A Policy-Based Rationalization of Collective Rules: Dimensionality, Specialized Houses, and Decentralized Authority

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Abstract

We offer a policy basis for interpreting, justifying, and designing $(3,3)$-political rules, a large class of collective rules analogous to those governing the selection of papers in peer-reviewed journals, where each referee chooses to accept, reject, or invite a resubmission of a paper, and an editor aggregates his own and referees' opinions into one of these three recommendations. We prove that any such rule is a weighted multicameral rule: a policy is collectively approved at a given level if and only if it is approved by a minimal number of chambers- the *dimension* of the rule-, where each chamber evaluates a different aspect of the policy using a weighted rule, with each evaluator's weight or authority possibly varying across chambers depending on his area(s) of expertise. Conversely, it is always possible to design a rule under which a policy is collectively approved at a given level if and only if it meets a certain number of predefined criteria, so that one can set the standards for policies first, and then design the rules that justify the passage of policies meeting those standards. These results imply that a given rule is only suitable for evaluating finite-dimensional policies whose dimension corresponds to that of the rule, and they provide a rationale for using different rules to pass different policies even within the same organization. We further introduce the concept of compatibility with a rule, and use it to propose a method to construct integer weights corresponding to evaluators' possible judgments under a given rule, which are more intuitive and easier to interpret for policymakers. Our findings shed light on multicameralism in political institutions and multi-criteria group decision-making in the firm. We provide applications to peer review politics, rating systems, and real-world organizations.

Keywords: $(3,3)$-political rules, multicameralism, multi-criteria group decision-making, decentralized authority, rule suitability and design.

JEL Codes: D71, D72, H40, K10

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1 Introduction

Collective organizational rules are central to the governance of countries, collectivities, clubs, and corporations. They define the procedure by which collective decisions are made, and contain mechanisms for how the preferences of different individuals are aggregated to yield a collective outcome. The simplest collective rules are *weighted rules*, where each individual has a certain number of votes (or seats), and a policy proposal is adopted if and only if the total number of votes it receives surpasses a certain threshold or quota. In the real world, however, many organizations use more complicated rules to make decisions. An important class of such rules is the class of $(3, 3)$-political rules, which are rules analogous to those governing the selection of papers in peer-reviewed journals, where each referee chooses to accept, reject, or invite a resubmission of a paper, and an editor aggregates his own and referees' opinions into one of these three recommendations. In this paper, we show that any such rule, no matter its complexity, can be written as a collection of perfectly *complementary* weighted rules, which has important policy implications for the interpretation, design and suitability of collective decision-making mechanisms.

The class of $(3, 3)$-political rules generalizes well-known classes of rules, including $(2, 2)$-political rules under which each voter either supports or opposes a policy proposal, the collective outcome being either the adoption of the proposal or its failure. These latter rules, which were introduced by von Neumann and Morgenstern (1944), have served as the cornerstone of the analysis of group decision-making in a broad range of important studies in game theory (e.g., Shapley (1953), Arrow (1963), Peleg (1978, 1984), Taylor and Zwicker (1999), Ray (2007), Laruelle and Valenciano (2008)), political economy (e.g., Barberà and Jackson (2006), Brams (1975), Acemoglu et al. (2011)), and corporate governance (e.g., Leech (1988, 2003)). But despite their influence, scholars have argued that $(2, 2)$-political rules are restrictive, as more than two levels of individual and collective approval are generally observed in real-life decisions (Fishburn (1973), Felsenthal and Machover (1997, 1998), Freixas and Zwicker (2003, 2009), Pongou et al. (2011), Tchantcho et al. (2008), Rubinstein (1980), Hsiao and Raghavan (1993)). For instance, in most real-world elections, individuals might vote *for* or *against* any of the candidates, or might *abstain*, a situation forcelluly brought to scholarly awareness by Felsenthal and Machover (1997, 1998). This compelling argument partly motivates our focus on more realistic collective decision-making mechanisms such as $(3, 3)$-political rules.

In this paper, we offer a policy basis for interpreting and designing collective rules. We state our main findings and discuss their policy implications.

1. First, we show that any $(3, 3)$-political rule can be decomposed into a hierarchical system of two weighted multicameral legislatures: the first legislature determines whether a proposal should be collectively approved at the highest level, such as accepting a paper, whereas the second legislature determines whether it should be collectively approved at the intermediate level, such as inviting a resubmission of a paper after revision. A proposal that fails in each of the two legislatures is simply collectively rejected. Each legislature consists of several specialized houses or chambers, and a policy proposal is adopted if and only if it is approved by each house, with each house using a weighted rule in that each decision-maker is assigned a (vector) weight that measures his political influence in that house, and a proposal passes the vote in that house if and only if the sum of points representing voters' opinions exceeds a predetermined quota. If we denote by d_1 the minimum

number of houses of the first legislature, and by d_2 the minimum number of houses of the second legislature, we say that the corresponding political rule is of *2-dimension* (d_1, d_2).

A practical implication of such a weighted multicameral representation of a $(3,3)$-political rule is that such a rule can be interpreted as a multi-criteria decision-making rule that explicitly asks decision-makers to evaluate different aspects of a proposal to be voted on. Each house evaluates one aspect, with each decision-maker's influence or authority possibly varying across houses depending on his area(s) of expertise. Importantly, since different rules generally have different dimensions, it follows that each rule is best suited for evaluating finite-dimensional policies whose dimension (number of aspects to be evaluated) corresponds to that of the rule, and that not all policies can be evaluated using the same rule. For instance, a one-dimensional policy such as tax rate should be evaluated using a different rule than policies of two or more dimensions.

2. Second, we show that any $(3,3)$-political rule can be written as the minimum of a finite number of $(3,3)$ "quasi-weighted" political rules (in a sense to be defined). This allows us to introduce the concept of quasi-dimension of a political rule, which generalizes the traditional concept of political dimension introduced by Taylor and Zwicker (1993).

3. Third, for any given pair of naturals (d_1, d_2), one can always design a $(3,3)$-political rule of 2-dimension (d_1, d_2). This finding has important implications for the design of constitutions that value the inputs of different experts for the passage of a finite-dimensional policy. For instance, if one wants to construct a rule under which a proposal is collectively approved at a given level if and only if it satisfies a finite set of criteria, this is always possible. In fact, the number of criteria determines d_1 and d_2. One can therefore set the standards for policies first, and then design the rules that would justify the passage of policies meeting those standards or criteria. For instance, one can always design a rule to select the winner of the election of Miss Universe where each contestant is judged regarding predetermined criteria such as beauty, self-confidence and ability to communicate.

   This finding theoretically provides a rationale for using different rules to pass different policies even within the same organization.

4. Fourth, based on a newly defined concept of *compatibility* with a rule, we show that there exist infinitely many weighted multicameral rules that are compatible with a $(3,3)$-political rule, and prove that these rules constitute a topologically open set. Further applying topological concepts, we propose a method to construct integer weights to record the possible judgments of an expert, which are more intuitive and much easier to interpret for policymakers who vote on a regular basis.

To illustrate our main findings, let us consider this "thought" process of selecting a paper for publication in a journal. Suppose that a paper has two parts, one theoretical and the other empirical. The editor (E) invites four scholars, two theorists ($T1$ and $T2$) and two empiricists ($E1$ and $E2$), to evaluate each aspect of the paper. We imagine a collective decision-making rule defined as follows:

- The paper is accepted if the following two situations both occur:

  - At least one of the referees with expertise in theory finds that the paper makes a theoretical contribution, and similarly, at least one of the empirical referees judges the paper to make an empirical contribution.

  - None of the four referees rejects the paper.

- The paper is rejected if it is judged by the two theoretical referees to make no theoretical contribution or it is judged by the two empirical referees to make no empirical contribution.

- The paper is invited to be resubmitted after revision in all other cases.

If we let $\mathcal{C}_1 = \{T1, T2\}$ and $\mathcal{C}_2 = \{E1, E2\}$, we can summarize this $(3,3)$-political rule by the following characteristic function $\mathcal{V}$:

$$\mathcal{V}(X_1, X_2, X_3) = \begin{cases} 2 \ \text{(paper is accepted) if } |X_1 \cap \mathcal{C}_i| = 2 \text{ or} \\ \qquad |X_1 \cap \mathcal{C}_i| \text{ and } |X_2 \cap \mathcal{C}_i| = 1 \text{ for any } i = 1,2 \\ 0 \ \text{(paper is rejected) if } |X_3 \cap \mathcal{C}_1| \text{ or } |X_3 \cap \mathcal{C}_2| = 2 \\ 1 \ \text{(resubmission is invited) otherwise} \end{cases}$$

where (X_1, X_2, X_3) is a vote profile in which X_1, X_2 and X_3 are respectively the sets of evaluators accepting the paper, inviting a resubmission, and rejecting the paper, and $|X_1|$, for instance, is the cardinality of X_1.

We note that in the example above, the editor could be one of the referees (e.g., $E = T1$ or $E = E1$), and if he has expertise in both aspects of the paper, he could play the role of two referees, which would allow him to express possibly different views on the theoretical and empirical contributions of the paper.[2]

As we can see, the formalization of this rule by its characteristic function $\mathcal{V}$ is economical, but a bit complex. However, we show that it can simply be represented as a pair of weighted two-house legislatures as in the following table:

	Legislature 1		Legislature 2	
	Theory	Empirics	Theory	Empirics
Referee T1	$(2,1,0)$	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$
Referee T2	$(2,1,0)$	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$
Referee E1	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$	$(2,1,0)$
Referee E2	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$	$(2,1,0)$
Quota	3	3	1	1

The first legislature (Legislature 1) determines whether the paper will be accepted. In this legislature, the theoretical contribution of the paper is evaluated in the first house (called Theory), and its empirical contribution in the second house (called Empirics). In each house, each voter has a three-component vector weight corresponding to the three possible judgments of the paper (accept, resubmit paper, reject). In the first house, each theoretical referee is assigned a vector weight of (2,1,0), which means that the paper receives 2 points if accepted by a theoretical referee, 1 point if invited to be revised and resubmitted, and 0 points otherwise. Each empirical referee is assigned a weight vector of (0,0,0), which means that their opinion on the theoretical aspect of the paper does not count. The paper passes the theoretical test if

[2] Note that under a different rule, the editor might have the right to overrule the recommandation of the referees. This situation correspond to the dictatorial political rule. In general, it is always possible to model a rule under which some referees have more power than others.

the sum of points received by the paper after each decision-maker evaluates its theoretical contribution is at least equal to 3. In the second house, each theoretical referee is assigned a weight vector of (0,0,0), and each empirical referee a weight vector of (2,1,0), and the paper passes the empirical test if it receives at least 3 points. So only the empirical referees wield power in the second house. The second legislature (Legislature 2) determines whether the paper will be invited to be revised and resubmitted. In this legislature, each evaluator has the same weight as in the first legislature; however, the paper can be resubmitted if and only if it receives at least 1 point in each of the two houses. The 2-dimension of this peer review rule is $(2, 2)$.

