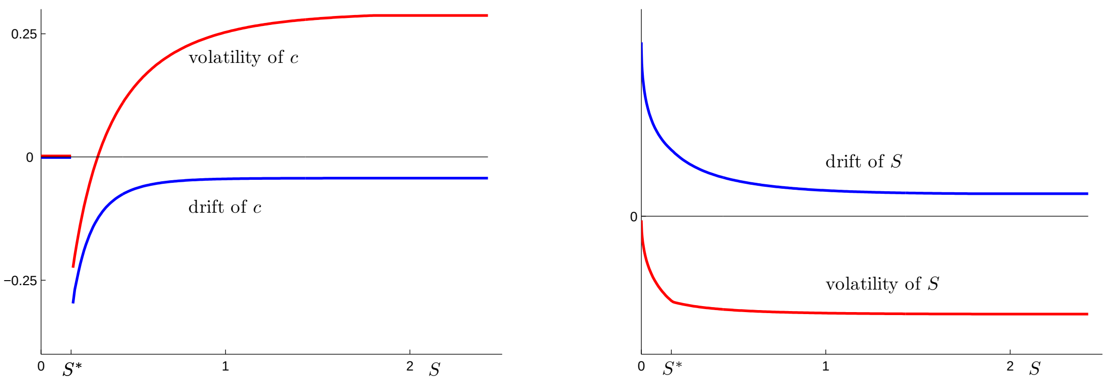
The drift of S_t, shown also in panel (b) of Figure 2, is positive at the boundary of the state space $S_t = 0$ and converges to its value from the model without quitting constraints when S_t goes to infinity. Similar to the volatility of S_t, these properties of its drift are explained by the fact that the quitting constraint is less likely to become binding when S_t is larger.

Moreover, note in Figure 2 that the drift in S_t at $S_t = 0$ is not only nonnegative, which is necessary to avoid violating the quitting constraint, but is actually strictly positive. Combined with the fact that S_t has zero volatility at zero, this implies that zero is a reflective barrier for the state variable in equilibrium. This property of our model with market-based incentives is different from the absorbing lower bound that appears in many dynamic moral hazard models with a fixed lower bound on the continuation utility (e.g., Sannikov (2008)).

These properties of the drift and the volatility of the state variable hold not only in the numerical example presented in Figure 2 but are true in our model in general. Formally, we have the following result.

Proposition 5 *Let $\alpha(S_t)$ and $\zeta(S_t)$ denote, respectively, the drift and the volatility of the state variable. In the equilibrium contract, $\alpha(S_t)$ is strictly decreasing with $\alpha(0) > 0$ and $\lim_{S_t \to \infty} \alpha(S_t) = -\mu - a_h$, and $\zeta(S_t)$ is strictly decreasing with $\zeta(0) = 0$ and $\lim_{S_t \to \infty} \zeta(S_t) = \rho\beta - \sigma$.*

Note that Proposition 5 implies that the volatility of S_t is always negative, but the sign of the drift in S_t is not pinned down. In particular, the direction in which S_t tends to move when it is large depends on the sign of $-\mu - a_h$. This value represents the drift of the state variable in the full-commitment version of our model as well as in the model with both frictions at large S_t. In the example presented in Figure 2, $-\mu - a_h < 0$ and the state variable has a unique



(a) Dynamics of compensation.

(b) Dynamics of S_t.

Figure 3: Example with $a_h < 0$. Threshold $S^* = 0.18$. No stationary point for S_t exists, i.e., the drift in S_t is everywhere positive.

stationary point, where its drift is zero. Because this stationary point is much smaller than S^* in this example, S_t tends to start to decrease toward zero before it reaches S^*, and thus it will only infrequently leave the region of strong market-based incentives.

Figure 3 modifies the parametrization used in Figure 2 by using a lower value of the drift parameter of the worker's productivity, $a_h < 0$, resulting in a positive asymptotic value for the drift of the state variable, $-\mu - a_h > 0$. Since, by Proposition 5, the drift of S_t is strictly positive at zero and monotonic in S_t, $-\mu - a_h > 0$ means that S_t has in this example a positive drift everywhere in the state space. In this modified parametrization, therefore, S_t tends to drift out of $(0, S^*)$. Over time it thus becomes less and less likely that market-based incentives are strong: market-based incentives are transient in this parametrization.[25] These observations lead us next to investigate more closely where the state variable tends to spend most time in equilibrium.

5.5 Market-based incentives in the long run

This section provides two results. The first result gives a sufficient condition for the existence of a stationary stochastic steady state (an invariant distribution) for the state variable S_t.

[25] Panel (a) of Figure 3 shows that dynamic properties of compensation in the parametrization with low a_h are qualitatively the same as those presented in panel (a) of Figure 2 for the case of high a_h.

Theorem 2 *If in the model with full commitment the drift of the state variable S_t is negative, i.e., if $-\mu - a_h < 0$, then in the model with both frictions there exists an invariant distribution for the state variable S_t.*

This result is intuitive because a negative drift in S_t when S_t is large prevents S_t from diverging. A strictly positive drift in S_t at zero makes the lower bound a reflecting barrier for S_t. These two forces give rise to a non-degenerate stationary distribution in S_t in the long run.

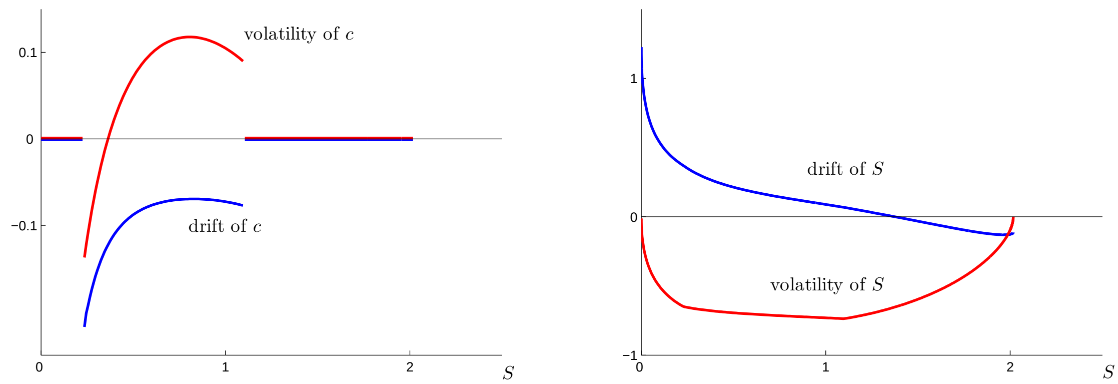
The second result uses the stationary distribution for S_t to examine the fraction of time that the optimal contract spends in the region with strong market-based incentives. Denote the invariant distribution of S_t by π.

Proposition 6 $\lim_{a_h \to \infty} \pi([S^*, \infty)) = 0$.

This proposition shows that if the worker's productivity has a sufficiently large drift under high effort, the optimal compensation contract will be free of pay-for-performance incentives most of the time. The argument for this result is that when a_h is large, the drift of the state variable is strongly negative at values of S_t strictly smaller than S^*. This makes events in which S_t leaves the region of strong market-based incentives very rare, and thus eliminates the need for pay-for-performance compensation incentives in equilibrium almost always.

6 Robustness

In this paper, we adopt the CARA utility function, a Brownian motion productivity process, and one-sided limited commitment for the tractability of this framework. In particular, in this framework we can show that the high effort recommendation is optimal everywhere, and we can characterize the region of strong market-based incentives analytically. However, our main result showing that market-based incentives have a strong impact on optimal compensation contracts is not specific to the CARA-normal model with one-sided lack of commitment. In this section, we examine robustness of our result by considering two extensions. First, we consider two-sided lack of commitment (i.e., firms can fire workers) in the CARA-normal model. Second, we consider a model with log preferences and a geometric Brownian motion productivity process in the one-sided lack of commitment case. The cost of departing from the CARA-normal



(a) Dynamics of compensation.

(b) Dynamics of S.

Figure 4: Example with two-sided lack of commitment.

framework in this section is that we are only able to provide numerical solutions for these two extensions.[26]

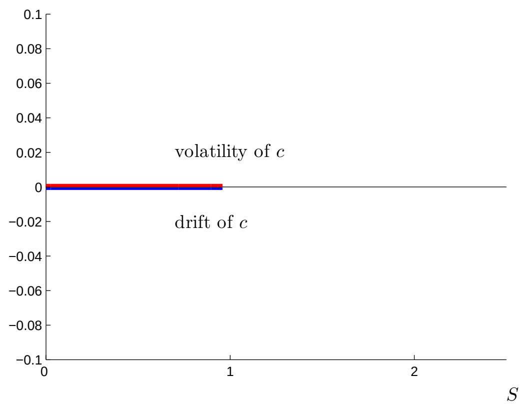
6.1 Two-sided lack of commitment

Following Phelan (1995), we assume in this section that firms can fire workers upon incurring a deadweight cost $F \geq 0$. This introduces a participation constraint on the side of the firm: $J(S_t) \leq F$ at all t. This constraint implies that $S_t \leq \bar{S}$ at all t, where $\bar{S} = J^{-1}(F)$. Our model in Section 5 is a special case with $F = \infty$.

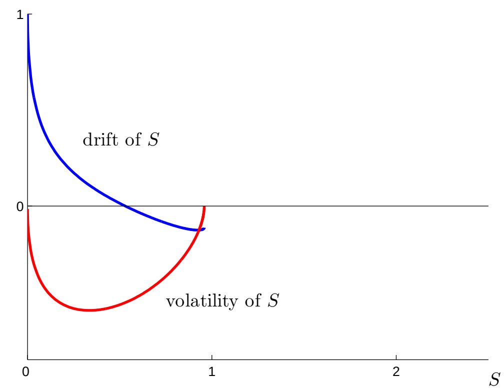
The numerical solutions we have obtained under various parameterizations show that market-based incentives become stronger when firm commitment becomes weaker. Figure 4 shows the equilibrium dynamics of compensation and the state variable in a typically parameterized example. In $[0, \bar{S}]$, there are two regions with strong market-based incentives, where compensation is constant, and one region in which short-term, pay-for-performance incentives are present.[27]

[26] Within the CARA-normal framework with one-sided limited commitment our analytical results can easily be extended to the case in which the absolute risk aversion parameter in the utility function is different from one. In this paper, we keep the absolute risk aversion parameter fixed at one because considering other values would make the notation less clear without adding any insight.

[27] Similar to the one-sided case, due to the non-separability of preferences between consumption and leisure, there is a discontinuity in the drift and in the volatility of compensation at the boundaries between the regions of strong and weak market-based incentives.



(a) Dynamics of compensation.



(b) Dynamics of S.

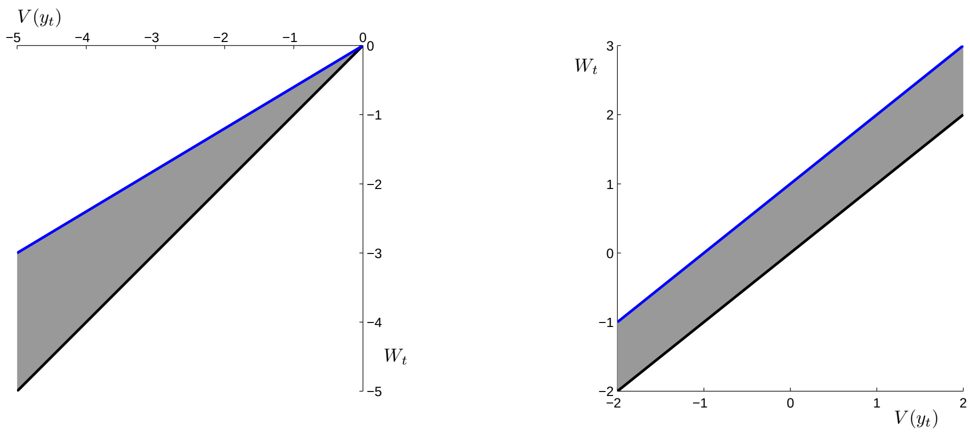
Figure 5: Example with two-sided lack of commitment and small firing cost F.

In the lower region of constant compensation, as in the baseline model, the worker is motivated by the prospect of the raise that the firm must give her to keep her from quitting when S_t hits zero. In the upper region of constant compensation, the worker is motivated by the wage cut that she will have to accept in order to keep the firm from firing her when S_t reaches $\bar{S}$.

Like quitting, firing of workers never actually happens in equilibrium. Panel (b) of Figure 4 shows that when S_t approaches the firing boundary $\bar{S}$, the drift of S_t is negative and its volatility goes to zero. Thus, like zero, $\bar{S}$ is a reflecting barrier for S_t.

In addition to the example presented in Figure 4, we have computed examples with different firing cost F. In these examples, we have examined the structure of equilibrium compensation. When F decreases, $\bar{S} = J^{-1}(F)$ decreases, so the interval $[0, \bar{S}]$ shrinks. The middle region of that interval, where market-based incentives are weak, shrinks as well. In fact, the middle region shrinks faster than the interval $[0, \bar{S}]$.

