Strategic real options with stochastic volatility in a duopoly model

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STRATEGIC REAL OPTIONS WITH STOCHASTIC VOLATILITY IN A DUOPOLY MODEL

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Abstract. The investment-timing problem has been considered by many authors under the assumption that the instantaneous volatility of the demand shock is constant. Recently, Ting et al. [9] carried out an asymptotic approach in a monopoly model by letting the volatility parameter follow a stochastic process. In this paper, we consider a strategic game in which two firms compete for a new market under an uncertain demand, and extend the analysis of Ting et al. to duopoly models under different strategic game structures. In particular, we investigate how the additional uncertainty in the volatility affects the investment thresholds and payoffs of players. Several numerical examples and comparison of the results are provided to confirm our analysis.

1. Introduction

In economics and finance, the traditional net present value (NPV) method has been well established. However, applying the NPV method to cases where an investment contains a significant share of irreversible costs and/or is tightly bounded to uncertain factors leads to sub-optimal behaviours. McDonald and Siegel [7] introduced a mathematical model for such cases by adopting the tools used to value financial options and showed that the option to invest has an intrinsic value that must be accounted for. Nowadays, the option to invest in a given project is called a *real option*. The real option analysis is a great tool for investments that are expensive, long term and affected by multiple risks. For example, Brennan and Schwartz [1] applied the real option analysis to the mining industry. Since Dixit and Pindyck provided a systematic treatment of the real option anlysis in [2], many researchers in applied mathematics, economics and finance have worked in this area. For instance, Mareguerra et al. [6] used the same framework to investigate how the conclusions from the monopoly model could be transferred into a competitive environment. Hsu and Lambrecht [5], and Graham [3] studied the strategic real options and the effects of private information on the investment time decisions. Recently, Ting et al. [9] presented an asymptotic analysis on real options under the fast mean-reverting and the geometric mean-reverting regimes of the stochastic volatility.

The motivation of this paper is to extend the results in [9] to a competitive environment. Specifically, we use Heston's stochastic volatility model to build a duopoly real option model. In our model, the firm who invests first is referred as the leader and the other is referred as the follower. When we determine the optimal investment thresholds and the payoff of each firm, we consider both competitive and non-competitive situations. To this end, we first modify the model in [9] by adding a demand function of the market, and find that follower's optimal strategy is similar to that in the monopoly model. Then, we investigate leader's optimal strategy by determining leader's project value. We consider two situations: The first one is that both firms are capable of becoming a leader, and the second one is that leader and the follower are pre-determined. In the first situation, firms must compromise the benefits between being the leader and capturing the value of waiting. In the second situation, the leader can invest in the project without any strategic consideration, and the follower only has one optimal threshold, whereas the leader's optimal threshold varies, depending on the initial level of the stochastic variable. Nevertheless, these optimal thresholds need to be determined by solving partial differential equations, due to the assumption on the additional uncertainty in the volatility.

We organise the paper as follows: In Section 2, we present a modified monopoly model and its asymptotic solution. In Section 3, we extend the analysis in Section 2 to a duopoly model, and analyse firms' strategic investment behaviours under competitive and non-competitive situations. In Section 4, we present some numerical examples to compare our results with the classical results. The conclusion is given in Section 5. Finally, the derivations of main equations and the proofs of selected propositions are presented in the appendices.

2. A monopoly model under stochastic volatility

In this section, we first present a basic continuous time model of irreversible investments, which combines those presented in [6] and [9]. Consider a firm with the potential to activate a project which produces a unit output flow by incurring a sunk cost I. Assume that there are no variable costs of the production, and all the uncertainties of the project are from firm-specific. Thus the monopoly profit flow of the firm is determined by its demand D and the demand shock Y_t. Assume that Y_t follows a diffusion process with a stochastic instantaneous variance S_t, i.e.,

$$\begin{aligned} dY_t &= \alpha Y_t dt + \sqrt{S_t} Y_t dZ_t, \\ dS_t &= k(m - S_t)dt + \sigma\sqrt{S_t} dW_t, \end{aligned}$$

where Z_t and W_t are two Brownian motions with $(dZ_t, dW_t) = \rho dt$, α is the drift rate, k is the mean reverting rate, m is the mean reverting level and σ is the so-called volatility of volatility. The goal of this model is to determine when the firm pays the sunk cost to activate the project so that its profit is maximized.

Let r and δ denote the risk-free interest rate and the dividend rate, respectively. By the existence theorem of an equivalent martingale measure, we are able to change the real probability measure $\mathbb{P}$ to a risk-neutral probability measure $\mathbb{Q}$ and describe the processes as

$$dY_t = (r - \delta)Y_t dt + \sqrt{S_t} Y_t d\widetilde{Z}_t, \tag{2.1}$$

$$dS_t = k^*(m^* - S_t)dt + \sigma\sqrt{S_t} d\widetilde{W}_t, \tag{2.2}$$

where

$$k^* = k + \lambda \quad \text{and } m^* = \frac{km}{k + \lambda}$$

are the risk-neutral parameters, and the new parameter λ is the premium of volatility risk, refer to [4]. For the rest of this paper, our analysis will be based on the risk-neutral probability measure $\mathbb{Q}$.

Let y and s be the values of Y_t and S_t. Assume that D is independent of y and s. The project value, denoted by $V(y, s)$, can be expressed as the sum of the current profit in the time interval $[t, t+dt]$ and the continuation value, i.e.,

$$V(y,s) = yDdt + \mathbb{E}\left[\frac{V(y+dy, s+ds|y,s)}{1+rdt}\right]. \tag{2.3}$$

By applying Itô's lemma, we first obtain

$$dV = \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(dY_t)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(dS_t)^2 + \frac{\partial^2 V}{\partial y \partial s}(dY_t, dS_t) + \frac{\partial V}{\partial y}dY_t + \frac{\partial V}{\partial s}dS_t,$$

which can help us to transform (2.3) into the following PDE

$$rV = \frac{1}{2}\left[y^2 s\frac{\partial^2 V}{\partial y^2} + s\sigma^2\frac{\partial^2 V}{\partial s^2} + 2\rho y s\sigma\frac{\partial^2 V}{\partial y\partial s}\right] + \alpha y\frac{\partial V}{\partial y} + k^*(m^* - s)\frac{\partial V}{\partial s} + yD \tag{2.4}$$

A meaningful particular solution we find for (2.4) is $V^p(y) = \frac{yD}{\delta}$. Note that this is also the only particular solution that is independent of the volatility. This is called the *fundamental component of the project value*, which is just the expected present value of revenue stream by keeping the project active forever. Other terms in the solution are speculative components which can be set to zero by invoking economic considerations. We denote the option value by $F(y,s)$. For any time t, the firm can either invest and take the immediate payoff $V(y) - I$, or wait for an small amount of time dt and take the continuation value

$$\mathbb{E}\left[\frac{F(y+dy, s+ds|y,s)}{1+rdt}\right].$$

It follows that $F(y,s)$ satisfies the following Bellman equation,

$$F(y,s) = \max\left\{V - I, \mathbb{E}\left[\frac{F(y+dy, s+ds|y,s)}{1+rdt}\right]\right\}. \tag{2.5}$$

Let y^* be the optimal threshold such that the continuation value is equal to the immediate payoff. Again applying Itô's lemma, we conclude that before y reaches y^*, F satisfies the following PDE

$$rF = \frac{1}{2}\left[y^2 s\frac{\partial^2 F}{\partial y^2} + s\sigma^2\frac{\partial^2 F}{\partial s^2} + 2\rho y s\sigma\frac{\partial^2 F}{\partial y\partial s}\right] + \alpha y\frac{\partial F}{\partial y} + k^*(m^* - s)\frac{\partial F}{\partial s} \tag{2.6}$$

with the boundary conditions

$$F(y^*, s) = V(y^*, s) - I, \tag{2.7}$$

$$F(0, s) = V(0, s), \tag{2.8}$$

$$\left.\frac{\partial F}{\partial y}\right|_{y=y^*} = \left.\frac{\partial V}{\partial y}\right|_{y=y^*}, \tag{2.9}$$

$$\left.\frac{\partial F}{\partial s}\right|_{y=y^*} = \left.\frac{\partial V}{\partial s}\right|_{y=y^*}, \tag{2.10}$$

where (2.7) is the value matching condition, (2.8) says that the value of option F becomes worthless if the firm does not expirence any demand shock, (2.9) and

