Identification and Estimation of a Discrete Game by Observing its Correlated Equilibria

Wang, Yafeng and Graham, Brett

China Center for Economic Research (CCER), Peking University, 100871 Beijing, China, The Wang Yanan Institute for Studies in Economics (WISE), Xiamen University, 361005 Xiamen, Fujian, China

8 October 2010

Identification and Estimation of a Discrete Game by Observing its Correlated Equilibria

Yafeng Wang* and Brett Graham†

Abstract

This paper studies the problem of identifying and estimating the normal-form payoff parameters of a simultaneous, discrete game of complete information where the equilibrium concept employed is correlated equilibrium rather than Nash equilibrium. We show that once we extend the equilibrium concept from Nash equilibrium to the correlated equilibrium, the identification and estimation of game-theoretic econometric models become simpler, since this extension avoids the usual requirement of computing all equilibria of games. To deal with the presence of multiple equilibria, unlike most other work on empirical games, we make use of the moment inequality restrictions induced by the underlying game-theoretic econometric models without making equilibrium selection assumptions. The resulting identified features of the model are sets of parameters such that the choice probabilities predicted by the econometric model are consistent with the empirical choice probabilities estimated from the data. The importance sampling technique is used to reduce computational burden and overcome the non-smoothness problems. We also show that the model selection tests for moment inequality models can be used to test equilibrium concepts such as correlated equilibrium versus Nash equilibrium.

Keywords: Game-Theoretic Econometric Models; Correlated Equilibrium; Partial Identification; Moment Inequality Restrictions; Importance Sampling.

JEL Classification Numbers: C35, C51, C72.

*Corresponding author. China Center for Economic Research (CCER), Peking University, 100871 Beijing, China. E-mail: econwang@pku.edu.cn.

†The Wang Yanan Institute for Studies in Economics (WISE), Xiamen University, 361005 Xiamen, Fujian, China. E-mail: bgraham@xmu.edu.cn.

1 Introduction

Game theory is one of the cornerstones of modern economic theory, and much progress has been made in clarifying the nature of strategic interaction in economic models. It is the benchmark theoretical model for analyzing strategic interactions among a handful of players. Given the importance of gaming in economic theory, the empirical analysis of games has been the focus of a recent literature in econometrics and industrial organization. Econometrically, a discrete game-theoretic econometric model is a generalization of a standard discrete choice model, such as the conditional logit or multinomial probit model. An agent's utility is often assumed to be a linear function of covariates and a random preference shock. However, unlike a discrete choice model, utility is also allowed to depend on the actions of other agents. Such modeling strategy was first suggested by the seminal work of Bresnahan and Reiss (1990, 1991).

Although there are numerous studies on both methodology and empirical applications of game-theoretic econometric models, the most widely studies is the class of incomplete information static games and dynamic games[1]. The complete information games received fewer studies due to its computational complexity, since it involves multidimensional integrals, moreover, complete information assumption will generally induce the presence of multiple equilibria Morris and Shihn (2003). Dealing with multiple equilibria is a difficult task because a particular realization of observables and a particular value of the payoff parameter vector may be consistent with difference outcomes of the model. For the presence of multidimensional integrals, Bajari, Hong, and Ryan (2010) and Ciliberto and Tamer (2009) provide simulation-based estimators for the static complete information game. For the presence of multiple equilibria, there are three main different approaches adopted by the existing research. The first one is that introducing a specific equilibrium selec-

[1]Studies of incomplete information static games include Sweeting (2005), Seim (2006), Aradillas-Lopez (2007, 2010) and Bajari, Hong, Krainer, and Nekipelov (2010), while the studies of dynamic game includes Aguirregabiria and Mira (2007) and Pesendorfer and Schmidt-Dengler (2008) among others.

tion mechanism to determine which equilibrium will be played among several equilibria, Bajari, Hong, and Ryan (2010) and Jia (2008) use such approach to studies empirical static games. While a specific equilibrium selection mechanism can provide enough information for identifying the underlying game, in general, we have limited knowledge about the equilibrium selection mechanism, and any misspecificaiton about it will lead inconsistent estimation. The second approach which first used by Bresnahan and Reiss (1990) is that transforming the outcome variable of the game from action profile to some other variable, which satisfies that, even if there are multiple equilibria in the game, it does a unique prediction of this particular variable, such as the number of entrants in the market used by Bresnahan and Reiss (1990). This is a useful method as long as we can find such a particular variable. The last approach proposed by Tamer (2003) is that doing inference of the empirical games without making any assumption about the equilibrium selection mechanism, the cost is that in general, one can only achieve the partial identification of the model, Berry and Tamer (2007) and Ciliberto and Tamer (2009) based on this method formally study the empirical static games. Although these approaches can handle the inference in the presence of multiple equilibria, a common practical issue is that all of them require the computation all the Nash equilibria of underlying game, which may result in heavy computational burden if not impossible when dealing with large games[2].

Here, we depart from the common used equilibrium concept – Nash equilibrium, and assume that the outcome of the game is generated by a more broad rational rule – correlated equilibrium which proposed by Aumann (1974, 1987). A most interesting feature of this extension of equilibrium concept is that the identification and estimation of empirical games become simpler, even if it spreads the corresponding equilibrium set[3].

[2] Mckelvey and Mclennan (1996) analyze the computational method for computing the Nash equilibrium set for general game and point the difficulty associated with this issue.

[3] Chwe (2007) also study the identification of games based on correlated equilibrium in a no randomness environment.

The advantages of correlated equilibrium in the context of identification comes from its convexity, that is, any aggregation of correlated equilibria is also a correlated equilibria which is not true for Nash equilibrium. We also adopt the partial identification approach to deal with the presence of multiple equilibria following Berry and Tamer (2007) and Ciliberto and Tamer (2009), where the identified set is characterized by moment inequality restrictions. But our approach does not require the computation of all the equilibria (either correlated or Nash), but only needs to compute some "extreme" equilibria, which can be obtained from simple linear programming. This does not mean that computing the whole set of correlated equilibria is simple[4], the key feature is that it does not need the whole set of equilibria. The importance sampling technique is used to approximate the multi-dimensional integrals. Given the existing research on empirical games based on Nash equilibrium, and the results established in this paper based on correlated equilibrium, we also provide a test framework for testing equilibrium concepts based on the moment inequality model selection test developed by Shi (2010). The nested relationship between Nash equilibrium and correlated equilibrium makes this test similar to the famous Hausman test (Hausman, 1978).

The paper is organized as follows. In section 2 we outline the general discrete simultaneous-move game with complete information to be estimated and formulate its equilibrium conditions based on correlated equilibrium, several important properties of correlated equilibrium are also presented. In section 3 we discuss the problem of partial identification of the model. Section 4 describes the procedure for estimating the identified set which formulated in section 3, which also includes the important issues about importance sampling approximation of the multiple integrals and the computation of "extreme" correlated equilibrium. Section 5 introducing a test procedure for testing correlated equilibrium versus Nash equilibrium. A simple Monte Carlo experiment is conducted in section

[4] Papadimitriou and Roughgarden (2008) develop a polynomial-time algorithm for finding correlated equilibria and also have a discussion about the difficulty in computing the whole set of correlated equilibria.

6. Section 7 simply concludes the paper.

2 The Model

There are T independent repetitions of a simultaneous-move (normal form) discrete game of complete information. In each game there are $i = 1, ..., N$ players. In each repetition of these games, each player i chooses a action a_i from the finite set of actions A_i simultaneously. Define $A = \times_i A_i$ and let $a = (a_1, ..., a_N)$ denote a generic element of A. Player i's von Neumann-Morgenstern (vNM) utility is a mapping $u_i : A^N \to R$, where R is the real line. We will sometimes drop the subscript t for simplicity when no ambiguity would arise.

Following Bresnahan and Reiss (1990, 1991), assume that the vNM utility of player i can be written as:

$$u_i(a, x_i, \epsilon_i; \theta_1) = \Pi_i(x_i, a; \theta_1) + \epsilon_i(a) \tag{1}$$

where $a \in A^N$. In Equation (1), player i's vNM utility from outcome a is the sum of two terms. The first term $\Pi_i(x_i, a; \theta_1)$ is a function which depends on action profile (outcome) a, the vector of actions taken by all of the players, x, the players' characteristics and some other variables which influence utility, and parameters θ_1, covariates x are observed by the econometrician. The second term $\epsilon_i(a)$, is a random preference shock which reflects the information about utility that is common knowledge to the players (since we study the game with complete information) but not observed by the econometrician. Unlike most other study on empirical games, here the preference shocks depend on the entire vector of actions a, not just the actions taken by player i. As argued by Bajari, Hong, and Ryan (2010), this is a more general setting. For example, in a simple entry game, the unobserved information about one player's payoff to the econometrician may not only vary across his own actions but also across the actions of other players. Let ϵ_i denote the vector of the individual shocks $\epsilon_i(a)$ and ϵ denote the vector of all preference shocks, $\epsilon_i(a)$ are assumed to be independent with a density $g_i(\epsilon_i(a)|\theta_2)$ and joint distribution $g(\epsilon|\theta_2) = \prod_i \prod_{a \in A} g_i(\epsilon_i(a)|\theta_2)$, where θ_2 denotes parameters of the distribution.