For a small enough firing cost F, the region of weak market-based incentives vanishes completely and, hence, equilibrium compensation never uses pay-for-performance incentives. In these cases, compensation is piecewise constant: c_t is constant when S_t fluctuates inside the interval $(0, \bar{S})$, c_t increases when S_t hits zero, and c_t decreases when S_t hits $\bar{S}$. Compensation, therefore, is the same as what it would be if workers' effort were observable. The commitment friction is strong enough to completely crowd out the private information friction in these cases.



(a) CARA-normal. (b) log-geometric.

Figure 6: Regions of strong market-based incentives.

Figure 5 presents one such example. In this example, F is smaller than in Figure 4, but greater than zero.[28] The equilibrium firing threshold $\bar{S}$ is also smaller than in Figure 4, but remains positive, i.e., the firm still provides insurance to the worker. Panel (a) shows that the drift and the volatility of c_t are both zero everywhere inside $(0, \bar{S})$. As in the previous example, we can see in panel (b) that 0 and $\bar{S}$ are reflecting barriers for S_t.

The lower the firing cost F, the less insurance firms provide in equilibrium. In the limiting case with $F = 0$, we have $\bar{S} = 0$ and firms provide no insurance, i.e., they simply pay to workers the output workers produce: $c_t = y_t$ at all t.

6.2 Log preferences and geometric Brownian motion

We have also studied numerically a version of our model with the log utility of consumption additively separable from the utility of leisure, with a geometric Brownian motion productivity process, and with one-sided commitment. In that framework, the high-effort action is not always optimal, but it is when slackness in the worker's quitting constraint is not too large. We have examined numerically the solution to the optimal contracting problem, and have found that strong market-based incentives also exist in this model.[29] Figure 6 shows the area of strong market-based incentives in our main CARA-normal model (panel (a)) and in a log-geometric

[28] All other parameters are the same as in Figure 4.

[29] Detailed solutions are available upon request.

model (panel (b)). The main conclusion of our previous analysis holds in the log-geometric framework: market-based incentives are strong when the quitting constraint is not very slack.

7 Conclusion

In this paper, we extend the classical dynamic principal-agent contracting framework by allowing the agent's outside option to depend on her to-date performance. We show that this extension gives rise to a new incentive for the agent to supply effort. That incentive is not driven by the response of contract-prescribed compensation to the observed performance, but rather by the increase in the agent's market value that occurs when the agent's performance is strong. The existence of this market-based incentive depends on the assumption of portability of at least a part of the agent's acquired skill or experience from one employer to another. We show that the market-based incentive can be strong in that it can provide sufficient incentives for maximum worker effort.

In the optimal long-term contract, the use of short-term compensation incentives, like bonuses, becomes necessary after an extended period of poor performance because market-based incentives are weak in this situation. At intermediate levels of performance, optimal compensation consists simply of constant pay with no incentives. Sustained strong performance leads to compensation increases, which can be interpreted as promotions giving the worker a new, better contract.

When we relax the assumption of full commitment on the side of the firm and allow for firing of workers upon paying a small firing cost, short-term compensation incentives become completely useless. When firms can fire workers, market-based incentives are stronger because workers are motivated not only by the prospect of a promotion but also by the risk of being fired.

Our theoretical analysis suggests that market-based incentives exist in principal-agent relationships beyond the particular setting of our model, as long as the agent's effort (or other desired action) improves her standing in the market outside the present principal-agent relationship. For this reason, we expect that market-based incentives play an important role in many firm-employee and, perhaps particularly so, firm-executive relationships. As well, market-based incentives may be important in lender-borrower relationships, where the borrower's outside option (e.g., refinancing terms) can depend on the performance of the loans she has held in the past.

In our model, maximum worker effort is optimal at all times. The existence of market-based incentives does not depend on this feature of the model. If less-than-maximum effort were to be implemented, however, we expect that the impact of market-based incentives on optimal compensation would be more complicated than the characterization we provide in this paper.

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Appendix A: proofs

Proof of Proposition 1

The proof proceeds in several steps.

(i) From the definition of equilibrium, we have $C(W,y) \geq 0$ for $W \geq V(y)$. This property and the quitting constraint $W_t \geq V(y_t)$ imply that the solution to the cost minimization problem (8) satisfies $\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s} - y_{t+s})ds|\mathcal{F}_t\right] \geq 0, \forall t \geq 0$.

(ii) Let us define $\tilde{C}(W,y)$ for any y and $W \geq V(y)$ as follows:

$$\tilde{C}(W,y) = \min_{(\boldsymbol{a},\boldsymbol{c})} \quad \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t - y_t)dt\right] \tag{27}$$

$$\text{subject to} \quad W_0 = W, \tag{28}$$

$$(\boldsymbol{a}, \boldsymbol{c}) \text{ is incentive compatible,} \tag{29}$$

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s} - y_{t+s})ds|\mathcal{F}_t\right] \geq 0, \forall t \geq 0, \tag{30}$$

where the process $\{y_t; t \geq 0\}$ starts from the initial condition $y_0 = y$. We now show that $C(W,y) = \tilde{C}(W,y)$ $\forall y, \forall W \geq V(y)$. Since the solution to (8) satisfies (30), $C(W,y) \geq \tilde{C}(W,y) \geq 0$. This implies $\tilde{C}(V(y),y) = 0$. If $W > V(y)$, denote the contract attaining the solution to (27) as $\tilde{\sigma}$. Define $\lambda \equiv \min\{t : W_t = V(y_t)\}$. Then a contract σ that is equal to $\tilde{\sigma}$ on $[0,\lambda)$ but switches to the market contract at λ has the same cost as $\tilde{\sigma}$, as both the market contract and the tail of $\tilde{\sigma}$ have zero cost starting at λ. Since σ satisfies (3), it is feasible in (8). Hence $C(W,y) \leq C(\sigma) = C(\tilde{\sigma}) = \tilde{C}(W,y)$.

(iii) If a contract $(\boldsymbol{a}, \boldsymbol{c})$ delivers utility W, then $(\boldsymbol{a}, \boldsymbol{c} + x)$ delivers We^{-x} for any $x \in \mathbb{R}$. This is because $W = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t, a_t)dt\right]$ if and only if

$$We^{-x} = e^{-x}\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t, a_t)dt\right] = \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}U(c_t + x, a_t)dt\right].$$

(iv) The incentive compatibility of $(\boldsymbol{a}, \boldsymbol{c})$ is equivalent to the incentive compatibility of $(\boldsymbol{a}, \boldsymbol{c} + x)$. In fact, the incentive compatibility of $(\boldsymbol{a}, \boldsymbol{c})$ requires that $\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty U(c_t, a_t)dt\right] \geq \mathbb{E}^{\boldsymbol{b}}\left[\int_0^\infty U(c_t, b_t)dt\right]$ for any deviation strategy $\boldsymbol{b}$, which is equivalent to

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty U(c_t + x, a_t)dt\right] \geq \mathbb{E}^{\boldsymbol{b}}\left[\int_0^\infty U(c_t + x, b_t)dt\right].$$

(v) We now verify that $\tilde{C}(W,y)=\tilde{C}(We^y,0)$. Suppose $(\boldsymbol{a},\boldsymbol{c})$ solves the problem in $\tilde{C}(W,y)$. We verify that $(\boldsymbol{a},\boldsymbol{c}-y)$ is feasible in the minimization problem defining $\tilde{C}(We^y,0)$. First, parts (iii) and (iv) imply that $(\boldsymbol{a},\boldsymbol{c}-y)$ delivers utility We^y and is incentive compatible. Second, if $\boldsymbol{y}=\{y_t;t\geq 0\}$ starts with the initial condition $y_0=y$ and $\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s}-y_{t+s})ds|\mathcal{F}_t\right]\geq 0$, then $\boldsymbol{y}^0=\{y_t^0;t\geq 0\}$ defined as $y_t^0=y_t-y\ \forall t$ starts with the initial condition $y_0^0=0$, and

$$\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s}-y-y_{t+s}^0)ds|\mathcal{F}_t\right]=\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}(c_{t+s}-y-(y_{t+s}-y))ds|\mathcal{F}_t\right]\geq 0.$$

Hence $(\boldsymbol{a},\boldsymbol{c}-y)$ satisfies (30) in $\tilde{C}(We^y,0)$ and so it is a feasible contract in this minimization problem.

Feasibility of $(\boldsymbol{a},\boldsymbol{c}-y)$ in this problem implies that

$$\tilde{C}(We^y,0)\leq\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t-y-y_t^0)dt\right]=\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t-y_t)dt\right]=\tilde{C}(W,y).$$

By a symmetric argument, we can show $\tilde{C}(We^y,0)\geq\tilde{C}(W,y)$. Thus, $\tilde{C}(W,y)=\tilde{C}(We^y,0)$, which by part (ii) implies (13).

(vi) To show $V(y)=V(0)e^{-y}$, suppose the equality does not hold. If $V(0)e^{-y}>V(y)$, then $0=C(V(0),0)=C(V(0)e^{-y},y)>C(V(y),y)=0$, which is a contradiction. If $V(0)<V(y)e^y$, then $0=C(V(0),0)<C(V(y)e^y,0)=C(V(y),y)=0$, which is again a contradiction.

(vii) If $(\boldsymbol{a},\boldsymbol{c})$ is optimal in the contracting problem for $C(y,V(y))$ defined in (8), then $(\boldsymbol{a},\boldsymbol{c}-y)$ is optimal in the contracting problem for $C(0,V(0))$. We first show that it is feasible in this problem. Parts (iii) and (iv) imply that the candidate contract $(\boldsymbol{a},\boldsymbol{c}-y)$ is incentive compatible and delivers utility $V(y)e^y=V(0)$. The candidate contract satisfies the quitting constraint (3) because

$$\begin{aligned}\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}U(c_{t+s}-y,a_{t+s})ds\,|\mathcal{F}_t\right] &= \exp(y)\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rs}U(c_{t+s},a_{t+s})ds\,|\mathcal{F}_t\right]\\ &\geq \exp(y)V(y_t)\\ &= V(y_t-y)\\ &= V\left(y_t^0\right),\end{aligned}$$

where, as before, the income process y_t starts at y, and $y_t^0=y_t-y$ starts at 0. Thus, the candidate contract $(\boldsymbol{a},\boldsymbol{c}-y)$ satisfies quitting, IC, and promise-keeping constraints, and so it is feasible in the contracting problem in a match with a worker whose initial productivity is 0 and whose market value is $V(0)$.

Next we show that the candidate contract $(\boldsymbol{a}, \boldsymbol{c} - y)$ attains $0 = C(0, V(0))$, and hence is optimal in this problem:

$$\begin{aligned}
\mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}((c_t - y) - y_t^0)dt\right] &= \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t - y - (y_t - y))dt\right] \\
&= \mathbb{E}^{\boldsymbol{a}}\left[\int_0^\infty re^{-rt}(c_t - y_t)dt\right] \\
&= 0.
\end{aligned}$$

■

Proof of Proposition 2

We first show that

$$J_{FC}(S) = J_{FC}(0) + S, \text{ for all } S \in \mathbb{R}. \tag{31}$$

Indeed, if an IC contract $(\boldsymbol{a}, \boldsymbol{c})$ delivers to the worker initial utility $V_{FC}(0)$, then for any $S \in \mathbb{R}$ the contract $(\boldsymbol{a}, \boldsymbol{c} + S)$ is also IC and delivers to the worker initial utility $V_{FC}(0)\exp(-S) = V_{FC}(S)$. Hence, for any y, the principal's cost function under full commitment satisfies $C_{FC}(V_{FC}(S), y) = C_{FC}(V_{FC}(0), y) + S$. Setting $y = 0$ in this equality and using definition $J_{FC}(S) = C_{FC}(V_{FC}(S), 0)$, we obtain $J_{FC}(S) = J_{FC}(0) + S$.