(2.10) are the smooth pasting conditions. Substituting V^p into (2.7)−(2.10) gives

$$\begin{aligned} F(y^*(s), s) &= \frac{y^* D}{\delta} - I, \\ F(0, s) &= 0, \\ \left.\frac{\partial F}{\partial y}\right|_{y=y^*} &= \frac{D}{\delta}, \\ \left.\frac{\partial F}{\partial s}\right|_{y=y^*} &= 0. \end{aligned}$$

Let $v^2 = m^* \sigma^2/2k$, $\epsilon = 1/k^*$. Following the asymptotic approach in [8] and [9], we define the following operators

$$\begin{aligned} \mathfrak{L}_0 &= \frac{v^2 s}{m^*}\frac{\partial^2}{\partial s^2} + (m^* - s)\frac{\partial}{\partial s}, \\ \mathfrak{L}_1 &= \frac{\rho v \sqrt{2}}{\sqrt{m^*}} ys \frac{\partial^2}{\partial y \partial s}, \\ \mathfrak{L}_2 &= \frac{1}{2} s y^2 \frac{\partial^2}{\partial y^2} + (r - \delta) y \frac{\partial}{\partial y} - r, \end{aligned}$$

Then, (2.6) can be rewritten as the following compact form,

$$\left(\frac{1}{\epsilon}\mathfrak{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathfrak{L}_1 + \mathfrak{L}_2\right) F = 0. \tag{2.11}$$

We expand $F(y, s)$ and $y^*(s)$ as

$$F(y, s) = F_0(y, s) + \sqrt{\epsilon} F_1(y, s) + \epsilon F_2(y, s) + \cdots, \tag{2.12}$$

$$y^*(s) = y_0^*(s) + \sqrt{\epsilon} y_1^*(s) + \epsilon y_2^*(s) + \cdots. \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11) and taking terms up to the order $\sqrt{\epsilon}$ give

$$\begin{aligned} &\frac{1}{\epsilon}\mathfrak{L}_0 F_0 + \frac{1}{\sqrt{\epsilon}}(\mathfrak{L}_0 F_1 + \mathfrak{L}_1 F_0) + (\mathfrak{L}_0 F_2 + \mathfrak{L}_1 F_1 + \mathfrak{L}_2 F_0) \\ &+ \sqrt{\epsilon}(\mathfrak{L}_0 F_3 + \mathfrak{L}_1 F_2 + \mathfrak{L}_2 F_1) = 0. \end{aligned} \tag{2.14}$$

Substituting (2.12) and (2.13) into the boundary conditions (2.7)−(2.10) yields

$$F_0(y^*(s), s) + \sqrt{\epsilon}\left(y_1^*(s) \left.\frac{\partial F_0}{\partial y}\right|_{y=y_0^*} + F_1(y_0^*(s), s)\right) = \frac{y_0^*(s) D}{\delta} + \sqrt{\epsilon}\frac{y_1^*(s) D}{\delta} - I \tag{2.15}$$

$$F_0(0, s) + \sqrt{\epsilon} F_1(0, s) = 0, \tag{2.16}$$

$$\left.\frac{\partial F_0}{\partial y}\right|_{y=y_0^*} + \sqrt{\epsilon}\left(y_1^*(s) \left.\frac{\partial^2 F_0}{\partial y^2}\right|_{y=y_0^*} + \left.\frac{\partial F_1}{\partial y}\right|_{y=y_0^*}\right) = \frac{D}{\delta}, \tag{2.17}$$

$$\left.\frac{\partial F_0}{\partial s}\right|_{y=y_0^*} + \sqrt{\epsilon}\left(y_1^*(s) \left.\frac{\partial^2 F_0}{\partial y \partial s}\right|_{y=y_0^*} + \left.\frac{\partial F_1}{\partial y}\right|_{y=y_0^*}\right) = 0. \tag{2.18}$$

Using the approach similar to that in [9], we derive the following proposition, whose the proof is provided in Appendix A.

Proposition 2.1. *Under the given assumptions, the payoff of the firm is given asymptotically by*

$$F(y) \approx \begin{cases} \left(1 - \frac{\rho\sigma\beta_1^2(\beta_1-1)}{k^*(\beta_2-\beta_1)}\ell n\left(\frac{y_0^*}{y}\right)\right)\left(\frac{y_0^* D}{\delta} - I\right)\left(\frac{y}{y_0^*}\right)^{\beta_1}, & \text{if} \quad y \leq y_0^*, \\ \frac{yD}{\delta} - I, & \text{if} \quad y \geq y_0^*, \end{cases} \tag{2.19}$$

where

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{m^*} + \sqrt{\left(\frac{r-\delta}{m^*} - \frac{1}{2}\right)^2 + \frac{2r}{m^*}}, \tag{2.20}$$

$$\beta_2 = \frac{1}{2} - \frac{r-\delta}{m^*} - \sqrt{\left(\frac{r-\delta}{m^*} - \frac{1}{2}\right)^2 + \frac{2r}{m^*}}, \tag{2.21}$$

$$y_0^* = \frac{\beta_1 I \delta}{(\beta_1 - 1)D}. \tag{2.22}$$

The optimal threshold y^* of the firm is given asymptotically by

$$y^* \approx \frac{\beta_1 I \delta}{(\beta_1 - 1)D}\left(1 + \frac{\rho\sigma\beta_1}{k^*(\beta_2 - \beta_1)}\right). \tag{2.23}$$

Thus the optimal strategy for the firm is to invest at time T^* given by

$$T^* = \inf\{t \geq 0 : y \geq y^*\}. \tag{2.24}$$

3. The duopoly case

In this section, we consider two firms, namely A and B, with potential to enter the market. The industry output, denoted by Q, can take values 0, 1 or 2, depending on the number of active firms in the market. If only one firm is active, the firm earns the monopoly profit flow $yD(1)$. If both A and B are active, they share the market and each of them earns the duopoly profit flow $yD(2)$. When potential competitors in the market are allowed, the fear of preemption by a rival reduces the value of waiting. The problem of determining the optimal stopping time for A and B in such a case is known as a stochastic duopoly game The firm, who invests first, is referred as the leader, and the other is referred as the follower. Being the leader has the advantage of earning the monopoly profit before the follower invests. When we analyse the strategic structure of our duopoly game, we need to consider two situations: one is that both A and firm B have the potential to become the leader, the other is that leader and the follower are pre-determined.

3.1. **The leader is not pre-determined.** In this case, we first determine follower's option value and then the leader's project value. If A sets its trigger point in the range where the payoff of being the leader is greater than that of being the follower, B can invest just before A and becomes the leader. Thus, to prevent from being preempted by B, A needs to invest just before B. So does B. Eventually, the optimal investing time for any firm to be the leader is to invest at the shock level where the leader's project value is equal to the follower's option value.