Given these T independent repetitions of the game with the above structure, the researcher can observe covariates x_t and the action profile a_t^o chosen by all players in each repetition these games. Unlike most other studies of empirical games, here we assume that all players choose their action according to the correlated equilibrium rather than Nash equilibrium, the studies of empirical games based on Nash equilibrium include Bajari, Hong, and Ryan (2010) and Ciliberto and Tamer (2009) among others. As a generalization of Nash equilibrium, Aumann (1974, 1987) provided a simple rationale for equilibrium – correlated equilibrium, which is formulated in a manner that does away with the dichotomy usually perceived between the "Bayesian" and the "game-theoretic" view of the world. The most notable feature of the correlated equilibrium is that it does not require explicit randomization on the part of the players. Formally, given the game structure defined above, and let (Ω, π) be a probability space, $\mathcal{P}_i$ be a partition of Ω, $i = 1, ..., N$, and let

$$Q_i = \{q_i : \Omega \to A | q_i \text{ is } P_i \text{ measurable}\} \tag{2}$$

where the partition can be expressed as $\mathcal{P}_i = \{\mathcal{P}_i(\omega)\}_{\omega \in \Omega}$, and $\mathcal{P}_i(\omega)$ is the element of the partition containing ω. Then the original definition of correlated equilibrium can be stated as:

Definition 2.1 (Correlated Equilibrium) *The collection $(\Omega, \pi, \{P\}_{i=1}^N, \{q_i\}_{i=1}^N)$ is a correlated equilibrium if $\forall i$,*

$$\sum_{\omega} u_i(q_{-i}(\omega), q_i(\omega))\pi(\omega) \geq \sum_{\omega} u_i(q_{-i}(\omega), \tau_i(\omega))\pi(\omega), \; \forall \tau_i \in Q_i \tag{3}$$

where for each i, q_i is constant on each member of $\mathcal{P}_i$.

The intuition is that given the information received through ω, players maximize their expected utility, and (3) are sufficient conditions for the utility maximization. The formulation of a correlated equilibrium in Definition 2.1 leads itself to a broad range of interpretations (e.g. sunspot equilibria), but from a computational point of view there is a more natural formulation which is known as canonical correlated equilibrium, where the state space is identified with the space of pure strategies, that is $\Omega = A = \times A_i$, and π is a distribution on A.

Definition 2.2 (Canonical Correlated Equilibrium) *If the draw of π, a_i is viewed as the "recommended" strategy and if this is the optimal choice for i–so that for each a_i, $\sum_{a-i} u_i(a_i', a_{-i})\pi(a_{-i}|a_i)$ is maximized by $a_i' = a_i$, that is*

$$\sum_{a_{-i}} u_i(a_{-i}, a_i)\pi(a_{-i}|a_i) \geq \sum_{a_{-i}} u_i(a_{-i}, a_i')\pi(a_{-i}|a_i),\ \forall a_i' \in A_i \tag{4}$$

then π is called a canonical correlated equilibrium.

The canonical correlated equilibrium can be summarized as follows. A point a in A is drawn according to the distribution π. Player i is informed of the ith component of a, a_i with the expectation that i will choose this action. Given π, player i can calculate the conditional distribution over A_{-i} and the conditional expected payoff from each choice $a_i' : \sum_{a_{-i}} u_i(a_{-i}, a_i')\pi(a_{-i}|a_i)$. The inequality (4) asserts that if i is told a_i, this is in fact a best choice for i. Clearly, the formulation (4) is more tractable than (3), and the following theorem states the strategic equivalence between the correlated equilibrium and canonical correlated equilibrium.

Theorem 1 *Let $(\Omega, \pi, \{P\}_{i=1}^N, \{q_i\}_{i=1}^N)$ be a correlated equilibrium. Then there is a canonical correlated equilibrium π^* yielding the same distribution on actions and the same expected payoff to each player.*

Proof. See Bergin (2005). ■

Based this strategic equivalence, we will use canonical correlated equilibrium instead of the correlated equilibrium formulated in Definition 2.1 and refer it as correlated equilibrium. Also, based on Theorem 2.1, one can show the following useful properties:

Proposition 2.1 *The set of Nash equilibrium payoffs is a subset of the set of correlated equilibrium payoffs.*

Proposition 2.2 *The set of correlated equilibrium payoffs is a convex set.*

Proposition 2.1 means that Nash equilibrium is the degenerated correlated equilibrium, since the mixed strategy used by players is independent over set A_i, while correlated equilibrium is the general distribution over the set $A = \times_i A_i$. The convexity will facilitate the computation of correlated equilibrium, since any convex combination of correlated equilibria is a correlated equilibrium, we will make use of these useful properties in the identification and estimation sections. Here we illustrate these properties through a simple 2×2 chicken game.

Example 2.1 (Chicken Game) *Consider the following chicken game, each with two actions:* $A_1 = A_2 = \{stop,\ go\}$. *The payoff matrix is:*

	S	G
S	$(4,4)$	$(1,5)$
G	$(5,1)$	$(0,0)$

The following five distributions over $A = \{(S,S),(S,G),(G,S),(G,G)\}$ *all are the corre-*

lated equilibria of this game:

$$\pi_1 = (0, 1, 0, 0)$$

$$\pi_2 = (0, 0, 1, 0)$$

$$\pi_3 = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})$$

$$\pi_4 = (0, \tfrac{1}{2}, \tfrac{1}{2}, 0)$$

$$\pi_5 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0)$$

where π_1 and π_2 both are pure strategy equilibria, while π_3 is a mixed strategy equilibrium corresponding to both players playing the mixed strategy $\{1/2, 1/2\}$. The last two correlated equilibria are the correlated equilibria which do not enter the set of Nash equilibrium. Actually, by the convexity of correlated equilibrium, we know that this chicken game has infinite correlated equilibria, the set of all the correlated equilibria is shown in figure (1), which is a polytope with five vertices. Calvo-Armengol (2006) studies the property of sets of all the correlated equilibria and Nash equilibria and the relationship between them in general 2×2 games.

The nested relationship between correlated equilibrium and Nash equilibrium makes our test of equilibrium concepts similar to the famous Hausman test (Hausman, 1978). If the equilibrium of the underlying game is generated by Nash equilibrium, but the researcher estimates the game based on correlated equilibrium, then the estimates is consistent but not efficient, while the equilibrium is generated by correlated equilibrium, but the researcher estimates the game based on Nash equilibrium, then the estimates is inconsistent.

Given the structure of the discrete normal form game discussed above, assume the outcome of such games are generated by correlated equilibrium, which is any solution for

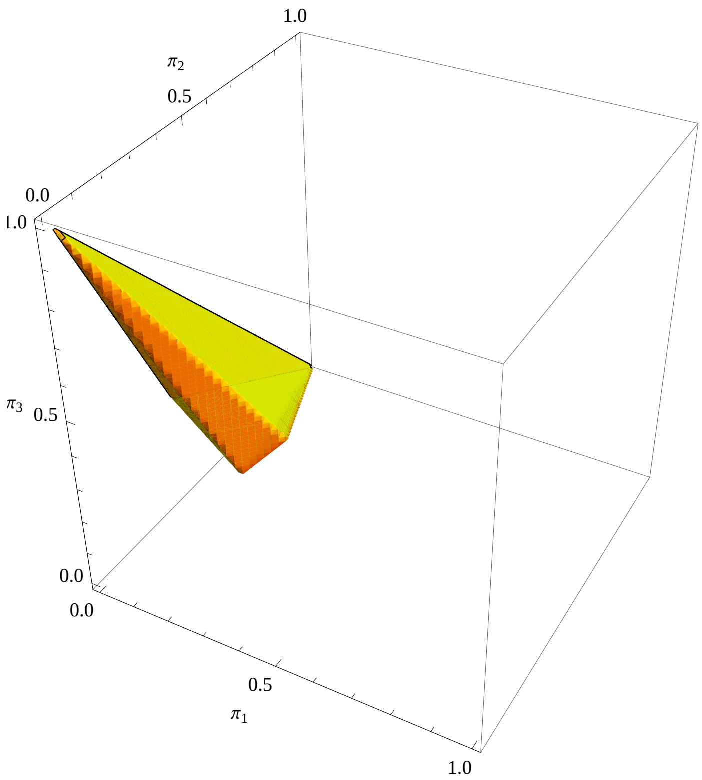

Figure 1: Correlated Equilibrium Set in the Chicken Game

the distribution π over action profile set A that satisfies:

$$\sum_{a_{-i}} u_i(a_{-i}, a_i, x_i, \epsilon_i; \theta_1)\pi(a_{-i}|a_i) \geq \sum_{a_{-i}} u_i(a_{-i}, a'_i, x_i, \epsilon_i; \theta_1)\pi(a_{-i}|a_i),\ \forall a'_i \in A_i, i = 1, ..., N \tag{5}$$

Our task is to estimate and draw an inference about the parameters of payoff functions, θ_1, and the parameters of the distribution of random preference shocks, θ_2, with the observation of the outcome a^o, some exogenous covariates which affect the payoffs, x. Note that the actual payoff levels are unobserved, i.e., they are latent variables. Before discussing the identification issues, we introduce several notations. For any distribution which satisfies the equilibrium condition (5), e.g. the correlated equilibrium of the underlying game, let

$$S_\pi(u(x, \epsilon; \theta_1)) \tag{6}$$

denote the collection of them, this will be the set of all the correlated equilibria, where $u(x, \epsilon; \theta_1)$ means that we can compute this set as long as we know the payoff functions of each player. Let $\pi(u(x, \epsilon; \theta_1))$ denote the generic elements of the set $S_\pi(u(x, \epsilon; \theta_1))$, for purposes of exposition, we will sometimes simply refer it as π, and $\pi \in S_\pi(u(x, \epsilon; \theta_1))$. Note that the correlated equilibrium relies on the exogenous covariates x, random preference shock ϵ, and structural parameter θ_1.