Substituting (31) into the HJB equation (21) and using $J'_{FC} = 1$ and $J''_{FC} = 0$, we obtain

$$r\left(S_t + J_{FC}(0)\right) = rS_t - r\log(-V_{FC}(0)) + \min_{\hat{u}_t, \hat{Y}_t}\left\{r(-\log(-\hat{u}_t)) + r\left(-1 - \hat{u}_t\right) + \frac{1}{2}\hat{Y}_t^2 - a_h\right\}.$$

Canceling rS_t on both sides, we obtain a static minimization problem (controls do not change over time) determining the value of $J_{FC}(0)$

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) + \min_{\hat{u}, \hat{Y} \geq -\hat{u}\beta}\left\{r(-\log(-\hat{u}_t)) + r\left(-1 - \hat{u}\right) + \frac{1}{2}\hat{Y}^2 - a_h\right\}. \tag{32}$$

Since the value minimized is quadratic in $\hat{Y}$ and $-\hat{u}\beta > 0$, the IC constraint will bind and optimal $\hat{Y} = -\hat{u}\beta$, which implies (24). The optimal $\hat{u}$ solves the convex problem

$$\min_{\hat{u}}\left\{-r\log(-\hat{u}) + r\left(-1 - \hat{u}\right) + \frac{1}{2}\left(-\hat{u}\right)^2\beta^2\right\}.$$

The first-order condition of this problem is a quadratic function in $(-\hat{u})$ given by

$$-1 + (-\hat{u}) + r^{-1}\beta^2\left(-\hat{u}\right)^2 = 0, \tag{33}$$

with a single positive root

$$-\hat{u} = \frac{\sqrt{1 + 4r^{-1}\beta^2} - 1}{2r^{-1}\beta^2}.$$

This root is in Proposition 2 denoted by ρ. Because $0 < \frac{\sqrt{1+4x}-1}{2x} < 1$ for all $x > 0$, we have that $0 < \rho < 1$. Substituting $-\hat{u} = \rho$ and $\hat{Y} = \rho\beta$ into (32) yields

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) - r\log(\rho) + r(-1+\rho) + \frac{1}{2}\rho^2\beta^2 - a_h. \tag{34}$$

To confirm that high effort is always optimal, note that the value of $J_{FC}(0)$ under low effort would be determined by

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) + \min_{\hat{u},\hat{Y}\leq -\hat{u}\beta}\left\{r(-\log(-\hat{u})) + r(-1-\hat{u}_t\phi) + \frac{1}{2}\hat{Y}^2 - a_l\right\}, \tag{35}$$

where the optimal $\hat{Y} = 0$ and the optimal $\hat{u}$ solves $\min_{\hat{u}}\{-r\log(-\hat{u}) + r(-1-\hat{u}\phi)\}$, which has a unique solution $-\hat{u} = \phi^{-1}$. This implies that $J_{FC}(0)$ under low effort would be

$$rJ_{FC}(0) = -r\log(-V_{FC}(0)) - r\log(\phi^{-1}) - a_l.$$

Thus, high effort is optimal if and only if $-r\log(\phi^{-1}) - a_l \geq -r\log(\rho) + r(-1+\rho) + \frac{1}{2}\rho^2\beta^2 - a_h$. To prove this inequality, note that Assumption 1 implies $\beta < \sigma$ and

$$\begin{aligned} a_h - a_l - r\log(\phi^{-1}) \geq \frac{1}{2}\beta\sigma \geq \frac{1}{2}\beta^2 &= \left(-r\log(-\hat{u}) + r(-1-\hat{u}) + \frac{1}{2}(-\hat{u})^2\beta^2\right)\Big|_{\hat{u}=-1} \\ &\geq -r\log(\rho) + r(-1+\rho) + \frac{1}{2}\rho^2\beta^2. \end{aligned}$$

We next show (23). From $u(c_t)/W_t = -\hat{u} = \rho$, we have $-\exp(-c_t) = W_t\rho$, which gives us that $dc_t = -d\log(-W_t) = d(S_t + y_t)$. Recalling (20), or using Ito's lemma again, we have

$$\begin{aligned} dc_t &= \left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2\right)dt + \hat{Y}dw_t^a \\ &= -\left(r(1-\rho) - \frac{1}{2}(\rho\beta)^2\right)dt + \rho\beta dw_t^a \\ &= -\mu dt + \rho\beta dw_t^a, \end{aligned}$$

where the second line uses optimal controls $-\hat{u} = \rho$ and $\hat{Y} = \rho\beta$, and the third line uses the definition of μ in Proposition 2. To see that $\mu > 0$ note that $r(1-\rho) - \frac{1}{2}(\rho\beta)^2 > r(1-\rho) - (\rho\beta)^2 = 0$, where the equality follows from (33). To obtain initial consumption c_0, note that $J_{FC}(0) = 0$ in equilibrium. This and (34) imply that

$$r(\log(-V_{FC}(0)) + \log(\rho)) = r(-1+\rho) + \frac{1}{2}\rho^2\beta^2 - a_h = -\mu - a_h.$$

From $-\exp(-c_0) = W_0\rho = V_{FC}(y_0)\rho = V_{FC}(0)e^{-y_0}\rho$ we have

$$c_0 = y_0 - (\log(-V_{FC}(0)) + \log(\rho)) = y_0 + \frac{\mu + a_h}{r}.$$

Solving $dc_t = -\mu dt + \rho\beta dw_t^a$ with this initial condition yields (23). ■

Proof of Proposition 3

We know from Grochulski and Zhang (2011) that the optimal compensation at time t is given by $c_t = u^{-1}(\bar{u}(m_t))$, where $\bar{u}$ is a strictly increasing function given by

$$\bar{u}(y) = V_{FI}(y) - \frac{V'_{FI}(y)}{\lim_{\varepsilon \downarrow 0} \frac{d}{d\varepsilon}\left(1 - \mathbb{E}_y^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right]\right)},$$

with $\tau_{y+\epsilon}$ denoting the hitting time of the level $y+\epsilon$. Because y_t is a Brownian motion with drift a_h, we know that $\mathbb{E}_y^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right] = e^{-\kappa\varepsilon}$. Thus, $\lim_{\varepsilon\downarrow 0} \frac{d}{d\varepsilon}\left(1 - \mathbb{E}^{a_h}\left[e^{-r\tau_{y+\varepsilon}}\right]\right) = \kappa$ and the function $\bar{u}$ is given by $\bar{u}(y) = V_{FI}(y) - \frac{-V_{FI}(y)}{\kappa} = \left(1+\frac{1}{\kappa}\right)V_{FI}(y) = u(y)\left(1+\frac{1}{\kappa}\right)(-V_{FI}(0))$. Therefore, optimal compensation satisfies $u(c_t) = u(m_t)\left(1+\frac{1}{\kappa}\right)(-V_{FI}(0))$. Applying u^{-1} to both sides, we obtain $c_t = m_t - \log\left(1+\frac{1}{\kappa}\right) - \log(-V_{FI}(0))$. From $J_{FI}(0) = 0$ we can compute $\log(-V_{FI}(0)) = \log\left(\frac{\kappa}{\kappa+1}\right) + \frac{\kappa\sigma^2}{2r}$, which gives us (25).

As in Grochulski and Zhang (2011), the worker's continuation value process satisfies

$$W_t = \left(1 - e^{-\kappa(m_t - y_t)}\right)\bar{u}(m_t) + e^{-\kappa(m_t-y_t)}V_{FI}(m_t) = \left(1 + \frac{1 - e^{-\kappa(m_t-y_t)}}{\kappa}\right)V_{FI}(m_t),$$

from which we can compute the volatility of W_t as $-V_{FI}(m_t)e^{-\kappa(m_t-y_t)}\sigma$, which, with $u(c_t) = V_{FI}(m_t)\left(1+\frac{1}{\kappa}\right)$ gives us (26). ■

Proof of Corollary 1

The proof follows immediately from (26). We only need to check that $\delta > 0$, or $\frac{\kappa\sigma}{\kappa+1} > \beta$. Indeed,

$$\frac{\kappa\sigma}{\kappa+1} > \frac{r\sigma\log(\phi^{-1})}{a_h - a_l} > \frac{r\sigma(1-\phi)}{a_h - a_l} = \beta,$$

where the first inequality follows from Assumption 1. ■

Preliminary analysis of the HJB equation

Below, we will often use $\hat{u}$, $\hat{Y}$, J' and J'' as shorthand notations for $\hat{u}(S)$, $\hat{Y}(S)$, $J'(S)$ and $J''(S)$, respectively.

Lemma A.1 *The IC constraint is slack if and only if* $\frac{\sigma J' J''}{J'+J''} \geq \beta$. *When it is slack,*

$$\hat{Y} = \frac{\sigma J''}{J' + J''}, \tag{36}$$

$$\hat{u} = -J'^{-1}. \tag{37}$$

Proof The first-order conditions for $\hat{Y}$ and $\hat{u}$ are

$$\hat{Y} \geq \frac{\sigma J''}{J' + J''}, \quad \hat{u} \geq -J'^{-1},$$

with equalities if the IC constraint is slack. Thus, if the IC constraint is slack, then (36) and (37) hold, and $\hat{Y} \geq -\hat{u}\beta$ implies $\beta \leq \frac{\hat{Y}}{-\hat{u}} = \frac{\sigma J' J''}{J'+J''}$. If the IC constraint binds, then $\hat{Y} > \frac{\sigma J''}{J'+J''}$ and $\hat{u} > -J'^{-1}$, and thus $\hat{Y} = -\hat{u}\beta$ implies $\beta = \frac{\hat{Y}}{-\hat{u}} > \frac{\sigma J' J''}{J'+J''}$. ■

Let $\mathscr{H}(S, J', J'')$ denote the right-hand side of the HJB equation (21), that is

$$\begin{aligned} \mathscr{H}(S, J', J'') \equiv \min_{\substack{\hat{u}, \hat{Y}, \\ \hat{Y} \geq -\hat{u}\beta}} \quad & r(S - \log(-V(0)) - \log(-\hat{u})) \\ & + J' \left(r(-1 - \hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h \right) + \frac{1}{2} J''(\hat{Y} - \sigma)^2, \end{aligned} \tag{38}$$

where J' and J'' are scalars. Whenever $\mathscr{H}(S, J', J'')$ is invertible in J'', we may rewrite the HJB equation as a second-order ordinary differential equation (ODE)

$$J''(S) = \mathscr{H}^{-1}(S, J, J'). \tag{39}$$

We study the invertibility of $\mathscr{H}(S, J', \cdot)$ next.

Lemma A.2 *If $J' \geq \frac{\kappa}{\kappa+1}$, then at any $J'' \in [0, \infty)$ the function $\mathscr{H}(S, J', J'')$ is strictly increasing in J'', and*

$$\hat{Y} < \sigma. \tag{40}$$

Proof The Envelope theorem states that $\frac{\partial \mathscr{H}}{\partial J''} = \frac{1}{2}(\hat{Y} - \sigma)^2$, which implies that $\mathscr{H}(S, J', J'')$ strictly increases in J'' whenever $\hat{Y} \neq \sigma$. It is then sufficient to show (40). Indeed, if the IC constraint is slack, then $\hat{Y} = \frac{\sigma J''}{J'+J''} < \sigma$. If the IC constraint binds, then

$$\hat{Y} = -\hat{u}\beta < \beta J'^{-1} \leq \frac{r(1-\phi)\sigma}{(a_h - a_l)\frac{\kappa}{\kappa+1}} < \frac{r(1-\phi)\sigma}{r\log(\phi^{-1})} < \sigma, \tag{41}$$

where the inequalities follow from $-\hat{u} < J'^{-1}$, $J' \geq \frac{\kappa}{\kappa+1}$, Assumption 1, and $\frac{1-\phi}{\log(\phi^{-1})} < 1$. ■

Lemma A.2 allows us to define the ODE (39) in the region

$$D \equiv \left\{ (S, J, J') \in \mathbb{R}^3 : \mathscr{H}(S, J', 0) \leq rJ \leq \mathscr{H}(S, J', \infty) \text{ and } J' \geq \frac{\kappa}{\kappa+1} \right\}.$$

Next we derive an explicit functional form for $\mathscr{H}^{-1}(S, J, J')$ when the IC constraint is slack.

Lemma A.3 *If the IC constraint is slack, then*

$$J'' = \left(\frac{\sigma^2/2}{r(J - S + \log(-V(0)) - \log(J') - 1) + (r + a_h)J'} - \frac{1}{J'} \right)^{-1}. \tag{42}$$

Proof If the IC constraint is slack, substituting (36) and (37) into the HJB equation yields

$$
\begin{aligned}
rJ \;=\; & rS - r\log(-V(0)) + r\log(J') + J'\left(r\left(-1+\frac{1}{J'}\right)+\frac{1}{2}\left(\frac{\sigma J''}{J'+J''}\right)^2 - a_h\right) \\
& + \frac{1}{2}J''\left(\frac{\sigma J''}{J'+J''}-\sigma\right)^2,
\end{aligned}
$$

which simplifies to

$$
rJ = rS - r\log(-V(0)) + r\log(J') - rJ' + r - a_h J' + \frac{1}{2}\sigma^2\frac{J'J''}{J'+J''}.
$$

Solving for J'' in the above yields (42). ■

The next lemma studies the HJB equation at the boundary $S=0$. Let $\alpha(S_t)$ and $\zeta(S_t)$ denote the drift and the volatility of S_t given in (20) evaluated at optimal controls $\hat{u}(S_t)$ and $\hat{Y}(S_t)$.

Lemma A.4 *In the model with two frictions,*

(i) $\frac{\kappa}{\kappa+1} \le J'(0) \le \frac{r}{r+a_h-\frac{1}{2}\sigma^2}$ *and* $0 \le \alpha(0) \le \frac{1}{2}(\kappa+1)\sigma^2$,

(ii) $J''(0) = \infty$ *and* $\zeta(0)=0$,

(iii) the IC constraint is slack when the quitting constraint binds.

Proof That $\alpha(0)\ge 0$ and $\zeta(0)=0$ follow from the nonnegativity of S_t at all t. In particular, $\zeta(0)\neq 0$ would imply $S_t<0$ shortly after $S_t=0$ because a typical Brownian motion sample path has infinite variation. From the law of motion (19) we have that $\alpha(0) = r\left(-1-\hat{u}(0)\right)+\frac{1}{2}\sigma^2 - a_h$ and $\zeta(0)=\hat{Y}(0)-\sigma$.