First, we calculate follower's option value. This can be done by assuming that one of the firms has already invested and then applying steps similar to those in Section 2. Thus, by Proposition 2.1, the follower's option value, $F^b(y)$, is given by

$$F^b(y) = \begin{cases} \left(1 - \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\right)\left(\frac{y_0^b D(2)}{\delta} - I\right)\left(\frac{y}{y_0^b}\right)^{\beta_1}, & \text{if} \quad y \le y_0^b, \\ \frac{yD(2)}{\delta} - I, & \text{if} \quad y \ge y_0^b, \end{cases} \tag{3.1}$$

where β_1 and β_2 are the same as those in (2.20) and (2.21), and

$$y_0^b = \frac{\beta_1 I\delta}{(\beta_1 - 1)D(2)}.$$

The optimal threshold is

$$y^b \quad = \quad \frac{\beta_1 I\delta}{(\beta_1 - 1)D(2)}\left(1 + \frac{\rho\sigma\beta_1}{k^*(\beta_2 - \beta_1)}\right), \tag{3.2}$$

and the corresponding investing time T^b is

$$T^b = \inf\left\{t \ge 0 : y \ge y^b\right\}. \tag{3.3}$$

Next, we calculate leader's option value and its optimal investing time. To this end, we need to find leader's project value V^a. Assume that the follower invests at time T^b. While the follower is waiting, the leader's project value can be expressed as the sum of the expected present discounted value of the monopoly future returns and the expected present discounted value of the duopoly future returns, i.e.,

$$V^a(y) = \mathbb{E}\left[\int_0^{T^b} e^{-rt} yD(1)dt\right] + \mathbb{E}\left[e^{-rT^b}\right]\frac{y_0^b D(2)}{\delta}$$

when $y \le y_0^b$. When $y \ge y_0^b$, A and B share the duopoly profit, and thus $V^a(y) = \frac{yD(2)}{\delta}$. Applying techniques similar to those in Section 2 to these two expectations leads to the following results, (refer to Appendix B)

$$V^a(y) = \begin{cases} \frac{yD(1)}{\delta} + \left(\frac{y}{y_0^b}\right)^{\beta_1}\left(1 - \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\right)\frac{y_0^b(D(2) - D(1))}{\delta}, & \text{if} \quad y \le y_0^b, \\ \frac{yD(2)}{\delta}, & \text{if} \quad y \ge y_0^b. \end{cases} \tag{3.4}$$

Following the proof of Proposition 4 in [6], we conclude the following proposition.

Proposition 3.1. *For the case when both firms can become leader, there exists a unique point $y_E \in (0, y^b)$ with the following properties:*

$$\begin{aligned} V^a - I &< F^b \quad \textit{for} \quad y < y_E, \\ V^a - I &= F^b \quad \textit{for} \quad y = y_E, \\ V^a - I &> F^b \quad \textit{for} \quad y_E < y < y^b, \\ V^a - I &= F^b \quad \textit{for} \quad y \ge y^b. \end{aligned}$$

In Figure 1, the green dash curve and the blue curve denote $F^b(y)$ and $V^a(y)$ for $y \le y^b$, respectively. We see that $V^a(y)$ is concave over the range $y < y^b$.

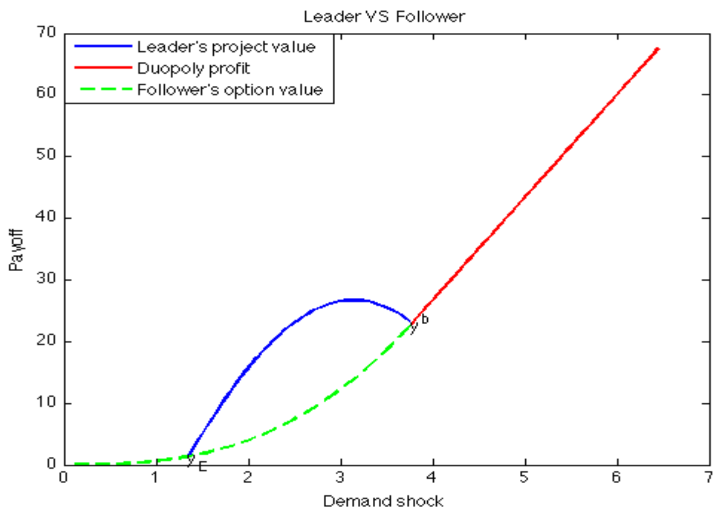

FIGURE 1. Values of Leader and Follower in Duopoly

3.2. **The leader is pre-determined.** In this case, the follower only has one strategy, which is to invest at time T^b. Before y reaches leader's first trigger point y_1^a, the leader's option value, denoted by F^a, satisfies the following PDE

$$rF^a = \frac{1}{2}\left[y^2 s\frac{\partial^2 F^a}{\partial y^2} + s\sigma^2\frac{\partial^2 F^a}{\partial s^2} + 2\rho y s\sigma\frac{\partial^2 F^a}{\partial y\partial s}\right] + \alpha y\frac{\partial F^a}{\partial y} + k^*(m^* - s)\frac{\partial F^a}{\partial s} \quad (3.5)$$

with the boundary conditions

$$F^a(y_1^a, s) = V^a(y_1^a, s) - I, \quad (3.6)$$

$$F^a(0, s) = 0, \quad (3.7)$$

$$\left.\frac{\partial F^a}{\partial y}\right|_{y=y_1^a} = \left.\frac{\partial V^a}{\partial y}\right|_{y=y_1^a}, \quad (3.8)$$

$$\left.\frac{\partial F^a}{\partial s}\right|_{y=y_1^a} = \left.\frac{\partial V^a}{\partial s}\right|_{y=y_1^a}. \quad (3.9)$$

Put

$$V_0^a(y) = \frac{yD(1)}{\delta} + Hy^{\beta_1} \text{ and } \sqrt{\epsilon}V_1^a(y) = \frac{2BH}{m^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)y^{\beta_1}.$$

Then, we can rewrite V^a in (3.4) as the following form,

$$V^a(y) = \begin{cases} V_0^a + \sqrt{\epsilon}V_1^a, & \text{if } y \le y_0^b, \\ \dfrac{yD(2)}{\delta}, & \text{if } y \ge y_0^b, \end{cases}$$

where

$$B = -\frac{\rho\sigma m^*}{2k^*}\beta_1^2(\beta_1 - 1) \text{ and } H = \left(\frac{1}{y_0^b}\right)^{\beta_1}\frac{y_0^b(D(2) - D(1))}{\delta}.$$

Applying techniques similar to those in Section 2 gives

$$\frac{1}{2}m^* y^2 \frac{d^2 F_0^a}{dy^2} + (r-\delta) y \frac{dF_0^a}{dy} - rF_0^a = 0$$

with the boundary conditions

$$\begin{aligned} F_0^a(y_{1,0}^a) &= V_0^a(y_{1,0}^a) - I, \\ F_0^a(0) &= 0, \\ \left.\frac{dF_0^a}{dy}\right|_{y_{1,0}^a} &= \left.\frac{dV_0^a}{dy}\right|_{y=y_{1,0}^a}. \end{aligned}$$

Note that the leader has different strategies corresponding to different shock levels. For the range of y that includes zero, $F_0^a = Jy^{\beta_1}$, where β_1 is given in (2.20) and

$$J = \left(\frac{y_1^a D(1)}{\delta} - I\right)\left(\frac{1}{y_{1,0}^a}\right)^{\beta_1} + \frac{y_0^b}{\delta}(D(2) - D(1))\left(\frac{1}{y_0^b}\right)^{\beta_1}, \tag{3.10}$$

(refer to Appendix A). The first optimal threshold $y_{1,0}^a$ is given by

$$y_{1,0}^a = \frac{\beta_1 I \delta}{(\beta_1 - 1) D(1)}. \tag{3.11}$$

Let $\overline{F}_1^a = \epsilon F_1^a$. Then $\overline{F}_1^a$ satisfies the following equation (refer to Appendix B)

$$\frac{1}{2}m^* y^2 \frac{d^2 \overline{F}_1^a}{dy^2} + (r-\delta) y \frac{d\overline{F}_1^a}{dy} - r\overline{F}_1^a = BJy^{\beta_1} \tag{3.12}$$

with the boundary conditions

$$\overline{F}_1^a(y_{1,0}^a) = \overline{V}_1^a(y_{1,0}^a), \tag{3.13}$$

$$\overline{F}_1^a(0) = 0. \tag{3.14}$$

Solving (3.12) gives (refer to Appendix A)

$$\overline{F}_1^a(y) = \frac{2BJ}{m^*(\beta_2 - \beta_1)} \ell n \left(\frac{y_{1,0}^a}{y}\right) y^{\beta_1} + \frac{2BH}{m^*(\beta_2 - \beta_1)} \ell n \left(\frac{y_0^b}{y_{1,0}^a}\right) y^{\beta_1}. \tag{3.15}$$

Now we determine the correction term $y_{1,1}^a$ for the first investment threshold. Let $\overline{V}_1^a = \epsilon V_1^a$. Applying Taylor's expansion theorem to equation (3.8) yields

$$\overline{y}_1^a = \left(\left.\frac{\partial \overline{V}_1^a}{\partial y}\right|_{y_0^a} - \left.\frac{\partial \overline{F}_1^a}{\partial y}\right|_{y_0^a}\right) \Big/ \left(\left.\frac{\partial^2 F_0^a}{\partial y^2}\right|_{y_0^a} - \left.\frac{\partial^2 V_0^a}{\partial y^2}\right|_{y_0^a}\right). \tag{3.16}$$

Substituting $\overline{V}_1^a$ and $\overline{F}_1^a$ into (3.16) gives

$$\overline{y}_{1,1}^a = \frac{\rho\sigma\beta_1 y_{1,0}^a}{k^*(\beta_2 - \beta_1)}. \tag{3.17}$$

We summarize the above results in the following proposition.