3 Identification

The general idea of the identification of game-theoretic econometric models is that matching the choice probabilities predicted by the model and the empirical choice probabilities estimated from the data, see Bajari, Hong, Krainer, and Nekipelov (2010), Bajari, Hong, and Ryan (2010) and Ciliberto and Tamer (2009). As usual, the empirical choice probabilities can be obtained from the data nonparametrically, but obtaining the choice probabilities predicted by the game where the equilibrium concept employed is correlated equilibrium is not practical without additional information on the equilibrium selection mechanism, this is due to the presence of multiple equilibria, dealing with multiple equilibria complicates the identification problem. Unlike Bajari, Hong, and Ryan (2010) which introduce a explicit equilibrium selection mechanism to achieve the point identification of structural parameters, here we identify the game without making any assumption on the equilibrium selection mechanism following the method proposed by Ciliberto and Tamer (2009) which study the same problem but use Nash equilibrium as the equilibrium concept, the reason is that we do not have enough information to specify any equilibrium selection mechanism Berry and Tamer (2007).

First enumerate the elements of A from $a = \{1, ..., \#A\}$. A is the set of pure strategy profiles and $a \in A$. Given a correlated equilibrium $\pi \in S_\pi(u(x, \epsilon; \theta_1))$ is a distribution over A, we have

$$\pi = (\pi(1), ..., \pi(a), ..., \pi(\#A))' \tag{7}$$

and

$$\sum_{a=1}^{\#A} \pi(a) = 1; 0 \leq \pi(a) \leq 1; \forall a \in A \tag{8}$$

Let $\mathcal{Y}$ be the set of *potentially observable outcomes*, since we assume that the observable outcome of the game is the equilibrium actions chosen by all the players, then $\mathcal{Y} = A$. Let $\Pr(y = a|x; \theta)$ denote the the probability that action profile a be the equilibrium action profile predicted by the model, where $\theta = (\theta_1, \theta_2)$, and $\Pr(y = a|x)$ be the empirical choice probability identified from the data which does not rely on the model, thus, does not rely on the parameters.

We introduce following assumptions:

Assumption 1 (Compactness of Parameter Space) *The parameter space Θ is compact.*

Assumption 2 (Scale and Location Normalizations) *The payoffs of one action for each player are fixed at a known constant.*

Assumption 3 (Regularity Conditions of Random Shocks) *The joint distribution of $\epsilon = (\epsilon_i(a))$, $G(\epsilon|\theta_2)$ is independent, independent of x, and known to all agents and the econometrician, and let $g(\epsilon|\theta_2)$* be the corresponding density.

Assumption 4 (Identification of $Pr(y|x)$) *The econometrician observes data that identify* $\Pr(y = a|x)$, $\forall a \in A$.

Assumption 1 is critical for the construction the large sample property of our estimator. The restriction in Assumption 2 is similar to the argument that we can normalize the mean utility from the outside good equal to a constant, usually zero, in a standard discrete choice model. One can clearly find that from the equilibrium condition (5) that adding a constant to all deterministic payoffs does not perturb the set of equilibria, so a location normalization is necessary. A scale normalization is also necessary, as multiplying all deterministic payoffs by a positive constant does not alter the equilibrium, and this restriction is subsumed in the Assumption 3, where we assume that the researcher know the joint distribution of random preference shocks, identification in this game with unspecified distribution is complicated if not impossible. Finally, Assumption 4 requires that researchers can identify the empirical choice probability from data, clearly, this is necessary since we match this probability with the choice probability predicted by the model to identify the model.

As discussed before, the set of correlated equilibrium, $S_\pi(u(\theta, x, \epsilon))$ may be a non-singleton set, and usually, it is a set with infinite elements. If $S_\pi(u(\theta, x, \epsilon))$ is non-singleton, in order to derive the choice probability predicted by the model, $\Pr(y = a|x; \theta)$, we need to introduce an equilibrium selection mechanism:

$$\psi(\cdot|x, \epsilon) : S_\pi(u(x, \epsilon; \theta_1)) \to [0, 1]^{d[S_\pi(u(x,\epsilon;\theta_1))]} \tag{9}$$

such that

$$\psi(\cdot|x, \epsilon) \geq 0 \tag{10}$$

$$\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x, \epsilon) = 1 \tag{11}$$

where $d[S_\pi(u(x, \epsilon; \theta_1))]$ is the dimension of $S_\pi(u(x, \epsilon; \theta_1))$. This equilibrium selection mechanism specifies the probability, $\psi(\pi|x, \epsilon)$, that any correlated equilibrium $\pi \in S_\pi(u(x, \epsilon; \theta_1))$ be the actual equilibrium. Since the $d[S_\pi(u(x, \epsilon; \theta_1))]$ can be infinite, we should use a continuos distribution to express this equilibrium selection mechanism, but for purposes of exposition, we use the discrete distribution.

Given the equilibrium selection mechanism (9), the choice probability implied by the model can be written as:

$$\Pr(y = a|x;\theta) = \int \left(\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a) \right) dG(\epsilon|\theta_2) \tag{12}$$

where $\pi(a)$ is the probability of action profile a associated with a correlated equilibrium π, and $\psi(\pi|x,\epsilon)$ is the probability that π be the final equilibrium, thus $\psi(\pi|x,\epsilon)\pi(a)$ is the probability that action profile a be the final equilibrium action profile associated with a correlated equilibrium π. Clearly, action profile a also can be chosen with other correlated equilibrium rather than π, thus the summation of these probability, $\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a)$, is the probability that action profile a be the final equilibrium action profile. Based on the choice probability implied by the model (12), the sharp identified set for parameter $\theta = (\theta_1, \theta_2)$ is defined as:

Definition 3.1 (Sharp Identified Set) *The sharp identified set for the parameter vector $\theta \in \Theta$ is given by:*

$$\Theta_I = \left\{ \begin{array}{ll} & \exists \psi,\ \forall a \in \mathcal{Y} \\ \theta \in \Theta : & \textit{such that: } E[\Pr(y = a|x)] = E[\Pr(y = a|x;\theta)] \\ & = E\left[\int \left(\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a) \right) dG(\epsilon|\theta_2) \right] \end{array} \right\} \tag{13}$$

Inference on the set Θ_I based on (13) is not practically feasible since one needs to deal with infinite dimensional nuisance parameters $\psi(\cdot|x,\epsilon)$, this is due to the presence of multiple equilibria, and note that the equilibrium selection mechanism also depends on the unobserved random preference shock ϵ. We may specify a parametric equilibrium selection mechanism as that it is characterized by finite parameters, such as Bajari, Hong, and Ryan (2010), which estimate a normal form game based on Nash equilibrium. The problem is that we do not have enough information to specify a particular equilibrium selection mechanism, and any misspecifiction of this mechanism will induce the inconsistent estimation. One may also try to use refined equilibrium concept, such as perfect correlated equilibrium (Dhillon and Mertens, 1996) or maximum entropy correlated equilibrium (Ortiz, Schapire, and Kakade, 2007), the problem is that even based on such refined equilibrium, one also can not guarantee the uniqueness of equilibrium. In the spirite of Ciliberto and Tamer (2009), here we leave the equilibrium selection mechanism unspecified but exploiting the fact that the equilibrium selection mechanism $\psi(\pi|x,\epsilon)$ is a probability and hence bounded between zero and one to derive a outer identified set for the model.

Since the equilibrium selection mechanism $\psi(\pi|x,\epsilon)$ is a probability distribution, then

$$0 \leq \psi(\pi|x,\epsilon) \leq 1, \forall \pi \in S_\pi(u(x,\epsilon;\theta_1)) \tag{14}$$

Based on this natural property of probability, we can derive an outer identified set for the parameter θ. Formally, let $H_1^a(\theta, X)$ denote the lower bound of the choice probability of action profile a implied by the model, $\Pr(y = a|x;\theta)$, and $H_2^a(\theta, X)$ the upper bound, then:

$$H_1^a(\theta, X) = \min \int \left[\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a) \right] dG(\epsilon|\theta_2) \tag{15}$$

$$H_2^a(\theta, X) = \max \int \left[\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a) \right] dG(\epsilon|\theta_2) \tag{16}$$