(i) First, to show $\frac{\kappa}{\kappa+1}\le J'(0)$, by contradiction, suppose $J'(0) < \frac{\kappa}{\kappa+1} = J'_{FI}(0)$. Then

$$
\begin{aligned}
& rJ_{FI}(0) + r\log(-V_{FI}(0)) \\
= \; & \min_{\hat{u}} r(-\log(-\hat{u})) + J'_{FI}(0)\left(r(-1-\hat{u}) - a_h + \frac{1}{2}\sigma^2\right) \\
= \; & r\log\left(\frac{\kappa+1}{\kappa}\right) + J'_{FI}(0)\left(r(-1+\frac{\kappa+1}{\kappa}) - a_h + \frac{1}{2}\sigma^2\right) \\
> \; & r\log\left(\frac{\kappa+1}{\kappa}\right) + J'(0)\left(r(-1+\frac{\kappa+1}{\kappa}) - a_h + \frac{1}{2}\sigma^2\right) \\
\ge \; & \min_{-\hat{u}\beta\le\sigma} r(-\log(-\hat{u})) + J'(0)\left(r(-1-\hat{u}) - a_h + \frac{1}{2}\sigma^2\right) \;=\; rJ(0) + r\log(-V(0)),
\end{aligned}
$$

where the first inequality follows from $\left(r(-1+\frac{\kappa+1}{\kappa}) - a_h + \frac{1}{2}\sigma^2\right) > 0$, and the second inequality follows from $\frac{\kappa+1}{\kappa}\beta < \sigma$. Because $J_{FI}(0)$ is the minimum cost to deliver utility

$V_{FI}(0)$ under one friction (limited commitment), the scalability in Proposition 1 implies that $J_{FI}(0) + \log(-V_{FI}(0)) - \log(-V(0))$ is the minimum cost to deliver utility $V(0)$ under one friction, which must be lower than $J(0)$, the cost to deliver the same utility $V(0)$ under two frictions. This contradicts the above inequality.

Second, since part (iii) shows that the IC constraint is slack, it follows from Lemma A.1 that $-\hat{u} = J'^{-1}$. Under this condition, $J'(0) \leq \frac{r}{r+a_h-\frac{1}{2}\sigma^2}$ is equivalent to $r\left(-1-\hat{u}(0)\right) + \frac{1}{2}\sigma^2 - a_h = \alpha(0) \geq 0$. Further, under $-\hat{u} = J'^{-1}$, $\frac{\kappa}{\kappa+1} \leq J'(0)$ is equivalent to $r\left(-1-\hat{u}(0)\right) + \frac{1}{2}\sigma^2 - a_h \leq r\left(-1+\frac{\kappa+1}{\kappa}\right) + \frac{1}{2}\sigma^2 - a_h = \frac{1}{2}(\kappa+1)\sigma^2$.

(ii) Suppose $J''(0) < \infty$ so that the assumptions of Lemma A.2 are met. But then (40) contradicts $\zeta(0) = \hat{Y}(0) - \sigma = 0$.

(iii) It follows from Assumption 1 and $J''(0) = \infty$ that $\beta = \frac{r\sigma(1-\phi)}{a_h-a_l} < \frac{r\sigma\log(\phi^{-1})}{a_h-a_l} < \sigma = \frac{\sigma J' J''}{J'+J''}$. By Lemma A.1, thus, the IC constraint is slack when $S_t = 0$.

■

Discussion. It is useful to briefly discuss the intuition behind Lemma A.4. As in the model with full information, the binding quitting constraint at $S_t = 0$ forces the firm to match the volatility of the worker's continuation value to that of her outside option, which implies that $\zeta(0) = 0$. This, in turn, is consistent with the firm's cost minimization if and only if the firm is infinitely averse to volatility in S_t at zero, hence $J''(0) = \infty$.

Because the firm matches the worker's continuation value volatility to her outside value volatility, the firm provides no insurance to the worker at $S_t = 0$. Because providing insurance is only feasible when $S_t > 0$, the firm induces a positive drift in S_t at $S_t = 0$ in order to be able to provide insurance. Comparing part (*i*) of Lemma A.4 with part (*i*) of Lemma B.1, however, we see that the firm's aversion to drift in the state variable, represented by the first derivative of the cost function, is larger in the two-friction model than in the full-information model. Accordingly, the positive drift in S_t at zero is smaller here than in the full-information model. This difference is due to the cost of future incentives. Part (*iii*) of Lemma A.4 shows that the IC constraint is slack when the quitting constraint binds. But we know from our analysis of the full-commitment model in Section 3 that the IC constraint binds when the quitting constraint is completely absent. Since the equilibrium contract in the two-friction model approximates the equilibrium contract of the full-commitment model when S_t is large, the IC constraint will bind in the two-friction model at S_t large enough. Inducing a positive drift in S_t in the two-friction model, therefore, has the disadvantage of making it more likely that the quitting constraint becomes sufficiently slack for the IC constraint to bind. This disadvantage is absent in the full-information model. The expected cost of future incentives, thus, makes positive drift in S_t more costly to the firm in the two-friction model, which is reflected in the firm's higher drift aversion $J'(0) \geq J'_{FI}(0)$ and lower drift of S_t at zero, as shown in part (i) of Lemma A.4.[30]

Closely related is the intuition for why the IC constraint is slack when the quitting constraint binds. Corollary 1 shows that the IC constraint is slack at $S_t = 0$ in the full-information model. This and the fact that the firm has a higher drift aversion in the two-friction model imply that the IC constraint must also be slack at $S_t = 0$ in the two-friction model. Indeed, a smaller drift in S_t at zero implies that the worker in the two-friction model receives a higher utility flow $\hat{u}_t$. Since the volatility of S_t is zero at $S_t = 0$, the normalized, market-induced sensitivity of the worker's continuation value, $\hat{Y}_t$, must equal σ in both models. With higher $\hat{u}_t$ and the same $\hat{Y}_t$, the IC constraint is more slack in the two-friction model than in the full-information model.

Proof of Theorem 1

The proof is organized into three lemmas: Lemma A.5, Lemma A.9 and Lemma A.10. Three auxiliary lemmas are also proved: Lemma A.6, Lemma A.7 and Lemma A.8.

We start out by noting that because $J''(0) = \infty$, the HJB equation at $S = 0$ reduces to

$$J(0) = -\log(-V(0)) + \log(J'(0)) + 1 - J'(0)\frac{r + a_h - \frac{1}{2}\sigma^2}{r}.$$

Treating the right-hand side of this equation as a function of $J'(0)$, denote its value by $h(J'(0))$. Lemma A.4 now implies a range of possible values for $J(0)$, $J'(0)$ and $J''(0)$ given by $\big(J(0), J'(0), J''(0)\big) = (h(J'(0)), J'(0), \infty)$ for $J'(0) \in \left[\frac{\kappa}{\kappa+1}, \frac{r}{r+a_h-\frac{1}{2}\sigma^2}\right]$. Thus, the knowledge of $J'(0)$ would be sufficient to pinpoint the values for $J(0)$ and $J''(0)$. Not knowing $J'(0)$, however, we will proceed as follows. Denote by $K(S)$ the function solving the HJB equation starting from an initial condition $K'(0) \in \left[\frac{\kappa}{\kappa+1}, \frac{r}{r+a_h-\frac{1}{2}\sigma^2}\right]$. This gives us a set of candidate solution curves $K(S)$, one for each starting value $K'(0) \in \left[\frac{\kappa}{\kappa+1}, \frac{r}{r+a_h-\frac{1}{2}\sigma^2}\right]$. The true cost function J has to coincide with one of these curves. The asymptotic condition $\lim_{S\to\infty} J'(S) = 1 = J'_{FC}(S)$ will determine which of the candidate solution curves represents the true cost function J.

In order to carry out this program, we need to first show that the solution to the HJB equation (42) exists in the neighborhood of zero despite the fact that the HJB does not satisfy the Lipschitz condition at $S = 0$ (because $J''(0) = \infty$).

Lemma A.5 *The HJB equation has a unique candidate solution K in a neighborhood of $S = 0$ with the boundary condition $(K(0), K'(0), K''(0)) = (h(K'(0)), K'(0), \infty)$ for any $K'(0) \in \left(\frac{\kappa}{\kappa+1}, B\right)$, where*

$$B = \begin{cases} 1, & \text{if } a_h < \frac{1}{2}\sigma^2, \\ \frac{r}{r+a_h-\frac{1}{2}\sigma^2}, & \text{if } a_h \geq \frac{1}{2}\sigma^2. \end{cases}$$

[30] Although conditions in part (i) of Lemma A.4 are given as weak inequalities, we show later that they are actually strict. Intuitively, the cost of future incentives is strictly positive because the IC constraint binds with strictly positive probability in equilibrium.

Proof Use a change of variable: define $x \equiv K'(S)$ and interpret both S and K as functions of x. Since $\frac{dS}{dx} = \frac{1}{K''(S)}$ and $\frac{dK}{dx} = \frac{dK}{dS}\frac{dS}{dx} = \frac{x}{K''(S)}$, we have the differential equation system

$$\begin{aligned} \frac{dS}{dx} &= \frac{1}{2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\,(r + a_h)\right)} - \frac{1}{x}, \\ \frac{dK}{dx} &= \frac{x}{2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\,(r + a_h)\right)} - 1. \end{aligned}$$

The solution exists and is unique in a neighborhood of $(x, S, K) = (K'(0), 0, h(K'(0)))$ because the local Lipschitz condition is satisfied. When x is close to $K'(0)$, S and K both strictly increase in x because

$$\begin{aligned} \left.\frac{dS}{dx}\right|_{x=K'(0)} &= 0, \\ \left.\frac{dK}{dx}\right|_{x=K'(0)} &= x\left.\frac{dS}{dx}\right|_{x=K'(0)} = 0, \\ \left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} &= \frac{2\sigma^{-2}\left(\frac{r}{x} - r - a_h\right)}{K'(0)^2} + \frac{1}{x^2} = \frac{2\sigma^{-2}\left(\frac{r}{K'(0)} - r - a_h + \frac{1}{2}\sigma^2\right)}{K'(0)^2} > 0, \\ \left.\frac{d^2K}{dx^2}\right|_{x=K'(0)} &= \left.\frac{dS}{dx}\right|_{x=K'(0)} + x\left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} > 0. \end{aligned}$$

Because the IC constraint is slack at $S = 0$ and $\frac{K'K''}{K'+K''} = \frac{x}{x\frac{1}{K''}+1} = \frac{x}{xS'(x)+1}$ is a continuous function of x, the IC constraint remains slack in a neighborhood of $x = K'(0)$. ■

We can now move on to studying global properties of candidate solutions to the HJB equation. For a given candidate solution K, define

$$\bar{S} \equiv \min_{S}\left\{S > 0 : K'(S) = 1 \text{ or } K''(S) = 0 \text{ or } K''(S) = \infty\right\},$$

with $\min\emptyset = \infty$.

The next three lemmas are auxiliary and will be used later.

Lemma A.6 *If $\bar{S} < \infty$ and $K''(\bar{S}) = 0$, then $K'(\bar{S}) < 1$.*

Proof By contradiction, suppose $K''(\bar{S}) = 0$ and $K'(\bar{S}) = 1$. Then the function $K(\cdot)$ that satisfies $K(S) = K(\bar{S}) + S - \bar{S}$ for all S solves the HJB equation. This violates the condition that $K''(0) = \infty$. ■

Lemma A.7 *If K is a candidate solution with $\bar{S} < \infty$ and $K''(\bar{S}) = \infty$, then $r(-1 - \hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h < 0$.*

Proof By contradiction, suppose $r(-1 - \hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h \geq 0$. The HJB equation at $S = \bar{S}$ is

$$rK(\bar{S}) = r(\bar{S} - \log(-V(0)) - \log(-\hat{u}(\bar{S}))) + K'(\bar{S})\left(r(-1 - \hat{u}(\bar{S})) + \frac{1}{2}\sigma^2 - a_h\right).$$

Because $r(-1-\hat{u}(\bar{S}))+\frac{1}{2}\sigma^2-a_h \geq 0$ and $K'(\bar{S})>K'(0)$,

$$\begin{aligned} rK(\bar{S}) &\geq r(\bar{S}-\log(-V(0))-\log(-\hat{u}(\bar{S})))+K'(0)\left(r(-1-\hat{u}(\bar{S}))+\frac{1}{2}\sigma^2-a_h\right) \\ &\geq r\bar{S}+\min_{\hat{u}} r(-\log(-V(0))-\log(-\hat{u}))+K'(0)\left(r(-1-\hat{u})+\frac{1}{2}\sigma^2-a_h\right) \\ &= r\bar{S}+rK(0), \end{aligned}$$

where the equality follows from the HJB equation at $S=0$. This contradicts the fact that $K'(S)<1$ for all $S\in[0,\bar{S}]$. ■

Lemma A.8 *If K is a candidate solution with $\bar{S}=\infty$, then $\lim_{S\to\infty} K'(S)=1$.*

Proof Suppose by contradiction $G\equiv\lim_{S\to\infty}K'(S)\neq 1$. Since $K'(S)<1$ for all S, $G<1$. Then

$$\begin{aligned} 0 &> rK(S)-r(K(0)+GS) \\ &= \min_{\hat{u},\hat{Y}}\left\{r(S-\log(-V(0))-\log(-\hat{u}))+K'(S)\left(r(-1-\hat{u})+\frac{1}{2}\hat{Y}^2-a_h\right)\right. \\ &\quad \left.+\frac{1}{2}K''(S)(\hat{Y}-\sigma)^2\right\}-r(K(0)+GS) \\ &\geq r(1-G)S+\min_{\hat{u}} r(-\log(-V(0))-\log(-\hat{u}))+K'(S)\left(r(-1-\hat{u})-a_h\right)-rK(0) \\ &\to \infty,\ \text{as } S\to\infty. \end{aligned}$$

This is a contradiction. ■

We now move on to two key lemmas of this proof.