Proposition 3.2. *Let $y_1^a = y_{1,0}^a + \overline{y}_{1,1}^a$, where $y_{1,0}^a$ and $\overline{y}_{1,1}^a$ are given in* (3.11) *and* (3.17) *respectively. When $y \leq y_1^a$, the optimal strategy for the leader is to wait until y first reaches the trigger level y_1^a and invest in the project.*

The optimal trigger level y_1^a is in fact the same as that in the monopoly case, since the fixed priority gives the leader the ability to fully capture the value of waiting without being preempted by the follower. However, this is only applicable when the shock level is low, under the assumption that the demand shock cannot suddenly jump to a high level. If the current shock level y is relatively high, we have to extend the analysis to a more complicated situation. For a range of y that includes neither zero nor infinity, the solution of the option value takes the form of

$$F_0^a(y) = J(1)y^{\beta_1} + J(2)y^{\beta_2}, \tag{3.18}$$

where β_1 and β_2 are the same as those in (2.20) and (2.21) (refer to Appendix A). We now determine two free boundaries y_2^a and y_3^a of the problem. When $y_1^a \leq y \leq y^b$, there is a boundary point $y_{2,0}^a$ satisfying the boundary conditions,

$$F_0^a(y_{2,0}^a) = V_0^a(y_{2,0}^a) - I, \tag{3.19}$$

$$\left.\frac{\partial F_0^a}{\partial y}\right|_{y=y_{2,0}^a} = \left.\frac{\partial V_0^a}{\partial y}\right|_{y=y_{2,0}^a}. \tag{3.20}$$

When $y \geq y^b$, there is a boundary point y_3^a, whose zero-order term $y_{3,0}^a$ satisfies the following boundary conditions

$$F_0^a(y_{3,0}^a) = \frac{y_{3,0}^a D(2)}{\delta} - I, \tag{3.21}$$

$$\left.\frac{\partial F_0^a}{\partial y}\right|_{y=y_{3,0}^a} = \frac{D(2)}{\delta}. \tag{3.22}$$

We can find the correction term F_1^a once $y_{2,0}^a$, $y_{3,0}^a$, $J(1)$ and $J(2)$ are determined. For a range of y that includes neither zero nor infinity, $\overline{F}_1^a$ satisfies

$$\frac{1}{2}m^* y^2 \frac{d^2\overline{F}_1^a}{dy^2} + (r-\delta)y\frac{d\overline{F}_1^a}{dy} - r\overline{F}_1^a = B(\beta_1)J(1)y^{\beta_1} + B(\beta_2)J(2)y^{\beta_2}, \tag{3.23}$$

(refer to Appendix A). We change the constant B to $B(\beta) = -\frac{\rho\sigma m^*}{2k^*}\beta^2(\beta-1)$, since it depends on β. The general solution to (3.23) takes the form of

$$\begin{aligned} \overline{F}_1^a(y) \quad &= \quad L(1)y^{\beta_1} + L(2)y^{\beta_2} \\ &\quad + \frac{2\ell n(y)}{m^*(\beta_1-\beta_2)}\left(B\left(\beta_1\right)J(1)y^{\beta_1} + B(\beta_2)J(2)y^{\beta_2}\right) \\ &\quad + \frac{2}{m^*(\beta_1-\beta_2)^2}\left(B(\beta_1)J(1)y^{\beta_1} - B(\beta_2)J(2)y^{\beta_2}\right). \end{aligned} \tag{3.24}$$

We can determine the constants $L(1)$ and $L(2)$ by the following boundary conditions

$$\begin{aligned} \overline{F}_1^a(y_{2,0}^a) \quad &= \quad \frac{2BH}{m^*(\beta_2-\beta_1)}\ell n\left(\frac{y_0^b}{y_{2,0}^a}\right)(y_{2,0}^a)^{\beta_1} \\ \overline{F}_1^a(y_{3,0}^a) \quad &= \quad 0. \end{aligned}$$

Again we use (3.16) to determine the trigger point expansions $y_{2,1}^a$ and $y_{3,1}^a$ numerically. The above analysis can be summarized in the following proposition.

Proposition 3.3. *Let $y_2^a = y_{2,0}^a + \overline{y}_{2,1}^a$, $y_3^a = y_{3,0}^a + \overline{y}_{3,1}^a$ where $y_{2,0}^a$, $y_{3,0}^a$ are solutions to* (3.19)*–*(3.22)*, and $\overline{y}_{2,1}^a, \overline{y}_{3,1}^a$ are determined by* (3.16)*. If $y \geq y_1^a$, the optimal investing strategy for the leader is to wait if $y \in [y_2^a, Y_3^a)$ and invest otherwise.*

The reason that the leader has to wait at a high demand shock level is that if the current demand shock is getting too close to follower's optimal threshold, the leader may not earn the monopoly profit. Even if the leader can earn the monopoly profit, the earning period is very short, since the follower will invest soon. Thus, the leader is recommended to wait until an even higher demand shock level is reached, and the leader and follower invest simultaneously and enjoy a higher duopoly profit.

4. Numerical examples

Let $r = 0.05, m^* = 0.65, \sigma = 0.6, k^* = 10, I = 40, \delta = 0.03, D(1) = 1$ and $D(2) = 0.5$. We first present the graph of the basic model with $\rho = 0$.

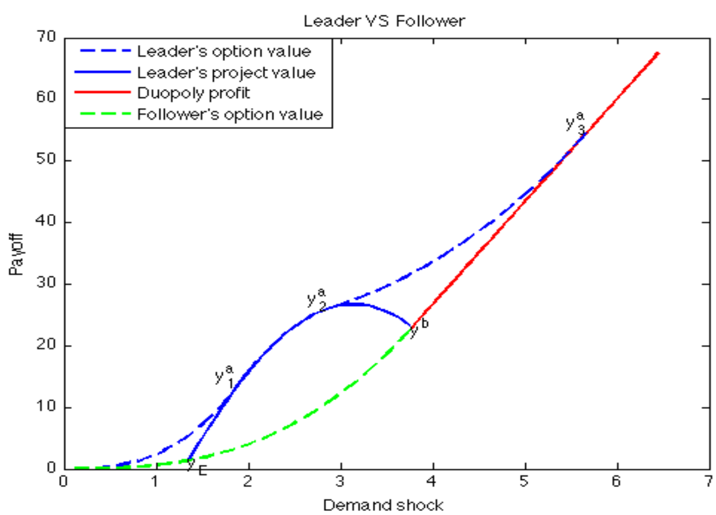

Figure 2. $r = 0.05, m^* = 0.65, \sigma = 0.6, k^* = 10, \rho = 0, I = 40, \delta = 0.03, D(1) = 1$ and $D(2) = 0.5$.

In Figure 2, the dash curves denote the option values of waiting, meaning that firms have not invested in the project yet. In the pre-determined leader-follower case, if y falls in the range $[0, y_1^a) \cap [y_2^a, y_3^a)$, we can see that waiting is more profitable than investing in the project for the leader. The follower will invest if y^b is reached. While the follower is waiting, its payoff is denoted by the green dash curve. If both firms invest in the project, they share the market and enjoy the duopoly profit, which is denoted by the red curve. The result is very similar to that in the classical model which takes the mean reverting rate m^* as a constant volatility. However, the volatility parameter σ of the volatility process S_t can also affect the result. We use figure 3 and figure 4 to demonstrate the effects of m^* and σ.