Given the exogenous covariates, X, and payoff parameter θ_1, we collect the value of random preference shocks ϵ such that the game admits π as unique equilibrium into the set $R_1^\pi(\theta_1, X)$, and collect other value of random preference shocks that the game admits multiple equilibria into the set $R_2^\pi(\theta_1, X)$. Thus we have:

$$\begin{aligned} & H_1^a(\theta, X) \\ & = \min \int \left[\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a) \right] dG(\epsilon|\theta_2) \\ & = \underbrace{\int_{R_1^\pi(\theta,X)} \pi(a) dG(\epsilon|\theta_2)}_{(1)} + \underbrace{\int_{R_2^\pi(\theta,X)} \min\{\pi(a) : \pi \in S_\pi(u(x,\epsilon;\theta_1))\} dG(\epsilon|\theta_2)}_{(2)} \end{aligned} \tag{17}$$

where part (1) means that when the game admits π as the unique equilibrium, then the choice probability of a implied by the model, $\Pr(y = a|x;\theta)$, is the probability of a associated with the unique correlated equilibrium π, that is $\pi(a)$. As long as the game admits unique equilibrium, the action profile a will appear as the equilibrium action profile with probability $\pi(a)$, thus it enters the lower bound $H_1^a(\theta, X)$. Part (2) in (17) means that when the game has multiple equilibria, we select one particular equilibrium from these equilibria as the final equilibria, which is the one that puts the lowest probability for action profile a. Clearly, the true equilibrium selection mechanism may put a lower probability for this particular equilibrium, but select this equilibrium with probability one do achieve the lower bound of $\Pr(y = a|x;\theta)$ given the information on hand. Similarly, the upper bound $H_2^a(\theta, X)$ can be derived as:

$$\begin{aligned} & H_2^a(\theta, X) \\ & = \max \int \left[\sum_{\pi \in S_\pi(u(x,\epsilon;\theta_1))} \psi(\pi|x,\epsilon)\pi(a) \right] dG(\epsilon|\theta_2) \\ & = \underbrace{\int_{R_1^\pi(\theta,X)} \pi(a) dG(\epsilon|\theta_2)}_{(1)} + \underbrace{\int_{R_2^\pi(\theta,X)} \max\{\pi(a) : \pi \in S_\pi(u(x,\epsilon;\theta_1))\} dG(\epsilon|\theta_2)}_{(2)} \end{aligned} \tag{18}$$

The meaning for the first part is same as (17), the probability associated with the unique equilibrium also enters the upper bound. When the game admits multiple equilibria, to derive the upper bound of choice probability, we select another particular equilibrium form these equilibria, that is the one puts the highest probability for the action profile a. Although the true equilibrium selection mechanism may selects this equilibrium with a lower probability, this selection does achieve the upper bound of $\Pr(y = a|x; \theta)$, since any other selection will select this particular equilibrium with probability no more than one.

Based on the lower bound and upper bound of the choice probability implied by the model, we have:

$$H_1^a(\theta, X) \leq \Pr(y = a|x; \theta) \leq H_2^a(\theta, X) \tag{19}$$

And when $\theta \in \Theta_I$

$$E[\Pr(y = a|x)] = E[\Pr(y = a|x; \theta)] \tag{20}$$

Thus we can define the outer identified set for the model parameter θ as:

Definition 3.2 (Outer Identified Region) *The outer identified set for model parameter $\theta = (\theta_1, \theta_2) \in \Theta$ is*

$$\Theta_O = \left\{ \theta \in \Theta : \begin{array}{c} \forall a \in \mathcal{Y} \\ \textit{such that:} \\ E[H_1^a(\theta, X)] \leq E[\Pr(y = a|x)] \leq E[H_2^a(\theta, X)] \end{array} \right\} \tag{21}$$

By introducing the following notations:

$$\boldsymbol{H}_1(\theta, X) = (H_1^1(\theta, X), ..., H_1^a(\theta, X), ..., H_1^{\#A}(\theta, X))'$$

$$\boldsymbol{H}_2(\theta, X) = (H_2^1(\theta, X), ..., H_2^a(\theta, X), ..., H_2^{\#A}(\theta, X))'$$

and

$$\Pr(\boldsymbol{y}|\boldsymbol{x}) = (\Pr(y = 1|x), ..., \Pr(y = a|x), ..., \Pr(y = \#A|x))'$$

Conditions for the outer identified set can be stated as:

$$E[\boldsymbol{H}_1(\theta, X)] \leq E[\Pr(\boldsymbol{y}|\boldsymbol{x})] \leq E[\boldsymbol{H}_2(\theta, X)] \tag{22}$$

Actually, the outer identified set Θ_O is broader than the sharp identified set Θ_I. Given that we do not have enough information about the equilibrium selection mechanism, the outer identified set Θ_O is the most thing we can learn about parameter θ from the underlying game and observation. In general, the set is not a singleton, as it is characterized by the moment inequality restrictions, and such model is called partial identified econometric models, corresponding to usual point identified case.

4 Estimation

The estimation problem is based on the moment inequality model

$$E[\mathbf{H}_1(\theta, \mathbf{X})] \leq E[\Pr(\mathbf{y}|\mathbf{x})] \leq E[\mathbf{H}_2(\theta, \mathbf{X})] \tag{23}$$

We follow Chernozhukov, Hong, and Tamer (2007) which provide a general framework for moment inequality models to build consistent estimator for the outer identified set Θ_O. Since the upper and lower bounds in the moment conditions (23) contain the multi-dimensional integrations, we first provide a simulation procedure to approximate these integrations. Due to the discreteness problem associated with the simple monte Carlo integration, in the spirit of Ackerberg (2009) and Bajari, Hong, and Ryan (2010), we make use of the importance sampling monte Carlo integration instead[5].

[5] McFadden (1989) noted the ability to use importance sampling to smooth simulations which is extended by Ackerberg (2009).

4.1 Importance Sampling Approximation

Importance sampling is most noted for its ability to reduce simulation error and computational burden, and was first used in game-theoretic models by Bajari, Hong, and Ryan (2010). First note that the correlated equilibrium depends on the parameter θ_1 only through the payoff function u, that is:

$$H_1^a(\theta, X)$$
$$= \int_{R_1^\pi(\theta,X)} \pi(a|u)dG(\epsilon|\theta_2) + \int_{R_2^\pi(\theta,X)} \min\{\pi(a|u) : \pi \in S_\pi(u(x,\epsilon;\theta_1))\}dG(\epsilon|\theta_2)$$

$$H_2^a(\theta, X)$$
$$= \int_{R_1^\pi(\theta,X)} \pi(a|u)dG(\epsilon|\theta_2) + \int_{R_2^\pi(\theta,X)} \max\{\pi(a|u) : \pi \in S_\pi(u(x,\epsilon;\theta_1))\}dG(\epsilon|\theta_2)$$

where $\pi(a|u)$ and the set of correlated equilibria $S_\pi(u(x,\epsilon;\theta_1))$ both determined by the payoff level u, and associated with θ_1 only through u. Based on this property, we can change variable of integration in and from ϵ to u. Let $h(u|X,\theta)$ denote the density of u, conditional on x and θ. Based on the utility function $u_i(a, x_i, \epsilon_i; \theta_1) = \Pi_i(x_i, a; \theta_1) + \epsilon_i(a)$ and the density for ϵ, $g(\epsilon|\theta_2)$, $h(u|X,\theta)$ can be derived as:

$$h(u|X,\theta) = \prod_i \prod_{a \in A} g(u_i(a) - \Pi_i(x_i, a; \theta_1)|\theta_2) \tag{24}$$

Then $H_1^a(\theta, X)$ and $H_2^a(\theta, X)$ can be converted as

$$
\begin{aligned}
&H_1^a(\theta, X) \\
&= \int\limits_{R_1^\pi(\theta,X)} \pi(a|u)dG(\epsilon|\theta_2) + \int\limits_{R_2^\pi(\theta,X)} \min\{\pi(a|u) : \pi \in S_\pi(u(x,\epsilon;\theta_1))\}dG(\epsilon|\theta_2) \qquad (25)\\
&= \int\limits_{R_1'} \pi(a|u)h(u|X,\theta)du + \int\limits_{R_2'} \min\{\pi(a|u) : \pi \in S_\pi(u)\}h(u|X,\theta)du
\end{aligned}
$$

$$
\begin{aligned}
&H_2^a(\theta, X) \\
&= \int\limits_{R_1^\pi(\theta,X)} \pi(a|u)dG(\epsilon|\theta_2) + \int\limits_{R_2^\pi(\theta,X)} \max\{\pi(a|u) : \pi \in S_\pi(u(x,\epsilon;\theta_1))\}dG(\epsilon|\theta_2) \qquad (26)\\
&= \int\limits_{R_1'} \pi(a|u)h(u|X,\theta)du + \int\limits_{R_2'} \max\{\pi(a|u) : \pi \in S_\pi(u)\}h(u|X,\theta)du
\end{aligned}
$$

where R_1' is collection of u such that the game admits unique equilibrium, and R_2' is the collection of u that the game admits multiple equilibria. By introducing a importance density $q(u)$, rewrite (25) and (26) as:

$$
\begin{aligned}
&H_1^a(\theta, X) \\
&= \int\limits_{R_1'} \pi(a|u)\frac{h(u|X,\theta)}{q(u)}q(u)du + \int\limits_{R_2'} \min\{\pi(a|u) : \pi \in S_\pi(u)\}\frac{h(u|X,\theta)}{q(u)}q(u)du \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
&H_2^a(\theta, X) \\
&= \int\limits_{R_1'} \pi(a|u)\frac{h(u|X,\theta)}{q(u)}q(u)du + \int\limits_{R_2'} \max\{\pi(a|u) : \pi \in S_\pi(u)\}\frac{h(u|X,\theta)}{q(u)}q(u)du \qquad (28)
\end{aligned}
$$

We can then simulate $H_1^a(\theta, X)$ and $H_2^a(\theta, X)$ by drawing random variables

$(u^1, ..., u^{ns}, ..., u^{NS})$ from the importance density $q(u)$, note that here u^{ns} is a vector, it contains the utility for all the players of the underlying game. Based on these simulated utility values, the importance sampling simulators for $H_1^a(\theta, X)$ and $H_2^a(\theta, X)$ are $\tilde{H}_1^a(\theta, X)$ and $\tilde{H}_2^a(\theta, X)$, respectively.