Like the characteristic function $\mathcal{V}$, most collective rules or constitutions are unfortunately simply defined as a distribution of voting power among the different subgroups of voters. In general, there is no rationale for the nature of policies that can be evaluated under these rules, which are also silent on the precise role of each voter in making decisions. A decomposition such as the one performed for $\mathcal{V}$, however, addresses those shortcomings, as it shows that $\mathcal{V}$ is best suited for the evaluation of two-dimensional policy proposals, where each dimension is evaluated using a weighted rule. It also clearly describes the weight given to the opinion of each voter along each dimension. Such an approach further suggests a transparent and policy-driven method to design rules. Indeed, instead of defining a rule by its characteristic function as it is usually done, one can simply fix the number of criteria to be met by a policy to be collectively adopted under such a rule, and define a weighted rule for each criterion, which is a simple exercise. Such a method would clearly value the opinion of experts, as the weight given to the judgment of a voter can be made to vary across the different dimensions.

Given that the class of $(3, 3)$-political rules generalizes well-known classes of rules, including $(2, 2)$-political rules, $(3, 2)$-political rules, and (2,3)-political rules, our findings extend to these latter classes of collective mechanisms as well. Within our framework, $(2, 2)$-political rules and $(3, 2)$-political rules have 2-dimension (or simply dimension) $(m, 0)$, where m is a natural number at least equal to 1. A well-known example of such a rule is the procedure to revise the Canadian Constitution (Kilgour (1983), Taylor and Zwicker (1999)). Under this procedure, a proposal to amend the Canadian Constitution becomes law only if it is approved by at least seven of the ten Canadian provinces, subject to the proviso that the approving provinces constitute at least half of Canada's population. Thus, this voting rule, which can be shown to be of dimension (2,0), assigns two weights to each province, with the first representing the vote cast on the proposal by the province, and the second representing its population share. The case where $m = 1$ represents the well-known class of weighted voting rules. These rules are the most appealing way of modelling voting power inequality in an assembly where each member retains a certain number of votes. Our analysis implies that such rules are most appropriate for evaluating one-dimensional policies.

In addition to generalizing well-known classes of political rules, our finding that $(3, 3)$-political rules can be viewed as systems of weighted multicameral legislatures provides a rationale for multicameralism in political institutions (Rowley and Schneider (2004), Buchanan and Tullock (1962), Bräuninger (2003)), and multi-criteria decision-making in market organizations. Multicameralism better captures the complexity of the notion of representation by its ability to aggregate diverse interests. The greater the number of houses, the greater the probability of the legislature to provide multiple perspectives on an issue, as our findings imply. Under multicameralism, representatives sharing a common interest are usually grouped in the same committee. Some interesting examples are: the Westminster system in

Britain with two houses (the house of Lords and the house of the commons), the South Africa's apartheid government in 1983 with three race-based houses (White, Colored, Asians), the Medieval Scandinavian deliberative assemblies with four houses (the nobility, the clergy, the burghers, and the peasants), and the Council of the International Seabed Authority (ISA) with four houses, each representing the interest of a specific group of agents (consumers, investors, producers of minerals, developing countries and some other countries).[3] Similarly, a firm's decision is sometimes determined by the different opinions expressed by its different departments as discussed in Baucells and Sarin (2003). In this case, a department (e.g., the marketing or the R&D department) plays the role of a specialized house.

Our paper also generalizes classical results on multicameral representation of $(2,2)$-political rules (Taylor and Zwicker (1992, 1993, 1999)). Taylor and Zwicker were the first to show that these latter rules have a weighted multicameral representation. They also introduced the algebraic notion of dimension, and argued that it is a "measure of the complexity" of a voting mechanism. They further showed that there exists a $(2,2)$-political rule of any given dimension. The concept of dimension has served as the cornerstone of many subsequent studies extending Taylor and Zwicker's results (see, e.g., Freixas and Puente (2008), Deineko and Woeginger (2006)). Laruelle and Valenciano (2011) extend the concept of dimension to voting rules allowing four inputs not necessarily ranked, and two possible collective outputs.

It follows from these studies that the focus has mainly been on rules that yield only two possible collective outcomes such as passing or failing a bill. However, many real-world group decisions are not binary. Our thought example of the process of selecting papers in a journal is a point in case. But there are other examples. A winner of a beauty contest may be the queen, the first runner-up, or the second runner-up. Hierarchy in the victory is also encountered in many sport competitions where the winners can receive the gold, silver, or bronze medal based upon judges' allotment of scores. Similarly, in a legislative assembly, a bill may pass or fail, or the decision on its passage may be adjourned, which might be a desired outcome for certain legislators who still have to make their mind. Freixas and Zwicker (2003) have proposed a model of games with multiple inputs and outputs that capture such situations.

In addition to generalizing the literature by focusing on rules with more than two inputs and outputs, our analysis yields new theoretical findings that have important policy implications for the interpretation and design of political rules. Following basic definitions in Section 2, we show that a $(3,3)$-political rule is a hierarchical system of two legislatures in Section 3. Building on this result, in Section 4, we introduce two new theories of dimension, namely the concepts of 2-dimension and quasi-dimension, which fully capture the complexity of $(3,3)$-political rules. These theories imply that most constitutions are implicit multi-criteria decision-making rules under which each voter can be seen as an expert who has the ability to judge of the pertinence of a proposal only in his area(s) of expertise. We also show that one can always design a decision-making rule that values the views of specialized experts, and that is only suited for the evaluation of policies of a given dimension. In section 5, we introduce the concept of compatibility with a rule, and show that every $(3,3)$-political rule has a weighted multicameral representation with integer weights and quotas. Our proofs are new. Section 6 provides some applications to real-world organizations, and Section 7 concludes.

[3] Note, however, that the fact that like-minded representatives are often grouped in the same chamber under multicameralism might lead to strategic voting, as shown by Dewan and Spriling (2011) with respect to the Westminster system.

2 Preliminary Definitions

$N = \{1, 2, ..., n\}$ is a non-empty set of evaluators, players, or voters. Subsets of N are coalitions. For any finite set S, we denote by $|S|$ the number of elements of S.

Voters are invited to vote on a proposal a. Each voter may vote *for* or *against* a, or may express an intermediate level of approval such as *abstaining*.[4] We denote by X_1 the set of voters who vote *for*, X_2 the set of voters who express the intermediate of approval, and X_3 the set of voters who vote *against*. We call $X = (X_1, X_2, X_3)$ a tripartition of N or a vote profile, and we denote by N^3 the set of all vote profiles of N.

2.1 $(3,3)$-Political Rules

A $(3,3)$-*political rule* is a pair $(N, \mathcal{V})$ where $\mathcal{V}$ is a function that maps each vote profile of N into one of three possible collective outcomes in the set $\{0, 1, 2\}$. 2 means that the proposal voted on is collectively accepted, 1 means that it is adjourned, and 0 means that it is rejected. When no confusion is possible, $\mathcal{V}$ will be used for $(N, \mathcal{V})$.

$(3,3)$-political rules might have different applications and interpretations depending on the context. In a legislature, a legislator might support or oppose a policy proposal or abstain, and the latter might pass, fail, or be adjourned. In the academic peer review context where each evaluator recommends either acceptance, rejection, or resubmission of a paper (the intermediate level of approval), the final collective outcome is usually one of these three recommendations. In the context of the comprehensive examination of Ph.D. candidates in the United States, a student can pass the exam both at the Master and Ph.D. levels, fail the exam both at the Master and Ph.D. levels, or pass the exam only at the Master level.

Two special classes of $(3,3)$-political rules are the classes of $(3,2)$-political rules and $(2,2)$-political rules. Under a $(3,2)$-political rule, voters have three options, but there are only two possible collective outcomes, such as adopting or failing a proposal. Under a $(2,2)$-political rule, voters have only two options, such as supporting or opposing a measure, and the collective outcome is either the adoption or the failure of the measure.

This paper uses the standard definition of a $(3,2)$-political rule as a rule that maps each vote profile of N into one of two possible collective outcomes in the set $\{0, 1\}$. In this case, 0 means the proposal has failed, and 1 means the proposal is accepted.

2.2 Monotonicity

One of the most intuitive properties of political rules is the monotonicity property. A $(3,3)$-political rule is monotonic if any increase in an individual's level of support for a proposal, ceteris paribus, cannot decrease the collectivity's suffrage for that proposal. To be more specific, let us consider two vote profiles X and Y on a proposal a. We say that X is a sub-tripartition of Y or Y is a super-tripartition of X (and we write $X \subseteq^3 Y$) to convey that each voter's level of support for a is weakly greater in Y than X. Formaly, X is a sub-tripartition of Y if either $X = Y$ or X can be transformed into Y by moving one or more voters to a higher level of approval, everything else being equal. On the other hand, if X is a sub-tripartition of Y, distinct to Y, we write $X \subset^3 Y$.

[4] The intermediate level of approval has several interpretations depending on the context, as we will see in Section 2.1.

We define $X_j \uparrow$ as the set of the voters who choose a level of approval higher than j , which means $X_1 \uparrow = X_1$, $X_2 \uparrow = X_1 \cup X_2$, and $X_3 \uparrow = X_1 \cup X_2 \cup X_3$.

We say that a $(3,3)$-political rule $\mathcal{V}$ is *monotonic* if for any vote profiles X, Y such that X is a sub-partition of Y, $\mathcal{V}(X) \leq \mathcal{V}(Y)$. This means that if X can make a proposal approved at a given level, so does Y. This is because voters have weakly higher level of support for the proposal in Y than in X.

