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November 1980

Article published in the *Financial Analysts Journal*, Vol. 36, No. 6, November/December 1980, pp. 71-74. The published version may be downloaded for free at: http://www.cfapubs.org/doi/abs/10.2469/faj.v36.n6.71

The Arc Sine Law and the Treasury Bill Futures Market

By Charles Dale and Rosemarie Workman[1]

A long succession of coin tosses in a game in which one person wins if a head turns up, the other if a tail turns up, is analogous to speculation in price movements of a financial asset, where the long investor wins if price moves up and the short investor wins if price moves down. If we predicted that, over the course of these two games, each player would be in the lead about half the time, we would be wrong.

The counterintuitive fact is that the odds overwhelmingly favor one player being in the lead the vast majority of the time. Indeed, each player being in the lead half the time is the least likely outcome. If a coin is tossed every second for a year, there is a 10 per cent chance that the lead will change hands for the last time before the end of the ninth day, remaining in the same hands for the next 356 days.

According to the so-called "arc sine law," mechanical trading rules applied to price movements in financial assets will result in long periods of cumulative success, but equally long periods of cumulative failure. The long periods of success will tempt investors to apply trading rules to actual decisions. The long periods of failure make it very likely that such application will eventually blow them out of the market.

As long as a trading rule produces a consistent profit over long time periods, its advocates are unlikely to be dissuaded by theoretical arguments about random walks. Such rules will probably be around indefinitely.

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In the long-standing debate between random walkers and chartists, the former have frequently turned to one or more nonparametric statistics tests to prove the existence of the random walk (or any of the related versions of the efficient market hypothesis), while the latter have demonstrated time and again that some type of mechanical trading rule can uncover profitable opportunities in speculative markets.[2] A very significant but poorly understood law of probability theory--the arc sine law--suggests that the random walkers may never be able to convince the technicians. This article examines the arc sine law in the context of the Treasury bill futures market.

The Arc Sine Law

Consider a two-person game involving the toss of a coin: One person wins if a head turns up, the other if a tail turns up. Suppose we had to predict the results of a large number of such games after a single coin toss; this would roughly correspond to examining the daily price change of a large number of different types of futures contracts. We would expect about half the coin tosses to be heads and half tails, and we would be more or less correct.

But what if, instead of examining the single results of many different games, we were to look at only one game over a long period of time? This would correspond to examining price movements in only one futures contract over an extended period. We might predict that, over the course of the game, each player would be in the lead about half the time. We would be wrong. The startling, baffling and counterintuitive fact is that the odds are overwhelmingly in favor of one player being in the lead the vast majority of the time.

A change in the lead from one player to the other requires that the two attain "equalization"-that is, return to their starting points of zero net winnings. Equalization can occur only after a coin has been tossed an even number of times. The probability of equalization (u_k) after k ($\equiv 2j$) tosses can be shown to be:[3]

$$u_k = {}^{(2j)!}/_{j!j!2^k}, \quad j = 1,2,3,\ldots. \qquad (1)$$

The probability that the last equalization during a given game of length n will occur at time k is given by:

[2] See, for example, Fred D. Arditti and W. Andrew McCollough, "Can Analysts Distinguish Between Real and Randomly Generated Stock Prices?" *Financial Analysts Journal*, November/December 1978, pp. 70-74, and references contained therein.

[3] See "Fluctuations in Coin Tossing and Random Walks," Chapter 3 of William Feller, *An Introduction to Probability Theory and its Applications*, 3rd edition (New York: John Wiley & Sons, Inc., 1968) pp. 67-97.

$$P_{k,n} = u_k u_{n-k} , \quad (2)$$

where the u's are given by Equation 1. Because Equation 2 may be approximated by the trigonometric arc sine function, it is called the "arc sine law for last visits," or simply the "arc sine law."

The arc sine function is a U-shaped curve, with the smallest value at the midpoint. Applied to Equation 2, this means that the probability that each player in our coin tossing game is in the lead exactly half the time is the *least* likely outcome! The shape of the relevant probability distribution is such that the odds overwhelmingly favor one person being in the lead most of the time. To use an example, if a coin is tossed every second for a year, there is a 10 per cent chance that the last equalization will occur before the end of the ninth day, with the lead remaining in the same hands for the next 356 days.[4]

An individual futures contract is similar to one of these coin tossing games. The odds greatly favor an upward or downward drift in the price movements of an individual contract. Since most mechanical trading rules are applied to individual contracts, it follows that these rules may work for a while, simply because of the seemingly strange expected behavior of individual contracts. But the same arc sine law that makes likely periods of extended success dictates the equal likelihood of periods of extended failure. With non-zero transactions costs, traders who religiously use mechanical trading rules will eventually get blown out of the market.

