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Billand, Pascal and Bravard, Christophe and Kamphorst, Jurjen J.A. and Sarangi, Sudipta

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Pascal Billand[†], Christophe Bravard[‡],
Jurjen Kamphorst[§], Sudipta Sarangi[¶]

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Abstract

Social networks, be it on the internet or in real life, facilitate information flows. We model this by giving agents incentives to link with others and receive information through those links. We consider networks where agents have an incentive to confirm the information they receive from others. Our paper analyzes the social networks that are formed. We first study the existence of Nash equilibria and then characterize the set of strict Nash networks. Next, we characterize the set of strictly efficient networks and discuss the relationship between strictly efficient networks and strict Nash networks. Finally, we check the robustness of our results by allowing for heterogeneity among agents, possibility of bilateral deviations of agents, and decay in the network.

JEL Classification: C72, D85.

Key Words: connections model, confirmation, two-way flow models.

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[†]Université de Lyon, Lyon, F-69003, France; F-42000, France; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France. email: pascal.billand@univ-st-etienne.fr.

[‡]Université Grenoble 2, UMR 1215 GAEL, F38000 Grenoble, France; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France. email: christophe.bravard@univ-st-etienne.fr.

[§]Erasmus School of Economics and Tinbergen Institute. email: kamphorst@ese.eur.nl.

[¶]DIW Berlin and Department of Economics, Louisiana State University, Baton Rouge, LA 70803, USA. email: sarangi@lsu.edu.

1 Introduction

Social networks are purveyors of information where its members use their direct and indirect connections to obtain information from others. A very substantial literature covers different aspects of this topic ranging from Granovetter [8], who studies transmission of information about job opportunities to Bala and Goyal [1], who focus on learning from one's neighbors, and Goyal and Galeotti [6] who examine how information (modeled as a public good) is gathered and shared within a network of individuals. In some situations the reliability of information acquired through the network matters, creating a need for the confirmation of information. This need can arise for several reasons, for instance agents might have issues with recall, or the information may have a subjective component to it. Our paper addresses network formation in this context − how does the need to confirm information obtained from an individual in the network affects network formation?

The need for the confirmation of information is especially true in situations where such information is used to make significant decisions. For instance, the *Book of Deuteronomy* states that "On the testimony of two or three witnesses a man [who has done an evil deed] shall be put to death, but no one shall be put to death on the testimony of only one witness" (17:6). Such confirmation is a key part of our judicial system where it is often necessary to have multiple witnesses who can corroborate a piece of evidence. In many instances, particularly if the information is subjective, researchers also have a need for confirming information. When writing a survey paper, one often reads the original source as well as other interpretations of the same work to write a more scholarly piece. While attending a conference we often talk to different researchers about the same paper to enhance our understanding of it. For the sake of credibility journalists typically attempt to confirm information in several different ways. Of course, government agencies also usually need to confirm information prior to acting on it.[1]

In this paper we model the desire for confirmation by allowing for the possibility of both *unconfirmed* and *confirmed* information based on the cost agents wish to incur. Agents in the model establish a network to acquire information by forming links with each other. Our objective is to identify the architecture of stable networks using the concepts of (strict) Nash

[1] All these cases require corroboration of information, that is using additional information to validate already obtained information. In other words, agents look for other information to support and reconconfirm (or to challenge or rebut) information they have found. There is an extensive literature on this subject (see for instance Miranda, Vercellesi and Bruno [16], and Jick [13]).

network when agents make decisions based on the benefits and costs of links. We establish when such equilibrium networks exist, and how they differ from efficient networks, *i.e.*, networks that maximize aggregate payoffs. Finally, we also consider some extensions of the original model to check the robustness of our findings.

Formally, we consider a setting where each agent, modeled as a node in the network, is a source of benefits of information that others can tap via the formation of costly links. Agents do not falsely report their information,[2] and we assume that a link with another agent allows access to the benefits available to the latter through all her (direct and indirect) links. In the model the costs of link formation are incurred only by the agent who sponsors the link, and these links taken together define a social network. We assume that information obtained through one path, or sequence of links, is said to be unconfirmed, while information that is obtained through one other distinct path in the network is said to be confirmed.[3] In our setting confirmed information is worth more to an agent than unconfirmed information which requires only one path and is therefore cheaper.[4] *Ex ante* all agents are assumed to be identical in the model. Consequently, the payoff obtained by each agent in a social network depends on (i) the number of confirmed resources she obtains from other agents, (ii) the number of unconfirmed resources she obtains from other agents, and (iii) the number of links she sponsors and so the costs she incurs. Observe that our formulation introduces heterogeneity endogenously in the model by allowing for the value of information to depend on the network structure.[5]

Our analysis of network formation in this model provides a number of interesting insights. First, we show that Nash networks in pure strategies may not exist under a general payoff function that incorporates the three elements mentioned above. However, we find that a Nash network always exists if the function that captures the costs of sponsoring links is convex.

Second, we characterize strict Nash networks and interestingly find that strict Nash net-

[2]This assumption is standard in the network formation literature.

[3]We can assume that agent i obtain confirmed information from agent j when she is directly linked with j. This does not affect our main results. We have chosen the current formulation.

[4]In computer science there is a also a body of literature that considers distinct paths in routing applications (see for instance Lee and Gerla [15] and Tsirigos and Haas [17]). Although this phenomenon called multipath routing is not used to confirm information, it increases the payoffs of players at higher costs by lowering delay, providing better security or improving fault tolerance. Thus our formal model can also be used to study situations where there is an explicit need for redundancy or alternate paths.

[5]The typical approach for introducing heterogeneity in the two-way flow model of Bala and Goyal has been through different exogenously given values and costs of links. See for instance Galeotti, Goyal and Kamphorst [7], and Billand, Bravard and Sarangi [5].

works need not be connected despite the fact that all agents are identical. We show that connected[6] strict Nash networks have simple architectures: they are either *minimally confirmed networks*, or *center sponsored stars*.[7] Then, we show that non-connected strict Nash networks contain wheels[8] and at most one subnetwork which is either empty, or minimally confirmed. Strict Nash networks have two interesting properties. In a strict Nash network a player *cannot* obtain both confirmed and unconfirmed resources; and there *do not exist* players who obtain unconfirmed resources and players who obtain confirmed resources.[9]

Third, we study efficient networks. As is often the case with such models, it is difficult to characterize efficient networks with a general payoff function; so we restrict attention to cases where the payoff function is linear. We show that an efficient network is either a minimal unconfirmed network, or a minimal network that is cyclic. The last three sections of the paper are intended as a robustness check of our paper. In section 4, we assume agents are heterogeneous in the way they value information. Some agents, say for instance journalists, derive no utility from information that is not confirmed. We study how network formation is affected by the introduction of such agents. In section 5, we examine situations where pair of players can bilaterally deviate in order to make a Pareto improvement in their payoffs and we use an equilibrium notion called *bilaterally rational network*. This serves as a robustness check of the strict Nash networks concept which is the most commonly used equilibrium concept. This refinement turns out to be interesting since it eliminates the earlier division of equilibrium networks into connected and non-connected and also supports a new type of architecture as an equilibrium network. Finally, in section 6, we briefly discuss the implications of a decay assumption in our framework, which takes into account distance along the alternate paths.

Our paper is inspired by the Nash networks model of Bala and Goyal [2] and here we expand the scope of the two-way flow version of their connections model (in the following we refer to this model as the standard BG model). In Bala and Goyal's model a set of agents simultaneously decide who they wish to link with, which in turn determines the network

[6]A network is connected if there is a path between every pair of agents.

[7]A minimally confirmed network is a network where each player obtains confirmed resources from every other player and if a link is removed from this network, then at least one player loses some confirmed resources. A center sponsored star is a network where a player sponsors a link with each other player while other players do not sponsor any links.

[8]In a wheel each player forms and receives one link.

[9]Though it is possible to have players who get no resources at all.

structure.[10] As in our model links are established as long as the agent initiating the link pays for it. Moreover, each agent obtains the information of agents she is directly or indirectly connected to; and agents do not report their information falsely. However, in the standard BG model there is no added benefit of getting information confirmed; in our model there is. This results in a very different set of equilibrium networks.

A number of variations of the standard BG model have also been developed in which there are additional benefits from having different paths. However, all of these rely on link imperfections of some type or the other. One of these considers the possibility that links can fail with an exogenous probability. See for instance Bala and Goyal [3] who introduce the basic model and Haller and Sarangi [9] who allow for exogenous heterogeneity in this model. Since links can fail, the incentive for alternate paths in this model is a type of insurance against link failure that provides an access to the same information. In another class of models introduced by Bala and Goyal [2] and generalized by Hojman and Szeidl [10] the value of information acquired from agents that are farther away in the network decreases in value.[11] Under certain situations this creates an incentive to establish an alternative path to an agent with whom a player is already linked. Note that in decay models the loss of information through the network is "continuous" with distance, while in our model the loss of information may be considered "discontinuous". The major difference between our model and decay models is that in these models the shortest path acts as the purveyor of information, while in our model two distinct paths are crucial.

The paper is organized as follows. In section 2 we present the model setup. In section 3 we study Nash networks, strict Nash networks, and efficient networks. In section 4, we assume that agents are heterogeneous with regard to the value they obtain from unconfirmed information. In section 5, we allow the possibility for players to make some bilateral deviations. In section 6 we discuss the role played by decay in a model with a confirmation assumption.

[10] Unlike the model of Jackson and Wolinsky [11], there are no consent issues here.

[11] Bala and Goyal [2] give a characterization of the equilibrium networks of diameters 2 and 3. This characterization is extended to equilibrium networks of all diameters by De Jaegher and Kamphorst [12]. Billand, Bravard and Sarangi [4] deal with the implications of exogenous heterogeneity in the presence of decay. Note that in models that allow for link imperfections and exogenously given heterogeneity, in equilibrium the set of Nash networks is quite large.

2 Model setup

Our model setup builds on the two-way flow connection model of Bala and Goyal [2].

Networks definitions. We begin by giving the formal definition of a directed network. A network $\boldsymbol{g}$ is an ordered pair of disjoint sets (N, A) where A is a subset of the set $N \times N$ of ordered pairs of N. The set N, with $|N| \geq 3$, is the set of vertices which corresponds with the players set and $A = A(\boldsymbol{g})$ is the set of arcs which corresponds with the relations or links between the players. We assume that there is no arc from a player i to herself. An ordered pair $(i, j) \in A(\boldsymbol{g})$ is said to be an arc directed from i to j and is denoted by ij. Here i is said to be the sponsor of ij and j the recipient of ij. Let $A_i(\boldsymbol{g}) = \{kj \in A(\boldsymbol{g}) : k = i \text{ and } j \in N\}$ be the set of arcs sponsored by player i in $\boldsymbol{g}$ and let $A_{-i}(\boldsymbol{g}) = A(\boldsymbol{g}) \setminus A_i(\boldsymbol{g})$ be the set of arcs sponsored by players $j \neq i$ in $\boldsymbol{g}$. The set of arcs of $\boldsymbol{g}$ can be written as $A(\boldsymbol{g}) = A_i(\boldsymbol{g}) \cup A_{-i}(\boldsymbol{g})$. To simplify notation, we write $A_i(\boldsymbol{g}) \cup \{i\ j\} = A_i(\boldsymbol{g}) + ij$ and $A_i(\boldsymbol{g}) \setminus \{i\ j\} = A_i(\boldsymbol{g}) - i\ j$. For consistency, we write $A(\boldsymbol{g}) \cup \{ij\} = A(\boldsymbol{g}) + ij$ and $A(\boldsymbol{g}) \setminus \{ij\} = A(\boldsymbol{g}) - i\ j$. We denote by $\boldsymbol{g}^{ij}$ the network $(N, A(\boldsymbol{g}) + ij)$, and by $\boldsymbol{g}^{-ij}$ the network $(N, A(\boldsymbol{g}) - ij)$. We say that $\overline{ij} \in A(\boldsymbol{g})$ if and only if $ij \in A(\boldsymbol{g})$, or $ji \in A(\boldsymbol{g})$. Let $\mathcal{G}$ be the set of directed networks with N as the set of vertices.

Let $V_i(A(\boldsymbol{g})) = \{j \in N \setminus \{i\} : \overline{ij} \in A(\boldsymbol{g})\}$ be the set of players with whom player i is directly linked in $\boldsymbol{g}$. If $|V_i(A(\boldsymbol{g}))| \geq 3$, then i is called a *key player* in $\boldsymbol{g}$.

For a directed network, $\boldsymbol{g}$, an *undirected path* (u-path) between player i and player j, $P_{ij}(\boldsymbol{g})$, is a sequence of players $(i_0,\ i_1,\ \ldots,\ i_L)$ such that $i_0 = i$ and $i_L = j$ and $\overline{i_\ell i_{\ell+1}} \in A(\boldsymbol{g})$, for $\ell \in \{0, \ldots, L-1\}$. Two u-paths between i and j are disjoint if the only players they have in common are i and j. The number of disjoint u-paths between i and j in $\boldsymbol{g}$ is the maximal number of disjoint u-paths between i and j in $\boldsymbol{g}$.[12] Let $N_i(A(\boldsymbol{g}))$ be the set of players $j \in N \setminus \{i\}$ such that there is at least one u-path between i and j in $\boldsymbol{g}$. In that case i and j are said to be connected in $\boldsymbol{g}$. For player i, we define the set of confirmed players as

$$N_i^C(A(\boldsymbol{g})) = \{j \in N \setminus \{i\} : \text{ There exist at least two disjoint u-paths between } i \text{ and } j \text{ in } \boldsymbol{g}\},^{13}$$

[12] In a network $\boldsymbol{g}$, the number of disjoint u-paths between i and j can be different according to the u-paths chosen.

[13] It is worth noting that in a component which contains only two players, say i and j, these players cannot receive confirmed resources from each other since there is only one u-path between them.

and the unconfirmed players set of player i as

$$N_i^U(A(\boldsymbol{g})) = \{j \in N \setminus (N_i^C(A(\boldsymbol{g})) \cup \{i\}) : \text{ There exists one u-path between } i \text{ and } j \text{ in } \boldsymbol{g}\}.$$

Obviously, $j \in N_i^C(A(\boldsymbol{g}))$ implies $i \in N_j^C(A(\boldsymbol{g}))$ and $j \in N_i^U(A(\boldsymbol{g}))$ implies $i \in N_j^U(A(\boldsymbol{g}))$.

There is an alternative way to model the confirmation of resources. We could assume that player i receives confirmed resources from player j when she is directly linked with j in addition to the case where there exist at least two disjoint u-paths between i and j. In other words, it is not necessary for player i who directly obtains resources from j to confirm it: resources are distorted because of the existence of intermediaries between players. It is worth noting that our main results concerning equilibrium networks (Propositions 4 and 5) are not qualitatively changed: we obtain the same architectures.[14]

A *cycle* consists of an u-path where there exists an arc between the terminal player and the initial player on the u-path. A *wheel* is a cycle where each player sponsors only one arc and receives only one arc.