2.3 Minimal Winning and Maximal Losing Vote Profiles

As stated earlier, a $(3,2)$-political rule $G = (N, \mathcal{V})$ is a voting rule with only two possible outcomes (such as accepting or rejecting a proposal). Vote profiles that lead to the approval of the proposal are called *winning vote profiles* or *winning tripartitions* ($\mathcal{V}(X) = 1$), and those that lead to the rejection of the proposal are *losing vote profiles* or *losing tripartitions* ($\mathcal{V}(X) = 0$). A vote profile X is said to be *minimal winning* if every sub-tripartition of X is losing, and X is said to be *maximal losing* if every super-tripartition of X is winning.

2.4 Monotonic Weighted Multicameral $(3,3)$-Political Rules

In the following, we provide a definition of a weighted multicameral $(3,3)$-political rule. Intuitively, a multicameral voting rule is a rule under which the vote takes place simultaneously in a set of houses or chambers, here denoted by $\mathcal{C}$. As more restrictive conditions should be fulfilled to shift a proposal to a higher level of approval, we split $\mathcal{C}$ into two disjoint sets of houses $\mathcal{C}_1$ and $\mathcal{C}_2$, where each set of houses constitutes a legislature. $\mathcal{C}_2$ represents the set of all the houses that must approve the proposal for it to earn the second level of collective approval. However, to earn the highest level of approval, the proposal should be approved in the additional number of houses that constitute $\mathcal{C}_1$. Let d be the total number of houses ($d = |\mathcal{C}|$). We use the letters i to index the d houses, and j to index the levels of approval of a voter ($1 \leq i \leq d$ and $1 \leq j \leq 3$). Each house i is characterized by a quota q_i; each legislator p in house i has a three-component vector weight $w_i(p) = (w_{i1}(p), w_{i2}(p), w_{i3}(p))$, meaning that the number of additional points credited to a proposal in that house is $w_{i1}(p)$ when p votes for it, $w_{i2}(p)$ when p abstains, and $w_{i3}(p)$ when p votes against it[5]; a proposal is approved in house i if the sum of points credited to it after everybody has voted is at least equal to the quota q_i. Our formal definition is below.

Definition 1 *Let $G = (N, \mathcal{V})$ be $(3,3)$-political rule. A weighted multicameral representation of G is a list $(\mathcal{C}_1, \mathcal{C}_2, w, q)$ that satisfies the three following conditions:*

(1) A vote profile X leads to the highest level of collective approval of a proposal if and only if the proposal is approved in all the voting chambers ($\mathcal{V}(X) = 2$ if and only if $w_i(X) \geq q_i$ for every house i in $\mathcal{C}_1 \cup \mathcal{C}_2$, where $w_i(X) = \sum \{\sum \{w_{ij}(p) | p \in X_j\} | 1 \leq j \leq 3\}$ is the sum of points credited to the proposal in house i after each voter has chosen a level of support, resulting in X).

(2) A vote profile X leads to the intermediate level of collective approval of a proposal if and only if the proposal is approved in all the chambers of $\mathcal{C}_2$, but is rejected in at least one of the chambers of $\mathcal{C}_1$ ($\mathcal{V}(X) = 1$ if and only if there is a house i in $\mathcal{C}_1$ such that $w_i(X) < q_i$ and for every house i in $\mathcal{C}_2$, $w_i(X) \geq q_i$).

[5] For instance, in the example in the introduction, the vector weights for theoretical referees and empirical referees in the Theory house are respectively: $w_{Theory}(T1) = w_{Theory}(T2) = (2,1,0)$, and $w_{Theory}(E1) = w_{Theory}(E2) = (2,1,0)$. In Legislature 1, the quota in the Theory house is $q_{Theory} = 3$.

(3) A vote profile X leads to the collective rejection of a proposal if and only if the proposal is rejected in at least one of the chambers of $\mathcal{C}_2$ ($\mathcal{V}(X) = 0$ if and only if there is a house i in $\mathcal{C}_2$ such that $w_i(X) < q_i$).

The table describing the peer review system in the introduction is a simple illustrative example of the definition of a weighted multicameral $(3,3)$-political rule. In that example, while the sets of houses in the two legislatures are the same ($\mathcal{C}_1 = \mathcal{C}_2 = \{Theory, Empirics\}$), this is not always the case in general. Similarly, while the vector weight of a voter does not vary across the two legislatures in that example, a voter's vector weight does vary across legislatures in general.

A political rule is said to be a *weighted multicameral political rule* if it has a weighted multicameral representation. Note that $\mathcal{C}_1 = \emptyset$ corresponds to the particular case of weighted multicameral $(3,2)$-political rules. If additionnally, $\mathcal{C}_2$ has only one house, the corresponding $(3,2)$-political rule is said to be *weighted*. We will say that a $(3,3)$-political rule G is weighted when the decision criteria are unidimensional, which means both $\mathcal{C}_1$ and $\mathcal{C}_2$ are singletons.

Under a weighted multicameral political rule, a voter's weight associated with a given level of approval might vary across houses, which implies that a voter may be more influential in certain houses than others, as illustrated in the example in the introduction. In each legislature, each theoretical referee has a greater weight in the Theory house than in the Empirics house. Similarly, each empirical referee has a greater weight in the Empirics house than in the Theory house. A house in which a voter has a high level of influence may be viewed as the house dealing with matters on which that voter is an expert. We therefore think of houses as being *specialized*, and of voters as experts who have *decentralized authority* and vote only on specific aspects of a proposal under consideration.

3 A $(3,3)$-Political Rule as a Hierarchical System of Legislatures

In this section, we show that a $(3,3)$-political rule can be viewed as a hierarchical system of legislatures, with each using a $(3,2)$-political rule. More precisely, we show that voting under a $(3,3)$-political rule G is equivalent to voting simultaneously under two $(3,2)$-political rules G^1 and G^2. The rule G^1 determines whether the proposal should be granted the highest level of approval while the rule G^2 decides whether or not the proposal should be granted the intermediate level of approval. The formal result is as follows:

Remark 1 *Let $G = (N, \mathcal{V})$ be a $(3,3)$-political rule. Recall that a $(3,2)$-political rule is entirely characterized by the set of all its winning vote profiles. Define the $(3,2)$-political rules $\mathcal{V}^1$ and $\mathcal{V}^2$ as follows: for every vote profile X, ($\mathcal{V}^1(X) = 1$ if and only if $\mathcal{V}(X) = 2$) and ($\mathcal{V}^2(X) = 1$ if and only if $\mathcal{V}(X) \in \{2,1\}$). This means $\mathcal{V}^1$ considers as winning vote profiles only those that lead to the highest level of collective approval of a proposal, whereas the winning vote profiles of $\mathcal{V}^2$ are those that lead to the approval level at least equal to the intermediate level.*

The decomposition of $\mathcal{V}$ as given in Remark 1 yields a unique pair of $(3,2)$-political rules $(\mathcal{V}^1, \mathcal{V}^2)$ [6]. We write $\mathcal{V} = \mathcal{V}^1 \star \mathcal{V}^2$ and we say that $\mathcal{V}^1 \star \mathcal{V}^2$ is a $(3,2)$-decomposition of $\mathcal{V}$ or $\mathcal{V}$ is a $(3,3)$-composition of $\mathcal{V}^1$ and $\mathcal{V}^2$. $\mathcal{V}^1$ is of a higher standard than $\mathcal{V}^2$ in the sense that it is more difficult for a policy proposal to

[6] Mathematically, $\mathcal{V}^1$ and $\mathcal{V}^2$ satisfy $\mathcal{V} = \mathcal{V}^1 + \mathcal{V}^2$.

pass under $\mathcal{V}^1$ than under $\mathcal{V}^2$ holding voters' opinions fixed[7]. We say that the pair $(\mathcal{V}^1, \mathcal{V}^2)$ is ordered[8].

The following theorem summarizes the point just made:

Theorem 1 *Every* $(3,3)$*-political rule has a unique* $(3,2)$*-decomposition.*

Remark 1 proves Theorem 1 by establishing a one-to-one mapping between the set of $(3,3)$-political rules $\mathcal{V}$ and the set of ordered pairs $(\mathcal{V}^1, \mathcal{V}^2)$ of $(3,2)$-political rules.

One of the implications of Theorem 1 is that some of the properties of $(3,2)$-political rules can be extend to the class of $(3,3)$-political rules using the $(3,2)$-decomposition of $(3,3)$-political rules.

The following result provides a necessary and sufficient condition for a $(3,3)$-political rule to be weighted.

Proposition 1 *Let* $G = (N, \mathcal{V})$ *be a* $(3,3)$*-politcal rule and* $\mathcal{V}^1 \star \mathcal{V}^2$ *the* $(3,2)$*-decomposition of* $\mathcal{V}$*.* $\mathcal{V}$ *is weighted if and only if* $\mathcal{V}^1$ *and* $\mathcal{V}^2$ *are weighted* $(3,2)$*-political rules.*

The proof of this result is constructed as follows: given a weighted $(3,3)$-political rule $\mathcal{V}$, from the weights of $\mathcal{V}$, we construct the weights of the two weighted $(3,2)$-political rules that compose $\mathcal{V}$. Conversely, if the weights of $\mathcal{V}^1$ and $\mathcal{V}^2$ are known, we show how to retrieve a system of weights for the rule $\mathcal{V}^1 \star \mathcal{V}^2$. See Appendix A for a formal proof.

Consider a pair of weighted $(3,2)$-political rules $(\mathcal{V}^1,\ \mathcal{V}^2)$. If $(\mathcal{V}^1,\ \mathcal{V}^2)$ is ordered, then the weighted $(3,3)$-political rule $\mathcal{V}^1 + \mathcal{V}^2 = \mathcal{V}^1 \star \mathcal{V}^2$ is said to be a *weighted* $(3,3)$*-political rule*. However, when $(\mathcal{V}^1,\ \mathcal{V}^2)$ is not ordered, the $(3,3)$-political rule $G^1 + G^2$ is not necessarily weighted. In this case, we said that $\mathcal{V}^1\ + \mathcal{V}^2$ is a *quasi-weighted* $(3,3)$*-political rule.*

We can notice that a weighted $(3,3)$-political rule is quasi-weighted although the converse is not necessarily true. Moreover, the notion of quasi-weighted rule coincides on the class of $(3,2)$-political rules with the concept of weighted rule. Any weighted $(3,2)$-political rule can be viewed as a $(3,2)$-composition of two weighted political rules $\mathcal{V}^1$ and $\mathcal{V}^2$ where $\mathcal{V} = \mathcal{V}^1$ and $\mathcal{V}^2$ is the trivial political rule which considers any vote profile as winning.