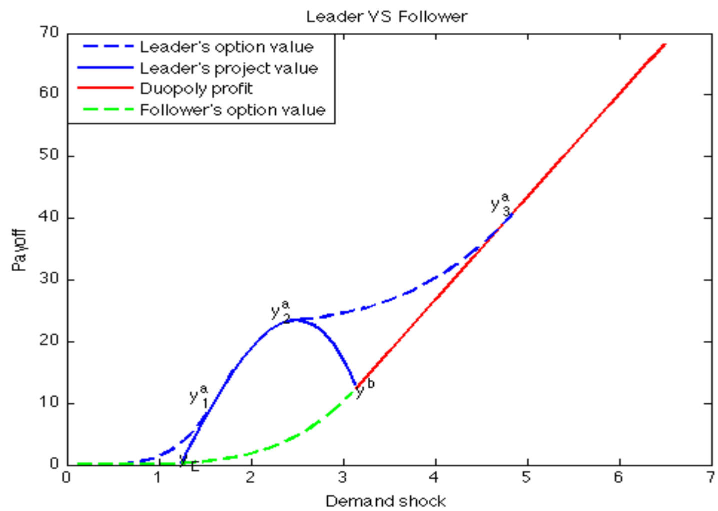

FIGURE 3. Change $m^* = 0.65$ to $m^* = 0.35$

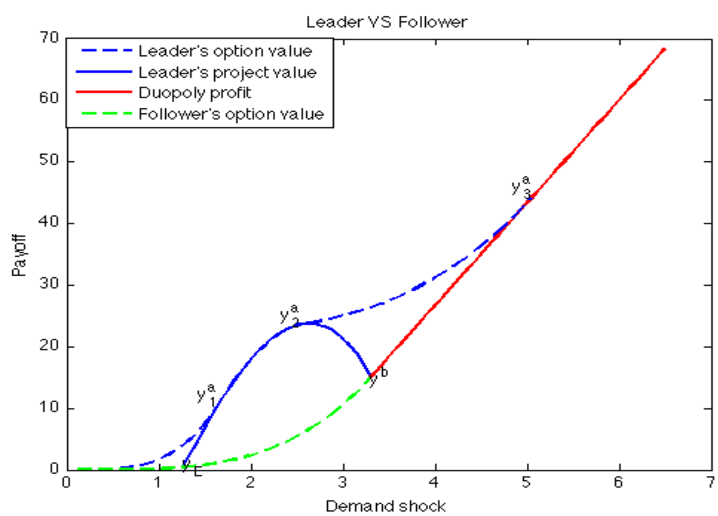

FIGURE 4. Change $\sigma = 0.6$ to $\sigma = 0.9$

In Figure 3, we fix other parameters, but decrease to $m^* = 0.35$. Comparing Figure 3 with Figure 2, we see that the option values of waiting decrease as m^* decreases. However, there is only a little impact on leader's project value and the duopoly profit, since in the option pricing theory, the smaller the volatility is, the less the call option is worth. In our model, the mean reverting level of the volatility process is similar to the mean of the volatility, thus the value of waiting decreases as m^* decreases. Then firms are recommended to invest earlier in both competitive and non-competitive situations. In Figure 4, we fix other parameters and increase σ, and the result appears to be similar to that in Figure 3.

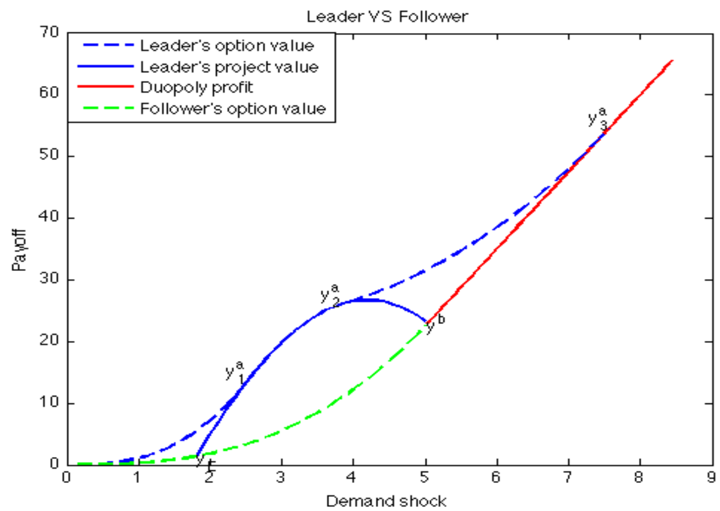

FIGURE 5. Change $\delta = 0.03$ to $\delta = 0.04$

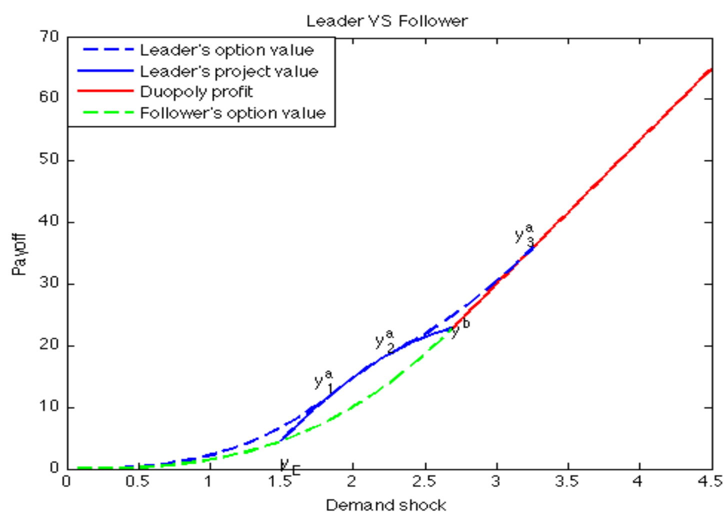

FIGURE 6. Change $D(2) = 0.5$ to $D(2) = 0.7$

We also investigate how the drift parameter α of the demand shock would affect the results. In Figure 5, we increase δ by 0.01, which is the same as decreasing α by 0.01. Comparing Figure 5 with Figure 2, the overall payoffs of firms decrease as α decreases. Thus, firms are recommended to delay the investment in both the competitive and the non-competitive situations. In Figure 6, we increase the duopoly demand $D(2)$. As expected, follower's payoff is increased. This makes being a leader less attractive, and results in a higher equilibrium threshold y_E in the competitive situation. Whereas in the non-competitive situation, the first investment threshold for the leader stays the same, since changing the duopoly profit does not affect leader's monopoly profit. However, the follower is recommended to invest earlier, since the duopoly profit is higher. This forces leader's second trigger point y_2^a to occur earlier, meaning that the leader is less likely to enjoy the monopoly profit, if the initial demand shock is high. If we let $D(2) = D(1)$, then y_1^a and y_2^a concide, and follower's payoff will increase to the same level as leader's payoff. This is exactly why we need to modify the monopoly model by adding a simple demand function to distinguish the payoffs between the leader and the follower.