A network $\boldsymbol{g}$ is *connected* if each player $i \in N$ is connected with every other player $j \in N \setminus \{i\}$. A network $\boldsymbol{g}$ is *minimally unconfirmed* if it is connected, and for any arc $ij \in A(\boldsymbol{g})$, the network $\boldsymbol{g}^{-ij}$ is not connected.[15] A network $\boldsymbol{g}$ is *confirmed*, if for all players $i \in N$ and $j \in N \setminus \{i\}$, there are at least two disjoint u-paths between i and j. A network $\boldsymbol{g}$ is *minimally confirmed* if it is confirmed, and for any arc $ij \in A(\boldsymbol{g})$, the network $\boldsymbol{g}^{-ij}$ is not confirmed.[16] A *sub-network* of $\boldsymbol{g} = (N, A)$, say $\boldsymbol{g}_{|X}$, is a network where the set of vertices, X, is a subset of N and an arc ij belongs to $A(\boldsymbol{g}_{|X})$ if and only if ij belongs to $A(\boldsymbol{g})$. A network $\boldsymbol{g}$ is a *minimal cycle network* if it is minimally confirmed and contains n arcs. A maximal connected sub-network of $\boldsymbol{g}$ is a *component*. Let $\mathcal{W}(\boldsymbol{g})$ be the set of players who belong to components that are wheels in $\boldsymbol{g}$ and let $\boldsymbol{g}_{|N\setminus\mathcal{W}}$ be the sub-network induced by the players in $N \setminus \mathcal{W}(\boldsymbol{g})$ in $\boldsymbol{g}$. It is obvious that if a component of $\boldsymbol{g}$, $\boldsymbol{g}_{|W}$, is a wheel of $\boldsymbol{g}$, then we have $W(\boldsymbol{g}) \subset \mathcal{W}(\boldsymbol{g})$. A *star* is a network where a player, say i, is involved in an arc with all the other players while the other players

[14] Sketch of Proofs of this statement are given in Appendix E. Note that in the alternative way to model the confirmation of resources, we have:
$N_i^C(A(\boldsymbol{g})) = \{j \in N \setminus \{i\} : j \in V_j(\boldsymbol{g})$ or there exist at least two disjoint u-paths between i and j in $\boldsymbol{g}\}$.

[15] In graph theory, a minimally unconfirmed network is a 1-arc-connected network and they are called trees. Bala and Goyal [2] called these networks minimally tw-connected networks.

[16] In graph theory, a minimally confirmed network is a 2-arc-connected network.

are involved only in the arc with i. If i sponsors all the arcs in the star, then the network is a *center sponsored star*; if i sponsors no arcs in the star, then the network is a *periphery sponsored star*. These architectures and a minimal cycle network are shown in Figure 1.
A network $\boldsymbol{g}$ is a *base network* if there does not exist a network $\boldsymbol{g}'$ such that

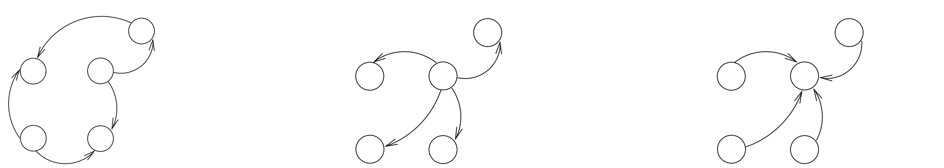
1. $N_i^C(\boldsymbol{g}') = N_i^C(\boldsymbol{g})$, $N_i^U(\boldsymbol{g}') = N_i^U(\boldsymbol{g})$, for all players $i \in N$;
2. There exists a player i such that $|A_i(\boldsymbol{g}')| < |A_i(\boldsymbol{g})|$ and for all players $j \in N \setminus \{i\}$, $A_j(\boldsymbol{g}') = A_j(\boldsymbol{g})$.

A network $\boldsymbol{g}$ is a *minimal base network* if there does not exist a base network $\boldsymbol{g}'$ such that

1. $N_i^C(\boldsymbol{g}') = N_i^C(\boldsymbol{g})$, $N_i^U(\boldsymbol{g}') = N_i^U(\boldsymbol{g})$, for all players $i \in N$;
2. $|A(\boldsymbol{g}')| < |A(\boldsymbol{g})|$.

A *bipartite network* is a network whose vertices can be divided into two disjoint sets X_1 and X_2 such that every arc connects a vertex in X_1 to one in X_2. A complete bipartite network is a bipartite network where there exists an arc between each vertex in X_1 and each vertex in X_2. Finally, a player who sponsors and receives no arcs is called *isolated player*.

We now illustrate some network architectures. In Figure 2, network $\boldsymbol{g}^1$ is not a base



Minimal cycle network | Center sponsored star | Periphery sponsored star

Figure 1: Networks architectures

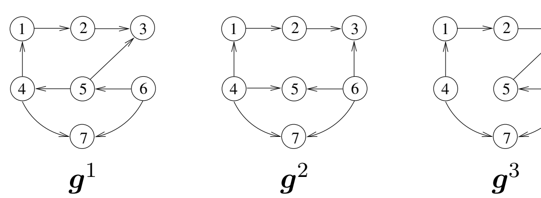


Figure 2: Base and minimal base networks

network since the arc 5 4 is not needed. Network $\boldsymbol{g}^2$ is a base network and $\boldsymbol{g}^3$ is a minimal base network.

Strategies of players. In this paper, we only use pure strategies. Let $G_i = \{ij : j \in N \setminus \{i\}\}$ be the set of arcs that player i can form with other players. In our context, each player $i \in N$ chooses a strategy which consists in forming arcs: $A_i \in 2^{G_i}$. It is worth noting that the set of arcs between distinct players of network $\boldsymbol{g}$ is $A(\boldsymbol{g}) = \bigcup_{i \in N} A_i(\boldsymbol{g})$. Given a network $\boldsymbol{g} \in \mathcal{G}$, $A_{-i}(\boldsymbol{g}) = \bigcup_{j \in N \setminus \{i\}} A_j(\boldsymbol{g})$ denotes the strategy profile played by all players except i.

Payoffs. To complete the definition of the normal-form game of network formation, we now specify the payoffs. When two players are connected, they gain access to each other's information. However, due to the characteristics of the network, information owned by each player is distorted. We assume that if two players are connected via at least two disjoint u-paths, they access each other's information and this information is more valuable. This is called *confirmed access* or *confirmed connection*. Conversely, if they are not connected via at least two disjoint u-paths, the information received is imprecise, and this reduces the value of the information. We call this *unconfirmed access* or *unconfirmed connection*. We assume that each player i prefers to obtain confirmed information instead of unconfirmed information.

An equivalent confirmation set (ECS) is a set of players who obtain confirmed information from every other player of this set. A maximal equivalent confirmation set (MECS) is a ECS which is not a subset of another ECS. Formally, we define the set of equivalent confirmation sets as follows: $E(\boldsymbol{g}) = \{X \subset N : i \in X, j \in X \Rightarrow j \in N_i^C(\boldsymbol{g})\}$. Likewise, the set of maximal equivalent confirmation sets is: $E^M(\boldsymbol{g}) = \{X \in E(\boldsymbol{g}) :$ there is no $X' \in E(\boldsymbol{g}), X \subset X'\}$. Let $M(\boldsymbol{g}) = \{j \in N : j \in X \cap X'$ with $X, X' \in E^M(\boldsymbol{g})\}$ be the set of players who belong simultaneously to several MECS. We illustrate the construction of these sets through the following example.

Example 1 In network $\boldsymbol{g}$ drawn in Figure 3, we have $E^M(\boldsymbol{g}) = \{A, B, C, D, E\}$, $M(\boldsymbol{g}) = \{1, 3, 5, 6\}$.

Finally, we assume that each arc is costly to form for its initiator. We now formally define the payoff function of each player i. Let f_1, f_2, f_3 be strictly increasing functions, with $f_k(0) = 0$ for $k \in \{1, 2, 3\}$.

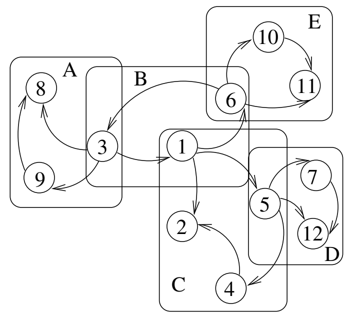


Figure 3: Maximal equivalent confirmation sets

The payoff function of each player i, given a network $\boldsymbol{g}$, is:

$$\pi_i(A(\boldsymbol{g})) = f_1(|N_i^C(A(\boldsymbol{g}))|) + f_2(|N_i^U(A(\boldsymbol{g}))|) - f_3(|A_i(\boldsymbol{g})|). \tag{1}$$

We assume that for all $x, x' \geq 0$ and $y \leq x'$, $f_1(x+y) + f_2(x'-y) > f_1(x) + f_2(x')$. This assumption, called (A1), implies that the payoff of agent i increases when simultaneously the number of confirmed resources obtained by i increases by y, and the number of unconfirmed resources obtained by i decreases by y. Moreover, since f_3 is increasing, the payoff function of player i decreases with the number of arcs she forms, given $|N_i^C(A(\boldsymbol{g}))|$ and $|N_i^U(A(\boldsymbol{g}))|$.
In the following, sometimes it is only possible to obtain results for the linear payoff function shown below:

$$\pi_i^L(A(\boldsymbol{g})) = V^C|N_i^C(A(\boldsymbol{g}))| + V^U|N_i^U(A(\boldsymbol{g}))| - c|A_i(\boldsymbol{g})|, \tag{2}$$

with $V^C > V^U > 0$, and $c > 0$.

Equillibrium and efficient networks. The strategy $A_i(\boldsymbol{g})$ is said to be a best response of player i against the strategy $A_{-i}(\boldsymbol{g})$ if:

$$\pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) \geq \pi_i(A', A_{-i}(\boldsymbol{g})), \text{ for all } A' \in 2^{G_i}. \tag{3}$$

The set of all best responses of player i's to $A_{-i}(\boldsymbol{g})$ is denoted by $\mathcal{BR}_i(A_{-i}(\boldsymbol{g}))$. A network $\boldsymbol{g}$ is said to be a Nash network if $A_i(\boldsymbol{g}) \in \mathcal{BR}_i(A_{-i}(\boldsymbol{g}))$ for each player $i \in N$. We define the set of all strict best responses of player i to $A_{-i}(\boldsymbol{g})$, as $s\mathcal{BR}_i(A_{-i}(\boldsymbol{g}))$, and a strict Nash network by replacing '$\geq$' by '$>$' and by setting $A' \in 2^{G_i} \setminus \{A_i(\boldsymbol{g})\}$ in inequality 3.

We define the total payoff function as $\Pi(\boldsymbol{g}) = \sum_{i \in N} \pi_i(A(\boldsymbol{g}))$. An *efficient network* $\boldsymbol{g}$ is a network such that $\Pi(\boldsymbol{g}) \geq \Pi(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{G}$.

3 Confirmation Model Analysis

3.1 Nash networks

First, we show that a Nash network in pure strategies does not always exist. This in itself is interesting since it differs from the result obtained in the standard BG model; in that model if the empty network is not Nash, then the periphery sponsored star is Nash. Second, we provide a condition which illustrates the importance of the convexity of the cost function for payoffs. This condition ensures the existence of Nash networks in pure strategies. Formally, the function f_3 is convex if $f_3(x+1) - f_3(x) \geq f_3(x) - f_3(x-1)$, for all $x \in \{1, \ldots, n-2\}$.

Proposition 1 *Suppose that the payoff function is given by equation 1. Then, there does not always exist a Nash network in pure strategies. If payoff function is given by equation 1 and f_3 is convex, then a Nash network in pure strategies will always exist.*

Proof First, we show through an example that if the payoff function is given by equation 1, then there does not always exist a Nash network. Suppose $N = \{1, 2, 3\}$, $f_1(2) = 7$, $f_2(1) = 5$, $f_2(2) = 5.5$, $f_3(1) = 4$ and $f_3(2) = 5$.[17] We show that no network can be Nash. Clearly a Nash network has at most 3 arcs. The empty network is not Nash since $f_1(0)+f_2(2)-f_3(2) = 0.5 > 0 = f_1(0)+f_2(0)-f_3(0)$. A network with one arc is not Nash because for the player who is not involved in the arc, we have: $f_1(0)+f_2(2)-f_3(1) = 1.5 > 0 = f_1(0)+f_2(0)-f_3(0)$. So she has an incentive to form an arc with one of the players. No network with two arcs can be Nash. More precisely, in such a network either a player receives two arcs, or such a player does not exist. In the former case, we have: $f_1(2)+f_2(0)-f_3(2) = 2 > 1.5 = f_1(0)+f_2(2)-f_3(1)$, and each player who sponsors an arc has an incentive to add one more arc. In the latter case, we have: $f_1(0)+f_2(1)-f_3(0) = 5 > 1.5 = f_1(0)+f_2(2)-f_3(1)$, and one of the players who has sponsored

[17]It is worth noting that $f_1(1)$ cannot appear in this model. Similarly, if $N = 3$, then $f_2(3)$ and $f_3(3)$ cannot appear in our example.

an arc has an incentive to remove it. No network with three arcs can be Nash. More precisely, in such a network there is at least one player who sponsors exactly one arc. This player is better off deleting this arc as $f_1(2) + f_2(0) - f_3(1) = 7 - 4 = 3 < 5.5 = f_1(0) + f_2(2) - f_3(0)$. This completes the proof of the first part.

For the second part, we show that a Nash network in pure strategies always exists when f_3 is convex. We will show that if the empty network and the periphery sponsored stars are not Nash networks, then a wheel is a Nash network.

Let us begin with the empty network $\boldsymbol{g}^e$. In $\boldsymbol{g}^e$ each player obtains a payoff equal to $f_1(0) + f_2(0) - f_3(0)$. There are two cases: either $\boldsymbol{g}^e$ is Nash and the proof is complete, or it is not. In the latter case, players have an incentive to form arcs. It follows that there exists x, $1 \leq x \leq n-1$, such that $f_1(0) + f_2(x) - f_3(x) > f_1(0) + f_2(0) - f_3(0)$. Since f_2 and f_3 are strictly increasing, we have $f_1(0) + f_2(n-1) - f_3(1) > f_1(0) + f_2(x) - f_3(x)$ for some x, $1 \leq x \leq n-1$. In this case, players in a periphery sponsored star, $\boldsymbol{g}^{ps}$, have no incentive to remove arcs. There are two cases: either $\boldsymbol{g}^{ps}$ is Nash and the proof is complete, or it is not. In the latter case, players have an incentive to form arcs in $\boldsymbol{g}^{ps}$. Consequently, we have: $f_1(x+1)+f_2(n-x-2)-f_3(1+x) > f_1(0)+f_2(n-1)-f_3(1)$, for some x, $1 \leq x \leq n-2$, that is $f_1(x+1)+f_2(n-x-2)-(f_1(0)+f_2(n-1)) > f_3(1+x)-f_3(1)$, for some x, $1 \leq x \leq n-2$. We show that if $\boldsymbol{g}^e$ and $\boldsymbol{g}^{ps}$ are not Nash networks, then a wheel is a Nash network. Indeed, we have $f_1(n-1) + f_2(0) - (f_1(0) + f_2(n-1)) \geq f_1(x+1) + f_2(n-x-2) - (f_1(0) + f_2(n-1)) > f_3(1+x) - f_3(1) \geq f_3(1) - f_3(0)$, for $1 \leq x \leq n-2$. The first inequality comes from assumption A1, and the last inequality comes from the convexity of f_3. It follows that $f_1(n-1) + f_2(0) - f_3(1) > f_1(0) + f_2(n-1) - f_3(0)$, that is no player in a wheel has an incentive to remove her arc. It follows that a wheel is a Nash network. $\square$

The intuition of the proof is as follows. If the empty network is not Nash, then each player has an incentive to form at least one arc. There are two cases with this: either each player accepts to maintain one arc when she obtains confirmed resources from all other players and the wheel is Nash, or each player does not accept to maintain one arc when she obtains confirmed resources from all other players and the periphery sponsored star is Nash.

Our second result highlights a general property of Nash networks: giving that linking is costly, in equilibrium there do not exist superfluous arcs.