4 New Theories of Dimension

Previous studies have characterized voting rules using the traditional concept of dimension introduced by Taylor and Zwicker (1993). They show that any monotonic $(2,2)$-political rule has a weighted multicameral representation with a minimal number of houses called *dimension*. Intuitively, the dimension of a monotonic $(2,2)$-political rule can be interpreted as the minimum number of houses that must accept a proposal in order for it to be adopted by the collectivity. This concept, however, does not apply to rules with more than two levels of collective approval, as the outcome of the rule can no longer be formulated as accepting or rejecting a proposal. Under a $(3,3)$-political rule, a proposal can be approved at two levels, the highest level and the intermediate level, the third level being its rejection. It seems reasonable

[7] Notice that $\mathcal{V}^1 \leq \mathcal{V}^2$ since $\mathcal{V}^1(X) \leq \mathcal{V}^2(X)$ for any vote profile X.

[8] In general, decomposing a $(3,3)$-political rule G as $\mathcal{V}^1 + \mathcal{V}^2$ does not necessarily provide a $(3,2)$-decomposition of G. A necessary and sufficient condition for $\mathcal{V}^1 + \mathcal{V}^2$ to be a $(3,2)$-decomposition of G is that the pair $(\mathcal{V}^1, \mathcal{V}^2)$ be ordered. We can also notice that the decomposition $\mathcal{V}^1 + \mathcal{V}^2$ preserves monotonicity in the sense that if G is monotonic, then $\mathcal{V}^1$ and $\mathcal{V}^2$ are monotonic too. Conversely we can verify that if $\mathcal{V}^1$ and $\mathcal{V}^2$ are monotonic, then $\mathcal{V}^1 + \mathcal{V}^2$ is monotonic.

to require that the first two levels of approval be incorporated in the definition of the dimension of a monotonic $(3,3)$-political rule, which we do in this section. To this effect, we introduce the concept of quasi-dimension and that of 2-dimension.

4.1 The Quasi-Dimension of a $(3,3)$-Political Rule

Earlier studies have shown that a $(2,2)$-political rule is a finite intersection of weighted $(2,2)$-political rules (*intersection* should be understood as *minimum*). According to Taylor and Zwicker (1993), the *dimension* of a $(2,2)$-political rule G is defined as the minimum integer m such that G can be written as the minimum of m weighted $(2,2)$-political rules. In the following, we represent a $(3,3)$-political rule as the minimum of a finite number of quasi-weighted $(3,3)$-political rules. We then define the quasi-dimension of a $(3,3)$-political rule G as the minimum number of quasi-weighted $(3,3)$-political rules necessary to construct G. A formal definition of the quasi-dimension of a $(3,3)$-political rule is provided below:

Definition 2 *The quasi-dimension of a $(3,3)$-political rule G is the minimum integer m such that G can be written as the minimum of m quasi-weighted $(3,3)$-political rules but cannot be written as the minimum of $m-1$ quasi-weighted $(3,3)$-political rules.*

The above definition is an extension of the concept of dimension as defined on the class of $(2,2)$-political rules. In the following, we therefore generalize some results obtained for $(2,2)$-political rules.

Theorem 2 *Every monotonic $(3,3)$-political rule has a weighted multicameral representation and can be written as the minimum of a finite number of (monotonic) quasi-weighted $(3,3)$-political rules.*

The proof of this theorem is Appendix B. First, we show that every monotonic $(3,2)$-political rule has a multicameral representation and is the minimum of a finite number of weighted $(3,2)$-political rules (the dimension of a $(3,2)$-political rule cannot exceed the number of maximal losing vote profiles). Then we propose a method to construct a multicameral representation of the $(3,3)$-political rule G from the system of weights of the $(3,2)$-political rules that compose G. Finally we show that the sub-political rules that compose G are $(3,3)$-quasi-weighted political rules.

The following result establishes that the concept of quasi-dimension is well defined on the class of monotonic $(3,3)$-political rules.

Proposition 2 *Every monotonic $(3,3)$-political rule has finite quasi-dimension.*

The proof of this result is straightforward given Theorem 2.

4.2 The 2-Dimension of a $(3,3)$-Political Rule

We introduce the concept of *2-dimension* of a monotonic $(3,3)$-political rule. It is a vector $(d_1, d_2) \in \mathbb{N} \times \mathbb{N} \cup \{0\}$ where d_1 is the minimum number of houses that must accept a proposal in order for it to be collectively approved at the highest level, and d_2 the minimum number of houses that must accept a proposal in order for it to be collectively approved at the intermediate level. Below, we provide a formal definition of the concept of 2-dimension.

Definition 3 *Let $G = (N, \mathcal{V})$ be a monotonic $(3,3)$-political rule and $\mathcal{V}^1 \star \mathcal{V}^2$ the $(3,2)$-decomposition of $\mathcal{V}$. The 2-dimension of $\mathcal{V}$ is the vector (d_1, d_2) where d_1 and d_2 are respectively the dimensions of $\mathcal{V}^1$ and $\mathcal{V}^2$.*

We note that the notion of 2-dimension generalizes the traditional notion of dimension since the two concepts coincide on the class of $(3,2)$-political rules and $(2,2)$-political rules. Indeed, a monotonic $(3,2)$-political rule of dimension d can be seen as having 2-dimension $(d, 0)$.

The following theorem establishes that the concept of 2-dimension is well defined on the class of monotonic $(3,3)$-political rules.

Theorem 3 *The 2-dimension of a monotonic $(3,3)$-political rule always exists.*

If one assumes that one-dimensional policies should be evaluated using weighted (or one-dimensional) rules, from a policy perspective, Theorem 3 implies that a given monotonic $(3,3)$-political rule is only suited for the evaluation of finite-dimensional policies whose dimension corresponds to that of the rule, with each aspect of the policy being evaluated in a different house or chamber using a weighted rule. The fact that any such rule has a weighted multicameral representation also implies that any $(3,3)$-political rule is a collection of *perfectly complementary* weighted rules in the sense that a policy proposal cannot be approved at a given level if it is not approved under each of the weighted rules of the legislature deciding on the approval of the proposal at that level (in fact, the rule used in each legislature is a minimum of weighted rules, which means that if a policy proposal fails under one of these rules thus yielding an outcome of zero for that rule, the minimum will therefore be zero, and so the proposal will fail overall in that legislature).

We also have the following proposition.

Proposition 3 *Let m be a positive integer. There always exists a monotonic $(3,2)$-political rule of dimension m (or 2-dimension $(m, 0)$).*

We now prove that there exists a monotonic $(3,3)$-political rule of any 2-dimension.

Theorem 4 *Let d_1 and d_2 be two positive integers. There exists a monotonic $(3,3)$-political rule of 2-dimension (d_1, d_2).*

Proposition 3 and Theorem 4 imply that one can always design a $(3,3)$-political rule for evaluating a policy that has to meet a set of d_1 criteria to be approved at the highest level and a set of d_2 criteria to be approved at the intermediate level, where the d_2 criteria in the second set altogether are somewhat weaker than the d_1 criteria in the first set so that a policy that satisfies the first set of criteria automatically satisfies the second set, while the converse is not true in general. This means that one can set the criteria for policies first, and then design the rules that rationalize the passage of policies meeting those criteria. This result also provides a rationale for using different rules to evaluate different policies, even within the same organization.

5 Useful Topology of Political Rules for Policymakers

In this section, we introduce the concept of compatibility with a rule, which we subsequently use to show that any $(3,3)$-political rule has a weighted multicameral representation in which weights and quotas are integers. Weights therefore represent the number of votes, which facilitates the interpretation of political rules for policymakers.

5.1 Compatibility

We introduce the concept of compatibility of a real vector with a rule, and topologically characterize the set of weighted multicameral rules that are compatible with a given political rule. We first establish our results on the class of $(3,2)$-political rules, then extend them to the class of $(3,3)$-political rules. Before defining the concept of compatibility, recall that any weighted multicameral representation of a rule $\varphi = (w_{ij}(p), q_i)_{1\leq i\leq d, 1\leq j\leq 3, 1\leq p\leq n}$ can be viewed as a vector of $\mathbb{R}^{d\times(3n+1)}$ (see Appendix F for the vector configuration). We have the following definition.

Definition 4 *Let $G = (N, \mathcal{V})$ be a $(3,2)$-political rule and $\varphi = (w_{ij}(p), q_i) \in \mathbb{R}^{d\times(3n+1)}$ a weighted multicameral representation of G.*

φ is $\mathcal{V}$- compatible if for every vote profile X, the following two conditions are satisfied:

(1) X is winning if and only if the weight of X in every house i is strictly higher than the corresponding quota q_i ($\mathcal{V}(X) = 1$ if and only if $w_i(X) > q_i$ for all $1 \leq i \leq d$).

(2) X is losing if and only if the weight of X is strictly lower than the quota in at least one house, and is always distinct from the quota in every house. ($\mathcal{V}(X) = 0$ if and only if $w_i(X) < q_i$ for some i and $w_i(X) \neq q_i$ for all i).

We show that the set of weighted multicameral representations of $\mathcal{V}$ that are $\mathcal{V}$-compatible is an open set.

Theorem 5 *Let $G = (N, \mathcal{V})$ be a $(3,2)$-political rule. The set of weighted multicameral representations of $\mathcal{V}$ that are $\mathcal{V}$-compatible is an open set of $\mathbb{R}^{d\times(3n+1)}$.*

This result is proved by showing that the set of weighted multicameral representations of $\mathcal{V}$, $\varphi = (w_{ij}(p), q_i) \in \mathbb{R}^{d\times(3n+1)}$, that are $\mathcal{V}$-compatible is the intersection of two open subsets of $\mathbb{R}^{d\times(3n+1)}$.

The immediate implication of this result is that all the properties of open sets of $\mathbb{R}^{d\times(3n+1)}$ can be applied to the set of $\mathcal{V}$-compatible weighted multicameral representations of the $(3,2)$-political rule G. For instance, if a weighted multicameral representation of a $(3,2)$-political rule is subject to some sufficiently small pertubations, the distribution of political power will remain unchanged. This finding will prove important in the next section where we show that any $(3,3)$-political rule has a weighted multicameral representation whose weights and quotas are integers, and propose a method to construct such a representation.