$$\begin{aligned}\tilde{H}_1^a(\theta, X) \quad = \quad & \tfrac{1}{NS}\sum_{ns} I(u^{ns} \in R_1')\pi(a|u^{ns})\tfrac{h(u^{ns}|X,\theta)}{q(u^{ns})}+ \\ & \tfrac{1}{NS}\sum_{ns} I(u^{ns} \in R_2')\min\{\pi(a|u^{ns}) : \pi \in S_\pi(u^{ns})\}\tfrac{h(u^{ns}|X,\theta)}{q(u^{ns})}\end{aligned} \tag{29}$$

$$\begin{aligned}\tilde{H}_2^a(\theta, X) \quad = \quad & \tfrac{1}{NS}\sum_{ns} I(u^{ns} \in R_1')\pi(a|u^{ns})\tfrac{h(u^{ns}|X,\theta)}{q(u^{ns})}+ \\ & \tfrac{1}{NS}\sum_{ns} I(u^{ns} \in R_2')\max\{\pi(a|u^{ns}) : \pi \in S_\pi(u^{ns})\}\tfrac{h(u^{ns}|X,\theta)}{q(u^{ns})}\end{aligned} \tag{30}$$

From the theory of importance sampling, $\tilde{H}_1^a(\theta, X)$ and $\tilde{H}_2^a(\theta, X)$ are unbiased simulators for $H_1^a(\theta, X)$ and $H_2^a(\theta, X)$, respectively. The most important property of these simulator is that they will generally be continuous in the parameter θ since it only depends on θ through $h(u|x, \theta)$ which is continuos in θ given that $g(\epsilon|\theta_2)$ is continuous.

Although the theory of importance sampling proves that are smooth and unbiased simulator for any choice of the importance density $q(u)$ which has sufficiently large support. However, as noted by Bajari, Hong, and Ryan (2010), as a practical matter, it is important to make sure that the tails of the importance density are not too thin in a neighborhood of the parameter which optimizes the objective function in our estimation procedure. We suggest to use some pre-estimated $\mathring{\theta}$ to construct the importance density

$$q(u) = h(u|X, \mathring{\theta}) \tag{31}$$

where can be obtained from the estimates of the game with incomplete information which is studied in Bajari, Hong, Krainer, and Nekipelov (2010), or through the generalized maximum entropy estimator for the static games of complete information

(Golan, Karp, and Perloff, 2000). Note that these two studies on empirical games both are based on the Nash equilibrium.

4.2 Estimation

Given the simulators obtained from the importance sampling, $\tilde{H}_1^a(\theta, X)$ and $\tilde{H}_2^a(\theta, X)$, for $H_1^a(\theta, X)$ and $H_2^a(\theta, X)$, respectively, introducing following notations:

$$\tilde{\boldsymbol{H}}_1(\theta, X) = (\tilde{H}_1^1(\theta, X), ..., \tilde{H}_1^a(\theta, X), ..., \tilde{H}_1^{\#A}(\theta, X))'$$

$$\tilde{\boldsymbol{H}}_2(\theta, X) = (\tilde{H}_2^1(\theta, X), ..., \tilde{H}_2^a(\theta, X), ..., \tilde{H}_2^{\#A}(\theta, X))'$$

From (23) we get the following simulated moment inequality restrictions:

$$E[\tilde{\boldsymbol{H}}_1(\theta, X_t)] \leq E[\Pr(\boldsymbol{y}|\boldsymbol{x}_t)] \leq E[\tilde{\boldsymbol{H}}_2(\theta, X_t)] \tag{32}$$

According to Chernozhukov, Hong, and Tamer (2007), our inferential procedures uses the objective function[6]:

$$\min_{\theta \in \Theta} Q(\theta) \equiv \int \left\| (\Pr(\boldsymbol{y}|\boldsymbol{x}) - \tilde{\boldsymbol{H}}_1(\theta, X))_- \right\|^2 + \left\| (\Pr(\boldsymbol{y}|\boldsymbol{x}) - \tilde{\boldsymbol{H}}_2(\theta, X))_+ \right\|^2 dF_x \tag{33}$$

this criterion function (32), that is if $\Pr(\boldsymbol{y}|\boldsymbol{x}) < \tilde{\boldsymbol{H}}_1(\theta, X)$, then $\left\| (\Pr(\boldsymbol{y}|\boldsymbol{x}) - \tilde{\boldsymbol{H}}_1(\theta, X))_- \right\|^2$ is strictly positive, and if $\Pr(\boldsymbol{y}|\boldsymbol{x}) > \tilde{\boldsymbol{H}}_2\ (\theta, X)$, then $\left\| (\Pr(\boldsymbol{y}|\boldsymbol{x}) - \tilde{\boldsymbol{H}}_2(\theta, X))_+ \right\|^2$ is strictly positive. It is easy to see that $Q(\theta) \geq 0$ for all $\theta \in \Theta$ and that $Q(\theta) = 0$ if and only if $\theta \in \Theta_O$.

To estimate the outer identified set Θ_O, we need to take a sample analog of $Q(\theta)$. First

[6] Let $\| x \|_+ = \| (x)_+ \|$ and $\| x \|_- = \| (x)_- \|$, where $(x)_+ := \max(x, 0)$, $(x)_- := \max(-x, 0)$ and $\| \cdot \|$ is the Euclidian norm.

replace $\Pr(\boldsymbol{y}|\boldsymbol{x})$ with a $\sqrt{T}$ consistent estimator $P_T(X)$[7]. The sample analog for $Q(\theta)$ is

$$Q_T(\theta) = \frac{1}{T^2}\sum_{t=1}^{T}\left[\left\|((P_T(X_t) - \tilde{\boldsymbol{H}}_1(\theta, X_t))_-\right\|^2 + \left\|P_T(X_t) - \tilde{\boldsymbol{H}}_2(\theta, X_t))_+\right\|^2\right] \tag{34}$$

Our estimation for Θ_O is any solution that minimizing (34), which can be obtained from:

$$\hat{\Theta}_O = \{\theta \in \Theta : TQ_T(\theta) \leq v_T\} \tag{35}$$

where $v_T \to \infty$ and $\frac{v_T}{T} \to 0$, Chernozhukov, Hong, and Tamer (2007) propose a resampling method to get suitable v_T.

Theorem 2 *Let Assumption 3 hold. Suppose that the regular conditions of the Theorem 3.1 in hold. Then we have that $\hat{\Theta}_O$ is a Hausdorff consistent estimator for Θ_O, that is $d_H(\hat{\Theta}_O, \Theta_O) = 0$ with probability one.*

The proof of Theorem is the same as the Theorem 3.1 in Chernozhukov, Hong, and Tamer (2007). To conduct inference about the above moment inequalities model, we use the methodology of Chernozhukov, Hong, and Tamer (2007) and Ciliberto and Tamer (2009). We conduct a set $\mathcal{C}_T$ for a prespecified $\alpha \in (0, 1)$ such that

$$\lim_{T\to\infty} (\theta_O \in \mathcal{C}_T) \geq \alpha \text{ for any } \theta_O \in \Theta_O \tag{36}$$

Which can be constructed as follows, Let

$$\mathcal{C}_T(c) = \left\{\theta \in \Theta : T\left(Q_T(\theta) - \min_z Q_T(z)\right) \leq c\right\} \tag{37}$$

Then do the following loop:

[7]The convergence rate of nonparametric estimates for $P_T(X)$ is slower than $\sqrt{T}$ when there are continuos variables in x, a useful method is to discretize all the variables in x and use the nonparametric frequency estimation.

1. compute a initial estimate for Θ_O as $\mathcal{C}_T(c_0)$, for example $\mathcal{C}_T(c_0) = \mathcal{C}_T(0)$, then subsampling the statistics $T\left(Q_T(\theta) - \min_z Q_T(z)\right)$ for $\theta \in \mathcal{C}_T(0)$ and obtain the estimate of its α-quantile, $c_1(\theta_0)$.

2. Update c through $c_1 = \sup_{\theta_0 \in \mathcal{C}_T(c_0)} c_1(\theta_0)$ and formulate $\mathcal{C}_T(c_1)$ as step 1.

3. Let $c_0 = c_1$ then obtain c_2.

Then $\mathcal{C}_T(c_2)$ will be our confidence region for $\hat{\Theta}_O$. See Chernozhukov, Hong, and Tamer (2007) and Ciliberto and Tamer (2009) for more on this. Such confidence region does not only have the desired coverage property, but will also be consistent in the sense of Theorem .