Proposition 2 *Suppose that the payoff function is given by equation 1 and $\boldsymbol{g}$ is a Nash network. Then, $\boldsymbol{g}$ is a base network.*

Proof Let $\boldsymbol{g}$ be a Nash network. To introduce a contradiction, suppose $\boldsymbol{g}$ is not a base network. Then, there is a costly arc, say ij, which can be deleted by player i such that the resulting network allows player i to obtain the same total resources. This implies that player i has a strict incentive to remove the arc ij. Consequently, $\boldsymbol{g}$ is not a Nash network, a contradiction. □

The proof of the next proposition is given in Appendix A.

Proposition 3 *Suppose that the payoff function is given by equation 1. Then, there exist functions f_1, f_2 f_3 such that any minimally unconfirmed network is a Nash network.*

Proposition 3 illustrates that the set of Nash networks is very large.[18] However, a Nash network in which a player, say i, has multiple best responses is likely to be unstable since i may decide to switch to another payoff equivalent strategy. This motivates the examination of strict Nash networks discussed in the next section.

3.2 Strict Nash networks

We now introduce the two main propositions about the characterization of strict Nash networks, which we separate into connected strict Nash networks and strict Nash networks which are not connected. Proposition 4 provides the architectures of connected strict Nash networks, while Proposition 5 provides the architectures of non-connected strict Nash networks. Proofs of these propositions are given in Appendix B. Recall that a key player is a player who is involved in at least three arcs.

[18] Bala and Goyal (pg. 1194, [2]) show that the number of Nash networks increases rapidly with the number of players in the standard BG model. In our framework, the number of Nash networks is larger than in the standard BG model, since in equilibrium we obtain minimally unconfirmed (or minimally connected in the language of BG) networks as well as confirmed networks.

Proposition 4 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a connected strict Nash network. Then,* $\boldsymbol{g}$ *is either (a) a minimally confirmed network where all key players are the sponsors of all arcs they are involved in, or (b) a center sponsored star.*

Recall that in the standard BG model only one player, the central player in the center sponsored star, obtains a payoff different from the others in a non-empty strict Nash network. However, once we introduce the notion of confirmed resources through independent paths, it is possible that several players are in asymmetric positions. This is illustrated through the example below.

Example 2 Suppose that $N = \{1, \ldots, 7\}$ and the payoff function is given by equation 2. Moreover, suppose that $V^C = 100$, $V^U = 1$ and $c = 10$. Then, network $\boldsymbol{g}$ in Figure 4 is a strict Nash network. In this network, all players obtain the same gross profit but players 4 and 6 incur the costs of 3 arcs, players 1 and 2 incur the costs of 1 arc and players 5 and 7 incur no cost at all.

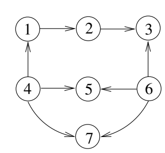


Figure 4: Network $\boldsymbol{g}$

The presence of key players in our model is an interesting finding. It is similar to the role played by the central player in a center sponsored star in the standard BG model: she incurs the cost of forming arcs and allows other players to be connected. More generally, the fact that two disjoint u-paths can raise payoffs, leads to a richer set of equilibrium outcomes in our model. We now turn to the characterization of another type of strict Nash networks where equilibrium networks among *ex ante* homogeneous players need not be connected.

Proposition 5 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a non-empty and non-connected strict Nash network. Then,* $\boldsymbol{g}$ *contains* x*,* $x \geq 1$*, wheels and a sub-network* $\boldsymbol{g}_{|_{N \setminus \mathcal{W}}}$*. Moreover,* $\boldsymbol{g}_{|_{N \setminus \mathcal{W}}}$ *is empty, or a minimally confirmed network where all key players are the sponsors of all arcs they are involved in.*

In the network literature, strict Nash networks are usually connected and with *ex ante* homogeneous agents, they are always connected. The typical reason is that by mimicking the strategy of a player who sponsors some arcs in a different component, one can obtain a higher net benefit just like that player. Therefore, if a network is not connected, there is one player who would strictly prefer to copy the strategy of a player in a different component. Consequently equilibrium networks are connected. In our model this is not the case. In a wheel for instance, each player i must form two arcs with players in a wheel to obtain confirmed resources from them while players in a wheel have formed only one arc in $\boldsymbol{g}$. We now illustrate through an example that there exist parameters such that a network $\boldsymbol{g}$ which contains several wheels is a strict Nash. In other words, a non-empty and non-connected network is a strict Nash network. This result differs from the standard BG model.

Example 3 Suppose the payoff function is given by equation 1 and $V^C = 5$, $V^U = 0.25$, $c = 9$. Suppose $N = \{1, \dots, 6\}$ and let $\boldsymbol{g}$ be such that $A(\boldsymbol{g}) = \{12, 23, 31, 45, 56, 64\}$. Then, $\boldsymbol{g}$ is a non-empty strict Nash network which is not connected.

Note that strict Nash networks have two interesting properties. In a strict Nash network

1. A player cannot obtain both confirmed and unconfirmed resources simultaneously; and
2. There cannot exist players who obtain unconfirmed resources along with players who obtain confirmed resources.

However, it is worth noting that players do not always obtain the same amount of resources. In the extreme case as stated in Proposition 5, some of them may receive no resources while others receive a strictly positive amount of confirmed resources.

3.3 Efficient networks

In the standard BG model, in general, an efficient network needs not be either connected or empty (see example pg. 1205, [2]). When the payoff function is linear, Bala and Goyal show that in the standard BG model, minimally unconfirmed networks and the empty network are the only candidates for efficient networks. In this section, we deal with efficient networks when the confirmation assumption is introduced.

Proposition 6 *Suppose that the payoff function is given by equation 1, and* $\boldsymbol{g}$ *is an efficient network. Then,* $\boldsymbol{g}$ *is a minimal base network. Moreover, if* $2[\min_{x\in\{0,\ldots,n-2\}}\{f_2(x+1)-f_2(x)\}] > \max_{y\in\{0,\ldots,n-2\}}\{f_3(y+1)-f_3(y)\}$*, then an efficient network is connected. Suppose in addition that* $3[\min_{x\in\{0,\ldots,n-3\},y\in\{2,\ldots,n-1\}}\{f_1(x+2)+f_2(y-2)-(f_1(x)+f_2(y))\}] > \max_{z\in\{0,\ldots,n-2\}}\{f_3(z+1)-f_3(z)\}$*, then an efficient network is a minimally confirmed network.*

Proof We prove successively the three parts of the proposition.

Suppose $\boldsymbol{g}$ is not a minimal base network. Then, there is a minimal base network $\boldsymbol{g}'$ which allows each player to obtain the same total resources and involves fewer arcs. Hence $\boldsymbol{g}$ is not an efficient network, a contradiction.

Suppose $MB_1 = 2[\min_{x\in\{0,\ldots,n-2\}}\{f_2(x+1)-f_2(x)\}] > \max_{y\in\{0,\ldots,n-2\}}\{f_3(y+1)-f_3(y)\} = MC$. To introduce a contradiction, suppose there exist two players i and j who are not connected in $\boldsymbol{g}$. Note that MB_1 gives the minimum benefit for two players from an arc connecting them, whereas MC gives the maximal possible cost of an arc. As arcs can only provide positive externalities to other players, this condition guarantees that an arc between these two players is welfare improving.

Let $MB_1 > MC$ and $MB_2 = 3\ [\min_{x\in\{0,\ldots,n-3\},y\in\{2,\ldots,n-1\}}\{f_1(x+2)+f_2(y-2)-(f_1(x)+f_2(y))\}] > MC$. Since $MB_1 > MC$ and $\boldsymbol{g}$ is efficient, $\boldsymbol{g}$ is connected and a minimal base network. To introduce a contradiction, suppose that $\boldsymbol{g}$ is not minimally confirmed. Then there exist three players i, i', and j such that $\overline{i'i} \in A(\boldsymbol{g})$, $\overline{ij} \notin A(\boldsymbol{g})$, and players i and i' do not obtain confirmed resources from j. There are two possibilities.

(a) Players i and i' do not obtain confirmed resources from each other in $\boldsymbol{g}$. In that case, either there exists a u-path between i and j which does not go through i', or there exists a u-path between i' and j which does not go through i. Wlog we assume that the latter is true. Since $\boldsymbol{g}$ is connected, if the link $\overline{ij}$ is added in $\boldsymbol{g}$, then each player obtains confirmed resources from the two other players and the additional benefits for the three players is at least MB_2. Moreover, the maximal additional cost associated with the link $\overline{ij}$ is MC. The contradiction follows from the fact that $MB_2 > MC$.

(b) Players i and i' obtain confirmed resources from each other in $\boldsymbol{g}$. Then there exists a player j' who obtains confirmed resources from i and i'. If j' does not obtain confirmed resources from j, then we use the same argument as in (a). If j' obtains confirmed resources from j, then there exists a player k who obtains confirmed resources from j and j', and unconfirmed resources from i and i' otherwise $\boldsymbol{g}$ is not a minimal base network. Thus, if the link $\overline{ij}$ is added in $\boldsymbol{g}$, then players i, i', j and k obtain confirmed resources from the others and the additional benefits for these players is $4[\min_{x\in\{0,\dots,n-3\},y\in\{2,\dots,n-1\}}\{f_1(x+2)+f_2(y-2)-(f_1(x)+f_2(y))\}] > MB_2$. The contradiction follows from the fact that $MB_2 > MC$. □

It is hard to say more about efficient networks under equation 1. However, it is possible to characterize efficient networks when the payoff function is linear and we do this next. Note that in each minimally unconfirmed network, there are $n-1$ arcs and each player obtains the unconfirmed information of each other. Hence, the total payoff obtained in such networks is $(n-1)(nV^U - c)$ when the payoff function is linear. Likewise, in each minimal cycle there are n arcs and each player obtains the confirmed information of each other. Hence, the total payoff obtained in such networks is $n((n-1)V^C - c)$ when the payoff function is linear.

Proposition 7 *Suppose that the payoff function is given by equation 2 and the empty network is not efficient. Let $\boldsymbol{g}$ be an efficient network. Then, $\boldsymbol{g}$ is either a minimal unconfirmed network, or a minimal cycle network. Moreover, if $n(n-1)(V^C - V^U) < c$, then $\boldsymbol{g}$ is a minimal unconfirmed network; and if $n(n-1)(V^C - V^U) > c$, then $\boldsymbol{g}$ is a minimal cycle network.*

Proof Let $\boldsymbol{g}$ be an efficient network. First, we show that $\boldsymbol{g}$ is either connected or empty. We know by Proposition 6 that an efficient network is a minimal base network. To introduce a contradiction, suppose that $\boldsymbol{g}$ is non-empty and non-connected. There are two cases: either $\boldsymbol{g}$ contains a cycle, or $\boldsymbol{g}$ does not contain any cycle.

1. Suppose that $\boldsymbol{g}$ contains a cycle, and so a MECS. We show that there is no player who does not belong to the MECS and who is directly connected with a player in the MECS in $\boldsymbol{g}$. To introduce a contradiction, suppose a MECS in $\boldsymbol{g}$, $T_1(\boldsymbol{g})$, which contains players i and j, with $ij \in A(\boldsymbol{g})$ and let player $\ell \notin T_1(\boldsymbol{g})$ such that $i\ell \in A(\boldsymbol{g})$. We define the network $\boldsymbol{g}'$ as follows $A(\boldsymbol{g}') = A(\boldsymbol{g}) - ij + \ell j$. We have

$$\Pi(\boldsymbol{g}') - \Pi(\boldsymbol{g}) = 2|T_1(\boldsymbol{g})|(V^C - V^U) > 0.$$

Consequently, $\boldsymbol{g}$ is not efficient, a contradiction. We now show that there does not exist a player ℓ who is not connected to the MECS in $\boldsymbol{g}$. Again, let players i and j belong to the

MECS $T_1(\boldsymbol{g})$ with $ij \in A(\boldsymbol{g})$. Since $\boldsymbol{g}$ is efficient, we have:

$$|T_1(\boldsymbol{g})|(|T_1(\boldsymbol{g})|-1)V^C - |T_1(\boldsymbol{g})|c \geq 0 \Rightarrow (|T_1(\boldsymbol{g})|-1)V^C - c \geq 0.$$

Since $\boldsymbol{g}$ is not connected, there exists a player ℓ who is not connected with player i. Let $\boldsymbol{g}'$ be a network such that $A(\boldsymbol{g}') = A(\boldsymbol{g}) - ij + i\ell + \ell j$. We have

$$0 \leq (|T_1(\boldsymbol{g})|-1)V^C - c < (2|T_1(\boldsymbol{g})|-1)V^C - c = (|T_1(\boldsymbol{g})|^2 - (|T_1(\boldsymbol{g})|-1)^2)V^C - c \leq \Pi(\boldsymbol{g}') - \Pi(\boldsymbol{g}).$$

It follows that $\boldsymbol{g}$ is not efficient, a contradiction.
2. Suppose that $\boldsymbol{g}$ contains no cycle. Since $\boldsymbol{g}$ is non-empty, there is a component $\boldsymbol{g}_{|_Y}$, $|Y| \geq 2$, in $\boldsymbol{g}$. We assume that players i and j belong to X with $ij \in A(\boldsymbol{g})$. Since $\boldsymbol{g}$ is efficient, we have

$$|Y|(|Y|-1)V^U - (|Y|-1)c \geq 0 \Rightarrow |Y|V^U - c \geq 0.$$

Since $\boldsymbol{g}$ is not connected, there is a player $\ell \notin Y$ who is not connected with i. Let $\boldsymbol{g}''$ be a network such that $A(\boldsymbol{g}'') = A(\boldsymbol{g}) + i\ell$. We have

$$0 \leq |Y|V^U - c < 2|Y|V^U - c \leq \Pi(\boldsymbol{g}'') - \Pi(\boldsymbol{g}).$$

It follows that $\boldsymbol{g}$ is not efficient, a contradiction.
Let $\boldsymbol{g}$ be a non-empty network. Clearly, $\boldsymbol{g}$ cannot contain more than n arcs since with n arcs it is possible to construct a minimal cycle network where all resources are confirmed. Moreover, $\boldsymbol{g}$ cannot have less than $n-1$ arcs since it is connected.
In a minimal cycle network, the total payoff is $n(n-1)V^C - nc$. Moreover, with $n-1$ arcs it is possible to construct a minimal unconfirmed network which gives access to unconfirmed information of all the other players. In this network, the total payoff is $n(n-1)V^U - (n-1)c$. It follows that if $n(n-1)(V^C - V^U) < c$, then $\boldsymbol{g}$ is a minimal unconfirmed network, and if $n(n-1)(V^C - V^U) > c$, then $\boldsymbol{g}$ is a minimal cycle network.

□

We now examine the relationship between strict Nash networks and efficient networks when the payoff function is linear. First, we establish through an example that strict Nash networks and efficient networks do not always coincide. The reason, of course, is the positive externalities of link formation.

Example 4 Suppose that the payoff function is given by equation 2 and $V^C < c$, $nV^U > c$ and $V^C - V^U < c/[n(n-1)]$. Center sponsored stars are efficient networks. Indeed, the total payoff obtained in center sponsored stars is greater than (i) the total payoff obtained in the empty network: $(n-1)(nV^U - c) > 0$ since $nV^U > c$; and (ii) the total payoff obtained in minimal cycles: $(n-1)(nV^U - c) > n((n-1)V^C - c)$ since $V^C - V^U < c/[n(n-1)]$. However, a center sponsored star is not a strict Nash network since the player who sponsors the arcs has no incentive to maintain them given that $V^U < c$.

Next, we provide conditions which ensure the coincidence of strict Nash networks and efficient networks under the linear payoff function.