5.2 Integer Weights and Quotas

We show that any $(3,3)$-political rule has a weighted multicameral representation in which weights and quota are integers. Such a representation is more intuitive, as it implies that the weights $w_{ij}(p)$ can now be interpreted as the number of votes held by voter p in the house i when he chooses the level of approval j. Such an interpretation is also more likely to be understood by policymakers, lawmakers and shareholders who vote on a regular basis, and to shed light on the real structure of power in an organization.

Theorem 6 *Let $G = (N, \mathcal{V})$ be a $(3,3)$-political rule. G has a weighted multicameral representation whose weights and quotas are integers.*

This result implies that one can always represent any $(3,3)$-political rule in a way that is understandable to any policymaker. An example is our example in the introduction where all weights and quotas are integers. See Section 6 for other examples as well.

We now extend below the concept of the weight monotonicity requirement introduced by Freixas and Zwicker (2003) for single-house rules. It says that the higher the level of approval of a voter for a given proposal, the higher should be his contribution to make that proposal approved by the society.

Definition 5 *Let $G = (N, \mathcal{V})$ be a monotonic $(3,3)$-political rule. A weighted multicameral representation (w, q) of G satisfies the* "weight monotonicity requirement" *if :*
for every voter $p \in N$ and for every $1 \leq i \leq d$, $w_{i1}(p) \geq w_{i2}(p) \geq w_{i3}(p)$.

We show below that any $(3,3)$-political rule admits a weighted multicameral representation that satisfies the weight-monotonicity requirement.

Theorem 7 *Any $(3,3)$-political rule has a weighted multicameral representation that satisfies the weight-monotonocity requirement.*

6 Applications

In this section, we apply our theoretical findings to some real-life organizations. For each organization, we define its decision-making rule and provide a weighted multicameral representation of that rule with integer weights and quotas, and satisfying the weight-monotonicity requirement.

6.1 The Council of the International Seabed Authority

The International Seabed Authority is an intergovernmental organization created in 1994 to monitor mineral-related activities in the international seabed territories outside of states regulated jurisdiction. The ISA has two principal organs: an assembly, which consists of all the ISA members, and a 36-member council, elected by the assembly. The council members are chosen to ensure equitable representation of countries from various groups. As described by Bräuninger (2003), the members of the council are distributed into four houses denoted C_1, C_2, C_3, C_4 as follows:

C_1 : four states elected from among the largest consumers of the minerals in question

C_2 : four states elected from among the largest investors in deep-sea mining

C_3 : four states from among the largest net exporters

C_4 : six developing countries and eighteen additional states to ensure a balanced geographical distribution of seats in the council.

In the ISA council, a decision on most issues requires a two-thirds majority approval of its members on the proviso that such decision is not opposed by a majority in any of the four houses. This voting rule can be modeled as a $(3,2)$-political rule $G = (N, \mathcal{V})$ as follows:

For any vote profile $X = (X_1, X_2, X_3)$,

$$\mathcal{V}(X) = 1 \text{ if and only if } \begin{cases} |X_1| \geq 24 \\ |X_3 \cap C_t| < \frac{1}{2}|C_t|, \quad 1 \leq t \leq 4 \end{cases}$$

This decision-making rule was studied by Bräuninger (2003), who proved that it is not of dimension one. In the following, we propose a weighted multicameral representation of the vote within the Council of ISA . In the matrix of weights below, the first, the second , the third and the fourth columns represent the house of consumers, investors, exporters and the fourth house, respectively. The last column represents the entire council.

	C_1	C_2	C_3	C_4	Council
w(consumer)	$(3,2,-1)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(1,0,0)$
w(investor)	$(0,0,0)$	$(3,2,-1)$	$(0,0,0)$	$(0,0,0)$	$(1,0,0))$
w(exporter)	$(0,0,0)$	$(0,0,0)$	$(3,2,-1)$	$(0,0,0)$	$(1,0,0)$
w(others)	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(2,2,-1)$	$(1,0,0)$
Quota	1	1	1	10	24

The vote in the ISA council is a monotonic $(3,2)$-political rule of dimension 5 (the 2-dimension is $(5,0)$ and the quasi-dimension is 5).

6.2 A Voting Rule in the United States Senate

We study the voting rule for electing the Vice-President of the United States when the Electoral College fails to do so. We show that the quasi-dimension of this rule is 1.

When electing the Vice-President of the U.S., if no candidate receives the majority of the electoral votes, the Senate chooses between the two candidates with the highest number of votes in the Electoral College. The U.S. Senate consists of 100 Senators and the current Vice-President. The voting rule is the absolute majority rule and does not explicitly specify the outcome of the rule if neither candidate receives 51 votes. Let a and b be the two candidates with the highest number of electoral votes. This voting mechanism can be modeled as a monotonic $(3,3)$-political rule $G = (N, \mathcal{V})$ as follows. For every vote profile $X = (X_1, X_2, X_3)$, let X_1, X_2, X_3 represent the set of voters who vote for a, abstain, and vote for b, respectively. The three possible outcomes of the vote are: a wins, the vote is inconclusive, b wins. The following model of this voting rule was proposed by Taylor and Zwicker (2003):

$$\mathcal{V}(X_1, X_2, X_3) = \begin{cases} a & \text{if } |X_1| \geq 51 \\ \text{inconclusive} & \text{if } |X_1| < 51 \text{ and } |X_3| < 51 \\ b & \text{if } |X_3| \geq 51 \end{cases}$$

We argue that the quasi-dimension of this rule is 1 and its 2-dimension is $(1,1)$. Let G^1 and G^2 be two monotonic $(3,2)$-political rules defined as follows:

$$\mathcal{V}^1(X_1,X_2,X_3)=1 \quad \text{if and only if } |X_1|\geq 51$$
$$\mathcal{V}^2(X_1,X_2,X_3)=1 \quad \text{if and only if } (|X_1|\geq 51) \text{ or } (|X_1|<51 \text{ and } |X_3|<51)$$

A weighted representation of G^1 and G^2 is respectively given by:

	Rule G^1	Rule G^2
w(Senator)	$(1,0,0)$	$(1,1,0)$
w(Vice-President)	$(1,0,0)$	$(1,1,0)$
Quota	51	51

Both G^1 and G^2 are of dimension 1. Since G^1 and G^2 provide a $(3,2)$-decomposition of G, the quasi-dimension of G is 1 and the 2-dimension of G is $(1,1)$.

6.3 Core Examination of Graduate Students in United States Universities

In some U.S. universities, Masters' students have two options when completing their Masters' degree. They can either write a thesis or take an exam. Students who choose the exam option are pooled with Ph.D. candidates to write a core examination. The core examination usually combines several subjects. Suppose that the core examination in a department of Economics is based on three subjects: microeconomics, macroeconomics and econometrics.

Assume that the core examination committee consists of 6 members divided into 3 subcommittees as follows: the microeconomics committee (2 members), the macroeconomics committee (2 members), and the econometrics committee (2 members).

To each student taking the core examination, each committee member assigns three decisions corresponding to their judgment on the student's performance on each of the three subjects. There are three possible decisions for each subject: pass, marginal pass, and fail. There are also three possible final outcomes for each student. A student can pass the exam both at the Master and Ph.D. levels, pass the exam only at the Master level, or fail the exam both at the Master and Ph.D. levels. Passing the core examination at a given level requires obtaining a certain grade on each subject. Assume that a student must have at least one "pass" and one "marginal pass" to pass a subject at the Ph.D. level, and at least two "marginal pass" or one "pass" to pass a subject at the Master's level.

The above described rule is a monotonic $(3,3)$-political rule. We denote the examination outcomes as follows: 2 = Ph.D., 1 = Master, 0 = fail. For any vote profile $X=(X_1,X_2,X_3)$, X_1, X_2 and X_3 are respectively the set of professors who vote "pass", "marginal pass" and "fail". Denote by C_1, C_2, C_3 the microeconomics committee, the macroeconomics committee, and the econometrics committee, respectively. On a given subject, only the corresponding subcommittee has the full competence to judge the performance of a student. Therefore, we assume that the decisions of the members outside of a subcommittee do not affect the students' outcome for that subject. The above described core examination can be modeled as a $(3,3)$-political rule $G=(N,\mathcal{V})$ as follows:

For any vote profile $X=(X_1,X_2,X_3)$,

$$\mathcal{V}(X) = \begin{cases} \text{Ph.D. if } (|X_1 \cap C_t| = 2 \text{ or } (|X_1 \cap C_t| = 1 \text{ and } |X_2 \cap C_t| = 1) \text{ for all } 1 \leq t \leq 3 \\ \text{Master if } (|X_2 \cap C_t| = 2) \text{ or } (|X_1 \cap C_t| = 1) \text{ for all } 1 \leq t \leq 3 \\ \text{Fail if } \exists 1 \leq t \leq 3,\ (|X_3 \cap C_t| \geq 1) \text{ and } (|X_1 \cap C_t| = 0). \end{cases}$$

Let G^1 (resp. G^2) be the monotonic $(3,2)$-political rule that determines whether a student will pass the core examination at the Ph.D. (resp. at least at the Master level). G^1 and G^2 can be defined as follows: for any vote profile $X = (X_1, X_2, X_3)$,

$$\mathcal{V}^1(X) = 1 \quad \text{if and only if } \mathcal{V}(X) = Ph.D.$$
$$\mathcal{V}^2(X) = 1 \quad \text{if and only if } \mathcal{V}(X) \in \{Ph.D., Master\}$$

The weighted representations of G^1 and G^2 are given by:

	Rule G^1			Rule G^2		
	C_1	C_2	C_3	C_1	C_2	C_3
w(microeconomist)	$(2,1,0)$	$(0,0,0)$	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$	$(0,0,0)$
w(macroeconomist)	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$
w(econometrician)	$(0,0,0)$	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$	$(0,0,0)$	$(2,1,0)$
Quota	3	3	3	2	2	2

Therefore, both G^1 and G^2 are of dimension 3. Since G^1 and G^2 provide a decomposition of G as in Remark 1, the 2-dimension of G is $(3,3)$.