4.3 Computation of the Equilibria

The simulated lower and upper bounds, $H_1^a(\theta, X)$ and $\tilde{H}_2^a(\theta, X)$, e.g., (29) and (30), contain the following equilibrium computation:

$$I(u \in R_1')\pi(a|u) \tag{38}$$

$$I(u \in R_2') \min\{\pi(a|u) : \pi \in S_\pi(u)\} \tag{39}$$

and

$$I(u \in R_2') \max\{\pi(a|u) : \pi \in S_\pi(u)\} \tag{40}$$

where u is a vector which contains the utility levels of all the players for each action profile. We first discuss the computation of (39) and (40), where the corresponding game admits multiple equilibria. First note that if we can identify the regions R_1' and R_2', then we only need to compute

$$\min_\pi\{\pi(a|u) : \pi \in S_\pi(u)\} \tag{41}$$

where the solution is the correlated equilibrium that puts the lowest probability for action profile a, and

$$\max_{\pi}\{\pi(a|u) : \pi \in S_\pi(u)\} \tag{42}$$

where the solution is the correlated equilibrium that puts the highest probability for action profile a. Both of them can be obtained through simple linear programming. For $\min\{\pi(a|u) : \pi \in S_\pi(u)\}$, it can be obtained from

$$\begin{aligned} &\min_{\pi} \;\; \pi(k) \\ &s.t. \begin{cases} \sum_{a_{-i}} u_i(a_{-i}, a_i)\pi(a_{-i}, a_i) \geq \sum_{a_{-i}} u_i(a_{-i}, \tilde{a}_i)\pi(a_{-i}, a_i), \; \forall i, a_i \text{ and } \tilde{a}_i \neq a_i \\ \sum_{a \in A} \pi(a) = 1, \; \pi(a) \geq 0 \end{cases} \end{aligned} \tag{43}$$

and $\max\{\pi(a|u) : \pi \in S_\pi(u)\}$ can be obtained from

$$\begin{aligned} &\max_{\pi} \;\; \pi(a) \\ &s.t. \begin{cases} \sum_{a_{-i}} u_i(a_{-i}, a_i)\pi(a_{-i}, a_i) \geq \sum_{a_{-i}} u_i(a_{-i}, \tilde{a}_i)\pi(a_{-i}, a_i), \; \forall i, a_i \text{ and } \tilde{a}_i \neq a_i \\ \sum_{a \in A} \pi(a) = 1, \; \pi(a) \geq 0 \end{cases} \end{aligned} \tag{44}$$

While the game only admits unique equilibrium, then the solution for the system of linear inequalities:

$$\begin{aligned} &\sum_{a_{-i}} u_i(a_{-i}, a_i)\pi(a_{-i}, a_i) \geq \sum_{a_{-i}} u_i(a_{-i}, \tilde{a}_i)\pi(a_{-i}, a_i), \; \forall i, a_i \text{ and } \tilde{a}_i \neq a_i \\ &\sum_{a \in A} \pi(a) = 1, \; \pi(a) \geq 0 \end{aligned} \tag{45}$$

is unique, thus either linear programming (43) or (44) both can provide this unique equilibrium, which means that in practice, we do not need to identify the regions based on whether there are multiple equilibria, the only computation we need to do is linear programming (43) or (44). Clearly, the computation of equilibria in our procedure is very simple, it only needs linear programming! While most other studies which focus on the empirical complete information games based on Nash equilibrium, need to compute all the Nash equilibrium of underlying games, for the general game, this will induce a heavy computational burden. See for example, Berry and Tamer (2007), Bajari, Hong, and Ryan (2010) and Ciliberto and Tamer (2009).

5 Test of Equilibrium Concepts

In this paper, we based on the correlated equilibrium study the identification and estimation of empirical static complete information game, where the multiple equilibria is an important issue to deal with. Actually, not only the equilibrium of a given game can be multiple, but also the equilibrium concept. Such as pure strategy Nash equilibrium, mixed strategy equilibrium, correlated equilibrium, evolutionary equilibrium and so on, besides the theoretical research, it is an interesting problem that testing which equilibrium concept is the most consistent with the real data. Given that there are existing research based on Nash equilibrium, and we study the same issue based on correlated equilibrium, we can conduct a formal test for the equilibrium concepts: Nash equilibrium versus correlated equilibrium.

Formally, Let $\mathcal{CE}$ denote the complete information static game based correlated equilibrium, and $\mathcal{NE}$ denote the same game based on Nash equilibrium. Since both of the two models can be characterized by moment inequality restrictions, then

$$\mathcal{CE} = \bigcup_{\theta \in \Theta} \mathcal{CE}_\theta; \ \mathcal{NE} = \bigcup_{\beta \in B} \mathcal{NE}_\beta \tag{46}$$

where

$$\mathcal{CE}_\theta = \{CE : E_{CE} m_j(X_i, \theta) \geq 0, \ j \in J_{CE}\} \tag{47}$$

$$\mathcal{NE}_\beta = \{NE : E_{NE} g_j(X_i, \beta) \geq 0, \, j \in J_{NE}\} \tag{48}$$

where $\{X_i \in \mathcal{X}\}_{i=1}^n$ is the sample generated from distribution μ, $m_j(X_i, \theta)$ and $g_j(X_i, \beta)$ are moment functions characterized by finite dimensional parameter θ and β, respectively. Then $E_{CE} m_j(X_i, \theta) \geq 0$ is equivalent to the moment inequalities (22), while $E_{NE} g_j(X_i, \beta) \geq 0$ can be explained as the moment conditions in Ciliberto and Tamer (2009)[8].

Given the above structure of the two models, we want to test the two distribution $\mathcal{CE}$ and $\mathcal{NE}$, which is closer to the true distribution μ. Since both of the two models are defined in terms of moment inequality restrictions, we can make use of the test for moment inequality models developed in Shi (2010). Consider the null hypothesis:

$$H_0 : d(\mathcal{CE}, \mu) = d(\mathcal{NE}, \mu) \tag{49}$$

where

$$d(\mathcal{CE}, \mu) = \inf_{CE \in \mathcal{CE}} d(CE, \mu); \, d(\mathcal{NE}, \mu) = \inf_{NE \in \mathcal{NE}} d(NE, \mu) \tag{50}$$

The distance $d(P, \mu)$ is defined as the Kullback-Leibler divergence measure:

$$d(P, \mu) = \int p_\mu \log p_\mu d\mu \tag{51}$$

where p_μ is the density of P with respect to μ. To construct the test statistics, For a data distribution μ, define the Lagrange multipliers:

$$\gamma_\mu^*(\theta) = \arg\min_\gamma \exp(\gamma' m(X_i, \theta)) \tag{52}$$

$$\lambda_\mu^*(\beta) = \arg\min_\lambda \exp(\lambda' g(X_i, \beta)) \tag{53}$$

[8] The moment conditions in Ciliberto and Tamer (2009) are based on the pure strategy Nash equilibrium, to obtain the moment conditions for Nash equilibrium, one needs to extend that result. Berry and Tamer (2007) briefly discussed the problem that arise when allowing for mixed strategies.

and criterion functions:

$$\mathcal{M}_{\mu}(\gamma,\theta) = E_{\mu}\exp(\gamma' m(X_i,\theta)) \tag{54}$$

$$\mathcal{G}_{\mu}(\lambda,\beta) = E_{\mu}\exp(\lambda' g(X_i,\beta)) \tag{55}$$

Shi (2010) prove that the null hypothesis (49) can be stated as:

$$H_0 : \max_{\theta\in\Theta}\mathcal{M}_{\mu}(\gamma_{\mu}^*(\theta),\theta) = \max_{\beta\in B}\mathcal{G}_{\mu}(\lambda_{\mu}^*(\beta),\beta) \tag{56}$$

The sample analog of $\mathcal{M}_{\mu}(\gamma_{\mu}^*(\theta),\theta)$ and $\mathcal{G}_{\mu}(\lambda_{\mu}^*(\beta),\beta)$ are:

$$\hat{\mathcal{M}}_n(\gamma,\theta) = \frac{1}{n}\sum_{i=1}^{n}\exp(\gamma' m(X_i,\theta));\ \hat{\mathcal{G}}_n(\lambda,\beta) = \frac{1}{n}\sum_{i=1}^{n}\exp(\lambda' g(X_i,\beta)) \tag{57}$$

where

$$\begin{aligned} \hat{\gamma}_n(\theta) &= \arg\min_{\gamma}\hat{\mathcal{M}}_n(\gamma,\theta), \quad & \hat{\lambda}_n(\beta) &= \arg\min_{\lambda}\hat{\mathcal{G}}_n(\lambda,\beta) \\ \hat{\Theta}_n &= \arg\max_{\theta\in\Theta}\hat{\mathcal{M}}_n(\hat{\gamma}_n(\theta),\theta) \quad & \hat{B}_n &= \arg\max_{\beta\in B}\hat{\mathcal{G}}_n(\hat{\lambda}_n(\beta),\beta) \end{aligned} \tag{58}$$

Then we can based on the quasi-likelihood ratio statistics

$$QLR_n = \max_{\theta\in\Theta}\hat{\mathcal{M}}_n(\hat{\gamma}_n(\theta),\theta) - \max_{\beta\in B}\hat{\mathcal{G}}_n(\hat{\lambda}_n(\beta),\beta) \tag{59}$$

to test the null hypothesis (56).

With several regular conditions, Shi (2010) prove that under H_0:

$$QLR_n \overset{d}{\rightsquigarrow} N(0,\varpi_n^2) \tag{60}$$

where $\varpi_n^2 = E_{\mu}[\exp(\gamma_{\mu}^*(\theta^*)' m(X_i,\theta^*)) - \exp(\lambda_n^*(\beta^*)' g(X_i,\beta^*))]^2$, $\theta^* \in \arg\max_{\theta\in\Theta}\mathcal{M}_{\mu}(\gamma_{\mu}^*(\theta),\theta)$, $\beta^* \in \arg\max_{\beta\in B}\mathcal{G}_{\mu}(\lambda_{\mu}^*(\beta),\beta)$. While in practice, ϖ_n^2 can be replaced with its sample analog $\hat{\varpi}_n^2$:

$$\hat{\varpi}_n^2 = \sup_{\theta \in \hat{\Theta}_n, \beta \in \hat{B}_n} \frac{1}{n} \sum_{\mu} [\exp(\hat{\gamma}_n(\theta)' m(X_i, \theta)) - \exp(\hat{\lambda}_n(\beta)' g(X_i, \beta))]^2 \tag{61}$$

Then the test criterion is

Test of Correlated Equilibrium versus Nash Equilibrium Let b_n be a sequence of positive numbers such that $b_n^{-1} + n^{-1} b_n \to 0$. Given the nominal size α and the $(1 - \alpha/2)$ quantile of the standard normal distribution, $z_{\alpha/2}$.