Lemma A.9 *There exists a unique $K'(0)\in(\frac{\kappa}{\kappa+1},B)$ such that the candidate solution K satisfies $\bar{S}=\infty$.*

Proof **Existence:** Suppose by contradiction that all candidate solutions have $\bar{S}<\infty$. The rest of the proof proceeds in several steps.

(i) The solution curves starting with different $K'(0)$ are ordered: higher $K'(0)$ leads to permanently higher solution curves. Suppose there are two curves K_1 and K_2 with initial conditions $K_1(0)<K_2(0)$ and $K_1'(0)<K_2'(0)$, then $K_1'(S)<K_2'(S)$ for all $S\in[0,\min\{\bar{S}_1,\bar{S}_2\}]$. If not, define $\underline{S}\equiv\min\{S: K_1'(S)=K_2'(S)\}$. Because $K_1'(S)<K_2'(S)$ for all $S\leq\underline{S}$, $K_1(\underline{S})<K_2(\underline{S})$. Hence the HJB equation and $K_1'(\underline{S})=K_2'(\underline{S})$ imply that $K_1''(\underline{S})<K_2''(\underline{S})$, which means that $K_1'(S)>K_2'(S)$ when $\underline{S}-S>0$ is small. This contradicts the definition of $\underline{S}$.

(ii) Define

$$\begin{aligned} U &\equiv \{K'(0) : \bar{S} < \infty, \text{ either } K'(\bar{S}) = 1 \text{ or } K''(\bar{S}) = \infty\}, \\ L &\equiv \{K'(0) : \bar{S} < \infty, K''(\bar{S}) = 0\}. \end{aligned}$$

It follows from Lemma A.6 that $U \cap L = \emptyset$. We show below that both U and L are nonempty and open, which generates a contradiction because $(\frac{\kappa}{\kappa+1}, B) = U \cup L$ is a connected set.

(iii) U is open. Take a $K'(0) \in U$. We will show that there exists a $\delta > 0$ such that if $|K_1'(0) - K'(0)| \leq \delta$, then $K_1'(0) \in U$. Since $K'(0) \in U$, $\bar{S} < \infty$. Two cases need to be considered: $K'(\bar{S}) = 1$, and $K''(\bar{S}) = \infty$. In the first case, because $K''(\bar{S}) > 0$, there exists a small $\epsilon > 0$ such that $K'(\bar{S} + \epsilon) > 1$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$, such if $|K_1'(0) - K'(0)| \leq \delta$, then

$$K_1'(\bar{S} + \epsilon) > 1, \tag{43}$$

$$\sup_{S \in [0, \bar{S}+\epsilon]} \left| \frac{1}{K_1''(S)} \right| < \infty. \tag{44}$$

Inequality (43) implies $\bar{S}_1 < \bar{S} + \epsilon$. It follows from (44) that $K_1''(\bar{S}_1) > 0$, hence $K_1'(0) \notin L$. Thus, $K_1'(0) \in U$.

In the second case, recall that the HJB equation is solved by a change of variable whenever $K''(S) = \infty$. Then

$$\left.\frac{dS}{dx}\right|_{x=K'(\bar{S})} = 0, \quad \left.\frac{d^2S}{dx^2}\right|_{x=K'(\bar{S})} = \frac{2\sigma^{-2}\left(r(-1-\hat{u}) - a_h + \frac{1}{2}\sigma^2\right)}{x^2} < 0,$$

where the inequality follows from Lemma A.7. Hence there exists a small $\epsilon > 0$ such that $\frac{dS}{dx}|_{x=K'(\bar{S})+\epsilon} < 0$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$ such that if $|K_1'(0) - K'(0)| \leq \delta$, then

$$\left.\frac{dS_1}{dx}\right|_{x=K'(\bar{S})+\epsilon} < 0, \tag{45}$$

$$\sup_{S \in [0, S_1(K'(\bar{S})+\epsilon)]} \left| \frac{1}{K_1''(S)} \right| = \sup_{x \in [K_1'(0), K'(\bar{S})+\epsilon]} \left| \frac{dS_1}{dx} \right| < \infty. \tag{46}$$

Inequality (45) implies $\bar{S}_1 < S_1(K'(\bar{S}) + \epsilon)$. It follows from (46) that $K_1''(\bar{S}_1) > 0$, hence $K_1'(0) \notin L$. Thus, $K_1'(0) \in U$.

(iv) L is open. Recall from Lemma A.6 that if $K'(0) \in L$, then $K''(\bar{S}) = 0$ and $K'(\bar{S}) < 1$. Differentiating the HJB equation and applying the Envelope theorem yield

$$0 = r + K''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}K'''(\sigma - \hat{Y})^2 - rK'.$$

Hence $K''(\bar S)=0$ and $K'(\bar S)<1$ imply

$$\frac{1}{2}K'''(\bar S)(\sigma-\hat Y)^2 = r(K'-1)-K''\left(r(-1-\hat u)+\frac{1}{2}\hat Y^2-a_h\right)<0.$$

Therefore $K'''(\bar S)<0$ and there exists a small $\epsilon>0$ such that $K''(\bar S+\epsilon)<0$.

Pick a small $\epsilon_1>0$, such that $K'(\epsilon_1)$ satisfies $r(-1+\frac{1}{K'(\epsilon_1)})+\frac{1}{2}\sigma^2-a_h>0$. Recall that the HJB equation is solved in a neighborhood of $S=0$ by a change of variable. For convenience, we denote the solution for an initial condition $K_1'(0)$ by $S_1(x)$ when $x\in[K'(0),K'(\epsilon_1)]$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta>0$, such that if $|K_1'(0)-K'(0)|\le\delta$, then

$$\begin{aligned} K_1''(\bar S+\epsilon) &< 0, && (47)\\ \sup_{S\in[0,\bar S+\epsilon]}\left|K_1'(S)\right| &< 1, && (48)\\ \sup_{S\in[S_1(K'(\epsilon_1)),\bar S+\epsilon]}\left|K_1''(S)\right| &< \infty. && (49)\end{aligned}$$

Inequality (47) implies $\bar S_1<\bar S+\epsilon$. If $\bar S_1\in(0,S_1(K'(\epsilon_1))]$ (i.e., $K_1'(\bar S_1)\le K'(\epsilon_1)$), because $r(-1+\frac{1}{K_1'(\bar S_1)})+\frac{1}{2}\sigma^2-a_h>0$, Lemma A.7 implies that $K_1''(\bar S_1)<\infty$. If $\bar S_1\in[S_1(K'(\epsilon_1)),\bar S+\epsilon]$, (49) implies that $K_1''(\bar S_1)<\infty$. It follows from (48) and $K_1''(\bar S_1)<\infty$ that $K_1'(0)\notin U$. Hence $K_1'(0)\in L$ if $|K_1'(0)-K'(0)|\le\delta$.

(v) $L\neq\emptyset$. We will show that $\frac{\kappa}{\kappa+1}\in L$. By contradiction, suppose $\frac{\kappa}{\kappa+1}\in U$. That is, if $K'(0)=\frac{\kappa}{\kappa+1}$, then either $K'(\bar S)=1$ or $K''(\bar S)=\infty$. The HJB equations for J_{FI} and K imply that if $J_{FI}(S)+\log(-V_{FI}(0))\ge K(S)+\log(-V(0))$ and $J_{FI}'(S)=K'(S)$, then $J_{FI}''(S)\ge K''(S)$. Hence, the same argument as in part (i) shows that $J_{FI}'(S)\ge K'(S)$ for all $S\le\bar S$. It follows from $J_{FI}'(S)<1,\forall S$ that $K'(\bar S)<1$ and $K''(\bar S)=\infty$. A contradiction arises as follows.

$$\begin{aligned} & rJ_{FI}(\bar S)+r\log(-V_{FI}(0))\\ = & \min_{\hat u,\hat Y} r(\bar S-\log(-\hat u))+J_{FI}'(\bar S)\left(r(-1-\hat u)+\frac{1}{2}\hat Y^2-a_h\right)+\frac{1}{2}J_{FI}''(\bar S)(\hat Y-\sigma)^2\\ < & \min_{\hat u} r(\bar S-\log(-\hat u))+J_{FI}'(\bar S)\left(r(-1-\hat u)+\frac{1}{2}\sigma^2-a_h\right)\\ \le & r(\bar S-\log(-\hat u(\bar S)))+J_{FI}'(\bar S)\left(r(-1-\hat u(\bar S))+\frac{1}{2}\sigma^2-a_h\right),\end{aligned}$$

where $\hat u(\bar S)=-(K'(\bar S))^{-1}$ is the optimal $\hat u$ at $\bar S$ for $K(\bar S)$. Because $J_{FI}'(\bar S)\ge K'(\bar S)$ and $r(-1-\hat u(\bar S))+\frac{1}{2}\sigma^2-a_h<0$ (shown by Lemma A.7),

$$\begin{aligned} & r(\bar S-\log(-\hat u(\bar S)))+J_{FI}'(\bar S)\left(r(-1-\hat u(\bar S))+\frac{1}{2}\sigma^2-a_h\right)\\ \le & r(\bar S-\log(-\hat u(\bar S)))+K'(\bar S)\left(r(-1-\hat u(\bar S))+\frac{1}{2}\sigma^2-a_h\right)=rK(\bar S)+r\log(-V(0)),\end{aligned}$$

which is a contradiction as $J_{FI}(0) + \log(-V_{FI}(0)) = K(0) + \log(-V(0))$ and $J'_{FI}(S) \geq K'(S)$ imply $J_{FI}(\bar{S}) + \log(-V_{FI}(0)) \geq K(\bar{S}) + \log(-V(0))$.

(vi) $U \neq \emptyset$. First, suppose $a_h < \frac{1}{2}\sigma^2$. If $1 - K'(0) > 0$ is sufficiently small, then $K(\cdot)$ will reach $K' = 1$. Second, suppose $a_h \geq \frac{1}{2}\sigma^2$. If $K'(0) = B$, then we show that $\frac{dS}{dx}|_{x=B+\epsilon} < 0$ for small $\epsilon > 0$. To prove this, note that, similar to the proof in Lemma A.5,

$$\begin{aligned}
\left.\frac{dS}{dx}\right|_{x=K'(0)} &= 0 = \left.\frac{dK}{dx}\right|_{x=K'(0)}, \\
\left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} &= \frac{2\sigma^{-2}\left(\frac{r}{K'(0)} - r - a_h + \frac{1}{2}\sigma^2\right)}{K'(0)^2} = 0, \\
\left.\frac{d^2K}{dx^2}\right|_{x=K'(0)} &= \left.\frac{dS}{dx}\right|_{x=K'(0)} + x\left.\frac{d^2S}{dx^2}\right|_{x=K'(0)} = 0.
\end{aligned}$$

The Taylor expansion of $2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\,(r + a_h)\right)$ is

$$\begin{aligned}
& K'(0) + 2\sigma^{-2}\left(r\left(\frac{dK}{dx} - \frac{dS}{dx} - \frac{1}{x}\right) + (r + a_h)\right)(x - K'(0)) \\
& +2\sigma^{-2}r\left(\frac{d^2K}{dx^2} - \frac{d^2S}{dx^2} + \frac{1}{x^2}\right)(x - K'(0))^2 + o((x - K'(0))^2) \\
= \;& x + \frac{2\sigma^{-2}r}{(K'(0))^2}(x - K'(0))^2 + o((x - K'(0))^2) > x,
\end{aligned}$$

where the inequality holds when $x - K'(0) > 0$ is small, since $\frac{2\sigma^{-2}r}{(K'(0))^2} > 0$. Therefore,

$$\left.\frac{dS}{dx}\right|_{x=B+\epsilon} = \frac{1}{2\sigma^{-2}\left(r(K - S + \log(-V(0)) - \log(x) - 1) + x\,(r + a_h)\right)} - \frac{1}{x} < 0,$$

for small $\epsilon > 0$. Because the solution of a differential equation depends continuously on its initial condition, there exists a small $\delta > 0$, such that if the initial condition $K'_1(0) \in (B - \delta, B)$, then

$$\left.\frac{dS_1}{dx}\right|_{x=B+\epsilon} < 0, \tag{50}$$

$$\sup_{S\in[0,S_1(B+\epsilon)]}\left|\frac{1}{K''_1(S)}\right| = \sup_{x\in[K'_1(0),B+\epsilon]}\left|\frac{dS_1}{dx}\right| < \infty. \tag{51}$$

It follows from $\frac{dS_1}{dx}|_{x=K'_1(0)} > 0$ and (50) that $\frac{dS_1}{dx} = 0$ for some $\hat{S} \in (0, S_1(B+\epsilon))$. Because $\frac{dS_1}{dx} = \frac{1}{K''_1(S)}$, we know that $K''_1(\hat{S}) = \infty$. Hence $\bar{S}_1 \leq \hat{S}$ must be finite. It follows from (51) that $K''_1(\bar{S}_1) > 0$, hence $K'_1(0) \notin L$ and $K'_1(0) \in U$.