One might want to modify the examination system to avoid situations where Masters' students are evaluated on econometrics as is the case in certain universities. Assume, for example, that every student is now required to have at least two "marginal pass" outcomes to pass microeconomics and macroeconomics. Ph.D. students are required to have one "pass" or one "marginal pass" to pass econometrics. The modified rule has the following weighted multicameral representation:

	Rule G^1			Rule G^2	
	C_1	C_2	C_3	C_1	C_2
w(microeconomist)	$(3,2,0)$	$(0,0,0)$	$(0,0,0)$	$(3,2,0)$	$(0,0,0)$
w(macroeconomist)	$(0,0,0)$	$(3,2,0))$	$(0,0,0)$	$(0,0,0)$	$(3,2,0)$
w(econometrician)	$(0,0,0)$	$(0,0,0)$	$(2,1,0)$	$(0,0,0)$	$(0,0,0)$
Quota	4	4	3	4	4

G^1 and G^2 are respectively of dimension 3 and 2, and the 2-dimension of G is $(3,2)$.

7 Concluding Remarks

We have provided a policy basis for rationalizing, that is, interpreting, justifying and designing $(3,3)$-political rules, a large class of collective organizational rules analogous to those governing the selection of papers in a peer-reviewed journal. The different examples investigated in the paper show that such rules can be extremely complex. Yet, we have shown that any such rule, no matter its complexity, is a hierarchical system of two weighted multicameral legislatures: the first legislature determines whether a proposal should be granted the highest level of collective approval, such as accepting a paper, and the second legislature determines whether it should only be approved at the intermediate level, such as inviting a resubmission. Such a representation is a very convenient way of modelling complex decision-making rules. It also allows us to view any $(3,3)$-political rule as a multi-criteria decision-making rule under which decision-makers rate different aspects of a finite-dimensional proposal, where each aspect is evaluated in a different specialized house using a weighted voting rule, and each evaluator's influence or weight possibly varies across houses depending on his area(s) of expertise. Evaluators therefore have decentralized authority, and each is seen as an expert in a particular area.

We have also shown that it is always possible to design a $(3,3)$-political rule of any given 2-dimension. Practically, this finding implies that it is always possible to design a rule under which a proposal is collectively approved at a given level if and only if it satisfies a certain number of "predefined criteria", so that one can set the criteria for policies first, and then design the rules that rationalize the passage of policies meeting those criteria. This shows that rules can be designed so as to be suitable only for the evaluation of certain policies, and suggests a rationale for using different rules to pass different policies. Accordingly, a unidimensional policy should be evaluated using a $(3,3)$-political rule that has only one house in each legislature, since only one aspect of the policy will be evaluated. However, multi-dimensional policy proposals should be evaluated under weighted multicameral legislatures, where each house examines one aspect of the policy, and each legislator is given more weight in the house evaluating the aspect that pertains to his area of expertise. We note that a legislator may be expert in many areas, in which case he should have influence in different houses.

Designing a collective rule based on a set of criteria to be met by an ideal policy is not only transparent, but practical. Indeed, it might be cumbersome to try to write a decision-making rule as we did in our example in the introduction. A simpler way of writing the same rule would be to consider the matrix of weights, which clearly show how much influence each evaluator wields along a given criterion. If one wants to change the rule at some point, perhaps because one wants to increase the standards of policies to be adopted, or because one wants to get more evaluators involved along each criterion, one can simply modify the weights and/or the quotas, which is simple (although this might not be so easy in practice). For instance, one might change the weight matrix of the two legislatures in that example as follows:

	Original Rule		Modified Rule	
	Theory	Empirics	Theory	Empirics
Referee T1	$(2,1,0)$	$(0,0,0)$	$(2,1,0)$	$(1,0,0)$
Referee T2	$(2,1,0)$	$(0,0,0)$	$(2,1,0)$	$(1,0,0)$
Referee E1	$(0,0,0)$	$(2,1,0)$	$(1,0,0)$	$(2,1,0)$
Referee E2	$(0,0,0)$	$(2,1,0)$	$(1,0,0)$	$(2,1,0)$
Quota for Legislature 1	3	3	5	5
Quota for Legislature 2	1	1	3	3

We note that under the modified rule, it is harder to get a paper accepted, or considered for resubmission, compared to the original rule. Under the modified rule, while experts still have more power in their area of expertise, their only opinion is no longer sufficient to determine that a paper passes the test in their area. For instance, while under the first rule, the coalition of theoretical referees $\{T1, T2\}$ has dictatorial power on acceptance along the theoretical dimension, it only has a veto right under the modified rule. Similarly, as regarding the empirical assessment of the paper, the coalition of empirical referees $\{E1, E2\}$ has only a veto right for acceptance under the modified rule, whereas they had dictatorial power under the first rule. However, each expert still has a veto right along his area of expertise since the paper cannot get accepted (even after revision) without him strongly backing the decision.

These two rules might also be used by two different journals or organizations. One lesson to be learned here is that writing such a rule explicitly, as we did in the introduction, could be very confusing in some instances and could subject the written rule to different interpretations. Similarly, writing the characteristic function of a rule may not let policymakers know whether the rule is really appropriate for evaluating a given proposal, and it may not be explicit on the precise role of each evaluator. The table, on the contrary, clearly shows the power structure among the different evaluators along each dimension of a potential proposal, and should therefore be preferred to a written rule. It also shows that the original and the modified rules are most appropriate for the evaluation of two-dimensional policies.

We have also proposed two new concepts of dimension (the quasi-dimension and the 2-dimension), and have shown that they generalize the literature. While the traditional theory is easily interpreted within our framework, we show that the 2-dimension measure does not only capture the number of possible outputs yielded by a $(3, 3)$-political rule, but it also takes into account the hierarchy in these outputs. Furthermore, we have introduced the concept of compatibility of a vector with a rule, which we have used to show that there exist infinitely many weighted rules that are strictly compatible with a $(3, 3)$-political rule, and that these rules form a topologically open set. We have also proposed a method to construct integer weights and quotas, which are more intuitive, more relevant for rule design and policy-making, and more transparent for policymakers. In sum, our results rationalize multicameralism in real-world organizations and multi-criteria group decision-making in market organizations (Baucells and Sarin (2003)), and suggest a simple and rational policy-driven approach to the design of collective rules.

Appendices

Appendix A: Proof of Proposition 1

If G is weighted, then $\{w_{ij}, i \in \mathcal{C}_1\}$ and $\{w_{ij}, i \in \mathcal{C}_2\}$ are the systems of weights of G^1 and G^2, respectively. Conversely, if $\{w_{1j}\}_{1\leq j\leq 3}$ and $\{w_{2j}\}_{1\leq j\leq 3}$ are respectively the systems of weights of the weighted rules G^1 and G^2, define the system of weights of G, $\{w_{ij}\}$, as $w_{ij} = w_{1j}$ if $i \in \mathcal{C}_1$ and $w_{ij} = w_{2j}$ if $i \in \mathcal{C}_2$ with $\mathcal{C}_1 = \{1\}$ and $\mathcal{C}_2 = \{2\}$.

Appendix B: Proof of Theorem 2

The following lemma is needed to prove our result.

Lemma 1 *Any monotonic* $(3,2)$*-political rule has a weighted multicameral representation. More precisely, every monotonic* $(3,2)$*-political rule can be written as the minimum of a finite number of weighted* $(3,2)$*-political rules.*

Proof:Let $G = (N, \mathcal{V})$ be a monotonic $(3,2)$-political rule and $(Y^1, ..., Y^d)$ the list of all the maximal losing tripartitions (or vote profiles) of N. For every voter $p \in N$, define the matrix of the weights of p as follows:

$$\text{for all } 1 \leq i \leq d \text{ and for all } 1 \leq j \leq 3, \quad w_{ij}(p) = \begin{cases} 0 & \text{if } p \in Y^i_j \uparrow \\ 1 & \text{otherwise.} \end{cases}$$

The quota for each house i is $q_i = 1$, $1 \leq i \leq d$

We have:

$$\begin{aligned} \mathcal{V}(X) = 1 \quad & \text{iff for all } \; 1 \leq i \leq d, \; X \not\subseteq^3 Y^i \\ & \text{iff for all } \; 1 \leq i \leq d, \; \text{there exists } 1 \leq j \leq 3, \quad X_j \uparrow \not\subseteq Y^i_j \uparrow \\ & \text{iff for all } \; 1 \leq i \leq d, \text{ there exist } 1 \leq j \leq 3 \text{ and } p \in X_j \uparrow \; \text{ such that } \; p \notin Y^i_j \uparrow \\ & \text{iff for all } 1 \leq i \leq d, \text{ there exist } 1 \leq j \leq 3 \text{ and } p \in X_j \uparrow \text{ such that } \; w_{ij}(p) = 1 \\ & \text{iff for all } 1 \leq i \leq d, \; w_i(X) \geq q_i. \end{aligned}$$

Therefore, $w = (w_{ij})_{1\leq i\leq d, 1\leq j\leq 3}$ and $q = (q_i)_{1\leq i\leq d}$ define a weighted multicameral representation of G.

For each $1 \leq i \leq d$, let $G_i = (N, \mathcal{V}_i)$ be the (monotonic) weighted $(3,2)$-political rule represented by (w_{ij}, q_i). It can be easily shown that for every $X \in N^3$, $\mathcal{V}(X) = \min\ \{\mathcal{V}_i(X),\ 1 \leq i \leq\ d\}$.