(1) If $n\hat{\varpi}_n^2 > b_n$ and $n^{\frac{1}{2}} QLR_n / \hat{\varpi}_n > z_{\alpha/2}$, then reject the null H_0 and accept correlated equilibrium as the equilibrium concept.

(2) If $n\hat{\varpi}_n^2 > b_n$ and $n^{\frac{1}{2}} QLR_n / \hat{\varpi}_n < -z_{\alpha/2}$, then reject the null H_0 and accept Nash equilibrium as the equilibrium concept.

(3) If $\hat{\varpi}_n^2$ and $n^{\frac{1}{2}} QLR_n / \hat{\varpi}_n$ do not satisfy the condition in (1) and (2), then accept the null H_0.

This test criterion is based on the nested model selection test of Shi (2010). Remember the Proposition 2.1, which states that Nash equilibrium is a subset of correlated equilibrium, which means that we have $\mathcal{NE} \subset \mathcal{CE}$ but not $\mathcal{CE} \subset \mathcal{NE}$. An interesting case is that if the test suggest us to accept the null hypothesis, which means that correlated equilibrium and Nash equilibrium do the same explanation for the real data, then all the correlated equilibrium of the underlying game are all the Nash equilibrium, according to Proposition 2.2, we can find that, in this case, the underlying game has unique correlated equilibrium, and thus, has unique Nash equilibrium. Finally, the property of nest between correlated equilibrium and Nash equilibrium means that based on correlated equilibrium to estimate empirical games is robust, while when the true equilibrium concept is Nash equilibrium, it is inefficient, while the true equilibrium concept is correlated equilibrium, except the special case that the game only admits unique equilibrium, based on Nash equilibrium to estimate the game will get the inconsistent estimation. This is similar to the choice between fixed effect and random effect in panel data models.

6 Monte Carlo Simulation

To demonstrate the performance of our estimates in finite samples, we conduct a Monte Carlo experiment using a simple 2×2 game. In each of the T repetitions of the simultaneous-move game with complete information, each has the following structure:

	0	1
0	$(0, 0)$	$(0, \epsilon_2(0, 1))$
1	$(\epsilon_1(1, 0), 0)$	$(\theta_1 + \epsilon_1(1, 1), \theta_2 + \epsilon_2(1, 1))$

Which can be explained as a static entry game, the action set of each player is $A_i = \{0, 1\}$, where 0 means no entry and 1 means entry. The utility function for player i is defined as:

$$u_i(a, \epsilon_i(a); \theta) = I(a_i = 1)(\theta_1 a_{-i} + \epsilon_i(a)) \quad (62)$$

As a simple experiment, we have not included any exogenous covariates x here. And according the location and scale normalization, we set the utility of no entry as 0 and the variance of the random preference shock as 1. Then we only need to estimate the strategic effect parameters θ_1 and θ_2.

All random preference shocks $\epsilon_{1t}(1, 0)$, $\epsilon_{2t}(0, 1)$, $\epsilon_{1t}(1, 1)$ and $\epsilon_{2t}(1, 1)$ are independently drawn from standard normal distribution. The parameter space Θ is set to $\Theta = [-5, 5]^2$, and there true values are:

$$\theta_1 = -0.5; \, \theta_2 = -1 \quad (63)$$

Which means that the entry of player i will decrease the payoff of player j given the entry of player j. Given these random shocks and parameters, we generate the outcome of each game, e.g., the observed action profiles, by a simple maximum entropy equilibrium selection mechanism:

$$\max_{\pi} - \sum_{a \in A} \pi(a) \ln \pi(a)$$
$$s.t. \begin{cases} \sum_{a_{-i}} u_i(a_{-i}, a_i)\pi(a_{-i}, a_i) \geq \sum_{a_{-i}} u_i(a_{-i}, \tilde{a}_i)\pi(a_{-i}, a_i), \ \forall i, a_i \text{ and } \tilde{a}_i \neq a_i \\ \sum_{a \in A} \pi(a) = 1, \ \pi(a) \geq 0 \end{cases} \tag{64}$$

Obviously, this maximum entropy equilibrium selection mechanism will generate a most dispersive correlated equilibrium π^* among all correlated equilibria. According to π^*, we use a simple random sampling to determine which action profile will be played. Based on the maximum entropy equilibrium selection mechanism, under the sample size 500, the $E[\Pr(\boldsymbol{y}_t)]$ is:

$$E[\Pr(\boldsymbol{y}_t)] = (0.291058, 0.274005, 0.35475, 0.080187) \tag{65}$$

Note that although we set the true value for parameters as $\theta_1 = -0.5$ and $\theta_2 = -1$, but only if we know the equilibrium selection mechanism, the point identification can be achieved, and thus we can compare our estimates with the true value of parameters. In practice, usually, we've no information about the equilibrium selection mechanism, thus there may a lot of values of the parameter that can generate the data we observed. We use the following procedure to do the set estimation (35), first, based on the simulated annealing algorithm to find the optimal solution of the minimization of (34), denote it as $\tilde{\theta}$; then in the parameter space Θ choosing rich directions[9] to the grid search until it condition (35)

[9]In this experiment, we choose 402 directions, which are randomly chosen according to a uniform distribution over $[0, 2\pi]$.

Table 1: The Results of Monte Carlo Simulation

	Initial Value for Set estimation	Interval Estimates
	$T = 500$	
θ_1	-0.5677	$[-2.3142, 0.9580]$
θ_2	-1.0273	$[-2.8319, 0.4027]$
	$T = 1000$	
θ_1	-0.5659	$[-2.2794, 0.9370]$
θ_2	-1.0267	$[-2.8184, 0.3816]$
	Monte Carlo Times: 1000	
	Importance Sampling Times: 999	

We generate 1000 samples of size $T = 500, 1000$ to assess the finite sample properties of our estimator, first use importance sampling simulator get simulated bounds of choice probability, then based on the above numerical procedure find the final estimates. The interval estimates are reported in Table 1, and the whole set estimators for $T = 500$ and 1000 are reported in Figure 1 and 2, both are compact sets in R^2 space. We also report the comparison between the two different sample size in Figure 3, where red denotes the for $T = 500$ and blue for $T = 1000$. Since we lack the information of the true range of the outer identified set, we can not say much about the performance of our estimator, while we conclude that the true value of the parameter lies in our estimated set. Moreover, from Figure 3 we can find that when the sample size increase, the range of $\hat{\Theta}_O$ decrease, which is similar to the convergence in the point identified case.

7 Conclusion

In this paper, we propose a framework for identifying and estimating the normal-form payoff parameters of a simultaneous, discrete game of complete information where the equilibrium concept employed is correlated equilibrium. Comparing with the existing studies based on Nash equilibrium, this extension of the equilibrium concepts simplifies the identification and estimation of game-theoretic econometric models, since our ap-