Proposition 8 *Suppose that the payoff function is given by equation 2.*

1. *If a minimal cycle network, $\boldsymbol{g}^{mc}$, is a strict Nash network, then $\boldsymbol{g}^{mc}$ is an efficient network.*
2. *If the empty network, $\boldsymbol{g}^e$, is an efficient network, then $\boldsymbol{g}^e$ is a strict Nash network.*
3. *Suppose $V^U > c$ and $V^C - V^U < c/[n(n-1)]$. Then a center sponsored star is both a strict Nash network and an efficient network.*

Proof We begin with an initial observation. Consider a network $\boldsymbol{g}$. Note that each arc in $\boldsymbol{g}$ creates positive externalities. More precisely, for a given arc $ij \in A(\boldsymbol{g})$, no player, except i, incurs any losses because of this arc: it costs them nothing but it may add to their unconfirmed resources or upgrade their unconfirmed resources to confirmed resources. Part 1 and Part 2 of the proposition follows this observation. We now prove Part 3 of the proposition.
Suppose that $V^U > c$ and $V^C - V^U < c/[n(n-1)]$. Let $\boldsymbol{g}^{ps}$ be a center sponsored star where player j is the sponsor of the arcs. First, we show that player j has no incentive to remove any of her arcs in $\boldsymbol{g}^{ps}$. If player j removes x arcs in $\boldsymbol{g}^{ps}$, she obtains a marginal payoff equal to $A_1 = x(c - V^U)$. A_1 is negative since $V^U > c$. Second, we show that it is inefficient to add arcs in $\boldsymbol{g}^{ps}$. To introduce a contradiction, suppose that some arcs are added in $\boldsymbol{g}^{ps}$. Then the incremental payoff associated with each of them is bounded above by $n(n-1)(V^C - V^U) - c$. By assumption $n(n-1)(V^C - V^U) - c < 0$, and we obtain a contradiction. We conclude by using the initial observation. $\square$

4 Heterogeneous players: Journalists

In this section we introduce heterogeneity among the players with respect to how they value information. We assume that a fraction of the players only value confirmed information and call them *journalists*. Since journalists are in the business of presenting information, they will tend to care more about confirmed information than the average businessman, or politician. This is simply because unconfirmed information cannot be published, while the others can still act upon unconfirmed information if it is too costly to acquire the confirmed information. In this section we capture this by introducing a subset of the population, say $J \subseteq N$ where J is the set of journalists, that does not value unconfirmed information. Note that this also serves to act as robustness check for our first model where all agents are identical. The introduction of journalists allows us to focus on network architectures that are primarily driven by confirmed information. Players in $N \backslash J$ are called *ordinary* players. We are interested whether such player heterogeneity will significantly alter the architectures of the strict Nash networks.

For simplicity we will base this extension on equation 1. Formally, for any $j \in J$ the payoff function becomes

$$\pi_j^J\left(A\left(\boldsymbol{g}\right)\right) = f_1\left(\left|N_j^C\left(A\left(\boldsymbol{g}\right)\right)\right|\right) - f_3\left(\left|A_j\left(\boldsymbol{g}\right)\right|\right), \tag{4}$$

while for any player $i \in N \backslash J$, the payoff function remains the same as in equation 1. To keep the proposition simple, we assume that the number of ordinary players is at least three, so $|N \backslash J| \geq 1$. It is worth noting that if $J = N$, then the empty network is a strict Nash network. No journalist in an empty network has an incentive to add arcs, as such a unilateral action does not give him confirmed information. Proof of the next proposition is given in Appendix C.

Proposition 9 *Suppose that the payoff function is given by equation 1 for ordinary players and by equation 4 for journalists, and let* $|J| > 0$. *Then,*

1. *if* $\boldsymbol{g}$ *is a connected strict Nash network,* $\boldsymbol{g}$ *contains one MECS, say* $X \subseteq N$. *If there exists some player* $k \in N \backslash X$, *then (a)* $V_k\left(\boldsymbol{g}\right)$ *is a singleton, say* $V_k\left(\boldsymbol{g}\right) = \{i\}$, *where* $i \in X$ *and* $ik \in A(\boldsymbol{g})$; *and (b)* $i, k \in N \backslash J$;

2. *if* $\boldsymbol{g}$ *is a non-connected, non-empty strict Nash network, then* $\boldsymbol{g}$ *contains* x, $x \geq 1$, *wheels and a sub-network* $g_{|N \backslash \mathcal{W}}$. *Moreover,* $\boldsymbol{g}_{|N \backslash \mathcal{W}}$ *is empty, or a minimally confirmed network*

where all key players are the sponsors of all arcs they are involved in.

The discussion above suggests that the presence of journalists in the population impacts the set of equilibrium networks in two ways. First, the center sponsored star cannot be a strict Nash network anymore for any $|J| > 0$. This of course depends on the f_2 function for journalists. If journalists would receive some small, positive benefit from unconfirmed information, then the center sponsored star could still be a strict Nash network, albeit for a smaller range of parameters. Second, there exist situations such that network $\boldsymbol{g}^1$ in Figure 5.a, with players $1, 2, 3 \in J$ and players $4, 5, 6 \in N \setminus J$, is a strict Nash network.[19] In this network a subset of players are confirmed connected, and all other players only receive unconfirmed information but are connected to the confirmed subset. In such a network all journalists are part of the confirmed connected subset, which seems intuitive.

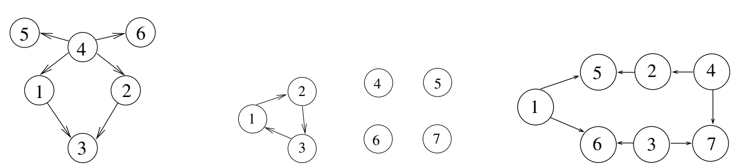
The introduction of journalists also has intuitive effects in another respect. In the proposition above, we have largely ignored the allocation of journalists and ordinary players over the different network positions. We only found that journalists cannot be involved in arcs through which no information is confirmed (part 2). However, it would seem natural to find journalists relatively often at key positions within the network, where several cycles are linked together. The reason is that journalists care more about getting information confirmed, and therefore they are more willing to sponsor the 'second' of the arcs in any particular cycle that they are part of in their role as key players. Example 5 illustrates that there exist parameter ranges where the role of key players is occupied only by journalists. It shows a network and parameter values, where in equilibrium there are only two key positions and these can only be held by journalists. Ordinary players will not occupy these positions, since in these positions they would strictly prefer to delete at least one arc. Clearly, this network can only be stable if there are at least two journalists in the population. To finalize our argument, note that the opposite cannot be the case: no journalist would prefer to delete an arc in a cycle that an ordinary player is willing to sponsor, as the journalist would lose more benefits from identical cost saving of forming arcs. Thus we obtain the intuitive result that journalists are relatively likely to be found in key positions of strict Nash networks.

[19] For instance, $\boldsymbol{g}^1$ is a strict Nash network if $f_1(x) = 10x$, $f_2(x) = 9x$, and $f_3(1) = 11$, $f_3(2) = 22$, $f_3(3) = 33$, $f_3(4) = 34$.

Example 5 We assume that the payoff function of each player in $N \setminus J$ is given by equation 2, where $V^C > 0$, $V^U > 0$, and $c > 0$. Similarly we assume that the payoff function of each player in J is given by equation 2, where $V^C > 0$, $V^U = 0$, and $c > 0$. Let $\boldsymbol{g}$ be the network given in Figure 4. Suppose that $V^C > c$ and $V^C - V^U < c < 6(V^C - V^U)$. Moreover, suppose that $J = \{4, 6\}$ and $N \setminus J = \{1, 2, 3, 5, 7\}$. Then $\boldsymbol{g}$ is a strict Nash network. Suppose now that $J = \{1, 6\}$ and $N \setminus J = \{2, 3, 4, 5, 7\}$. Then player 4 does not play a strict best response since she has an incentive to remove her arc with player 5. Consequently, $\boldsymbol{g}$ is not a strict Nash network.

Note that the presence of journalists has an ambiguous effect on social welfare. Welfare may go up because arcs that create cycles induce positive externalities, and therefore the fact that journalists facilitate cycles can be beneficial. However, the presence of journalists may allow for networks with small cycles to belong to the set of strict Nash networks. Networks with small cycles are less efficient than those with large cycles. We illustrate these points in Example 6.

Example 6 Let $N = \{1, \ldots, 7\}$ be the set of players. We assume that the payoff function of each player in $N \setminus J$ is given by equation 2, where $V^C > 0$, $V^U > 0$, and $c > 0$. Similarly we assume that the payoff function of each player in J is given by equation 2, where $V^C > 0$, $V^U = 0$, and $c > 0$. First we assume that $2(V^C - V^U) < c < 3(V^C - V^U)$, $3V^U < c$, and $c < 2V^C$. If $J = \{1, 2, 3\}$, then network $\boldsymbol{g}^2$ drawn in Figure 5.b is a strict Nash network, while a cycle which belongs to a strict Nash network would contain more players if all players were ordinary players. Second, we assume that $6(V^C - V^U) < c < 3V^C$. If $J = \{1, 2, 3, 4\}$, then network $\boldsymbol{g}^3$ drawn in 5.c is a strict Nash network, while if all players were ordinary players, then a minimally confirmed network is not a strict Nash network.



a. Network $\boldsymbol{g}^1$ b. Network $\boldsymbol{g}^2$ c. Network $\boldsymbol{g}^3$

Figure 5: Networks $\boldsymbol{g}^1$, $\boldsymbol{g}^2$ and $\boldsymbol{g}^3$

5 Bilaterally rational networks

The network formation literature on Nash networks mostly uses strict Nash networks to study the equilibrium networks. But there has been almost no attempts to study their robustness as an equilibrium concept. Hence in this section we undertake this task. Moreover we feel that this is important since it will allow us to check the robustness of our results which happen to include the results of the standard Bala and Goyal model in the form of unconfirmed networks. We now allow the possibility for players to make some bilateral deviations. To capture the possibility of bilateral deviations in our setting, we use *bilaterally rational networks* which are inspired by Kim and Wong (see definition 2, pg. 540, [14]). A bilaterally rational network is a strict Nash network where given all other players strategies, no pair of players can propose a joint change in their own strategies that strictly improves the payoff of one of them without reducing the payoff of the other. This assumption is intuitively reasonable and can be as a minimal assumption that captures the cooperative or joint behavior of players. We now define this notion formally. Let $A_{-ij}(\boldsymbol{g}) = \bigcup_{\ell \in N \setminus \{i,j\}} A_\ell(\boldsymbol{g})$ denotes the strategies profile played by all players except i and j. The pair of strategies $A_{ij}(\boldsymbol{g}) = (A_i(\boldsymbol{g}), A_j(\boldsymbol{g}))$[20] is said to be a bilateral best response of players i and j against the strategy $A_{-ij}(\boldsymbol{g})$ if there is no pair of strategies $A'_{ij} \in 2^{G_i} \times 2^{G_j} \setminus A_{ij}(\boldsymbol{g})$, such that

$$\pi_i(A_{ij}(\boldsymbol{g}), A_{-ij}(\boldsymbol{g})) \leq \pi_i(A'_{ij}, A_{-ij}(\boldsymbol{g})) \text{ and } \pi_j(A_{ij}(\boldsymbol{g}), A_{-ij}(\boldsymbol{g})) < \pi_j(A'_{ij}, A_{-ij}(\boldsymbol{g})). \quad (5)$$

Inequalities 5 capture the idea that no pair of agents can make a joint deviation to create a Pareto improvement for themselves. The set of all bilateral best responses of players i and j to $A_{-ij}(\boldsymbol{g})$ is denoted by $\mathcal{BBR}_{ij}(A_{-ij})$. A network $\boldsymbol{g}$ is said to be a bilaterally rational network if $(A_i(\boldsymbol{g}), A_j(\boldsymbol{g})) \in \mathcal{BBR}_{ij}(A_{-ij})$ for each pair of players $(i,j) \in N \times N$ and if $A_i(\boldsymbol{g}) \in s\mathcal{BR}_i(A_{-i}(\boldsymbol{g}))$.

Finally, we introduce strict bilaterally rational networks. In this case, Inequalities 5 are replaced by the following one:

$$\pi_i(A_{ij}(\boldsymbol{g}), A_{-ij}(\boldsymbol{g})) \leq \pi_i(A'_{ij}, A_{-ij}(\boldsymbol{g})) \text{ and } \pi_j(A_{ij}(\boldsymbol{g}), A_{-ij}(\boldsymbol{g})) \leq \pi_j(A'_{ij}, A_{-ij}(\boldsymbol{g})). \quad (6)$$

[20] $A_{ij}(\boldsymbol{g})$ can be interpreted as $A_i(\boldsymbol{g}) \cup A_j(\boldsymbol{g})$.

Inequalities 6 capture the idea that no pair of agents can make a joint deviation to create a weak Pareto improvement for themselves. More precisely, in absence of coordination costs, Inequalities 6 capture the notion that players should strictly prefer not to deviate. Thus strict bilaterally rational networks play the same role relative to bilaterally rational networks as the role played by strict Nash networks relative to Nash networks.

We present the main propositions that characterize the architectures of bilaterally rational and strict bilaterally rational networks. Proofs of Propositions 10 and 11 are provided in Appendix D. First, we characterize bilaterally rational networks.

Proposition 10 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a non-empty bilaterally rational network. Then,* $\boldsymbol{g}$ *is a minimally confirmed network where all key players are the sponsors of all arcs they are involved in, or a center sponsored star.*

Note that non-empty bilaterally rational networks are always connected, whereas non-empty strict Nash networks need not be: an equilibrium network may have multiple components provided that at most one component is not a wheel (Proposition 5). The reason a bilaterally rational network is connected is as follows. Consider a non-empty and non-connected strict Nash network $\boldsymbol{g}$. Such a network $\boldsymbol{g}$ contains a wheel. Let jj' be an arc in the wheel, and let player i belong to another component. Then i can become part of the wheel at a cost of one arc, by sponsoring ij' and asking player j to replace his arc to j' by an arc to player i. Player j would not lose by doing so, making this an eligible deviation. Second, we characterize strict bilaterally rational networks.

Proposition 11 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a non-empty strict bilaterally rational network. Then,* $\boldsymbol{g}$ *is a center sponsored star, or a minimal cycle, or a complete bipartite network containing two key players who are linked with all non key players. These non key players are at least* 3.

Strict bilaterally rational networks allow for a refinement of the set of strict Nash networks. Indeed, the strict Nash network drawn in Figure 4 is not a strict bilaterally rational network. We illustrate complete bipartite networks containing two key players who are linked with at least three non key players in Figure 6.