Uniqueness: By contradiction, suppose there are two initial conditions $K_1'(0) < K_2'(0)$ with $\bar{S}_1 = \bar{S}_2 = \infty$. Subtracting one HJB equation from the other yields

$$\begin{aligned} & r(K_2(S) - K_1(S)) \\ = \ & \min_{\hat{u},\hat{Y}} \left\{ -r\log(-\hat{u}) + K_2'(S)\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}K_2''(S)(\hat{Y}-\sigma)^2 \right\} \\ & -\min_{\hat{u},\hat{Y}} \left\{ -r\log(-\hat{u}) + K_1'(S)\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}K_1''(S)(\hat{Y}-\sigma)^2 \right\}. \end{aligned}$$

The left-hand side is positive at $S = 0$ and is strictly increasing with S, as shown in part (i) of the proof of existence. Lemma A.8 implies that $\lim_{S\to\infty} K_1'(S) = \lim_{S\to\infty} K_2'(S) = 1$. For any $\epsilon > 0$, there exists a large S such that $0 < K_1''(S) + K_2''(S) < \epsilon$. Therefore, the right-hand side can be made as small as needed if S is large. This is a contradiction. ■

Lemma A.10 *The candidate solution K with $\bar{S} = \infty$ is the true cost function J.*

Proof Because the technique of using the HJB equation to verify the optimality of K is standard, we omit the details of the steps involved. We verify two things:

(i) The cost of any IC contract is weakly higher than $K(S)$.

(ii) There exists an IC contract whose cost equals $K(S)$.

To see (i), pick an IC contract starting at $S_0 = S \geq 0$ and consider the stochastic process $\{S_t; t \geq 0\}$ in this contract. Define

$$M_t \equiv \int_0^t (c_s - y_s) r e^{-rs} ds + e^{-rt} K(S_t). \tag{52}$$

The HJB equation implies that M_t is a submartingale (i.e., it has a nonnegative drift), hence

$$K(S) = M_0 \leq \mathbb{E}\left[M_\infty\right] = \mathbb{E}\left[\int_0^\infty (c_s - y_s) r e^{-rs} ds\right]. \tag{53}$$

To see (ii), construct a stochastic process $\{S_t; t \geq 0\}$ using $S_0 = S$ and the policy functions implied by the HJB equation for K. Denote the contract generated by $\{S_t; t \geq 0\}$ and the policy functions as σ^*. Then M_t defined in (52) is a martingale, and the inequality in (53) is replaced with an equality. This shows that the cost of σ^* is $K(S)$. ■

Proof of Proposition 4

We will show the existence of a unique $S^* > 0$ such that

$$\frac{\sigma J' J''}{J' + J''} \quad \begin{cases} > \beta, & \text{if } S < S^*, \\ = \beta, & \text{if } S = S^*, \\ < \beta, & \text{if } S > S^*. \end{cases}$$

By Lemma A.1, this will show that the IC constraint is slack if and only if $S_t < S^*$.

Existence of S^*: We have shown in the proof of Lemma A.5 that $\frac{\sigma J'(S)J''(S)}{J'(S)+J''(S)} > \beta$ when S is small. If S^* does not exist, then $\frac{\sigma J'(S)J''(S)}{J'(S)+J''(S)} > \beta$ for all S. This implies that

$$\sigma J''(S) > \frac{\sigma J'(S)J''(S)}{J'(S)+J''(S)} \geq \beta, \text{ for all } S,$$

which contradicts the fact that $J'(S) < 1$ for all S.

Uniqueness of S^*: It is sufficient to show that if $\frac{\sigma J'(S^*)J''(S^*)}{J'(S^*)+J''(S^*)} = \beta$ for some S^*, then $\frac{\sigma J'(S)J''(S)}{J'(S)+J''(S)} > \beta$ for $S < S^*$.

First, we show that if $\frac{\sigma J'J''}{J'+J''} \geq \beta$, then

$$rJ' + a_h \frac{J'J''}{J'+J''} < r. \tag{54}$$

If $a_h \leq 0$, then (54) is obvious because $J' < 1$. When $a_h > 0$, by contradiction, suppose that $\frac{\sigma J'J''}{J'+J''} \geq \beta$ and $rJ' + a_h \frac{J'J''}{J'+J''} \geq r$ at some $\hat{S}$. Starting from $\hat{S}$, solve the differential equation

$$rJ = r(S - \log(-V(0)) + \log(J')) + J'(-r - a_h) + r + \frac{\sigma^2}{2}\frac{J'J''}{J'+J''}. \tag{55}$$

Differentiating with respect to S in the above yields

$$\begin{aligned}
\frac{\sigma^2}{2}\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS} &= rJ' - r\left(1 + \frac{J''}{J'}\right) + J''(r + a_h) \\
&= \frac{J'+J''}{J'}\left(rJ' + a_h \frac{J'J''}{J'+J''} - r\right) \geq 0.
\end{aligned} \tag{56}$$

Hence either $\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS} > 0$, or $\frac{d\left(\frac{J'J''}{J'+J''}\right)}{dS} = 0$. In the latter case, $rJ' + a_h \frac{J'J''}{J'+J''} - r = 0$ and it follows from $J'' > 0$ that $\frac{d^2\left(\frac{J'J''}{J'+J''}\right)}{dS^2} > 0$. In both cases, there exists a small $\epsilon > 0$ such that $\frac{J'J''}{J'+J''}$ is strictly increasing in $[\hat{S}, \hat{S}+\epsilon]$. Hence the solution J to (55) does satisfy the HJB equation on $[\hat{S}, \hat{S}+\epsilon]$ and the IC constraint is slack. If we extend the solution beyond $\hat{S}+\epsilon$, $\frac{J'J''}{J'+J''}$ is always strictly increasing, because J' is increasing and a_h is positive in (56). Hence,

$$J'' > \frac{J'J''}{J'+J''} > \frac{J'(\hat{S})J''(\hat{S})}{J'(\hat{S})+J''(\hat{S})}, \text{ for all } S > \hat{S},$$

contradicting the fact that $J'(S) < 1$ for all S.

Second, we show that $\frac{\sigma J'(S)J''(S)}{J'(S)+J''(S)} > \beta$ for all $S < S^*$. Solve the differential equation (55) backward on $[0, S^*]$. Equations (54) and (56) show that $\frac{J'(S)J''(S)}{J'(S)+J''(S)}$ is strictly decreasing in S. Hence the solution J to (55) does satisfy the HJB equation and the IC constraint is slack.

This completes the proof of the second statement in Proposition 4. To prove the first statement, we now show that both the drift and the volatility of compensation are zero when $S \leq S^*$.

It follows from $\hat{u} = \frac{u(c)}{-W}$ and $S = \log(\frac{V(y)}{W})$ that

$$c = -\log(-\hat{u}) - \log(-W) = -\log(-\hat{u}) + S + y - \log(-V(0)).$$

If $S \leq S^*$, then $-\hat{u} = (J')^{-1}$, and $c = \log(J') + S + y - \log(-V(0))$. According to Ito's lemma, the drift of compensation is

$$\begin{aligned}
& \frac{J''}{J'}\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 + \frac{1}{2}\frac{J'''J' - J''^2}{J'^2}(\hat{Y}-\sigma)^2 \\
= \; & \frac{J''}{J'}\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 + \frac{1}{2}\frac{J'''}{J'}(\hat{Y}-\sigma)^2 - \frac{1}{2}\frac{J''^2}{J'^2}(\hat{Y}-\sigma)^2 \\
= \; & \frac{J''}{J'}\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + r(-1+\frac{1}{J'}) + \frac{1}{2}\frac{J'''}{J'}(\hat{Y}-\sigma)^2 \\
= \; & \left(J''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + r + \frac{1}{2}J'''(\hat{Y}-\sigma)^2 - rJ'\right)J'^{-1},
\end{aligned}$$

where the second equality follows from $\hat{Y} = \frac{\sigma J''}{J'+J''}$ and $\hat{Y}^2 = \frac{J''^2}{J'^2}(\hat{Y}-\sigma)^2$. Differentiating the HJB equation with respect to S and applying the Envelope theorem yield

$$rJ' = r + J''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}J'''(\hat{Y}-\sigma)^2.$$

Therefore, the drift of compensation is zero. The volatility of compensation is

$$\frac{J'+J''}{J'}(\hat{Y}-\sigma) + \sigma = \frac{J'+J''}{J'}\left(-\frac{J'\sigma}{J'+J''}\right) + \sigma = 0.$$

Verification of optimality of high effort

Lemma A.11 *Under Assumption 1, it is optimal to implement high effort for all $S \geq 0$.*

Proof The law of motion for S_t under low effort is

$$dS_t = \left(r\left(-1-\hat{u}_t\phi\right) + \frac{1}{2}\hat{Y}_t^2 - a_l\right) + \left(\hat{Y}_t - \sigma\right)dw_t^{a_l}.$$

To show that low effort is suboptimal, we need to verify that

$$\begin{aligned}
& \min_{\substack{\hat{u},\hat{Y},\\ \hat{Y}\geq -\hat{u}\beta}} r(-\log(-\hat{u})) + J'(S)\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}J''(S)(\hat{Y}-\sigma)^2 \\
\leq \; & \min_{\substack{\hat{u},\hat{Y},\\ \hat{Y}\leq -\hat{u}\beta}} r(-\log(-\hat{u})) + J'(S)\left(r(-1-\hat{u}\phi) + \frac{1}{2}\hat{Y}^2 - a_l\right) + \frac{1}{2}J''(S)(\hat{Y}-\sigma)^2. \qquad (57)
\end{aligned}$$

First, if $S \leq S^*$, then the IC constraint $\hat{Y} \geq -\hat{u}\beta$ is slack according to Proposition 4. Inequality (57) is equivalent to

$$J'(S)(a_h - a_l) \geq r\log(\phi^{-1}),$$

which follows from $J'(S) \geq J'(0) > \frac{\kappa}{\kappa+1}$ and Assumption 1.

Second, if $S > S^*$ (i.e., the IC constraint binds), then $\frac{\sigma J' J''}{J'+J''} \leq \beta$. We have

$$\begin{aligned}
&\min_{\substack{\hat{u},\hat{Y},\\ \hat{Y}\geq -\hat{u}\beta}} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}J''(\hat{Y}-\sigma)^2 \\
\leq\ & \left(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) + \frac{1}{2}J''(\hat{Y}-\sigma)^2\right)\Big|_{\hat{u}=\frac{-1}{J'},\hat{Y}=\frac{\beta}{J'}} \\
=\ & \big(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) - a_h\right)\big)\big|_{\hat{u}=\frac{-1}{J'}} + \frac{1}{2J'^2}(J'\beta^2 + J''(\sigma J' - \beta)^2) \\
\leq\ & \big(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) - a_h\right)\big)\big|_{\hat{u}=\frac{-1}{J'}} + \frac{1}{2}\beta\sigma,
\end{aligned}$$

where the last inequality follows from $\sigma J' \geq \frac{r\sigma\log(\phi^{-1})}{a_h - a_l} \geq \frac{r\sigma(1-\phi)}{a_h - a_l} = \beta$ and

$$\begin{aligned}
\beta\sigma J'^2 - J'\beta^2 - J''(\sigma J' - \beta)^2 &= (\sigma J' - \beta)(J'\beta - J''(\sigma J' - \beta)) \\
&= (\sigma J' - \beta)(J' + J'')\left(\beta - \frac{\sigma J' J''}{J' + J''}\right) \geq 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\big(r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}) - a_h\right)\big)\big|_{\hat{u}=\frac{-1}{J'}} + \frac{1}{2}\beta\sigma \\
\leq\ & \min_{\hat{u}} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}\phi) - a_h\right) + r\log(\phi^{-1}) + \frac{1}{2}\beta\sigma \\
\leq\ & \min_{\substack{\hat{u},\hat{Y},\\ \hat{Y}\leq -\hat{u}\beta}} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}\phi) + \frac{1}{2}\hat{Y}^2 - a_l\right) + \frac{1}{2}J''(\hat{Y}-\sigma)^2 \\
& + J'(-a_h + a_l) + r\log(\phi^{-1}) + \frac{1}{2}\beta\sigma \\
\leq\ & \min_{\substack{\hat{u},\hat{Y},\\ \hat{Y}\leq -\hat{u}\beta}} r(-\log(-\hat{u})) + J'\left(r(-1-\hat{u}\phi) + \frac{1}{2}\hat{Y}^2 - a_l\right) + \frac{1}{2}J''(\hat{Y}-\sigma)^2,
\end{aligned}$$

where the last inequality follows from Assumption 1. Thus (57) is verified. ■

Proof of Proposition 5

We start with the following auxiliary lemma:

Lemma A.12 *$J'''(S) < 0$ for all $S \geq 0$. Further, $\lim_{S\to\infty} J''(S) = 0$.*