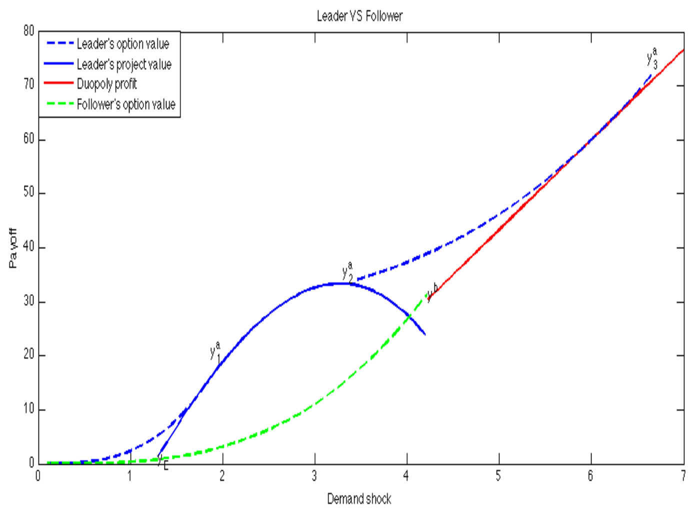

FIGURE 7. Change $\rho = 0$ to $\rho = -1$

The next important parameter that we must investigate is ρ. We consider two extreme values $\rho = -1$ and $\rho = 1$. Comparing Figure 7 with Figure 2, we see that not only the investment thresholds but also some of the payoffs increase. Typically, we see that there is a significant increase in leader's project values, which results in an increase in the option values of waiting after y_2^a. However, there is no much difference in the option values of waiting before y_2^a. Comparing Figure 8 with Figure 2, we find the opposite results. Note that the previous results we have on other parameters when $\rho = 0$ still hold here, and yet the results are also affected by ρ. For instance, in Figure 9, the changes of option values as m^* decreases are not essentially different from those in Figure 2. But a positive correlation also decreases the payoffs and suggests firms to invest earlier. From Figure 7 − Figure 9, we see that when there is correlation between y and s, the investment thresholds do not lead to optimal results. In fact, we don't expect F^a and F^b to be accurate when the mean reverting rate k^* is small, since $\epsilon = \frac{1}{k^*}$ in these asymptotic solutions. Thus, to make the asymptotic results accurate, we must assume that the volatility process S_t is under fast-mean reversion, i.e., the mean reverting rate k^* is big.

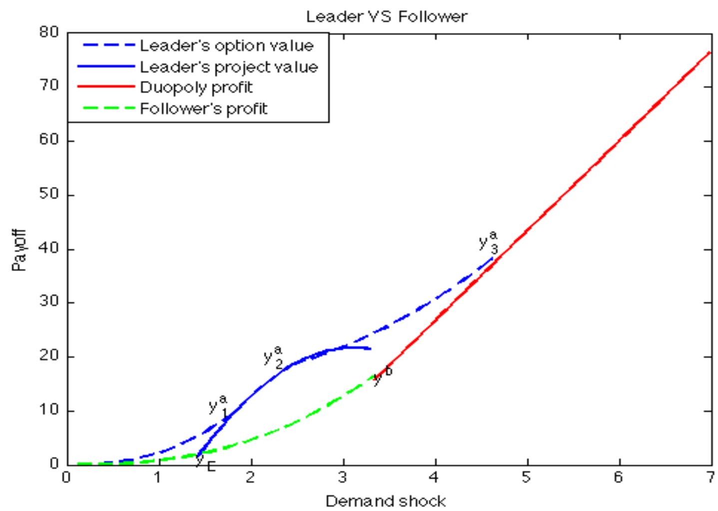

FIGURE 8. Change $\rho = 0$ to $\rho = 1$

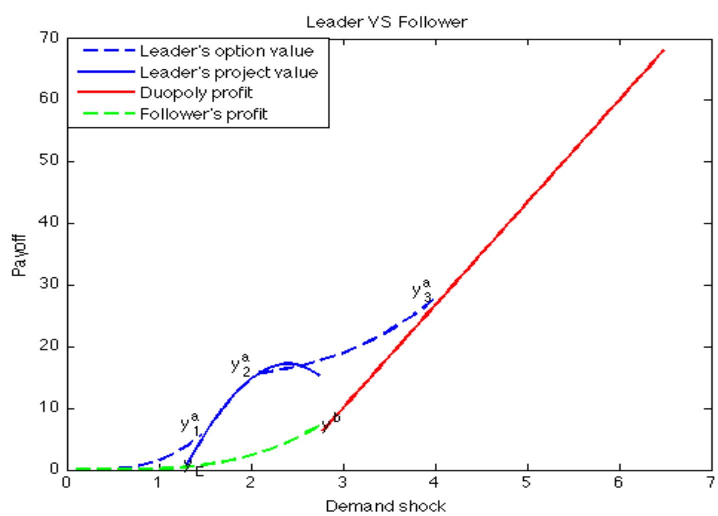

FIGURE 9. Change $m^* = 0.65$, $\rho = 0$ to $m^* = 0.35$, $\rho = 1$

In Figure 10 and Figure 11, we see neither of them is much different from Figure 2. We refer the reader to [9] for an analysis of the relation between the mean reverting rate and the optimal investment thresholds.

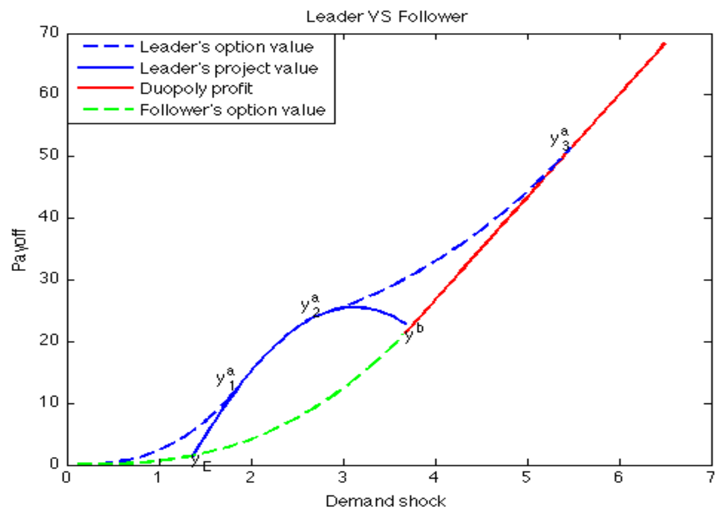

FIGURE 10. Change $k^* = 10$, $\rho = 0$ to $k^* = 50$, $\rho = 1$

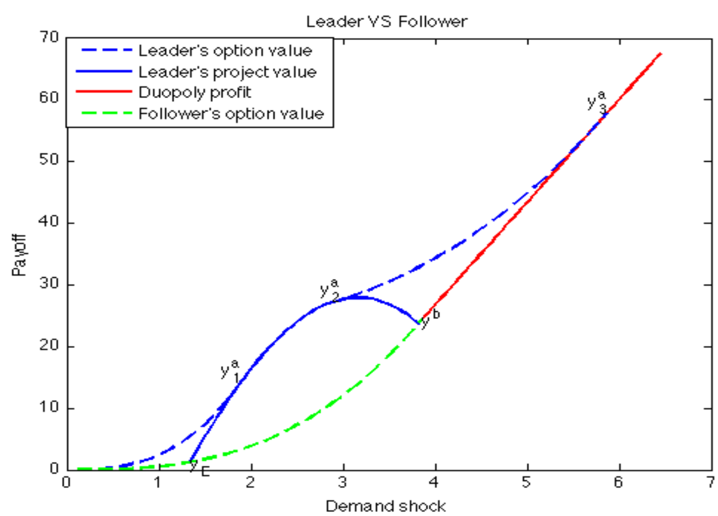

FIGURE 11. Change $k^* = 10$, $\rho = 0$ to $k^* = 50$, $\rho = -1$

5. Concluding remarks and further extensions

In this paper, we investigate a stochastic duopoly game of the investment timing problem. In our model, we assume that the project value depends on the demand shock which follows Heston's model in [4]. We start with a monopoly model, and then extend it to a duopoly model for the competitive and the non-competitive situations. Both situations lead to PDEs which need to be solved numerically. The asymptotic solutions we obtained for real option values or optimal investment thresholds are made up of the first two terms of asymptotic expansions. We find that the first terms of these solutions are the same as the solutions of the classical real option problem with a constant volatility.

From theoretical results and numerical examples, we conclude the following. The option values of waiting and the optimal thresholds of firms increase (decrease) as the mean reverting level m^* of the volatility process increases. In contrast, we find that decreasing the drift parameter α will decrease forms' payoffs but increase the optimal thresholds for the investment. We have also shown how the demand function contributes to the difference between being the leader and being the follower in the model. We find that the positive (resp. negative) correlation between the demand shock and the volatility decreases (resp. increases) the expected payoffs and optimal thresholds for the leader and the follower. Finally, the asymptotic result converges to the classical one with constant volatility as the mean reverting rate k^* increases.