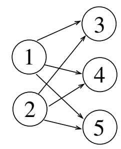


Figure 6: Network $\boldsymbol{g}$

6 Decay and Confirmation Networks

In the payoff functions used till now, reliability is generally used when we have node or link failure. Here we assumed that unconfirmed resources obtained through indirect arcs have the same value as those obtained through direct arcs. This assumption is strong; in general, there may be a reduction in value (due to loss in accuracy or distortion of information), as resources are transmitted through a series of players. More precisely, if player i obtains unconfirmed resources from player j through a long sequence of intermediate players, then she should have a greater incentive to confirm information, than if she obtains unconfirmed resources from player j through fewer intermediaries. Formally this is akin to requiring confirmation in Jackson and Wolinky's [11] "connections" model. To formalize this idea, define the distance $d_{\boldsymbol{g}}(i,j)$ between players i and j to be the number of arcs along the shortest u-path between i and j in $\boldsymbol{g}$. We denote the number of non-confirmed players at distance m from i in the network $\boldsymbol{g}$ by $n_i^m(A(\boldsymbol{g}))$. The payoff of player i in $\boldsymbol{g}$ is given by:[21]

$$\pi_i(A(\boldsymbol{g})) = |N_i^C(A(\boldsymbol{g}))| + \sum_{w=1}^{n-1} \alpha_w n_i^w(A(\boldsymbol{g})) - c|A_i(\boldsymbol{g})|. \tag{7}$$

The positive weights $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ measure the relative importance of neighbors at different distances. We assume that $1 > \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{n-1}$, so that more distant players yield less benefits. Since $1 > \alpha_1$ each player prefers to obtain confirmed resources over unconfirmed resources, given the number of arcs she forms. Note that the payoff function given by Equation 1 assumes that resources obtained through the network become more valuable on confirmation. However, the architecture of the network does not affect the value of the resources transmitted through it. By contrast, the payoff function given by Equation 7 says that unconfirmed

[21]This function is inspired by the payoff function in Hojman and Szeidl [10]. It generalizes the payoff function with decay introduced by Bala and Goyal [2].

resources obtained through more intermediaries is worth less than resources obtained through fewer intermediary players. In other words, the architecture of the network plays a role by being able to affect the value of the resources transmitted through it. We now briefly sum up the impact of the decay assumption in our confirmation model on equilibrium architectures.[22] First, under Equation 7 we can obtain a result similar to Proposition 4. In other words non-empty acyclic strict Nash networks are minimally unconfirmed networks. The intuition is the same in that linking is costly and hence leads to minimally unconfirmed networks.
Moreover, using continuity we can construct a situation similar to Example 3 where strict Nash networks are not always connected. This happens when costs of linking are high relative to the value of confirmed information. Then it is possible to construct scenarios consisting of unconfirmed networks that are minimal. Of course when the costs of linking are lower than the benefits of a direct arc, strict Nash networks will always be connected. In fact it is easy to identify conditions that make stars strict Nash. Moreover, if $c < \min\{\alpha_1, 1-\alpha_2\}$, then strict Nash networks are minimally confirmed networks. Basically if $c < 1-\alpha_2$, then it is worthwhile to initiate an arc to a player who is two steps away (decay creates strong incentives to avoid long u-paths) allowing for confirmation. Similar results hold for bilaterally rational networks under decay.

7 Conclusion

In this model of network formation we consider the need for alternate paths between the same agents in the network. This is motivated by the desire to have more valuable information. Information obtained from two different paths is considered to be confirmed and hence is of higher value. We characterize the strict Nash and efficient networks in the game. Interestingly, we find that although two independent paths may be desirable, there exist parameter ranges where equilibrium networks are non-empty and non-connected. We check the robustness of our results in three ways. First, we assume that players are heterogenous with regard to the value they obtain from unconfirmed information. We establish that who care more about confirmation are more likely to be found in key positions of networks. Second, we introduce the

[22]Detailed proofs for this section can be found in the working paper version at `http://bus.lsu.edu/McMillin/Working_Papers/pap12_02.pdf`.

notion of strict bilateral rational network. This acts as a robustness check for the commonly used notion of strict Nash networks. Our main finding here is that by taking into account the decisions of a pair of agents, this criterion ensures that non-empty equilibrium networks are always connected. A third extension considers the role of decay and its implication for confirmed information. Decay creates incentives for shorter u-paths between agents, we find that again it is possible to have both minimally unconfirmed networks (stars) and minimally confirmed networks as strict Nash networks.

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Appendix

Appendix A. Nash networks

Proof of Proposition 3. We first define three properties that the functions f_1, f_2 and f_3 need to satisfy.

- (P1) for all $x \in \{1, \ldots, n-1\}$ and $z \in \{0, \ldots, n-2\}$ we have:

$$f_1(x) + f_2(n-1-x) - [f_1(0) + f_2(n-1)] < f_3(z+1) - f_3(z).$$

- (P2) for all $x \in \{1, \ldots, n-1\}$, $z \in \{0, \ldots, n-1\}$, with $x < x'$, $z \leq z'$ and $x' - x > z' - z$, we have:

$$f_1(x) + f_2(n-1-x') - [f_1(0) + f_2(n-1)] < f_3(z') - f_3(z).$$

- (P3) We have for all $x \in \{0, \dots, n-2\}$:

$$f_2(x+1) - f_2(x) > f_3(x+1) - f_3(x).$$

Suppose that the payoff function is given by equation 1 and f_1, f_2, f_3 satisfy P1, P2 and P3.[23] To introduce a contradiction, consider a minimally unconfirmed network $\boldsymbol{g}$ which is not a Nash network. Since $\boldsymbol{g}$ is a minimally unconfirmed network, it is obvious that $\boldsymbol{g}$ is a base network where each player obtains $n-1$ unconfirmed resources from others. Since $\boldsymbol{g}$ is not a Nash network there exists a player, say i, who has a strict incentive to modify her strategy. Let E_i be the alternative strategy chosen by player i. There exist three kinds of alternative strategies for player i in $\boldsymbol{g}$: (a) $|E_i| = |A_i(\boldsymbol{g})|$, with $E_i \neq A_i(\boldsymbol{g})$, (b) $|E_i| < |A_i(\boldsymbol{g})|$, (c) $|E_i| > |A_i(\boldsymbol{g})|$. We deal successively with these three cases.

(a) Suppose $|E_i| = |A_i(\boldsymbol{g})|$, with $E_i \neq A_i(\boldsymbol{g})$. There are two possibilities concerning the resources that player i obtains in $\boldsymbol{g}'$ with $A(\boldsymbol{g}') = E_i \cup A_{-i}(\boldsymbol{g})$.

- Player i does not obtain confirmed resources in $\boldsymbol{g}'$. Since player i obtains $n-1$ unconfirmed resources in $\boldsymbol{g}$, the amount of unconfirmed resources in $\boldsymbol{g}'$ cannot exceed the amount of resources she obtains in $\boldsymbol{g}$. Moreover, player i incurs the same costs in $\boldsymbol{g}$ and $\boldsymbol{g}'$. Consequently, player i obtains a payoff in $\boldsymbol{g}'$ which is smaller or equal to the payoff she obtains in $\boldsymbol{g}$.

- Player i obtains x, $x > 0$, confirmed resources in $\boldsymbol{g}'$. Since $\boldsymbol{g}$ is a base network, in $\boldsymbol{g}'$ player i obtains a number of unconfirmed resources equal to $n-1-x'$ with $x' > x$. Moreover, player i incurs the same costs in network $\boldsymbol{g}$ and in network $\boldsymbol{g}'$. We conclude by P2 that $\pi_i(\boldsymbol{g}') - \pi_i(\boldsymbol{g}) = f_1(x) + f_2(n-1-x') - (f_1(0) + f_2(n-1)) < 0$.

To sum up, if $|E_i| = |A_i(\boldsymbol{g})|$, with $E_i \neq A_i(\boldsymbol{g})$, then E_i cannot strictly improve the payoff of player i.

(b) Suppose $|E_i| < |A_i(\boldsymbol{g})|$. There are two possibilities concerning the resources that player i obtains in $\boldsymbol{g}'$ with $A(\boldsymbol{g}') = E_i \cup A_{-i}(\boldsymbol{g})$.

- Player i does not obtain confirmed resources in $\boldsymbol{g}'$. Since $\boldsymbol{g}$ is a base network and player i forms $|E_i| < |A_i(\boldsymbol{g})|$ arcs, in $\boldsymbol{g}'$ she obtains $n-1-x'$ unconfirmed resources, with $x' \geq |A_i(\boldsymbol{g})| - |E_i|$. By P3 and the fact that f_2 and f_3 are increasing, we have $\pi_i(\boldsymbol{g}') = f_1(0) + f_2(n-1-x') - f_3(|E_i|) < f_1(0) + f_2(n-1) - f_3(|A_i(\boldsymbol{g})|) = \pi_i(\boldsymbol{g})$.

[23] For example, let $N = \{1, \dots, 9\}$ and $f_1(x) = 2.1x + 3$, $f_2(x) = 2x$, and $f_3(x) = x$. Then these functions satisfy P1, P2 and P3.

- Player i obtains x, $x > 0$, confirmed resources in $\boldsymbol{g}'$. Since $\boldsymbol{g}$ is a base network, in $\boldsymbol{g}'$ player i obtains $n-1-x'$ unconfirmed resources, with $x' > x$ and $x'-x > |A_i(\boldsymbol{g})|-|E_i|$. Consequently, by P2 we have $\pi_i(\boldsymbol{g}') = f_1(x)+f_2(n-1-x')-f_3(|E_i|) < f_1(0)+f_2(n-1)-f_3(|A_i(\boldsymbol{g})|) = \pi_i(\boldsymbol{g})$.
To sum up, if $|E_i| < |A_i(\boldsymbol{g})|$, then E_i cannot strictly improve the payoff of player i.
(c) Suppose $|E_i| > |A_i(\boldsymbol{g})|$. There are two possibilities concerning the resources that player i obtains in $\boldsymbol{g}'$ with $A(\boldsymbol{g}') = E_i \cup A_{-i}(\boldsymbol{g})$.
- Player i does not obtain confirmed resources in $\boldsymbol{g}'$. Player i does not obtain more unconfirmed resources in network $\boldsymbol{g}'$ than in network $\boldsymbol{g}$. Moreover, player i incurs higher costs in $\boldsymbol{g}'$ than in $\boldsymbol{g}$. Consequently, E_i does not improve the payoff of player i.
- Player i obtains x, $x > 0$, confirmed resources in $\boldsymbol{g}'$. Since player i obtains $n-1$ unconfirmed resources in $\boldsymbol{g}$, she obtains at most $n-1-x$ unconfirmed resources in $\boldsymbol{g}'$. By P1 and the fact that f_3 is increasing, we have $\pi_i(\boldsymbol{g}') \leq f_1(x)+f_2(n-1-x)-f_3(|E_i|) < f_1(0)+f_2(n-1)-f_3(|A_i(\boldsymbol{g})|) = \pi_i(\boldsymbol{g})$.
To sum up, if $|E_i| > |A_i(\boldsymbol{g})|$, then E_i cannot strictly improve the payoff of player i. It follows that there does not exist an alternative strategy which allows player i to strictly improve her payoff, a contradiction. $\square$

Appendix B. Strict Nash networks

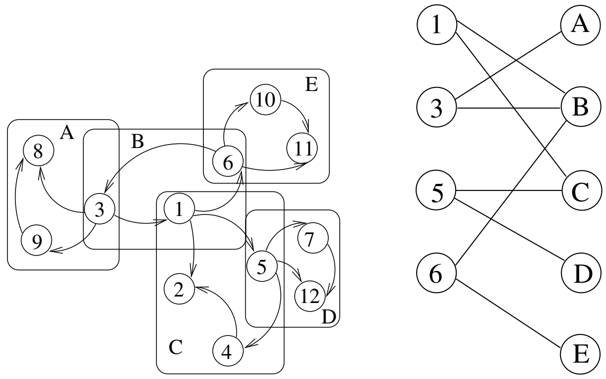
We present five lemmas that allow us to characterize the strict Nash networks. To prove the first one, we need to construct from $\boldsymbol{g}$ a bipartite network called $\boldsymbol{g}^M$. The set of vertices of $\boldsymbol{g}^M$ is $M(\boldsymbol{g}) \cup E^M(\boldsymbol{g})$. There is a link between $i \in M(\boldsymbol{g})$ and $X(\boldsymbol{g}) \in E^M(\boldsymbol{g})$ if i belongs to $X(\boldsymbol{g})$ in $\boldsymbol{g}^M$.[24] In the next example, we illustrate the construction of the network $\boldsymbol{g}^M$ from $\boldsymbol{g}$.

Example 7 We construct from network $\boldsymbol{g}$ in Figure 4.a, the network $\boldsymbol{g}^M$ associated with $\boldsymbol{g}$ in Figure 4.b.

Notice that by construction the network $\boldsymbol{g}^M$ contains no cycle, otherwise the set $E^M(\boldsymbol{g})$ is not well defined.

Lemma 1 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a strict Nash network. Then,* $M(\boldsymbol{g}) = \emptyset$.

[24] In $\boldsymbol{g}^M$ the direction of the arcs plays no role. Consequently, we use "link" instead of "arc" for the network $\boldsymbol{g}^M$.



(a) Network $\boldsymbol{g}$ (b) Network $\boldsymbol{g}^M$

Figure 7: Construction of $\boldsymbol{g}^M$

Proof We prove the lemma in two steps. First, we establish that each MECS has at most one link in $\boldsymbol{g}^M$. Second, we use this result to show that $M(\boldsymbol{g}) = \emptyset$.

1. We establish that each MECS has at most one link in $\boldsymbol{g}^M$. To introduce a contradiction suppose that there is a MECS, say $X(\boldsymbol{g})$, that has two links in $\boldsymbol{g}^M$. In the following, we focus on the component, say $\mathcal{Z}(\boldsymbol{g})$, which contains $X(\boldsymbol{g})$ in $\boldsymbol{g}^M$. We know that $\boldsymbol{g}^M$ is acyclic, so $\mathcal{Z}(\boldsymbol{g})$ is acyclic. Consequently, there are two vertices in $\mathcal{Z}(\boldsymbol{g})$ which have only one link. Moreover, by construction, each vertex in $M(\boldsymbol{g})$ which belongs to $\mathcal{Z}(\boldsymbol{g})$ has links with at least two vertices in $\mathcal{Z}(\boldsymbol{g})$. It follows that there are two vertices in $E^M(\boldsymbol{g})$ which have only one link in $\mathcal{Z}(\boldsymbol{g})$. We conclude that there exist two MECS which belong to $\mathcal{Z}(\boldsymbol{g})$, say $X_m(\boldsymbol{g}) \in E^M(\boldsymbol{g})$ and $X'_m(\boldsymbol{g}) \in E^M(\boldsymbol{g})$, which contain only one player who belongs to $M(\boldsymbol{g})$. We denote by i_m the unique player in $M(\boldsymbol{g})$ who belongs to $X_m(\boldsymbol{g})$ and by i'_m the unique player in $M(\boldsymbol{g})$ who belongs to $X'_m(\boldsymbol{g})$. Since a MECS contains at least 3 players, $X_m(\boldsymbol{g})$ contains at least two players, say $j \notin M(\boldsymbol{g})$ and $k \notin M(\boldsymbol{g})$, such that $jk \in A(\boldsymbol{g})$. By using the same arguments $X'_m(\boldsymbol{g})$ contains two players $j' \notin M(\boldsymbol{g})$ and $k' \notin M(\boldsymbol{g})$ such that $j'k' \in A(\boldsymbol{g})$.

   We now show that players j and j' cannot play simultaneously a strict best response in $\boldsymbol{g}$. We define $\boldsymbol{g}'$ as the network such that $A(\boldsymbol{g}') = A(\boldsymbol{g}) + jk' - jk$. Likewise, we define $\boldsymbol{g}''$ as the network such that $A(\boldsymbol{g}'') = A(\boldsymbol{g}) + j'k - j'k'$.

   Let $x < |X_m(\boldsymbol{g})| - 1$ be the number of confirmed resources that player j obtains when she removes her arc with player k in $\boldsymbol{g}$ and let $x' < |X'_m(\boldsymbol{g})| - 1$ be the number of confirmed resources that player j' obtains when she removes her arc with player k' in $\boldsymbol{g}$. Likewise, let y be the number of confirmed resources that player j obtains in $\boldsymbol{g}'$ from players who belong neither to $X_m(\boldsymbol{g})$, nor to $X'_m(\boldsymbol{g})$ (it is also the number of confirmed resources that player j' obtains in $\boldsymbol{g}''$ from players who belong neither to $X_m(\boldsymbol{g})$, nor to $X'_m(\boldsymbol{g})$).