The Treasury Bill Futures Market

Trading in 90-day Treasury bill futures began in 1976, on the International Monetary Market Division of the Chicago Mercantile Exchange. Each futures contract calls for delivery of one million dollars worth of 90, 91 or 92-day Treasury bills. Typically, the margin requirement has been about $800 per contract and commissions have been about $60 per round trip.

While several types of mechanical trading rule have been applied to futures trading, we will discuss one of the most popular techniques-the use of moving averages.[5] Moving averages work as follows: The trader sums the closing prices over x days and then

[4] Ibid.

[5] We obtained similar results using other trading rules, i.e., stop-loss orders, trailing stop-loss orders and filter rules. Tables of these results may be obtained from the authors. See also Charles Dale, "Brownian Motion in the Treasury Bill Futures Market," *Business Economics*, May 1981, pp. 47-54, for a different approach to random price movements that confirms our results.

divides by x to obtain an average price. Each day, he drops the oldest price and uses the new closing price to compute a new average. Whenever a price goes above the average computed at the end of the previous day, he makes a purchase. Whenever the price on a given day drops below the previous day's computed average, he both liquidates his long contract and sells a contract short. Any position held at the end of the trading period is liquidated.

We assume that the speculator would trade only one contract at a time, hence would require an initial margin deposit of $800. However, since futures markets all have a daily settlement, a speculator may be forced to put up more money to hold an unprofitable position.[6] We assume he will always put up whatever additional funds are necessary. (While perhaps unrealistic, this assumption better illustrates the working of the arc sine law.)

Table I shows the results from trading on the basis of the moving average rules. For the first contract, only the 10-day moving average showed a profit; using the 10-day moving average would have produced good results until 1978. While it is true that all 12 contracts taken together exhibited net profits, the reader may ask himself, after viewing the results, just how confident he would be in starting from scratch and using the 10-day moving average rule on new contracts.[7]

The popular 30-day moving average showed five consecutive winners. The decision to use a trading rule is typically made after just such a run of winners. Yet, in this case, the winners were followed by five consecutive losers. Putting the 30-day rule into practice after mid-1977 would have produced enormous losses.[8]

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[6] A theoretical model which utilizes this feature of daily settlement, or "marking to the market," is given by Fischer Black, "The Pricing of Commodity Contracts*," Journal of Financial Economics*, April 1976, pp. 167-179.

[7] See Peter D. Praetz, "On the Methodology of Testing for Independence in Future Prices: Comment," *Journal of Finance*, June 1976, pp. 977-979; Robert M. Bear and Richard A. Stevenson, "Reply (to a Comment by P.D. Praetz*)," Journal of Finance*, June 1976, pp. 980-983; and Raymond M. Leuthold, "Reply (to a Comment by P.D. Praetz*)," Journal of Finance*, June 1976, pp. 984-985, for a discussion of this point. Leuthold in particular makes the point that many speculators would like to trade and obtain profits, regardless of whether prices are moving randomly or not. This is precisely the reason that mechanical trading rules, which are widely misunderstood, can be such a dangerous trap for the unsophisticated investor.

[8] Various forms and combinations of moving averages are frequently limped together with the phrase, "momentum analyses." For a dramatic display of the erratic results obtained when using such models for forecasting foreign exchange rates, see Stephen H. Goodman, "No Better than the Toss of a Coin," *Euromoney*, December 1978, pp. 75-85.