Proof When $S < S^*$, recall that we have shown that $\frac{J'(S)J''(S)}{J'(S)+J''(S)} = \frac{1}{\frac{1}{J'(S)}+\frac{1}{J''(S)}}$ is strictly decreasing in S in the proof of Proposition 4 . Hence either $J'(S)$ or $J''(S)$ must be strictly decreasing. Since $J'(S)$ increases with S, $J''(S)$ strictly decreases with S when $S < S^*$. If $J''(S)$ is not globally decreasing, then there is a $\bar{S} \geq S^*$ at which $J'''(\bar{S}) = 0$. When the IC constraint binds at $S > S^*$, we have $\hat{Y} = -\hat{u}\beta$ and the HJB equation takes the form of

$$\begin{aligned} rJ(S) &= rS - r\log(-V(0)) \\ &\quad + \min_{\hat{u}} \left\{ -r\log(-\hat{u}) + J'(S)\left(r\left(-1-\hat{u}\right) + \frac{1}{2}(-\hat{u}\beta)^2 - a_h \right) + \frac{1}{2}J''(S)\left(-\hat{u}\beta - \sigma\right)^2 \right\}. \end{aligned}$$

The first-order condition for the optimal $\hat{u}$ is

$$rc'(\hat{u}) + J''(S)\sigma\beta = rJ'(S) + (J'(S) + J''(S))\beta^2(-\hat{u}). \tag{58}$$

Because J' increases with S while J'' is stationary at $S = \bar{S}$, equation (58) implies that $(-\hat{u})$ and $\hat{Y}$ decrease with S, when S is close to $\bar{S}$. Differentiating the HJB equation yields

$$0 = r + J''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h \right) + \frac{1}{2}J'''(\sigma - \hat{Y})^2 - rJ'.$$

Because the term $J''\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) - rJ'$ decreases with $S \in (\bar{S}-\epsilon, \bar{S}+\epsilon)$ for a small ϵ, $J'''(S) < 0$ for $S \in (\bar{S}-\epsilon, \bar{S})$ and $J'''(S) > 0$ for $S \in (\bar{S}, \bar{S}+\epsilon)$. Because of these two inequalities, J''' cannot be zero again for any $S > \bar{S}$. That is, $J''' > 0$ for all $S > \bar{S}$. Then J'' increases with S and J' will reach one eventually, a contradiction. ■

Now we can prove the proposition.

First, we show that $(-\hat{u})$ and $\hat{Y}$ decrease with S. If $S \leq S^*$, then $-\hat{u} = \frac{1}{J'(S)}$ and $\hat{Y} = \frac{\sigma J''}{J'+J''}$ decrease with S, because J' increases and J'' decreases with S. If $S \geq S^*$, then $(-\hat{u})$ and $\hat{Y}$ decrease with S, because in the first-order condition (58), J' increases and J'' decreases with S, and $\sigma > \beta(-\hat{u})$. Further, because $\lim_{S\to\infty} J'(S) = 1$ and $\lim_{S\to\infty} J''(S) = 0$, the first-order condition (58) approaches condition (33), which means that $\lim_{S\to\infty}(-\hat{u}) = \rho$ and $\lim_{S\to\infty} \hat{Y} = \rho\beta$.

Second, we show the properties of the drift and the volatility of S. That $\alpha(S) = r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h$ and $\xi(S) = \hat{Y} - \sigma$ are decreasing in S is because $-\hat{u}$ and $\hat{Y}$ decrease with S. That $\alpha(0) > 0$ follows from $\hat{Y}(0) = \sigma$, $-\hat{u}(0) = \frac{1}{J'(0)}$, and $J'(0) < \frac{r}{r+a_h-\frac{1}{2}\sigma^2}$ in Lemma A.9. That $\lim_{S_t\to\infty} \alpha(S_t) = -\mu - a_h$ follows from $\lim_{S\to\infty}(-\hat{u}) = \rho$, $\lim_{S\to\infty} \hat{Y} = \rho\beta$, and the definition of μ. That $\lim_{S\to\infty} \zeta(S) = \rho\beta - \sigma$ follows from $\lim_{S\to\infty} \hat{Y} = \rho\beta$. ■

Proof of Theorem 2

First, plug optimal policies $-\hat{u}_t = \rho$ and $\hat{Y}_t = \rho\beta$ into (20) to obtain the equilibrium dynamics of the state variable S_t in the full-commitment model:

$$dS_t = -\left(\mu + a_h\right)dt - \left(\sigma - \rho\beta\right)dw_t^a. \tag{59}$$

Thus, by assumption we have

$$\mu + a_h > 0. \tag{60}$$

Second, construct a function $f : [J'(0), \infty) \to [0, \infty)$ such that $f'(x) > 0$ for $x > J'(0)$, $f''(x)$ is continuous for $x \geq J'(0)$, and $f(x) = S(x)$ for $x \approx J'(0)$ and $f(x) = x$ for large x. We can extend the domain of f to $(-\infty, \infty)$ by defining $f(x) \equiv f(2J'(0) - x)$ for $x < J'(0)$. Because $S'(x)|_{x=J'(0)} = 0$, the left derivative and right derivative of f are equal at $x = 0$. Hence, f is still continuously differentiable after the extension.

Third, construct a diffusion process for $x \in (-\infty, \infty)$ as follows. Because S is a diffusion process, so is $x = f^{-1}(S)$ whenever $x > J'(0)$. The drift $\bar{\alpha}(x)$ and volatility $\bar{\zeta}(x)$ of x are, respectively,

$$\begin{aligned} \bar{\alpha}(x) &= (f^{-1})'(S)\alpha(S) + \frac{1}{2}(f^{-1})''(S)(\zeta(S))^2, \\ \bar{\zeta}(x) &= (f^{-1})'(S)\zeta(S), \end{aligned}$$

where $S = f(x)$. Symmetrically, if $x < J'(0)$, then $x = 2J'(0) - f^{-1}(S)$ is also a diffusion process.

Fourth, for S to have an invariant distribution it is sufficient to show that x has an invariant distribution. To show that x has an invariant distribution on $(-\infty, \infty)$ we verify the sufficient conditions in Karatzas and Shreve (1991, Exercise 5.40, page 352).

(i) Nondegeneracy. The volatility $\bar{\zeta}(x) \neq 0$ at $x > J'(0)$ because $f'(x) > 0$ for $x > J'(0)$ and $\zeta(S) \neq 0$ for $S > 0$. Although $\zeta(0) = 0$, $\bar{\zeta}(J'(0)) \neq 0$ because $f^{-1}(S) = J'(S)$ for $S \approx 0$ and

$$\lim_{x \downarrow J'(0)} \bar{\zeta}(x) = \lim_{S \downarrow 0} J''(S)(\hat{Y} - \sigma) = \lim_{S \downarrow 0} \frac{J''(S)J'(S)}{J'(S) + J''(S)} > 0.$$

The volatility $\bar{\zeta}(x) \neq 0$ at $x < J'(0)$ due to symmetry.

(ii) Local integrability. Because $\zeta(x)$ is continuous in x and is always nonzero, it is bounded away from zero. That is, there exists $\epsilon > 0$ such that $(\zeta(x))^2 \geq \epsilon$ for all x.

(iii) $p(-\infty) = -\infty$ and $p(\infty) = \infty$, where the *scale function* $p(x)$ is defined as

$$p(x) \equiv \int_c^x \exp\left(-2\int_c^\xi \frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2} d\theta\right) d\xi,$$

where c is a fixed number. We will only show $p(\infty) = \infty$ as the proof for $p(-\infty) = -\infty$ is similar. Since $f(\theta) = \theta$ for large θ, $\lim_{\theta \to \infty} \frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2} = \lim_{\theta \to \infty} \frac{\alpha(\theta)}{\zeta(\theta)^2} = \frac{-\mu - a_h}{\sigma^2} < 0$, where the inequality follows from (60). Therefore, $\lim_{\xi \to \infty} -2\int_c^\xi \frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2} d\theta = \infty$, and $\lim_{x \to \infty} p(x) = \infty$.

(iv) $m(-\infty,\infty)<\infty$, where the *speed measure* m is defined as

$$m(dx)\equiv\frac{2dx}{p'(x)\bar{\zeta}(x)^2}.$$

Because $\lim_{\theta\to\infty}\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}=\frac{-\mu-a_h}{\sigma^2}<0$ and $\lim_{\theta\to\infty}\bar{\zeta}(\theta)^2=\sigma^2$, there is a large $\bar{x}$, such that $\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}<\frac{-\mu-a_h}{2\sigma^2}$ and $\bar{\zeta}(\theta)^2>\frac{\sigma^2}{2}$ for $\theta\geq\bar{x}$. Hence, if $x\geq\bar{x}$, then

$$\begin{aligned}p'(x)\bar{\zeta}(x)^2 &= \exp\left(-2\int_c^x\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}d\theta\right)\bar{\zeta}(x)^2\\ &\geq \exp\left(-2\int_c^{\bar{x}}\frac{\bar{\alpha}(\theta)}{\bar{\zeta}(\theta)^2}d\theta\right)\exp\left(\frac{\mu+a_h}{\sigma^2}(x-\bar{x})\right)\frac{\sigma^2}{2},\end{aligned}$$

which implies that $m([\bar{x},\infty))=\int_{\bar{x}}^{\infty}\frac{2dx}{p'(x)\bar{\zeta}(x)}dx$ is finite. That $m((-\infty,2J'(0)-\bar{x}])<\infty$ follows from symmetry.

■

Proof of Proposition 6

We start with the following auxiliary lemma:

Lemma A.13 *Let* $S_1=1-\log(1+\frac{1-e^{-\kappa}}{\kappa})$ *and* $S_2=2-\log(1+\frac{1-e^{-2\kappa}}{\kappa})$. *For large* a_h, $r(-1+\frac{1}{J'_{FI}(S_1)})+\frac{1}{2}\sigma^2-a_h<-\frac{a_h}{3}$ *and* $S_1<S_2<S^*$.

Proof First, we compute $J'_{FI}(S_1)$ in the full-information model. From the proof of Proposition 3, we know that $u_t=\frac{\kappa+1}{\kappa}V_{FI}(m_t)$ and $W_t=(1+\frac{1-e^{-\kappa(m_t-y_t)}}{\kappa})V_{FI}(m_t)$. Therefore, $\hat{u}(S_t)=\frac{u_t}{-W_t}=-\frac{\frac{\kappa+1}{\kappa}}{1+\frac{1-e^{-\kappa(m-y)}}{\kappa}}$. The first-order condition in the HJB equation implies $\hat{u}(S)=-\frac{-1}{J'_{FI}(S)}$, which together with $m-y=1$ at S_1 imply that

$$r\left(-1+\frac{1}{J'_{FI}(S_1)}\right)=r\left(-1+\frac{\frac{\kappa+1}{\kappa}}{1+\frac{1-e^{\kappa(y-m)}}{\kappa}}\right)=\frac{re^{-\kappa}}{\kappa+1-e^{-\kappa}}.$$

It follows from $\lim_{a_h\to\infty}a_h\kappa=r$ that

$$\lim_{a_h\to\infty}\frac{\frac{re^{-\kappa}}{\kappa+1-e^{-\kappa}}}{a_h}=\lim_{a_h\to\infty}\frac{re^{-\kappa}}{(\kappa+1-e^{-\kappa})a_h}=\frac{1}{2}.$$

Therefore, $r\left(-1+\frac{1}{J'_{FI}(S_1)}\right)+\frac{1}{2}\sigma^2-a_h<-\frac{a_h}{3}$ for large a_h.

Second, $S_1<S_2$ because $\lim_{a_h\to\infty}S_1-S_2=-1-\lim_{\kappa\to0}\log(1+\frac{1-e^{-\kappa}}{\kappa})+\lim_{\kappa\to0}\log(1+\frac{1-e^{-2\kappa}}{\kappa})=-1+\log(\frac{3}{2})<0$.

Third, $S_2<S^*_{FI}$, where S^*_{FI} denotes the smallest S at which the IC constraint is violated in the full-information model. At S_2, $m-y=2$. It follows from $e^{\kappa(-2)}>\frac{r(1-\phi)}{a_h-a_l}\frac{\kappa+1}{\kappa}$ that $S_2<S^*_{FI}$ for large a_h.