Let K be the number of players from whom player j obtains unconfirmed resources and who do not belong to $X'_m(\boldsymbol{g})$ (it is also the number of players from whom player j' obtains unconfirmed resources and who do not belong to $X_m(\boldsymbol{g})$). We now establish that either j, or j' can shift an arc such that her number of confirmed resources increases, while her number of unconfirmed resources she receives and the number of arcs she forms are the same. Suppose that player j replaces the arc jk by the arc jk'. We have:

$$\begin{aligned}\Delta_j = \pi_j(A(\boldsymbol{g}')) - \pi_j(A(\boldsymbol{g})) &= f_1(x+y+|X'_m|) + f_2(K + (|X_m| - 1 - x - y)) \\ &\quad - f_1(|X_m| - 1) - f_2(K + |X'_m|).\end{aligned}$$

We obtain:

$$0 > \Delta_j \geq f_1(|X'_m|) + f_2(K + |X_m| - 1) - f_1(|X_m| - 1) - f_2(K + |X'_m|). \qquad (8)$$

The first inequality comes from the strict Nash property of $\boldsymbol{g}$ and the second inequality comes from the assumption (A1) made on the payoff function.

Suppose that player j' replaces the arc $j'k'$ by the arc $j'k$. We have:

$$\begin{aligned}\Delta_{j'} = \pi_{j'}(A(\boldsymbol{g}')) - \pi_{j'}(A(\boldsymbol{g})) &= f_2(K + (|X'_m(\boldsymbol{g})| - 1 - x' - y)) \\ &\quad + f_1(x' + y + |X_m(\boldsymbol{g})|) \\ &\quad - f_1(|X'_m(\boldsymbol{g})| - 1) - f_2(K + |X_m(\boldsymbol{g})|).\end{aligned}$$

We obtain:

$$0 > \Delta_{j'} \geq f_1(|X_m(\boldsymbol{g})|) + f_2(K + |X'_m(\boldsymbol{g})| - 1) - f_1(|X'_m(\boldsymbol{g})| - 1) - f_2(K + |X_m(\boldsymbol{g})|). \quad (9)$$

The first inequality comes from the strict Nash property of $\boldsymbol{g}$ and the second inequality comes from the assumption (A1) made on the payoff function. By Assumption (A1), we have $f_1(|X'_m(\boldsymbol{g})|) + f_2(K + (|X_m(\boldsymbol{g})| - 1)) > f_1(|X'_m(\boldsymbol{g})| - 1) + f_2(K + (|X_m(\boldsymbol{g})|))$ and $f_1(|X_m(\boldsymbol{g})|) + f_2(K + (|X'_m(\boldsymbol{g})| - 1)) > f_1(|X_m(\boldsymbol{g})| - 1) + f_2(K + (|X'_m(\boldsymbol{g})|))$. It follows that equations 8 and 9 are not compatible. A contradiction.

2. We now show that $M(\boldsymbol{g}) = \emptyset$. Suppose $\boldsymbol{g}$ is a strict Nash network and $i \in M(\boldsymbol{g})$. We call X and X' two MECS which contain i. Since a MECS has at most one link in $\boldsymbol{g}^M$, players $i \in M(\boldsymbol{g})$ are not connected in $\boldsymbol{g}^M$. Hence, there is no player $j \in M(\boldsymbol{g})$ who belongs to X or X'. Since each MECS contains at least 3 players, there are two players $j \notin M(\boldsymbol{g})$ and $k \notin M(\boldsymbol{g})$ in X and two players $j' \notin M(\boldsymbol{g})$ and $k' \notin M(\boldsymbol{g})$ in X' such that $jk \in A(\boldsymbol{g})$ and $j'k' \in A(\boldsymbol{g})$. Hence, we can use the arguments given in the proof of the previous point to show that it is not possible that player j does not have any incentive to replace the arc jk by the arc jk' and that simultaneously player j' does not have any incentive to replace the arc $j'k'$ by the arc $j'k$. Consequently, $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

□

Lemma 2 *Let $\boldsymbol{g}$ be a strict Nash network and let X be a MECS in $\boldsymbol{g}$. Suppose $i \in X$, $j \notin X$ and $\overline{ij} \in A(\boldsymbol{g})$. Then, (a) no player $k \in X$ sponsors an arc with i in $\boldsymbol{g}$, (b) $ij \in A(\boldsymbol{g})$, and (c) $V_j(\boldsymbol{g}) = \{i\}$.*

Proof Let $\boldsymbol{g}$ be a strict Nash network and let X be a MECS in $\boldsymbol{g}$. Suppose $i \in X$, $j \notin X$. We prove successively the three parts of the Lemma.

1. To introduce a contradiction suppose there is $k \in X$ such that $ki \in A(\boldsymbol{g})$. By construction, player k does not obtain confirmed resources from player j otherwise X is not a MECS. If player k replaces the arc ki by the arc kj, she increases the number of confirmed resources by 1, and decreases the number of unconfirmed resources by 1. Therefore, player k does not play a strict best response in $\boldsymbol{g}$ by Assumption (A1), and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

2. To introduce a contradiction suppose that $ji \in A(\boldsymbol{g})$. Since $i \in X$, there exists another player $k \in X$. If player $j \notin X$ replaces the arc ji by jk, then she obtains the same payoff

as in $\boldsymbol{g}$. Consequently, j does not play a strict best response in $\boldsymbol{g}$, and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

3. To introduce a contradiction, suppose that there is a player $j' \in V_j(\boldsymbol{g}) \setminus \{i\}$. Clearly, there is no other u-path between i and j' in $\boldsymbol{g}$ other than the one going through j. Otherwise $i \in M(\boldsymbol{g})$ which is impossible by Lemma 1. Consequently, if player i replaces the arc ij by the arc ij', then she obtains the same payoff as in $\boldsymbol{g}$. It follows that i does not play a strict best response in $\boldsymbol{g}$, and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

□

Lemma 3 *Suppose that the payoff function is given by equation 1 and $\boldsymbol{g}$ is a strict Nash network. Let X be a MECS of $\boldsymbol{g}$. If $i \in X$ and i is a key player, then i is the sponsor of all the arcs she is involved in.*

Proof Let $\boldsymbol{g}$ be a strict Nash network. Let X be a MECS of $\boldsymbol{g}$ and let i be a key player who belongs to X. Suppose player i sponsors an arc with a player $k \notin X$, then i has sponsored all the arcs in which she is involved by Lemma 2 (a).

Suppose now that player i is linked only to players in X. To introduce a contradiction, suppose i does not sponsor all her arcs. Then there are three distinct players, say j, j_1 and j_2 in X, such that $\overline{ij}$, $\overline{ij_1}$, $\overline{ij_2} \in A(\boldsymbol{g})$, where at least one of these arcs is not sponsored by player i. Without loss of generality we assume that $ji \in A(\boldsymbol{g})$. Since $\boldsymbol{g}$ is a strict Nash network, and hence a base network either $\overline{jj_1} \notin A(\boldsymbol{g})$, or $\overline{jj_2} \notin A(\boldsymbol{g})$. Wlog we assume that $\overline{jj_1} \notin A(\boldsymbol{g})$. We will show that player j has an incentive to replace the arc ji by the arc jj_1. Let $\boldsymbol{g}'$ be the network such that $A(\boldsymbol{g}') = A(\boldsymbol{g}) + jj_1 - ji$.

First, we show that $N_j^C(A(\boldsymbol{g})) \subset N_j^C(A(\boldsymbol{g}'))$. To establish this result, *(i)* we first show that if $k \in N_j^C(A(\boldsymbol{g})) \setminus N_j^C(A(\boldsymbol{g}'))$, then there is an u-path between j and k which does not contain any player $j' \in V_i(\boldsymbol{g}) \setminus \{j\}$. *(ii)* Then, we establish that in this case, there are two disjoint u-paths between j and k in $\boldsymbol{g}'$ which contradicts the assumption concerning the existence of a player k who belongs to $N_j^C(A(\boldsymbol{g})) \setminus N_j^C(A(\boldsymbol{g}'))$.

(i) Suppose $k \in N_j^C(A(\boldsymbol{g})) \setminus N_j^C(A(\boldsymbol{g}'))$. Then, the resources of k are confirmed in $\boldsymbol{g}$ by player j due to an u-path, say $P^1_{j,k}(\boldsymbol{g})$, which contains player i. This u-path also contains a player, say $j'_1 \in V_i(\boldsymbol{g}) \setminus \{j\}$. We establish that there is an u-path between j and k in $\boldsymbol{g}$ disjoint from $P^1_{j,k}(\boldsymbol{g})$, say $P^2_{j,k}(\boldsymbol{g})$, which does not contain any player $j' \in V_i(\boldsymbol{g}) \setminus \{j\}$. To introduce a

contradiction, suppose that such an u-path does not exist, that is $P^2_{j,k}(\boldsymbol{g})$ contains a player, say $j_2 \in V_i(\boldsymbol{g}) \setminus \{j, j'_1\}$. In this situation, players i, j'_1, k, j_2, and j belong to a cycle in $\boldsymbol{g}$. It follows that player i (or player j_2) has no incentive to maintain the arc ij_2 (or the arc $j_2 i$) in $\boldsymbol{g}$ and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

To sum up, we know that (a) there is an u-path $P^2_{j,k}(\boldsymbol{g})$ which does not contain any player $j' \in V_i(\boldsymbol{g}) \setminus \{j\}$, (b) the u-path $P^1_{j,k}(\boldsymbol{g})$ contains player i, and (c) $P^1_{j,k}(\boldsymbol{g})$ and $P^2_{j,k}(\boldsymbol{g})$ are disjoint.

(ii) We now show that there are two disjoint u-paths between j and k in $\boldsymbol{g}'$. There are two cases: either $P^1_{j,k}(\boldsymbol{g})$ contains player j_1, or $P^1_{j,k}(\boldsymbol{g})$ does not contain player j_1. If $P^1_{j,k}(\boldsymbol{g})$ contains player j_1: $P^1_{j,k}(\boldsymbol{g}) = j, i, j_1, \ldots, k$ in $\boldsymbol{g}$ with $ji \in A(\boldsymbol{g})$, then there is an u-path between j and k in $\boldsymbol{g}'$: $P^1_{j,k}(\boldsymbol{g}') = j, j_1, \ldots, k$ in $\boldsymbol{g}'$, with $jj_1 \in A(\boldsymbol{g})$. Clearly, $P^1_{j,k}(\boldsymbol{g}')$ and $P^2_{j,k}(\boldsymbol{g})$ are disjoint, so $k \in N^C_j(\boldsymbol{g}')$, that is $k \notin N^C_j(\boldsymbol{g}) \setminus N^C_j(\boldsymbol{g}')$: a contradiction. If $P^1_{j,k}(\boldsymbol{g})$ does not contain player j_1, then it contains a player, say j_2, in $V_i(\boldsymbol{g})$: $P^1_{j,k}(\boldsymbol{g}) = j, i, j_2, \ldots, k$ in $\boldsymbol{g}$ with $ji \in A(\boldsymbol{g})$. In this case, there is an u-path between k and j in $\boldsymbol{g}'$: $P^{1'}_{j,k}(\boldsymbol{g}') = j, j_1, i, j_2 \ldots, k$ with $jj_1 \in A(\boldsymbol{g}')$. $P^{1'}_{j,k}(\boldsymbol{g}')$ and $P^2_{j,k}(\boldsymbol{g})$ are disjoint since $j_2 \in V_i(\boldsymbol{g})$, and so $j_2 \notin P^2_{j,k}(\boldsymbol{g})$. Consequently, $k \in N^C_j(A(\boldsymbol{g}'))$, that is $k \notin N^C_j(\boldsymbol{g}) \setminus N^C_j(\boldsymbol{g}')$: a contradiction.

Second, since $\overline{ij_1} \in A(\boldsymbol{g}) \cap A(\boldsymbol{g}')$, we have $N^U_j(A(\boldsymbol{g})) \subset N^U_j(A(\boldsymbol{g}'))$.

Third, as the costs of each arc are the same, this implies that j is indifferent between sponsoring ji and jj_1. Since $N^C_j(A(\boldsymbol{g})) \subset N^C_j(A(\boldsymbol{g}'))$, it follows that player j does not play a strict best response in $\boldsymbol{g}$. This concludes the proof. □

Lemma 4 *Suppose that the payoff function is given by equation 1 and $\boldsymbol{g}$ is a non-empty strict Nash network. No player i in a MECS X forms an arc with a player $j \notin X$.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network which contains a MECS X. To introduce a contradiction suppose that a key player $i \in X$ is linked with $j \notin X$. Since i belongs to a MECS, there exist i_1 and i_2 such that $\overline{ii_1}, \overline{i_2 i_1} \in A(\boldsymbol{g})$. By Lemma 2, $V_j(\boldsymbol{g}) = \{i\}$, $ij \in A(\boldsymbol{g})$ and $ii_1 \in A(\boldsymbol{g})$. Let K be the number of resources of unconfirmed players that each player who belongs to X obtains in $\boldsymbol{g}$. Since $\boldsymbol{g}$ is a strict Nash network player j has no incentive to form an arc with player i_1. It follows that

$$\pi_j(A(\boldsymbol{g})) = f_1(0) + f_2(|X| + K - 1) - f_3(0) > f_1(|X|) + f_2(K-1) - f_3(1) = \pi_j(A(\boldsymbol{g}) + ji_1) \quad (10)$$

There are two cases: either $i_1i_2 \in A(\boldsymbol{g})$, or $i_2i_1 \in A(\boldsymbol{g})$. We deal successively with the two possibilities.

1. Suppose $i_1i_2 \in A(\boldsymbol{g})$. Player i_1 cannot sponsor more than one arc otherwise she is a key player and she cannot receive an arc from player i by Lemma 3.

   Since $\boldsymbol{g}$ is a strict Nash network, player i_1 has no incentive to remove her arc. We have:

   $$\pi_{i_1}(A(\boldsymbol{g})) = f_1(|X|-1)+f_2(K)-f_3(1) > f_1(0)+f_2(|X|+K-1)-f_3(0) = \pi_j(A(\boldsymbol{g})-i_1i_2). \tag{11}$$

   Due to A1, inequalities 10 and 11 are not compatible, a contradiction.

2. Suppose $i_2i_1 \in A(\boldsymbol{g})$. Since player i_1 is not a key player $|V_{i_1}(\boldsymbol{g})| = 2$. We have $V_{i_1}(\boldsymbol{g}) = \{i, i_2\}$, and

   $$\pi_{i_2}(A(\boldsymbol{g})) = f_1(|X|-1) + f_2(K) - f_3(|A_{i_2}(\boldsymbol{g})|) = \pi_{i_2}(A(\boldsymbol{g}) - i_2i_1 + i_2j). \tag{12}$$

   Consequently, player i_2 does not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

□

Lemma 5 *Suppose that the payoff function is given by equation 1 and $\boldsymbol{g}$ is a non-empty strict Nash network. Then, $\boldsymbol{g}$ is either connected, or it contains wheels and a sub-network $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$ where $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$ is either empty, or connected.*

Proof Let $\boldsymbol{g}$ be a non-empty strict Nash network.

1. First, we consider situations where $\boldsymbol{g}$ does not contain a wheel. To introduce a contradiction suppose that $\boldsymbol{g}$ is not connected. There are two situations, (i) either $\boldsymbol{g}$ contains no cycle, or (ii) $\boldsymbol{g}$ contains a cycle (which is not a wheel).