Table I Moving Average Rules, 90-Day T-Bill Futures

Moving Average (Days)	Mar 76	Jun 76	Sep 76	Dec 76	Mar 77	Jun 77	Sep 77	Dec 77	Mar 78	Jun 78	Sep 78	Dec 78
60	0[a]	145	1170	5435	270	-2440	-2830	-800	-2300	-1250	-1990	2730
	(0)[b]	(180)	(480)	(240)	(780)	(540)	(780)	(300)	(600)	(600)	(840)	(420)
50	0	-1505	1910	5010	2855	-3695	-1660	-425	-1330	-2565	-1065	3030
	(0)	(480)	(240)	(240)	(420)	(1020)	(660)	(300)	(780)	(1140)	(840)	(420)
40	-95	-1910	-1875	5175	3030	380	-4015	-290	-3795	-2585	-1005	2805
	(120)	(360)	(600)	(300)	(420)	(420)	(1140)	(540)	(1020)	(1260)	(1080)	(1420)
30	-1000	1285	3660	5920	3980	885	-4825	-1610	-2555	-2285	-3675	960
	(300)	(240)	(240)	(180)	(420)	(540)	(900)	(960)	(1080)	(1260)	(1500)	(840)
20	-515	1910	6615	5370	4410	-880	-2150	-910	-1130	-4500	775	-2325
	(240)	(240)	(360)	(780)	(540)	(780)	(900)	(960)	(1080)	(1500)	(900)	(1500)
10	25	760	4570	3255	455	-470	1385	1680	-985	-2825	-105	195
	(300)	(840)	(1080)	(1620)	(1620)	(1620)	(1440)	(1020)	(1560)	(1800)	(1980)	(1980)
9	-175	725	3645	3545	785	-705	1870	-30	-1680	-2410	-130	-705
	(300)	(900)	(1380)	(1380)	(1740)	(1980)	(1380)	(1380)	(1680)	(2160)	(2280)	(1980)
8	-890	1095	3220	2580	640	-1275	-335	320	-1680	-1860	205	295
	(540)	(780)	(1380)	(1620)	(1680)	(2100)	(1860)	(1380)	(1680)	(2160)	(1920)	(1980)
7	-990	150	805	765	-415	-395	-480	420	-410	-1410	120	305
	(540)	(900)	(1620)	(1860)	(2340)	(2220)	(1980)	(1380)	(1560)	(2160)	(2280)	(2220)
6	-1610	-820	1885	575	-30	2280	1305	1350	-800	2235	300	545
	(660)	(1020)	(1740)	(2100)	(2580)	(2220)	(2220)	(1500)	(1800)	(2460)	(2400)	(2580)
5	-1610	-400	1080	1335	-2050	3545	1035	670	305	-3010	-1690	1150
	(660)	(1500)	(2220)	(2340)	(2700)	(2280)	(2340)	(1980)	(1920)	(2760)	(2940)	(2700)

a. Net profits in dollars.
b. Commissions.

The Treasury bill futures market may prove to be a useful vehicle for arbitrage or for hedging.[9] In the long run, it will not prove a profitable vehicle for speculators. On the other hand, since in the long run almost any type of price movement can be achieved by a random walk series, it may not be possible to show that a particular trading rule won't work. As long as one or another type of trading rule produces consistent profits over a given time period, its advocates are unlikely to be dissuaded by arcane theoretical arguments about random walks; technical analysis will probably be around indefinitely.

Conclusion

The fact that mechanical trading rules may show profits for long periods of time is not inconsistent with the possibility that price changes in the markets are random. One conclusion to be drawn from the random walk hypothesis is that, if a speculator keeps playing the game long enough, he will lose. Nothing in our study of the Treasury bill futures market would contradict this conclusion.

One of the most important implications of the arc sine law is that mechanical trading rules such as moving averages will not be profitable in the long run. In addition to its particular application to financial analysis, however, the law holds some broader implications for the way the world works. For example, when two sports teams play, even though they have equal ability, the arc sine law dictates that one team will probably be in the lead most of the game. But the law also says that games with a close final score are surprisingly likely to be "last minute, come from behind" affairs, in which the ultimate winner trailed for most of the game.

The result holds if the process generating the difference in score between the teams is symmetric with respect to time. Given such a process, and a close final score, a first reversal close to the ending buzzer (or gun, etc.) is as probable as a last reversal close to the beginning. Over a series of games in which close final scores are common, one team could easily achieve a string of several last minute victories. The coach of such a team might be credited with being brilliantly talented, for having created a "second half" team. While the coach could indeed be exceptional, there is nonetheless a good possibility that he owes his success to chance.

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[9] For example, both William Poole, "Using T-Bill Futures to Gauge Interest Rate Expectations," *Federal Reserve Bank of San Francisco Economic Review*, Spring 1978, pp. 7-19, and Don Puglisi, "Is the Futures Market for Treasury Bills Efficient?" *Journal of Portfolio Management*, Winter 1978, pp. 64-67, believe that arbitrage possibilities are nonexistent. For the opposite view, see Anthony J. Vignola and Charles J. Dale, "Is the Futures Market for Treasury Bills Efficient?" *Journal of Portfolio Management*, Winter 1979, pp. 78-81, and Richard W. Lang and Robert H. Rasche, "A Comparison of Yields on Futures Contracts and Implied Forward Rates," *Federal Reserve Bank of St. Louis Monthly Review*, December 1978, pp. 21-30.

Or consider identical twins with equal intelligence and ambition. The arc sine law says that, in any given course in school, one of the two may consistently lag the other. As a result, the hapless laggard may find himself being punished for his relatively poor performance. His indignant parents may forbid him to attend his high school basketball games, thereby depriving him of the opportunity of seeing a team that may be headed by a promising young coach who has a reputation for achieving last minute victories with his great second half team!