There are some other possible research directions that we can consider in the future. For instance, we can extend our current model from the complete information competition to the incomplete information competition and find pure/mixed strategies for firms. It would be interesting to consider our current model under other stochastic processes. We can also investigate the situation in which firms decide to abandon a project. Such a situation has been considered in the classical real option analysis.

Appendix A

Proof of Proposition 2.1. The asymptotic solution consists of two parts, the zero order term and the correction term. Taking the $\frac{1}{\epsilon}$ term from (2.14) and the corresponding boundary conditions give

$$\mathfrak{L}_0 F_0(y,s) = 0, \qquad \text{if} \quad y \le y_0^*, \tag{A.1}$$

$$F_0(y,s) = \frac{yD}{\delta} - I, \qquad \text{if} \quad y \ge y_0^*, \tag{A.2}$$

$$\frac{\partial F_0}{\partial y} = \frac{D}{\delta}. \tag{A.3}$$

Since $\mathfrak{L}_0$ takes derivatives with respect to s only, (A.1) implies that F_0 is independent of s when $y \le y_0^*$. (A.2) shows that F_0 is also independent of s on the other side of the y_0^*. F_0 being independent of s also implies that y_0^* is independent of y. Then taking the $\frac{1}{\sqrt{\epsilon}}$ term from (2.14) and the corresponding boundary conditions give

$$\mathfrak{L}_0 F_1(y,s) = 0, \qquad \text{if} \quad y \le y_0^*, \tag{A.4}$$

$$F_1(y,s) = 0, \qquad \text{if} \quad y \ge y_0^*, \tag{A.5}$$

$$y_1^* \frac{\partial^2 F_0}{\partial y^2}\bigg|_{y=y_0^*} + \frac{\partial F_1}{\partial y}\bigg|_{y=y_0^*} = 0. \tag{A.6}$$

(A.4) follows from the fact that F_0 is independent of s, and it implies that F_1 is also independent of s. From (A.5), we conclude that $\mathfrak{L}_1 F_1 = 0$. Thus,

$$\mathfrak{L}_0 F_2 + \mathfrak{L}_2 F_0 = 0, \qquad \text{if} \quad y \le y_0^*, \tag{A.7}$$

$$F_2(y,s) = 0, \qquad \text{if} \quad y \ge y_0^*. \tag{A.8}$$

(A.7) is a Poisson equation for F_2 with respect to the operator $\mathfrak{L}_0$. According to [8], a solution exists if and only if $\mathfrak{L}_2 F_0$ is centred with respect to the invariant distribution of the diffusion whose infinitesimal generator is $\mathfrak{L}_0$. Thus, $\langle \mathfrak{L}_2 F_0 \rangle = 0$, where the angled brackets indicate taking the average of the argument with respect to the invariant distribution of the diffusion whose infinitesimal generator is $\mathfrak{L}_0$. Since F_0 does not depend on s, the centering condition becomes $\langle \mathfrak{L}_2 \rangle F_0 = 0$, which is equivalent to

$$\frac{1}{2} m^* y^2 \frac{d^2 F_0}{dy^2} + (r-\delta) y \frac{dF_0}{dy} - r F_0 = 0. \tag{A.9}$$

Note that the invariant distribution is in fact a Gamma distribution and hence $\langle s \rangle = m^*$. This ODE is similar to that in the classical real option problem whose volatility is given by a constant $\sqrt{m^*}$. Following [2], we obtain the zero order term of the problem as follows:

$$F_0(y) = \begin{cases} \left(\dfrac{y_0^* D}{\delta} - I \right) \left(\frac{y}{y_0^*} \right)^{\beta_1}, & \text{if} \quad y \le y_0^*, \\ \\ \dfrac{yD}{\delta} - I, & \text{if} \quad y \ge y_0^*, \end{cases}$$

where

$$y_0^* = \frac{\beta_1 I \delta}{(\beta_1 - 1) D} \quad \text{and} \quad \beta_1 = \frac{1}{2} - \frac{r-\delta}{m^*} + \sqrt{\left(\frac{r-\delta}{m^*} - \frac{1}{2} \right)^2 + \frac{2r}{m^*}}.$$

Next, we are to find the correction term. Since $\mathfrak{L}_1 F_1 = 0$, the order 1 terms of (2.14) give

$$\begin{aligned} \mathfrak{L}_0 F_2 &= -\mathfrak{L}_2 F_0 \\ &= -(\mathfrak{L}_2 F_0 - \langle \mathfrak{L}_2 F_0 \rangle) \\ &= -\frac{1}{2}(s - m^*) y^2 \frac{d^2 F_0}{dy^2}, \end{aligned} \tag{A.10}$$

Let $\overline{c}(t, y)$ be a function independent of s. Let $\phi(s)$ be the solution to the Poisson equation $\mathfrak{L}_0(\phi) = s - m^*$, where $\phi(s)$ satisfies (refer to [8])

$$\begin{aligned} \phi(s) &= \int_0^\infty \mathbb{E}(m^* - S_t | S_0 = s) dt \\ &= \int_0^\infty m^* - s_0 e^{-t} - m^*(1 - e^{-t}) dt \\ &= \int_0^\infty (m^* - s) e^{-t} dt \\ &= m^* - s. \end{aligned} \tag{A.11}$$

From (A.10) and (A.11), we get

$$\begin{aligned} F_2(t,y,s) &= -\frac{1}{2}\mathfrak{L}_0^{-1}(s-m^*)y^2\frac{d^2F_0}{dy^2} \\ &= -\frac{1}{2}(\phi(s)+\overline{c}(t,y))y^2\frac{d^2F_0}{dy^2} \\ &= \frac{1}{2}(s-m^*-\overline{c}(t,y))y^2\frac{d^2F_0}{dy^2} \\ &= \frac{1}{2}(s+c(t,y))y^2\frac{d^2F_0}{dy^2} \end{aligned} \tag{A.12}$$

where $c(t,y) = -(\overline{c}(t,y)+m^*)$ is independent of s. Collecting the order $\sqrt{\epsilon}$ terms of (2.14) yields

$$\begin{aligned} \mathfrak{L}_0F_3 + \mathfrak{L}_1F_2 + \mathfrak{L}_2F_1 = 0, \qquad & \text{if} \quad y \le y_0^*, \\ F_3(y,s) = 0, \qquad & \text{if} \quad y \ge y_0^*, \end{aligned}$$

which lead to a Poisson equation for F_3. The corresponding centering condition is

$$\langle \mathfrak{L}_1F_2 + \mathfrak{L}_2F_1 \rangle = 0.$$

since $\mathfrak{L}_1$ takes derivatives with respect to s, and $c(t,y)$ is independent of s, combining the above equation with (A.12) gives

$$\begin{aligned} \mathfrak{L}_2(\sqrt{m^*})F_1 &= \langle \mathfrak{L}_2F_1 \rangle \\ &= -\langle \mathfrak{L}_1F_2 \rangle \\ &= \langle \mathfrak{L}_1 sy^2\frac{d^2F_0}{dy^2} \rangle \\ &= -\frac{\rho\sigma m^*}{2\sqrt{k}}\left(2y^2\frac{d^2F_0}{dy^2} + y^3\frac{d^3F_0}{dy^3}\right), \end{aligned} \tag{A.13}$$

Let $\overline{F}_1 = \sqrt{\epsilon}F_1$. Substituting $\mathfrak{L}_2$ and F_0 into (A.13) gives

$$\frac{1}{2}m^*y^2\frac{d^2\overline{F}_1}{dy^2} + (r-\delta)y\frac{d\overline{F}_1}{dy} - r\overline{F}_1 = BAy^{\beta_1}, \tag{A.14}$$

where

$$B = -\frac{\rho\sigma m^*}{2k}\beta_1^2(\beta_1-1) \quad \text{and} \quad A = \left(\frac{y_0^*D}{\delta} - I\right)\left(\frac{1}{y_0^*}\right)^{\beta_1}.$$