   (i) Suppose $\boldsymbol{g}$ contains no cycle. Since $\boldsymbol{g}$ is non-empty, there are two players i and j such that $ij \in A(\boldsymbol{g})$. Since $\boldsymbol{g}$ is not connected, there are players $\ell \in N \setminus \{i\}$ such that $\ell \notin N_i^U(A(\boldsymbol{g}))$. Either (i.a) $|A_\ell(\boldsymbol{g})| = 0$ for all $\ell \notin N_i^U(A(\boldsymbol{g}))$, or (i.b) there exists a player $\ell \notin N_i^U(A(\boldsymbol{g}))$ such that $|A_\ell(\boldsymbol{g})| > 0$.

   (i.a) Suppose $|A_\ell(\boldsymbol{g})| = 0$ for all $\ell \notin N_i^U(A(\boldsymbol{g}))$, that is, players $\ell \notin N_i^U(A(\boldsymbol{g}))$ are isolated players in $\boldsymbol{g}$. Then there are two cases: $\pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) \leq \pi_\ell(A_\ell(\boldsymbol{g}), A_{-\ell}(\boldsymbol{g}))$ or

$\pi_i(A_i(\boldsymbol{g}), A_{-i}(\boldsymbol{g})) \geq \pi_\ell(A_\ell(\boldsymbol{g}), A_{-\ell}(\boldsymbol{g}))$. In the first case, player i has an incentive to remove all her arcs and $\boldsymbol{g}$ is not a strict Nash network. In the second case, let network $\boldsymbol{g}'$ be such that $A_\ell(\boldsymbol{g}') = \{\ell i\}$ and $A_j(\boldsymbol{g}') = A_j(\boldsymbol{g})$ for all $j \in N \setminus \{\ell\}$. We have $\pi_\ell(A(\boldsymbol{g}')) = f_1(0) + f_2(|N_i^U(A(\boldsymbol{g})) + 1|) - f_3(1) > f_1(0) + f_2(|N_i^U(A(\boldsymbol{g}))|) - f_3(1) \geq f_1(0) + f_2(|N_i^U(A(\boldsymbol{g}))|)) - f_3(|(A_i(\boldsymbol{g})|) = \pi_i(A(\boldsymbol{g})) \geq \pi_\ell(A(\boldsymbol{g}))$. Consequently, player ℓ does not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network.

(i.b) Suppose there is a player $\ell \notin N_i^U(A(\boldsymbol{g}))$ such that $|A_\ell(\boldsymbol{g})| > 0$. Since $\boldsymbol{g}$ contains no cycle there exists a player i', with $i \in N_{i'}^U(\boldsymbol{g})$, and a player ℓ', with $\ell \in N_{\ell'}(\boldsymbol{g})$, who receive no arcs in $\boldsymbol{g}$. In the following we deal with players i' and ℓ'. Wlog suppose that $\pi_{i'}(A_{i'}(\boldsymbol{g}), A_{-i'}(\boldsymbol{g})) \geq \pi_{\ell'}(A_{\ell'}(\boldsymbol{g}), A_{-\ell'}(\boldsymbol{g}))$. By the same reasoning as in the second case of point (i.a) above, we can check that player ℓ' does not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network.

(ii) Suppose $\boldsymbol{g}$ contains a cycle (which is not a wheel). Since there is a cycle, there is a MECS, say X. Moreover, since $\boldsymbol{g}$ contains no wheel, the considered cycle contains a player, say i, who has formed arcs with at least two players. By Lemma 3, i receives no arcs. Since $\boldsymbol{g}$ is not connected there is a player $\ell \in N \setminus \{i\}$ such that i and ℓ are not connected in $\boldsymbol{g}$. Either (ii.a) $|A_\ell(\boldsymbol{g})| = 0$ for all $\ell \notin N_i(A(\boldsymbol{g}))$, where $N_i(A(\boldsymbol{g})) = N_i^U(A(\boldsymbol{g})) \cup N_i^C(A(\boldsymbol{g}))$, or (ii.b) there exist players $\ell \notin N_i(A(\boldsymbol{g}))$ such that $|A_\ell(\boldsymbol{g})| > 0$.

(ii.a) $|A_\ell(\boldsymbol{g})| = 0$ for all $\ell \notin N_i(A(\boldsymbol{g}))$. By the same type of reasoning as in point (i.a) above we can check that player i or each player $\ell \notin N_i(A(\boldsymbol{g}))$ do not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network.

(ii.b) Suppose there exist players $\ell \notin N_i(A(\boldsymbol{g}))$ such that $|A_\ell(\boldsymbol{g})| > 0$.

Suppose that there exist players $\ell \notin N_i(A(\boldsymbol{g}))$ such that $|A_\ell(\boldsymbol{g})| > 0$ who belong to a cycle. We consider a player ℓ who is not connected with i in $\boldsymbol{g}$, who belongs to a cycle and who has formed arcs with at least two players. Note that ℓ receives no arcs by Lemma 3. Suppose wlog that $\pi_i(A(\boldsymbol{g})) \geq \pi_\ell(A(\boldsymbol{g}))$. Let $\boldsymbol{g}''$ be the network such that $A_\ell(\boldsymbol{g}'') = A_i(\boldsymbol{g})$ and $A_j(\boldsymbol{g}'') = A_j(\boldsymbol{g})$ for all $j \in N \setminus \{\ell\}$. We have $\pi_\ell(A(\boldsymbol{g}'')) = f_1(|N_i^C(A(\boldsymbol{g}))|+1) + f_2(|N_i^U(A(\boldsymbol{g}))|) - f_3(|A_i(\boldsymbol{g})|) > f_1(|N_i^C(A(\boldsymbol{g}))|) + f_2(|N_i^U(A(\boldsymbol{g}))|) - f_3(|A_i(\boldsymbol{g})|) = \pi_i(A(\boldsymbol{g})) \geq \pi_\ell(A(\boldsymbol{g}))$, hence player ℓ does not play a strict best response, and $\boldsymbol{g}$ is not a strict Nash network.

Suppose that no player ℓ, $\ell \notin N_i(A(\boldsymbol{g}))$ and$\ell \notin N_i(A(\boldsymbol{g}))$ and $|A_\ell(\boldsymbol{g})| > 0$ $|A_\ell(\boldsymbol{g})| > 0$, belongs to a cycle. We consider player $\ell \notin N_i(A(\boldsymbol{g}))$ with $|A_\ell(\boldsymbol{g})| > 0$. If $\pi_i(A(\boldsymbol{g})) \geq \pi_\ell(A(\boldsymbol{g}))$, then by the same reasoning as in the case (ii.b.1), we can check that player ℓ does not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network. If $\pi_i(A(\boldsymbol{g})) \leq \pi_\ell(A(\boldsymbol{g}))$, then by the same reasoning as in the second case of point (i.a) above, we can check that player i does not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network.

2. Second, consider situations where $\boldsymbol{g}$ contains wheels. In such a case, we focus on $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$. More precisely, we use the same arguments as in the previous point to obtain the result.

□

Proof of Proposition 4. Let $\boldsymbol{g}$ be a connected strict Nash network. By Lemma 2 and 4 either (1) $\boldsymbol{g}$ contains a MECS, say X, and all players belong to X, or (2) $\boldsymbol{g}$ does not contain any MECS. We deal successively with these two cases.

1. Suppose $\boldsymbol{g}$ contains a MECS, say X and all players belong to X. We show that $\boldsymbol{g}$, which is a confirmed network, is a minimal confirmed network where all key players are the sponsors of all arcs they are involved in. Suppose $\boldsymbol{g}$ is a confirmed network which is not a minimal confirmed network. Then, there is a costly arc which can be deleted such that the resulting network is still confirmed. This implies that the sponsor of the arc is better off deleting the arc, hence $\boldsymbol{g}$ is not a strict Nash network, a contradiction. Finally, by Lemma 3, we know that in a minimally confirmed strict Nash network, all key players are the sponsors of all arcs they are involved in.

2. The proof given here is inspired by the proof given by Bala and Goyal (Proposition 4.2, pg. 1204, [2]). Suppose $\boldsymbol{g}$ does not contain any MECS. It follows that $\boldsymbol{g}$ contains no cycle. Since $\boldsymbol{g}$ is connected, it is minimally unconfirmed. Moreover, if $\overline{ij} \in A(\boldsymbol{g})$, then there is no player k such that $ki \in A(\boldsymbol{g})$ (or $kj \in A(\boldsymbol{g})$), since k can replace the arc ki by kj (kj by ki) and obtains the same payoff as in $\boldsymbol{g}$. It follows that if $ji \in A(\boldsymbol{g})$, then j has formed arcs with all players in $V_j(A(\boldsymbol{g}))$. Moreover, no player, except j can form arcs with players in $V_j(A(\boldsymbol{g}))$. Consequently, player j forms arcs with all players in $N \setminus \{j\}$ in $\boldsymbol{g}$ since this network is minimally unconfirmed. It follows that $\boldsymbol{g}$ is a center sponsored star.

□

Proof of Proposition 5. Let $\boldsymbol{g}$ be a non-empty and non-connected strict Nash network. We know by Lemma 5 that there is a unique sub-network, $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$, which is either empty, or connected. By using the same arguments as in Proposition 4 we obtain the result: $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$ is empty, or a center sponsored star, or a minimally confirmed network. We now show that $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$ is not a center sponsored star. To introduce a contradiction, suppose a strict Nash network $\boldsymbol{g}$ which contains both a wheel $\boldsymbol{g}_{|_W}$, with $W(\boldsymbol{g}) \subset \mathcal{W}(\boldsymbol{g})$ and a center sponsored star $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$. Let $i \in N \setminus \mathcal{W}(\boldsymbol{g})$ be the player who forms arcs in $\boldsymbol{g}_{|_{N\setminus\mathcal{W}}}$, let $j \in N \setminus (\mathcal{W}(\boldsymbol{g}) \cup \{i\})$ be a player in the center sponsored star and let $\ell \in W$. By construction, $|W(\boldsymbol{g})| \geq 3$. Finally, we set $A(\boldsymbol{g}') = A(\boldsymbol{g}) - ij + i\ell$. We have $\pi_i(A(\boldsymbol{g})) = f_1(0) + f_2(|N \setminus \mathcal{W}(\boldsymbol{g})| - 1) - f_3(|N \setminus \mathcal{W}(\boldsymbol{g})| - 1) < f_1(0) + f_2(|N \setminus \mathcal{W}(\boldsymbol{g})| - 2 + |W(\boldsymbol{g})|) - f_3(|N \setminus \mathcal{W}(\boldsymbol{g})| - 1) = \pi_i(A(\boldsymbol{g}'))$. The inequality comes from the strict increasing property of f_2. It follows that player i does not play a strict best response and $\boldsymbol{g}$ is not a strict Nash network, a contradiction.

□

Appendix C. Heterogenous players: Journalists

Proof of Proposition 9. We prove successively the different parts of the proposition.

1. We first show that $\boldsymbol{g}$ will contain exactly one MECS. Suppose not, then either $\boldsymbol{g}$ contains zero MECS or at least two MECS. If g contains zero MECS, then $\boldsymbol{g}$ would be minimally unconfirmed connected. By standard arguments (see the proof of Proposition 4), $\boldsymbol{g}$ would be a center sponsored star, $\boldsymbol{g}^{CSS}$. Let k be the central player of $\boldsymbol{g}^{CSS}$. Clearly k has to be an ordinary player, and earns payoff $f_2\,(n-1) - f_3\,(n-1) > 0$. However, as $|J| > 0$, there is a journalist among the peripheral players, who could earn $f_1\,(n-1) - f_3\,(n-2)$ by sponsoring an arc to each of the other peripheral players. Since $f_3\,(n-2) < f_3\,(n-1)$, and $f_1\,(n-1) > f_2\,(n-1)$ by (A1) and $f_1(0) = f_2(0)$, we obtain a contradiction.

Now suppose that $\boldsymbol{g}$ contains more than one MECS. W.l.o.g. let X and X' be two distinct MECS of $\boldsymbol{g}$. Note that Lemma's 1-3 also apply in the model with journalists as all the arguments in these lemmas are about switching arcs, rather than about the willingness to add or maintain arcs. By Lemma 1, no player is part of multiple MECS. Moreover, by Lemma 2 if any player i in an MECS X is connected to some player $j \notin X$, then $V_j\,(\boldsymbol{g}) = \{i\}$. These two observations combined imply that no player i in MECS X can be connected to any player $j' \in X'$. By the connectedness of $\boldsymbol{g}$, it cannot contain more than one MECS.

Now we show the remainder of part 1. Let $\boldsymbol{g}$ be a connected strict Nash network. Suppose that there exists a player $j \in N \backslash X$. By Lemma 2, there exists $i \in X$ such that $V_j(\boldsymbol{g}) = \{i\}$, and $ij \in A(\boldsymbol{g})$. Clearly, i would strictly prefer to delete ij if $i \in J$. Thus $i \notin J$. Now we prove that $j \notin J$. Following the proof of Lemma 4, there exist $i_1, i_2 \in X$ such that $ii_1, i_1i_2 \in A(\boldsymbol{g})$. Moreover player i_1 sponsors exactly one arc. For the sake of contradiction, suppose that $j \in J$. Then j would gain $f_1(|X|)$ at the cost of a single arc by sponsoring ji_1, whereas i_1 gains at most $f_1(|X-1|)$ by his single arc. Thus we obtain our contradiction: either i_1 would strictly prefer to delete i_1i_2, or j would prefer to add ji_1 neither of that is consistent with $\boldsymbol{g}$ being a strict Nash network.

2. We first show that if $\boldsymbol{g}$ is non-empty and non-connected, then all arcs are part of a cycle. Consider the following two observations. First in any strict Nash network $\boldsymbol{g}$, a player who sponsors an arc that is not part of a cycle, does so to gain the maximal number of additional connections. Second, any arc which is not part of a cycle will result in exactly one additional unconfirmed connection, otherwise the sponsor would be indifferent regarding the recipient of the arc. The second observation implies that if $ij \in A(\boldsymbol{g})$ and ij does not belong to a cycle then $V_j(\boldsymbol{g}) = \{i\}$. In that case player i (weakly) prefers to replace her arc ij by an arc ij', with $j' \notin N_i(\boldsymbol{g})$. It follows that if a non-connected network $\boldsymbol{g}$ contains an arc which is not part of a cycle, then $\boldsymbol{g}$ must be connected, a contradiction.

Note that, by Lemma 1, this implies that all components are MECS. Lemma 5 tells us that if any players in $\boldsymbol{g}$ are isolated, then all non-empty components must be wheels, as isolated journalists will have no more incentives to sponsor two arcs to some non-empty component than isolated ordinary players. Finally, if no players are isolated in $\boldsymbol{g}$, then no two components (both MECS) can have a player who sponsors at least two arcs. Suppose not. Then there exist two components X and X' and two players $i \in X$ and $i' \in X'$ such that i and i' sponsor at least two arcs each. W.l.o.g. let $|X| \geq |X'|$. Then i' gains confirmed access to at most $|X'| - 1$ players by sponsoring at least two arcs. Therefore i' would strictly improve his payoffs if he would replace all his current arcs in $\boldsymbol{g}$ by arcs to two players in X, as he would get confirmed access to $|X| > |X'| - 1$ players at the cost of only two arcs. Thus, if no players are isolated, then at most one component in $\boldsymbol{g}$ is not a wheel. $\square$

Appendix D. Strict bilaterally rational networks

The proof of the next lemma is an adaptation of the proof given by Bala and Goyal ([2], Proposition 4.2, pg 1204).