Proof of Theorem 2 Let $G = (N, \mathcal{V})$ be a (monotonic) $(3,3)$-political rule and $\mathcal{V}^1 \star \mathcal{V}^2$ the $(3,2)$-decomposition of G (see Proposition 1). Following Lemma 1, $\mathcal{V}^1$ and $\mathcal{V}^2$ can respectively be decomposed as the minimum of a finite number of weighted $(3,2)$-political rules $\{\mathcal{V}^1_i\}_{1\leq i\leq\ d_1}$ and $\{\mathcal{V}^2_{i'}\}_{1\leq i'\leq\ d_2}$. For every $1 \leq r \leq 2$, $1 \leq i \leq d_1$ or $1 \leq i \leq d_2$, let $w^r_i = (w^r_{ij})_{1\leq j\leq 3}$ be the matrix functions of voters' weights in the rule $\mathcal{V}^r_i$, and q^r_i the quota of $\mathcal{V}^r_i$. Define $\mathcal{C}_1 = \{1, ..., d_1\}$, $\mathcal{C}_2 = \{d_1 + 1, ..., d_1 + d_2\}$, and w and q as follows: for every voter p,

$$w_{ij}(p) = \begin{cases} w^1_{ij}(p) & \text{if } i \in \mathcal{C}_1 \\ w^2_{i-d_1 j}(p) & \text{if } i \in \mathcal{C}_2 \end{cases}$$

$$q_i = \begin{cases} q^1_i & \text{if } i \in \mathcal{C}_1 \\ q^2_{i-d_1} & \text{if } i \in \mathcal{C}_2 \end{cases}$$

For every vote profile $X \in N^3$:

$$\begin{aligned} \mathcal{V}(X) = 2 \quad & \Longleftrightarrow \mathcal{V}^1(X) = 1 \text{ and } \mathcal{V}^2(X) = 1 \\ & \Longleftrightarrow \mathcal{V}^1_i(X) = 1 \text{ and } \mathcal{V}^2_i(X) = 1, \ \forall\ 1 \leq i \leq d_1, 1 \leq i \leq d_2 \\ & \Longleftrightarrow w_i(X) \geq q_i, \ \forall\ \ i \in \mathcal{C}_1 \cup \mathcal{C}_2. \end{aligned}$$

$$\begin{aligned} \mathcal{V}(X) = 1 \quad & \Longleftrightarrow \mathcal{V}^1(X) = 0 \text{ and } \mathcal{V}^2(X) = 1. \\ & \Longleftrightarrow \exists\, 1 \leq i \leq d_1,\ \mathcal{V}^1_i(X) = 0 \text{ and } \mathcal{V}^2_{i'}(X) = 1,\ \forall 1 \leq i' \leq d_2 \\ & \Longleftrightarrow \min\{\mathcal{V}^1_i(X) + \mathcal{V}^2_{i'}(X),\ 1 \leq i \leq d_1,\ 1 \leq i' \leq d_2. \end{aligned}$$

$$\begin{aligned} \mathcal{V}(X) = 0 \quad & \Longleftrightarrow \mathcal{V}^1(X) = 0 \text{ and } \mathcal{V}^2(X) = 0 \\ & \Longleftrightarrow \exists\, 1 \leq i \leq d_1,\ 1 \leq i \leq d_2, \mathcal{V}^1_i(X) = 0 \text{ and } \mathcal{V}^2_i(X) = 0 \\ & \Longleftrightarrow \exists\ i \in \mathcal{C}_2, w_i(X) < q_i \end{aligned}$$

Futhermore, each $\mathcal{V}^1_i + \mathcal{V}^2_{i'}$ is a quasi-weighted $(3,3)$-political rule, and it is easy to show that $\mathcal{V} = min\{\mathcal{V}^1_i + \mathcal{V}^2_{i'}, 1 \leq i \leq\ d_1, 1 \leq i' \leq\ d_2\}$.

Appendix C: Proof of Theorem 3

The $(3,2)$-decomposition of a monotonic $(3,3)$-political rule as described in Remark 1 yields a unique pair of monotonic $(3,2)$-political rules. Since the notion of dimension is well-defined on the class of monotonic $(3,2)$-political rules (following Lemma 1), the 2-dimension of any monotonic $(3,3)$-political rule always exists.

Appendix D : Proof of Proposition 3

Let m be a positive integer and $N = \{1, 2, ..., 2m\}$. Define a monotonic $(3,2)$-political rule $G = (N, \mathcal{V})$ by: for every vote profile $X = (X_1, X_2, X_3)$,

$$\mathcal{V}(X) = 1 \text{ iff } X_1 \cap \{2i-1, 2i\} \neq \emptyset, \forall 1 \leq i \leq m.$$

Below is a two-step proof that the dimension of the above-described monotonic $(3,2)$-political rule is m.

Step 1 : First, we show that G is a minimum of m weighted monotonic $(3,2)$-political rules.

For $1 \leq i \leq m$, let $G_i = (N, \mathcal{V}_i)$ be the weighted monotonic $(3,2)$-political rule represented by $(w_{ij}, q_i)_{1 \leq j \leq 3}$ defined as follows: for every $p \in N$,

$$w_{ij}(p) = \begin{cases} 1 & \text{if } j = 1 \text{ and } p \in \{2i-1, 2i\} \\ 0 & \text{otherwise} \end{cases}$$

$$q_i = 1.$$

Let $X \in N^3$ be a winning vote profile.

$$\begin{array}{lll} X \in \mathcal{V}^{-1}\{1\} & \text{iff} & X_1 \cap \{2i-1, 2i\} \neq \emptyset, \quad \text{for all} \quad 1 \leq i \leq m \\ & \text{iff} & \text{there is} \quad p \in N \quad \text{such that} \quad w_{ij}(p) = 1, \ \text{for all} \quad 1 \leq i \leq m \\ & \text{iff} & w_i(X) \geq q_i, \quad \text{for all } 1 \leq i \leq m \\ & \text{iff} & X \in \mathcal{V}_i^{-1}\{1\}\ , \quad \forall 1 \leq i \leq m \\ & \text{iff} & X \in \cap \left\{ \mathcal{V}_i^{-1}\{1\} \ \mid\ 1 \leq i \leq m \right\}. \end{array}$$

Thus $\mathcal{V}^{-1}\{1\} = \cap \left\{ \mathcal{V}_i^{-1}\{1\} \mid 1 \leq i \leq m \right\}$.

step 2: Our next goal is to show that G is not the minimum of $m-1$ weighted monotonic $(3,2)$-political rules. Assume on the contrary that G can be written as the minimum of $m-1$ weighted monotonic $(3,2)$-political rules $G_i^{'}$ represented by $(w_{ij}^{'}, q_i^{'})_{1 \leq j \leq 3}$.

For every $1 \leq i \leq m$, define $Y^i = (N - \{2i-1, 2i\}, \{2i-1, 2i\}, \emptyset)$.

Each Y^i is a losing vote profile of G. Therefore, we can pick one of the $m-1$ weighted monotonic $(3,2)$-political rules in which Y^i has a weight less than the quota of that monotonic $(3,2)$-political rule. By the pigeonhole principle, we can assume without losing generality that we have a similar weighted monotonic $(3,2)$-political rule $G_t^{'}$ that makes both Y^1 and Y^2 losing vote profiles.

Let X^1 and X^2 be two vote profiles defined as follows:

$X^1 = (N - \{2,3\}, \{2,3\}, \emptyset)$ and $X^2 = (N - \{1,4\}, \{1,4\}, \emptyset)$.

Since X^1 and X^2 (resp. Y^1 and Y^2) are two winning (resp. losing) vote profiles of the monotonic $(3,2)$-political rule $G_t^{'}$, it must be the case that $w_t^{'}(X^1) + w_t^{'}(X^2) \geq 2q_t^{'}$ and $w_t^{'}(Y^1) + w_t^{'}(Y^2) < 2q_t^{'}$. However, this cannot hold because $w_t^{'}(X^1) + w_t^{'}(X^2) = w_t^{'}(Y^1) + w_t^{'}(Y^2)$. Therefore, it is impossible to write G as the minimum of $m-1$ monotonic $(3,2)$-political rules.

Appendix E : Proof of Theorem 4

Let $N = \{1, 2, ..., 2d_1, 2d_1+1, ..., 2d_1+2d_2\}$. Define the monotonic $(3,3)$-political rule $G = (N, \mathcal{V})$ by: for every vote profile $X = (X_1, X_2, X_3)$,

$$\mathcal{V}(X) = 2 \quad \text{iff} \quad (\forall\ 1 \leq i \leq d_1,\ \forall\ 1 \leq i' \leq d_2\ X_1 \cap \{2i-1, 2i\} \neq \emptyset) \text{ and } (X_1 \cap \{2d_1+2i'-1, 2d_1+2i'\} \neq \emptyset)$$

$$\mathcal{V}(X) = 1 \quad \text{iff} \quad (\exists\ 1 \leq i \leq d_1,\ X_1\ \cap \{2i-1, 2i\} = \emptyset) \text{ and } (X_1 \cap \{2d_1+2i'-1, 2d_1+2i'\} \neq \emptyset, \forall\ 1 \leq i' \leq d_2)$$

$$\mathcal{V}(X) = 0 \quad \text{iff} \quad (\exists\ 1 \leq i \leq d_1,\ \exists\ 1 \leq i' \leq d_2,\ X_1 \cap \{2i-1, 2i\} = \emptyset) \text{ and } (X_1 \cap \{2d_1+2i'-1, 2d_1+2i'\} = \emptyset)$$

Let $G^1 = (N, \mathcal{V}^1)$ and $G^2 = (N, \mathcal{V}^2)$ be two monotonic $(3,2)$-political rules defined as: for every vote profile $X = (X_1, X_2, X_3)$,

$\mathcal{V}^1(X) = 1 \quad$ iff $\quad X_1 \cap \{2i-1, 2i\} \neq \emptyset \ \ \forall 1 \leq i \leq d_1$.

$\mathcal{V}^2(X) = 1 \quad$ iff $\quad X_1 \cap \{2d_1+2i'-1, 2d_1+2i'\} \neq \emptyset \ \ \forall\ 1 \leq i' \leq d_2$.

The method used to construct G^1 and G^2 is the same as the one used in the proof of Theorem 3. Therefore, the monotonic $(3,2)$-political rules G^1 and G^2 are respectively of dimension d_1 and d_2. Furthermore, $G^1 \star G^2$ is the $(3,2)$-decomposition of G. This allows us to conclude that the 2-dimension of G is (d_1, d_2).

Appendix F : Configuration of the Matrix of Weights of a Multicameral Rule as a Vector of $\mathbb{R}^{d\times(3n+1)}$

Let u and v be two positive integers. It is always possible to define a one-to-one function between the set $\mathcal{M}_{u\times v}(\mathbb{R})$ of all the $u\times v$ real matrices and $\mathbb{R}^{u\times v}$. Consider the one-to-one function f defined from $\mathcal{M}_{d\times(3n+1)}(\mathbb{R})$ to $\mathbb{R}^{d\times(3n+1)}$ that maps the matrix $w=(w_{i,j}(p),q_i)_{1\le i\le d,1\le j\le 3,1\le p\le n}$ to the vector $f(w)$ defined by:

$$\begin{aligned} f(w) = \ & (w_{11}(1), w_{12}(1), w_{13}(1), ..., w_{11}(n), w_{12}(n), w_{13}(n), q_1, ..., \\ & w_{d1}(1), w_{d2}(1), w_{d3}(1), ..., w_{d1}(n), w_{d2}(n), w_{d3}(n), q_d). \end{aligned}$$

f allows to view a weighted multicameral representation of a $(3,2)$-political rule rule as a vector of $\mathbb{R}^{d\times(3n+1)}$.