The homogeneous equation associated with (A.14) is indentical to (A.9). To solve (A.14), we only need to find a particular solution by the method of variation of parameters. Let $\overline{F}_{1p}(y) = C(y)y^{\beta_1} + D(y)y^{\beta_2}$ be a particular solution, where β_1 and β_2 are the same as those in (2.20) and (2.21), and $C(y)$ and $D(y)$ are parameters to be determined. Taking the first derivative gives

$$\frac{d\overline{F}_{1p}}{dy} = \beta_1C(y)y^{\beta_1-1} + \beta_2D(y)y^{\beta_2-1},$$

where we make

$$\frac{dC}{dy}y^{\beta_1} + \frac{dD}{dy}y^{\beta_2} = 0.$$

Taking the second derivative gives

$$\frac{d^2\overline{F}_{1p}}{dy^2} = \beta_1(\beta_1-1)C(y)y^{\beta_1-2} + \beta_2(\beta_2-1)D(y)y^{\beta_2-2} + \beta_1 y^{\beta_1-1}\frac{dC}{dy} + \beta_2 y^{\beta_2-1}\frac{dD}{dy}.$$

We substitute these equations back to (A.14) and get

$$\begin{aligned} \frac{dC}{dy}y^{\beta_1} + \frac{dD}{dy}y^{\beta_2} &= 0, \\ \frac{1}{2}m^*\left(\beta_1\frac{dC}{dy}y^{\beta_1+1} + \beta_2\frac{dD}{dy}y^{\beta_2+1}\right) &= BAy^{\beta_1}. \end{aligned}$$

Solving this system of equations gives

$$C(y) = \frac{-2BA\ell n(y)}{m^*(\beta_2-\beta_1)} \text{ and } D(y) = \frac{-2BAy^{\beta_1-\beta_2}}{m^*(\beta_2-\beta_1)^2}.$$

Thus the general solution is of the form

$$\begin{aligned} \overline{F}_1(y) &= C_1 y^{\beta_1} + C_2 y^{\beta_2} + C(y)y^{\beta_1} + D(y)y^{\beta_2} \\ &= C_1 y^{\beta_1} + C_2 y^{\beta_2} - \frac{2BAy^{\beta_1}}{m^*(\beta_2-\beta_1)}\left(\ell n(y) + \frac{1}{\beta_2-\beta_1}\right), \end{aligned}$$

where the constants C_1 and C_2 are yet to be determined. The boundary conditions for $\overline{F}_1$ requires that at $y=0$, $\overline{F}_1(0)=0$. This leads to $C_2=0$. At $y=y_0^*$, we must have $\overline{F}_1(y_0^*)=0$. This leads to

$$C_1 = \frac{2BA}{m^*(\beta_2-\beta_1)}\left(\ell n(y_0^*) + \frac{1}{\beta_2-\beta_1}\right).$$

Hence, the correction term is given by

$$\sqrt{\epsilon}F_1(y) = \overline{F}_1(y) = \frac{2BA}{m^*(\beta_2-\beta_1)}\ell n\left(\frac{y_0^*}{y}\right)y^{\beta_1}.$$

Finally, we isolate y_1^* from (2.17) and get

$$y_1^* = -\left.\frac{dF_1}{dy}\right|_{y=y_0^*} \Big/ \left.\frac{d^2F_0}{dy^2}\right|_{y=y_0^*},$$

which is equivalent to

$$\sqrt{\epsilon}y_1^* = -\left.\frac{d\overline{F}_1}{dy}\right|_{y=y_0^*} \Big/ \left.\frac{d^2F_0}{dy^2}\right|_{y=y_0^*}.$$

Thus we can get

$$\sqrt{\epsilon}y_1^* = \frac{\rho\sigma\beta_1 y_0^*}{k^*(\beta_2-\beta_1)}.$$

Combining F_0, $\overline{F}_1$, y_0^* and $\overline{y}_1^*$ gives the result in (2.19). Note that similar approach is also applied to (3.15), (3.18), and (3.24).

Appendix B

Derivation of (3.4)**.** Let $L(y,s) = \mathbb{E}\left[\int_0^{T^b} e^{-rt} D(1)dt\right]$. By dynamic programming, we get the following PDE

$$\begin{aligned} rL &= \frac{1}{2}\left[y^2 s\frac{\partial^2 L}{\partial y^2} + s\sigma^2\frac{\partial^2 L}{\partial s^2} + 2\rho y s\sigma\frac{\partial^2 L}{\partial y\partial s}\right] \\ &\quad +\alpha y\frac{\partial L}{\partial y} + k^*(m^* - s)\frac{\partial L}{\partial s} + yD(1). \end{aligned} \tag{B.1}$$

with the boundary conditions

$$L(y_0^b, s) = 0, \tag{B.2}$$

$$L(0, s) = 0, \tag{B.3}$$

$$\left.\frac{\partial L}{\partial s}\right|_{y=y_0^b} = 0. \tag{B.4}$$

Note that we only need three boundary conditions to work out L, since y_0^b is already given in (3.2). Condition (B.2) says that as y approaches y_0^b, T^b is likely to be small and so $L(y_0^b, s) = 0$. (B.3) says that when y is very small, T^b is likely to be large and therefore $L(0,s) = 0$. (B.4) is derived from the fact that the project value does not depend on the volatility after the follower invests. Again we follow the steps similar to those in Appendix A and get the following non-homogeneous ODE

$$\frac{1}{2}m^* y^2\frac{d^2 L_0}{dy^2} + (r-\delta)y\frac{dL_0}{dy} - rL_0 + yD(1) = 0. \tag{B.5}$$

with the boundary conditions

$$\begin{aligned} L_0(0) &= 0, \\ L_0(y_0^b) &= 0, \\ \left.\frac{\partial L_0}{\partial s}\right|_{y=y_0^b} &= 0. \end{aligned}$$

Solving (B.5) by following steps similar to those in Appendix A gives

$$L_0(y) = -\frac{y_0^b D(1)}{\delta}\left(\frac{y}{y_0^b}\right)^{\beta_1} + \frac{yD(1)}{\delta}, \tag{B.6}$$

and the correction term for $L(y,s)$ is given by

$$\sqrt{\epsilon}L_1(y) = \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\frac{y_0^b D(1)}{\delta}\left(\frac{y}{y_0^b}\right)^{\beta_1}. \tag{B.7}$$

Combining (B.6) with (B.7) gives

$$\mathbb{E}\left[\int_0^T e^{-rt} yD(1)\right] = \frac{yD(1)}{\delta} - \frac{y_0^b D(1)}{\delta}\left(\frac{y}{y_0^b}\right)^{\beta_1}\left(1 - \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\right).$$

Now, let $G(y,s) = \mathbb{E}\left[e^{-rT}\right]$. Then the following ODE for zero order term holds,

$$\frac{1}{2}m^* y^2\frac{d^2 G_0}{dy^2} + (r-\delta)y\frac{dG_0}{dy} - rG_0 = 0. \tag{B.8}$$

The corresponding boundary conditions are

$$G_0(0) = 0, \tag{B.9}$$

$$G_0(y_0^b) = 1. \tag{B.10}$$

(B.9) says when y is very small, T^b is likely to be large and e^{-rT^b} is close to 0 therefore $G_0(0) = 0$. (B.10) says that as y approaches y_0^b, T^b is likely to be small and thus $G_0(y_0^b) = 1$. Solving (B.8) gives

$$G_0(y) = \left(\frac{y}{y_0^b}\right)^{\beta_1}. \tag{B.11}$$

Similarly the correction term for $G(y,s)$ is given by (refer to Appendix A)

$$\sqrt{\epsilon}G_1(y) = -\frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\left(\frac{y}{y_0^b}\right)^{\beta_1}. \tag{B.12}$$

Combining (B.11) with (B.12) gives

$$\mathbb{E}\left[e^{-rT}\right]\frac{y_0^b D(2)}{\delta} = \left(\frac{y}{y_0^b}\right)^{\beta_1}\left(1 - \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k^*(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\right)\frac{y_0^b D(2)}{\delta}$$

The above results lead to (3.4).

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