Lemma 6 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a non-empty strict bilaterally rational network. Suppose* $\boldsymbol{g}_{|X}$ *is an acyclic connected component of* $\boldsymbol{g}$*. Then,* $\boldsymbol{g}_{|X}$ *is a center sponsored star.*

Proof Let $\boldsymbol{g}$ be a non-empty strict bilaterally rational network and let $\boldsymbol{g}_{|X}$ be an acyclic connected component of $\boldsymbol{g}$. Since $\boldsymbol{g}$ is a strict bilaterally rational network, it is a strict Nash network. Consequently, $\boldsymbol{g}_{|X}$ is minimally unconfirmed. To introduce a contradiction, suppose that $\boldsymbol{g}_{|X}$ is not a center sponsored star and that player i has formed an arc with player j in this network. No player $\ell \in X$ forms an arc with player j or player i in $\boldsymbol{g}$, otherwise player ℓ can replace her arc ℓj (or ℓi) by the arc ℓi (respectively ℓj) and all players obtain the same payoff as in $\boldsymbol{g}$. Consequently, $\boldsymbol{g}$ is not a strict bilaterally rational network, a contradiction. $\square$

Lemma 7 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a non-empty bilaterally rational network. Then,* $\boldsymbol{g}$ *is connected.*

Proof Let $\boldsymbol{g}$ be a non-empty bilaterally rational network. To introduce a contradiction, suppose that $\boldsymbol{g}$ is not connected.

First, suppose that $\boldsymbol{g}$ is acyclic. By Lemma 6, there exist two players, say i and j such that $ij \in A(\boldsymbol{g})$ and $V_j(\boldsymbol{g}) = \{i\}$. Suppose that player ℓ is not connected with player i in $\boldsymbol{g}$. Let $\boldsymbol{g}'$ be the network where player i replaces the arc ij by the arc $i\ell$, that is $A(\boldsymbol{g}') = A(\boldsymbol{g}) + i\ell - ij$. We have: $\pi_i(A(\boldsymbol{g})) \leq \pi_i(A(\boldsymbol{g}'))$ and $\pi_\ell(A(\boldsymbol{g})) < \pi_\ell(A(\boldsymbol{g}'))$, the former inequality comes from the fact that player ℓ can be a non-isolated player. Consequently, $\boldsymbol{g}$ is not a bilaterally rational network, a contradiction.

Second, suppose that $\boldsymbol{g}$ contains a MECS, say $X_1(\boldsymbol{g})$, such that players i and j are members of $X_1(\boldsymbol{g})$ and $ij \in A(\boldsymbol{g})$. Since $\boldsymbol{g}$ is not connected, there exists a player ℓ who does not obtain any resources from player i. We show that player ℓ is not an isolated player. Indeed, if ℓ is an isolated player, then we have either $\pi_\ell(A(\boldsymbol{g})) \geq \pi_i(A(\boldsymbol{g}))$, or $\pi_\ell(A(\boldsymbol{g})) < \pi_i(A(\boldsymbol{g}))$. Suppose $\pi_\ell(A(\boldsymbol{g})) \geq \pi_i(A(\boldsymbol{g}))$. Then, $\boldsymbol{g}$ is not a strict Nash network, and so it is not a bilaterally rational network, since player i should remove all her arcs to improve her payoff. Suppose

$\pi_\ell(A(\boldsymbol{g})) < \pi_i(A(\boldsymbol{g}))$. Let $\boldsymbol{g}'$ be the network such that $A(\boldsymbol{g}') = A(\boldsymbol{g}) + i\ell + \ell j - ij$. In $\boldsymbol{g}'$ player i obtains a higher payoff than in $\boldsymbol{g}$ since she obtains an additional confirmed resource. Moreover, we have $\pi_\ell(A(\boldsymbol{g}')) \geq \pi_i(A(\boldsymbol{g}')) > \pi_i(A(\boldsymbol{g})) > \pi_\ell(A(\boldsymbol{g}))$, and $\boldsymbol{g}$ is not a bilaterally rational network.

Since ℓ is not an isolated player, (i) either player ℓ belongs to a MECS, (ii) or she belongs to an acyclic component. (i) Suppose player ℓ belongs to a MECS, say $X_2(\boldsymbol{g})$. Then there exist two players in $X_2(\boldsymbol{g})$, say i' and j' such that $i'j' \in A(\boldsymbol{g})$. We consider the network $\boldsymbol{g}'$ such that $A(\boldsymbol{g}') = A(\boldsymbol{g}) + ij' + i'j - ij - i'j'$. In $\boldsymbol{g}'$, players i and i' obtain confirmed resources from all players who belong to $X_1(\boldsymbol{g}) \cup X_2(\boldsymbol{g})$ and incur the same costs as in $\boldsymbol{g}$. Consequently, we have: $\pi_i(A(\boldsymbol{g}')) - \pi_i(A(\boldsymbol{g})) \geq f_1(|X_1(\boldsymbol{g})| + |X_2(\boldsymbol{g})|) - f_1(|X_1(\boldsymbol{g})|) > 0$ and $\pi_{i'}(A(\boldsymbol{g}')) - \pi_{i'}(A(\boldsymbol{g})) \geq f_1(|X_1(\boldsymbol{g})| + |X_2(\boldsymbol{g})|) - f_1(|X_2(\boldsymbol{g})|) > 0$. The inequalities come from the fact that f_1 is increasing. Consequently, $\boldsymbol{g}$ is not a bilaterally rational network, a contradiction. (ii) Suppose player ℓ belongs to an acyclic component. Then by Lemma 6, player ℓ belongs to a center sponsored star. We consider the player who sponsors the arcs in this center sponsored star, say ℓ_0. This player has formed an arc in $\boldsymbol{g}$ with a player, say ℓ_1, such that $V_{\ell_1} = \{\ell_0\}$. Since f_2 is an increasing function and a MECS contains at least three players, we have: $\pi_\ell(A(\boldsymbol{g}) + \ell_0 i - \ell_0\ell_1) > \pi_\ell(A(\boldsymbol{g}))$. Consequently, $\boldsymbol{g}$ is not a bilaterally rational network, a contradiction. □

Proof of Proposition 10. Let $\boldsymbol{g}$ be a non-empty bilaterally rational network. By Lemma 7, $\boldsymbol{g}$ is connected. Since $\boldsymbol{g}$ is a bilaterally rational network, it is a strict Nash network. Consequently, by Proposition 4 only minimally confirmed networks, where all key players are the sponsors of all arcs they are involved in, and center sponsored stars are candidates to be bilaterally rational networks. □

Since a strict bilaterally rational network is a strict Nash network, Lemmas 4 and 5 are satisfied for strict bilaterally rational networks. Hence we have Lemmas 8 and 9.

Lemma 8 *Suppose that the payoff function is given by equation 1 and $\boldsymbol{g}$ is a strict bilaterally rational network. Let X be a MECS of $\boldsymbol{g}$. If $i \in X$ and i is a key player, then i is the sponsor of all arcs she is involved in.*

Lemma 9 *Suppose that the payoff function is given by equation 1 and $\boldsymbol{g}$ is a non-empty strict bilaterally rational network. No player i in a MECS X forms an arc with a player $j \notin X$.*

Lemma 10 *Suppose that the payoff function is given by equation 1 and* $\boldsymbol{g}$ *is a minimally confirmed strict bilaterally rational network which is not a minimal cycle. Then* $\boldsymbol{g}$ *is a complete bipartite network which contains two key players who are linked with at least three non key players.*

Proof Let $\boldsymbol{g}$ be a minimally confirmed strict bilaterally rational network which is not a minimal cycle.

1. We show that there are at least two key players in $\boldsymbol{g}$. Since $\boldsymbol{g}$ is not a minimal cycle, there exists a player, say i_0, who is involved in 3 arcs in $\boldsymbol{g}$. By Lemma 8, i_0 sponsors all these arcs. We now show that there exists another player, say j_0, who is involved in 3 arcs in $\boldsymbol{g}$. Let $\{i_1, i_2, i_3\} \subset V_{i_0}$. Since $\boldsymbol{g}$ is a minimally confirmed network, there exist two disjoint u-paths between i_1 and i_2: $P^1_{i_1 i_2}(\boldsymbol{g}) = i_1, i_0, i_2$ with $i_0 i_1, i_0 i_2 \in A(\boldsymbol{g})$ and $P^2_{i_1 i_2}(\boldsymbol{g})$. Likewise there exist two disjoint u-paths between i_1 and i_3: $P^1_{i_1 i_3}(\boldsymbol{g}) = i_1, i_0, i_3$ with $i_0 i_1, i_0 i_3 \in A(\boldsymbol{g})$ and $P^2_{i_1 i_3}(\boldsymbol{g})$. There always exists a player, say $j_0 \in N \setminus \{i_1\}$, who belongs simultaneously to $P^2_{i_1 i_2}(\boldsymbol{g})$ and $P^2_{i_1 i_3}(\boldsymbol{g})$ otherwise player i_0 obtains confirmed resources from i_1 through i_2 and i_3 and $\boldsymbol{g}$ is not a minimally confirmed network (the arc $i_0 i_1$ is not needed). Likewise, it is worth noting that $j_0 \notin \{i_2, i_3\}$ otherwise $\boldsymbol{g}$ is not a minimally confirmed network. It follows that j_0 is involved in 3 arcs: an arc which linked two players who belong to a u-path between i_1 and j_0, an arc which linked two players who belong to a u-path between i_2 and j_0, and an arc which linked two players who belong to a u-path between i_3 and j_0. By Lemma 8, player j_0 sponsors all her arcs in $\boldsymbol{g}$.

2. We show that key players sponsor arcs with the same players. To introduce a contradiction, suppose that key players i_0 and j_0 do not sponsor arcs with the same players. Since $\boldsymbol{g}$ is a minimally confirmed network, there exists an u-path between i_0 and j_0, say $P_{i_0, j_0}(\boldsymbol{g}) = i_0, j_1, \ldots, j_{m-1},\ j_m, j_0$ where $j_1 \neq j_m$. Suppose players i_0 and j_0 modify their strategies: player i_0 replaces the arc $i_0 j_1$ by $i_0 j_m$ and player j_0 replaces the arc $j_0 j_m$ by $j_0 j_1$. Then, we obtain the network $\boldsymbol{g}'$ such that $A(\boldsymbol{g}') = A(\boldsymbol{g}) + i_0 j_m - i_0 j_1 + j_0 j_1 - j_0 j_m$. Players i_0 and j_0 obtain the same payoff in $\boldsymbol{g}$ and in $\boldsymbol{g}'$: they obtain the same resources and incur the same costs in $\boldsymbol{g}'$ and in $\boldsymbol{g}$. Consequently, $\boldsymbol{g}$ is not a strict bilaterally rational network, a contradiction.

3. We show that each key player sponsors links with all non key players. Let i and k be key players and let j be a non key player. Players i and k are both confirmed connected to non-key player j, so i and k both have two disjoint u-paths to player j. Consequently, there is a u-path

between i and k on which j lies. By Lemma 3, i and k sponsor the first link on the u-paths connecting them to j. By Part 2, this means that the only other player on this path is player j.

4. We show that there are no links between non key players in $\boldsymbol{g}$. Indeed, since there are at least two key players in $\boldsymbol{g}$, and each of them are linked to all non key players, the latter receives at least 2 links and cannot be involved in more links.

5. We show that there are 2 key players and at least 3 non key players in $\boldsymbol{g}$. First, since key players have formed at least 3 links, and cannot be linked, it follows that there are at least 3 non key players in $\boldsymbol{g}$. Second, we know from point 3. that each non key player receives a link from each key player. Since a non key player is involved in at most 2 links, there are at most 2 key players. By point 1 it follows that there are 2 key players. □

Proof of Proposition 11. Let $\boldsymbol{g}$ be a non-empty strict bilaterally rational network. By Lemma 7, $\boldsymbol{g}$ is connected. Suppose $\boldsymbol{g}$ is acyclic. Then, by Lemma 6, it is a center sponsored star. Suppose that $\boldsymbol{g}$ contains a cycle. Then, Lemma 10 establishes that if $\boldsymbol{g}$ is not a minimal cycle, then it is a complete bipartite network containing two key players who are linked with at least three non key players. □

In the following, we call the framework, where each player i confirm resources from player j when she is directly linked with j in addition to the the case where there exist at least two u-path between i and j, the direct link confirmation framework (DLCF). We show that the results of our original framework and the DLCF are identical regarding strict Nash networks.

Appendix E. The Direct Link Confirmation Model: Sketch of proofs

The main difference between the framework adopted in the paper and the DLCF model is that we now have to replace the notion of MECS with cycles. Consequently to establish that the results are the same in the two models, we first deal with results concerning cycles. Thus, in all the following proofs we have to replace MECS by cycles.

Lemma 1'.

In DLCF, Lemma 1 should state that it is not possible that there exists a player i who belongs simultaneously to two distinct maximal (relative to the vertices) cycles. By using similar arguments as those given in Lemma 1, we obtain the required result.

Lemma 2'.

Lemma 2 can be adapted in the DLCF:

(a) no player j who belongs to a cycle will sponsor an arc with a player i in the cycle who has sponsored an arc with a player i' who does not belong to the cycle. Indeed, player j would have an incentive to replace her arc with i by an arc with i';

(b) no player i' who does not belong to a cycle sponsors an arc with player i who belongs to a cycle. Indeed, the argument given in Lemma 2 is preserved: The payoff of i' does not change if she replaces her current arc with i in the cycle by an arc with another player of the cycle say j;

(c) player i', who does not belong to a cycle and is directly linked with a player i who belongs to the cycle, has no other direct neighbor j', otherwise player i could replace her arc with i' by an arc with j'.

Lemma 3'.

We have to show that in the DLCF, a key player sponsors all the arcs she is involved in.

By Lemma 2, we know that player i who belongs to a cycle sponsors arcs with players who do not belong to the cycle. Arguments which show that a key player sponsors arc with players who belong to the cycle in the DLCF are the same as in our original framework: If a player j has formed a link with a key player, then she can replace this link by a link with a neighbor of the key player without reducing her payoff.

Lemma 4'.

Assume that players i, j, i_1 , i_2 are in the same position as in the original Lemma 4. We have to show that a player in a cycle does not sponsor an arc with a player who does not belong to this cycle. Let K be the number of unconfirmed resources obtained by player i_1. Recall that i_1 cannot be a key player in Lemma 4. Hence she is involved only in the two links with i and

i_2. This is true in the DLCF.

In the proof of Lemma 4, equation 10 becomes:

$$\begin{aligned}\pi_j(A(g)) &= f_1(1) + f_2(|X| + K - 2) - f_3(0)\\ &> f_1(|X|) + f_2(K-1) - f_3(1)\\ &= \pi_j(A(g) + ji_1).\end{aligned}$$

Let us examine the two possibilities given in the proof of Lemma 4.

- Assume $i_1i_2 \in A(g)$. Equation 11 becomes:

$$\begin{aligned}\pi_{i_1}(A(g)) &= f_1(|X| - 1) + f_2(K) - f_3(1)\\ &> f_1(1) + f_2(|X| + K - 2) - f_3(0)\\ &= \pi_{i_1}(A(g) - i_1i_2).\end{aligned}$$

We obtain a contraction as in the original framework.

- Assume $i_2i_1 \in A(g)$. Equation 12 becomes:

$$\begin{aligned}\pi_{i_2}(A(g)) &= f_1(|V_{i_2}(g) \setminus X| + |X| - 1) + f_2(K - |V_{i_2}(g) \setminus X|)\\ &\quad - f_3(|A_{i_2}(g)|)\\ &= \pi_{i_2}(A(g) - i_2i_1 + i_2j).\end{aligned}$$

We obtain a contraction as in the original framework.

Lemma 5'.

Lemma 5 is not altered by the assumptions of DLCF.

To sum up, the assumption that the confirmation can happen when two players are directly linked does not change the fact that non-acyclic components in a strict Nash network are cycles. The properties of these cycles are not altered by the fact that the confirmation can happen when two players are directly linked.