Appendix G: Proof of Theorem 5

Define the sets U_G and V_G as follows:

$U_G=\cap\{U_X,\ X$ is a winning vote profile$\}$ where: $U_X=\left\{\varphi\in\mathbb{R}^{d\times(3n+1)}\ /\ w_i(X)>q_i,\ \forall\ 1\le i\le d\right\}$.

$V_G=\cap\{V_X,\ X$ is a losing vote profile$\}$ where: $V_X=\{\varphi\in\mathbb{R}^{d\times(3n+1)}/\exists 1\le i\le d, w_i(X)<q_i\}$.

The following two lemmas are needed to establish our result.

Lemma 2 *Let $G=(N,\mathcal{V})$ be a $(3,2)$-political rule. The sets U_G and V_G are open sets of $\mathbb{R}^{d\times(3n+1)}$.*

Proof: For any vote profile X, define:

$$\begin{array}{rlcl} h_X: & \mathbb{R}^{d\times(3n+1)} & \longrightarrow & \mathbb{R}^d \\ & \varphi=(w_{ij}(p),q_i) & \longmapsto & (w_i(X)-q_i)_{1\le i\le d} \end{array}$$

The function h_X is well-defined and continuous.

Let us show that U_G is an open set of $\mathbb{R}^{d\times(3n+1)}$.

U_X is an open set of $\mathbb{R}^{d\times(3n+1)}$ because U_X is the inverse image of an open subset of $\mathbb{R}^d$ by the continuous function h_X, as shown below:

$$\begin{aligned} U_X &= \left\{\varphi\in\mathbb{R}^{d\times(3n+1)}\ /w_i(X)>q_i\ \forall 1\le i\le d\right\} \\ &= \left\{\varphi\in\mathbb{R}^{d\times(3n+1)}\ /\ h_X(\varphi)\in]0,+\infty[^d\right\} \\ &= h_X^{-1}(]0,+\infty[^d). \end{aligned}$$

U_G is the intersection of a finite number of open sets of $\mathbb{R}^{d\times(3n+1)}$. Therefore U_G is an open set of $\mathbb{R}^{d\times(3n+1)}$.

Now, let us show that V_X is an open subset of $\mathbb{R}^{d\times(3n+1)}$.

For every $\varphi\in V_X$, write: $d_{1\varphi}=|\{i,\ w_i(X)>q_i\}|$ and $d_{2\varphi}=|\{i,\ w_i(X)<q_i\}|$. Notice that $h_X(\varphi)\in V_\varphi$ where V_φ is the cartesian product of $d_{1\varphi}$ sets $]0,+\infty[$ and $d_{2\varphi}$ sets $]-\infty,0[$. V_φ is an open set of $\mathbb{R}^{d\times(3n+1)}$. Furthermore,

$$\begin{aligned} V_X &= \{\varphi\in\mathbb{R}^{d\times(3n+1)}\mid h_X(\varphi)\in\ V_\varphi\} \\ &= \cup\{h_X^{-1}(V_\varphi),\varphi\in V_X\}. \end{aligned}$$

V_X is a union of open sets of $\mathbb{R}^{d\times(3n+1)}$, which implies that V_X is an open set of $\mathbb{R}^{d\times(3n+1)}$. Since V_G is the intersection of a finite number of open sets of $\mathbb{R}^{d\times(3n+1)}$, V_G is an open set of $\mathbb{R}^{d\times(3n+1)}$.

Lemma 3 *Let $G = (N, \mathcal{V})$ be a $(3,2)$-political rule. The set $\Gamma(G)$ of all the vectors $\varphi \in \mathbb{R}^{d\times(3n+1)}$ that are $\mathcal{V}$-compatible is the intersection of U_G and V_G.*

Proof: Let $\varphi \in \mathbb{R}^{d\times(3n+1)}$ a vector that is $\mathcal{V}$-compatible and X a vote profile of N.

If X is a winning vote profile, then $w_i(X) > q_i$ for all $1 \leq i \leq d$. Therefore $\varphi \in U_X$. This implies that $\varphi \in U_G = \cap\{U_X,\ X \text{ winning vote profile }\}$

If X is a losing vote profile, then there exists $1 \leq i \leq d$ such that $w_i(X) < q_i$. Therefore $\varphi \in V_X$. This implies that $\varphi \in V_G = \cap\{V_X,\ X \text{ losing vote profile}\}$.

Therefore, $\Gamma(G)$ is the intersection of U_G and V_G.

Now, we can proof our main result.

Proof of Theorem 5: Because $\Gamma(G)$ can be decomposed as an intersection of open sets of $\mathbb{R}^{d\times(3n+1)}$ (Lemma 3), $\Gamma(G)$ is an open set of $\mathbb{R}^{d\times(3n+1)}$.

Appendix H: Proof of Theorem 6

To proof our main result, we will need the following lemmas that are also interesting on their own.

Lemma 4 *Let $G = (N, \mathcal{V})$ be a $(3,2)$-political rule. If $\varphi = (w_{ij}(p), q_i) \in \mathbb{R}^{d\times(3n+1)}$ is a weighted multicameral representation of G, then there exists $q' \in \mathbb{R}^d$ such that $\varphi' = (w_{ij}(p), q_i')$ is $\mathcal{V}$-compatible.*

Proof: Define μ_i as the lower bound of the set $\{w_i(X)/\ w_i(X) \geq q_i, X \in N^3\}$ and λ_i as the upper bound of the set $\{w_i(X)/\ w_i(X) < q_i, X \in N^3\}$. Notice that $\lambda_i < q_i \leq \mu_i$. Define $q_i' = \frac{\mu_i+\lambda_i}{2}$ and consider a vote profile X.

If X is a winning vote profile, then $w_i(X) \geq \mu_i > q_i'$, for all $1 \leq i \leq d$.

If X is a losing vote profile, then there exists $1 \leq i \leq d$ such that $w_i(X) \leq \lambda_i < q_i'$.

Thus $\varphi' = (w_{ij}(p), q_i')$ is $\mathcal{V}$-compatible .

It follows that a $\mathcal{V}$-compatible vector can be obtained from a weighted multicameral representation of a $(3,2)$-political rule G. Altering the weights of voters is unnecessary. A judicious alteration of the quotas of G is sufficient. A corollary to the previous lemma is the following.

Corollary 1 *Let G be a $(3,2)$-political rule. G is a weighted multicameral rule if and only if there exists $\varphi \in \mathbb{R}^{d\times(3n+1)}$ that is $\mathcal{V}$-compatible.*

Lemma 5 *Let G be a $(3,2)$-political rule. G has a weighted multicameral representation if and only if there exists $\varphi \in \mathbb{Q}^{d\times(3n+1)}$ that is $\mathcal{V}$-compatible.*

Proof: If G has a weighted multicameral representation, then $\Gamma(G)$ is a nonempty open subset of $\mathbb{R}^{d\times(3n+1)}$ (Theorem 5). The denseness of $\mathbb{Q}^{d\times(3n+1)}$ in $\mathbb{R}^{d\times(3n+1)}$ yields a $\varphi \in \Gamma(G) \cap \mathbb{Q}^{d\times(3n+1)}$. The converse is trivial.

Lemma 6 *Let G be $(3,2)$-political rule. G has a weighted multicameral representation if and only if there exists $\varphi \in \mathbb{Z}^{d\times(3n+1)}$ that is $\mathcal{V}$-compatible.*

Proof: If G has a weighted multicameral representation, then $\Gamma(G) \cap \mathbb{Q}^{d\times(3n+1)} \neq \emptyset$ (Lemma 5). Pick $\chi = (\frac{a_{ij}(p)}{b_{ij}(p)}, \frac{c_i}{d_i}) \in \Gamma(G) \cap \mathbb{Q}^{d\times(3n+1)}$ where $(b_{ij}(p), d_i) \in \mathbb{N}^{d\times(3n+1)}$. Let σ be the least common multiple of the set $\{b_{ij}(p), d_i\}_{1\leq i\leq d, 1\leq j\leq 3, 1\leq p\leq n}$ and $\varphi = \sigma\chi$. It readily follows that $\varphi \in \Gamma(G) \cap \mathbb{Z}^{d\times(3n+1)}$. The converse is trivial.

Proof of Theorem 6

Let $G^1 \star G^2$ the $(3,2)$-decomposition of a $(3,3)$-political rule G. The monotonicity of G insures the monotonicity of G^1 and G^2. We can find weighted multicameral representations of G^1 and G^2 with integer weights and quotas (Lemma 6). Use these systems of weights to construct a weighted multicameral representation of G with integer weights and quotas . For the method of construction, see the proof of Proposition 2.

Appendix I: Proof of Theorem 7

Let $w = (w_{ij}, q_i)_{1\leq i\leq d, 1\leq j\leq 3}$ be a weighted multicameral representation of G that violates the weight-monotonocity requirement. This implies there is a voter $x \in N$ and a house t such that $w_{tl}(x) < w_{tl+1}(x)$, with $1 \leq t \leq d, 1 \leq l \leq 2$.

Construct the new weighted multicameral representation u of G as follows:

$$u_{ij}(p) = \begin{cases} w_{tl+1}(p) & \text{if } p = x, i = t \text{ and } j = l \\ w_{tl}(p) & \text{if } p = x, i = t \text{ and } j = l+1 \\ w_{ij}(p) & \textit{otherwise.} \end{cases}$$

We want to show that both u and w identically distribute the elements of N^3 into the group of winning and losing vote profiles. Let X be any vote profile with $x \in X_{l+1}$ and Y the vote profile derived from X by shifting voter x to X_l. We see that:

1. $X \leq_{\mathcal{V}} Y$ (by monotonicity), and for every $1 \leq i \leq d$, $w_i(X) > w_i(Y)$.

2. $u_i(X) = w_i(Y)$ and $u_i(Y) = w_i(X)$.

Therefore, for every $1 \leq i \leq d$, $w_i(X) \geq q_i$ if and only if $w_i(Y) \geq q_i$ (from (1)), which happens if and only if $u_i(X) \geq q_i$ (from (2)).

The technique may be repeated for each voter whose weights violate the weight-monotonicity requirement.

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