Fourth, $S^*_{FI} < S^*$. We show that $J'(S^*_{FI}) > J'_{FI}(S^*_{FI})$ and $J''(S^*_{FI}) > J''_{FI}(S^*_{FI})$. An argument similar to part (i) in the proof of Lemma A.9 shows the former. To see the latter, suppose by contradiction $J''(S^*_{FI}) \leq J''_{FI}(S^*_{FI})$. The HJB equation for J_{FI} is

$$\begin{aligned} rJ_{FI}(S^*_{FI}) + r\log(-V_{FI}(0)) &= r(S^*_{FI} - \log(-\hat{u})) + J'_{FI}(S^*_{FI})\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) \\ &\quad + \frac{1}{2}J''_{FI}(S^*_{FI})(\hat{Y}-\sigma)^2. \end{aligned}$$

Hence $J'(S^*_{FI}) > J'_{FI}(S^*_{FI})$, $J''(S^*_{FI}) \leq J''_{FI}(S^*_{FI})$, and $r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h < 0$ imply that

$$\begin{aligned} rJ_{FI}(S^*_{FI}) + r\log(-V_{FI}(0)) &= r(S^*_{FI} - \log(-\hat{u})) + J'_{FI}(S^*_{FI})\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) \\ &\quad + \frac{1}{2}J''_{FI}(S^*_{FI})(\hat{Y}-\sigma)^2 \\ &> r(S^*_{FI} - \log(-\hat{u})) + J'(S^*_{FI})\left(r(-1-\hat{u}) + \frac{1}{2}\hat{Y}^2 - a_h\right) \\ &\quad + \frac{1}{2}J''(S^*_{FI})(\hat{Y}-\sigma)^2 \\ &\geq rJ(S^*_{FI}) + r\log(-V(0)), \end{aligned}$$

which contradicts $J_{FI}(S) + \log(-V_{FI}(0)) \geq J(S) + \log(-V(0))$ for all $S \geq 0$. That $S^*_{FI} < S^*$ follows from

$$\frac{\sigma J'(S^*_{FI})J''(S^*_{FI})}{J'(S^*_{FI}) + J''(S^*_{FI})} > \frac{\sigma J'_{FI}(S^*_{FI})J''_{FI}(S^*_{FI})}{J'_{FI}(S^*_{FI}) + J''_{FI}(S^*_{FI})} = \beta.$$

■

Now we can prove the proposition.

Because the trend of S is negative in $[S_1, \infty)$, the derivative of the scale function, $p'(S)$, is strictly increasing in S. Further,

$$\begin{aligned} \big(\log(p'(S))\big)' &= -2\left(\frac{r(-1+\frac{1}{J'(S)}) + \frac{1}{2}\hat{Y}(S)^2 - a_h}{(\hat{Y}(S)-\sigma)^2}\right) \\ &\geq -2\left(\frac{r(-1+\frac{1}{J'_{FI}(S_1)}) + \frac{1}{2}\sigma^2 - a_h}{(\hat{Y}(S)-\sigma)^2}\right) \geq \frac{2a_h}{3\sigma^2}, \quad \text{for } S \geq S_1, \end{aligned}$$

where the first inequality follows from $J'(S) \geq J'(S_1) > J'_{FI}(S_1)$ and the second inequality follows from $r(-1+\frac{1}{J'_{FI}(S_1)}) + \frac{1}{2}\sigma^2 - a_h < -\frac{a_h}{3}$, which is shown in Lemma A.13. This implies that $p'(S) \geq p'(S^*)\exp(\frac{2a_h}{3\sigma^2}(S-S^*))$ for $S \geq S^*$. We have

$$\frac{m[S^*,\infty)}{m[S_1,S_2]} = \frac{\int_{S^*}^{\infty}\frac{1}{p'(S)(\hat{Y}(S)-\sigma)^2}dS}{\int_{S_1}^{S_2}\frac{1}{p'(S)(\hat{Y}(S)-\sigma)^2}dS} \leq \frac{\int_{S^*}^{\infty}\frac{1}{p'(S)}dS}{\int_{S_1}^{S_2}\frac{1}{p'(S)}dS},$$

which follows from $\hat{Y}(S) > \hat{Y}(\tilde{S})$ for all $S < S^* < \tilde{S}$. This inequality is shown by $\hat{Y}(S) = \frac{J''(S)}{J'(S)+J''(S)} = \frac{J'(S)J''(S)}{J'(S)+J''(S)}(J'(S))^{-1} > \beta(J'(S))^{-1} \geq \beta(J'(\tilde{S}))^{-1} = \beta(-\hat{u}(\tilde{S})) = \hat{Y}(\tilde{S})$. Further,

$$\frac{\int_{S^*}^{\infty} \frac{1}{p'(S)} dS}{\int_{S_1}^{S_2} \frac{1}{p'(S)} dS} \leq \frac{\int_{S^*}^{\infty} \frac{1}{p'(S)} dS}{(S_2 - S_1)\frac{1}{p'(S^*)}} \leq \frac{\int_{S^*}^{\infty} \frac{1}{\exp(\frac{2a_h}{3\sigma^2}(S-S^*))} dS}{S_2 - S_1} = \frac{1}{\frac{2a_h}{3\sigma^2}(S_2 - S_1)}.$$

Hence $\lim_{a_h \to \infty} \pi([S^*, \infty)) = \lim_{a_h \to \infty} \frac{m[S^*,\infty)}{m[0,\infty)} \leq \lim_{a_h \to \infty} \frac{m[S^*,\infty)}{m[S_1,S_2]} = 0$. ■

Appendix B: Properties of the cost function J_{FI} and dynamics of the state variable S_t in the model with full information

Lemma B.1 *In the model with full information,*

(i) J'_{FI} *is everywhere positive and strictly increasing with*

$$J'_{FI}(0) = \frac{\kappa}{\kappa+1} \quad \text{and} \quad \lim_{S_t \to \infty} J'_{FI}(S_t) = 1.$$

(ii) J''_{FI} *is everywhere positive and strictly decreasing with*

$$J''_{FI}(0) = \infty \quad \text{and} \quad \lim_{S_t \to \infty} J''_{FI}(S_t) = 0.$$

(iii) The drift of the state variable, α*, is strictly decreasing with*

$$\alpha(0) = \frac{1}{2}(\kappa+1)\sigma^2 > 0 \quad \text{and} \quad \lim_{S_t \to \infty} \alpha(S_t) = -a_h.$$

(iv) The volatility of the state variable, ζ*, is everywhere negative and strictly decreasing with*

$$\zeta(0) = 0 \quad \text{and} \quad \lim_{S_t \to \infty} \zeta(S_t) = -\sigma.$$

Proof It is useful to derive policies $\hat{u}(S_t)$ and $\hat{Y}(S_t)$ as functions of m_t and y_t. From the proof of Proposition 3, we know that $u_t = \frac{\kappa+1}{\kappa} V_{FI}(m_t)$, $Y_t = -V_{FI}(m_t)e^{-\kappa(m_t - y_t)}\sigma$ and $W_t = (1 + \frac{1-e^{-\kappa(m_t-y_t)}}{\kappa})V_{FI}(m_t)$. Therefore,

$$\begin{aligned} \hat{u}(S_t) &= \frac{u_t}{-W_t} = -\frac{\frac{\kappa+1}{\kappa}}{1 + \frac{1-e^{-\kappa(m-y)}}{\kappa}}, \\ \hat{Y}(S_t) &= \frac{Y_t}{-W_t} = \frac{e^{-\kappa(m_t-y_t)}}{1 + \frac{1-e^{-\kappa(m_t-y_t)}}{\kappa}}\sigma. \end{aligned}$$

This implies that $\hat{u}(S_t)$ increases and $\hat{Y}(S_t)$ decreases in S_t. Further, $\hat{u}(0) = -\frac{\kappa+1}{\kappa}$, $\hat{Y}(0) = \sigma$, $\lim_{S_t \to \infty} \hat{u}(S_t) = \lim_{(m_t - y_t) \to \infty} \hat{u}(S_t) = -1$, and $\lim_{S_t \to \infty} \hat{Y}(S_t) = \lim_{(m_t-y_t) \to \infty} \hat{Y}(S_t) = 0$.

(i) Since $J'_{FI}(S_t) = (-\hat{u}(S_t))^{-1}$, the property of $J'_{FI}(S_t)$ follows from that of $\hat{u}(S_t)$ in the above.

(ii) It follows from $J'_{FI}(S_t) = (-\hat{u}(S_t))^{-1} = (1+\frac{1-e^{-\kappa(m_t-y_t)}}{\kappa})/\frac{\kappa+1}{\kappa}$ and $S_t = m_t - y_t - \log(\kappa + 1 - e^{-\kappa(m_t-y_t)})$ that

$$J''_{FI}(S_t) = \frac{\kappa}{\kappa+1}\frac{e^{-\kappa(m_t-y_t)}}{1-\frac{e^{-\kappa(m_t-y_t)}\kappa}{\kappa+1-e^{-\kappa(m_t-y_t)}}} = \frac{\kappa}{\kappa+1}\frac{1}{e^{\kappa(m_t-y_t)}-\frac{\kappa}{\kappa+1-e^{-\kappa(m_t-y_t)}}},$$

which decreases in $m_t - y_t$. If $S_t = 0$, then $m_t - y_t = 0$ and clearly $J''_{FI}(0) = \infty$. Moreover, $\lim_{S_t\to\infty} J''_{FI}(S_t) = \lim_{m_t-y_t\to\infty} J''_{FI}(S_t) = 0$.

(iii) It follows from $\alpha(S_t) = r(-1-\hat{u}(S_t)) + \frac{1}{2}(\hat{Y}(S_t))^2 - a_h$ that $\alpha(S_t)$ decreases in S_t. Further,

$$\begin{aligned}
\alpha(0) &= r(-1+\frac{\kappa+1}{\kappa}) + \frac{1}{2}\sigma^2 - a_h = \frac{1}{2}(\kappa+1)\sigma^2,\\
\lim_{S_t\to\infty}\alpha(S_t) &= r(-1+1) + \frac{1}{2}0^2 - a_h = -a_h.
\end{aligned}$$

(iv) It follows from $\zeta(S_t) = \hat{Y}(S_t) - \sigma$ that $\zeta(S_t)$ decreases in S_t. Further,

$$\zeta(0) = \hat{Y}(0) - \sigma = 0, \quad \lim_{S_t\to\infty}\zeta(S_t) = 0 - \sigma = -\sigma.$$

■

Discussion. The quitting constraint is the only friction in the full-information version of our model. If this friction were absent, the contracting environment would be the so-called first best: firms would fully insure workers against fluctuations in their productivity by giving them permanently constant compensation and workers would be committed to never quitting or shirking. In the first best, W_t is constant, so, as evident from (18), the dynamics of the state variable S_t reduce to $dS_t = -dy_t$, which means that $\alpha(S_t) = -a_h$ and $\zeta(S_t) = -\sigma$ at all S_t. With the worker's compensation constant, the firm's profit simply follows the random changes in the output produced by the worker. The first-best cost function, denoted as J_{FB}, therefore satisfies $J'_{FB}(S_t) = 1$, as a larger drift of the worker's output process would reduce the firm's cost one-to-one.[31] Also, since firms are risk-neutral and never run into quitting or incentive constraints in the first best, they are indifferent to volatility in S_t. This means that $J''_{FB}(S_t) = 0$ for all S_t.[32]

[31]Recall from (22) that the first derivative of the cost function represents the impact of the state variable's drift on the firm's total cost. In the first best, the drift of the state variable is the negative of the drift of the worker's output.

[32]Recall again from (22) that the second derivative of the cost function represents the impact of the state variable's volatility on the firm's total cost.

Lemma B.1 shows that the equilibrium cost function and the dynamics of the state variable in the model with the quitting constraint converge to the first best when slackness S_t in the quitting constraint becomes large. This convergence is intuitive. When S_t is large, the expected time until the quitting constraint binds again is large, and so the equilibrium contract (25) is expected to provide full insurance to the worker far into the future. Since the equilibrium contract at this point approximates the first-best contract very closely, its cost is close to the first-best cost function.

On the other extreme, when the quitting constraint binds (i.e., at $S_t = 0$), ensuring that it continues to be satisfied under all realizations of the shock to the worker's productivity is only possible if, first, the volatility of S_t at $S_t = 0$ is zero, and, second, the drift of S_t at $S_t = 0$ is nonnegative. The optimal contract, as we see in Lemma B.1, does induce $\zeta(0) = 0$. Consistently, $J''_{FI}(0) = \infty$, which reflects the fact that the firm is infinitely averse to the volatility in S_t when the quitting constraint binds, as any nonzero volatility would lead to a violation of the quitting constraint with probability one immediately after S_t hits zero.

Note that zero volatility of S_t means that the volatility of the worker's continuation value inside the contract is the same as the volatility of her outside option, which means that locally at $S_t = 0$ the firm cannot provide any insurance to the worker. To avoid violating the quitting constant, clearly, the drift of S_t at $S_t = 0$ must be nonnegative. A strictly positive drift of S_t at $S_t = 0$ is beneficial in that it relaxes the quitting constraint, which allows the firm to provide insurance to the worker as soon as S_t becomes strictly positive. But positive drift in S_t is also costly because in order to obtain it the contract must back-load compensation and produce a strictly positive drift in the worker's continuation value W_t. Positive drift in W_t is costly as it means that intertemporal smoothing of the worker's consumption is poor. (Recall that drift of W_t at the first best is zero.) The optimal drift $\alpha(0)$ given in the above lemma is the outcome of balancing this trade-off. It is strictly positive, so zero is a reflecting rather than absorbing barrier for the state variable and insurance is provided to the worker. Its size is limited, however, by the intertemporal inefficiency of excessive compensation back-loading. Consistently, $J'_{FI}(0) = \frac{\kappa}{\kappa+1} < 1 = J'_{FB}(0)$ reflects the fact that positive drift of S_t has a benefit in the full-information-limited-enforcement model that it does not have in the first best: it helps relax the quitting constraint. As a consequence, the firm is less averse to drift in S_t than it is at the first best, which means that J'_{FI} is everywhere smaller than $J'_{FB} \equiv 1$.