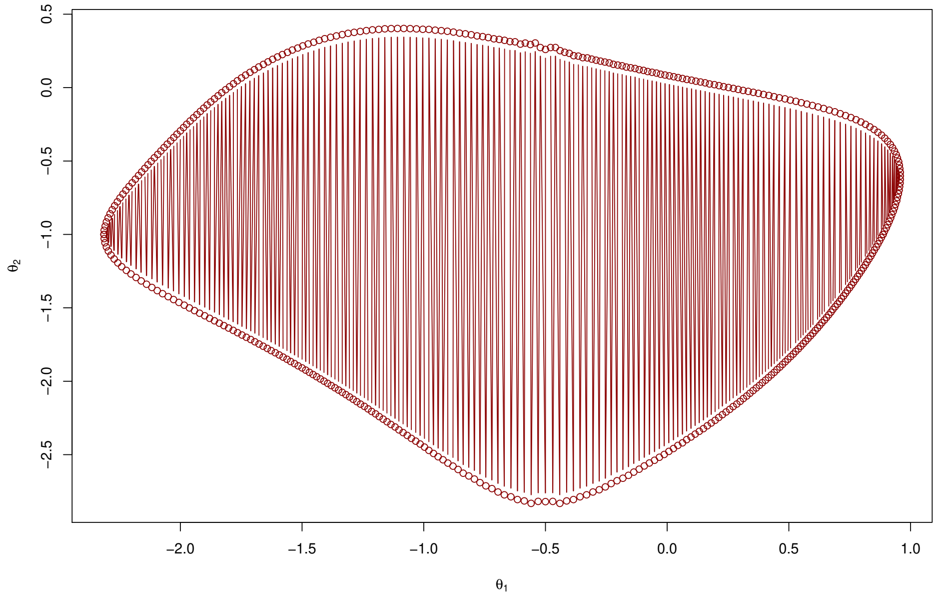

Figure 2: The Set Estimate of $\hat{\Theta}_O$ when $T = 500$

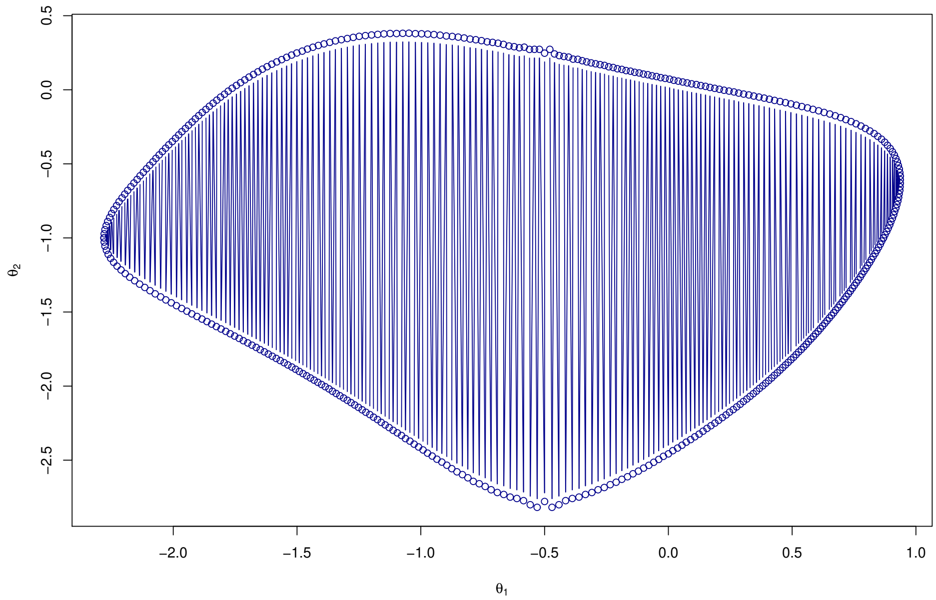

Figure 3: The Set Estimate of $\hat{\Theta}_O$ when $T = 1000$

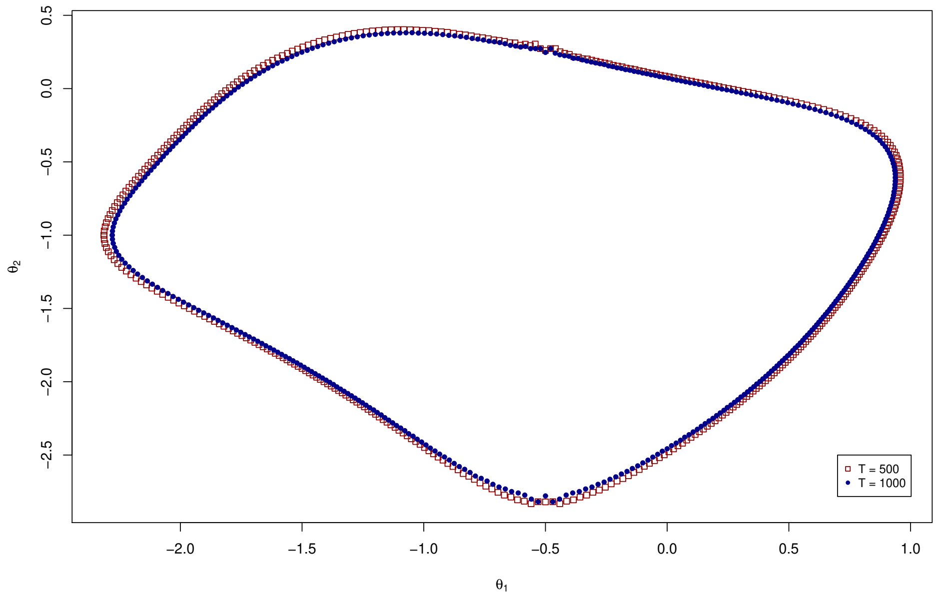

Figure 4: The Comparison of $\hat{\Theta}_O$ between Different Sample Size

proach does not require the computation of the whole set of equilibria, it only needs some "extreme" equilibria which can be obtained through linear programming. To deal with the presence of multiple equilibria, we make use of the moment inequality restrictions induced by the underlying game-theoretic econometric models without making equilibrium selection assumptions, which avoids the misspecification of equilibrium selection mechanisms, and leads to a partial identified model. Given the outer identified set characterized by moment restrictions, the set estimator developed by Chernozhukov, Hong, and Tamer (2007) is used to obtain its estimates. The importance sampling technique is used to reduce computational burden and overcome the non-smoothness problems. We also show that the model selection tests for moment inequality models developed by Shi (2010) can be used to test equilibrium concepts such as correlated equilibrium versus Nash equilibrium. The most limitation of our estimation is that it relies on the known distribution of random preference shocks which is rarely known to researchers, working with the unknown distribution is an important topic for future research. Another possible extension is that updating our estimator to the one based on conditional moment restriction.

References

Ackerberg, D. A. (2009): “A New Use of Importance Sampling to Reduce Computational Burden in Simulation Estimation,” *Quantitative Marketing and Economics*, 7(4), 343 – 376.

Aguirregabiria, V., and P. Mira (2007): “Sequential Estimation of Dynamic Discrete Games,” *Econometrica*, 75, 1–53.

Aradillas-Lopez, A. (2007): “Pairwise Difference Estimation of Incomplete Information Games,” *Working Paper*.

——— (2010): “Semiparametric Estimation of a Simultaneous Game with Incomplete Information,” *Journal of Econometrics*, 157(2), 409–431.

Aumann, R. J. (1974): “Subjectivity and Correlation in Randomized Strategies,” *Journal of Mathematical Economics*, 1(1), 67–96.

——— (1987): “Correlated Equilibrium as an Expression of Bayesian Rationality,” *Econometrica*, 55(1), 1–18.

Bajari, P., H. Hong, J. Krainer, and D. Nekipelov (2010): “Estimating Static Models of Strategic Interactions,” *Journal of Business and Economic Statistics*, 28(4), 469–482.

Bajari, P., H. Hong, and S. Ryan (2010): “Identification and Estimation of a Discrete Game of Complete Information,” *Econometrica*, 78(5), 1529–1568.

Bergin, J. (2005): *Microeconomic Theory: A Concise Course*. Oxford University Press.

Berry, S., and E. Tamer (2007): "Identification in Models of Oligopoly Entry," in *Advanced in Economics and Econometrics: Theory and Application*, vol. II, chap. 2, pp. 46–85. Cambridge University Press.

Bresnahan, T. F., and P. C. Reiss (1990): "Entry in Monopoly Markets," *Review of Economic Studies*, 57, 531–553.

——— (1991): "Empirical models of Discrete Games," *Journal of Econometrics*, 48, 57–81.

Calvo-Armengol, A. (2006): "The Set of Correlated Equilibria of 2×2 Games," *Working Paper*.

Chernozhukov, V., H. Hong, and E. Tamer (2007): "Estimation and Confidence Regions for Parameter Sets in Econometric Models," *Econometrica*, 75(5), 1243–1284.

Chwe, M. S. (2007): "Identifying a Game by Observing its Correlated Equilibria," *Working Paper*.

Ciliberto, F., and E. Tamer (2009): "Market Structure and Multiple Equilibria in Airline Markets," *Econometrica*, 77(6), 1791–1828.

Dhillon, A., and J. F. Mertens (1996): "Perfect Correlated Equilibria," *Journal of Economic Theory*, 68(2), 279–302.

Golan, A., L. S. Karp, and J. M. Perloff (2000): "Estimating Coke's and Pepsi's Price and Advertising Strategies," *Journal of Business and Economic Statistics*, 18(4), 398–409.

Hausman, J. A. (1978): "Specification Tests in Econometrics," *Econometrica*, 46(6), 1251–71.

Jia, P. (2008): "What Happens When Wal-Mart Comes to Town: An Empirical Analysis of the Discount Retailing Industry," *Econometrica*, 76, 1263–1316.

McFadden, D. (1989): "A Method of Simulated Moments for Estimation of Discrete Response Models without Numerical Integration," *Econometrica*, 57(5), 995–1026.

Mckelvey, R., and A. Mclennan (1996): "The computation of equilibrium in finite games," in *The Handbook of Computational Economics*, ed. by H. Amman, D. A. Kendrick, and J. Rust, vol. 1996. Elsevier Science, The Netherlands.

Morris, S., and H. Shihn (2003): "Global Games: Theory and Applications," in *Advances in Economics and Econometrics*, ed. by M. Dewatripont, L. Hansen, and S. Turnovsky, pp. 56–114. Cambridge University Press.

Ortiz, L. E., R. E. Schapire, and S. M. Kakade (2007): "Maximum Entropy Correlated Equilibria," *Eleventh International Conference on Artificial Intelligence and Statistics*.

Papadimitriou, C., and T. Roughgarden (2008): "Computing Correlated Equilibria in Multi-Player Games," *The Proceedings of the 37th Annual Symposium on the Theory of Computing*.

Pesendorfer, M., and P. Schmidt-Dengler (2008): "Asymptotic Least Squares Estimators for Dynamic Games," *Review of Economic Studies*, 75, 901–928.

Seim, K. (2006): "An Empirical Model of Firm Entry with Endogenous Product-type Choices," *RAND Journal of Economics*, 37(3), 619–640.

Shi, X. (2010): "Model Selection Tests for Nonnested Moment Inequality Models," *Job Market Paper of Yale University*.

Sweeting, A. (2005): "Coordination Games, Multiple Equilibria and the Timing of Radio Commercials," *Working Paper*.

Tamer, E. T. (2003): "Incomplete Bivariate Discrete Response Model with Multiple Equilibria," *Review of Economic Studies*, 70, 147–167.