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**Modelling of stochastic fat-tailed
auto-correlated processes: an application
to short-term rates**

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The Model of the Short Rate Stochastic Dynamics

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Abstract

This paper proposes the short rate model with account of the stochastic dynamics of the short rates. Widely used financial products sensitive to the daily rate changes dictate importance of the adequate modelling of short rates. Their intrinsic properties are investigated based on the historical market data. We introduce the new model with the non-gaussian random driver and the auto-correlation factor. Special calibration procedures for the model are presented. Short rate stochastic dynamics and its display in an overnight indexed swap were investigated in several numerical experiments. ¹

1. Introduction

An overnight rate (OR) modelling experiences a common problem how to impose simultaneously non-gaussian statistics and significant interdependence of daily changes. There are different approaches to address these issues: jump processes, fat-tail parameters, limiting boundaries (James 2000), mixture of normal distributions (Lee 1999 and Kim 2000), correlations in quiet and hectic days (Kim 2000) and (Embrechts 1999), volatility and correlation interdependence (Loretan 2000), using the Student t-distribution with non-integer values of degree of freedom as a simulation tool to approximate fat-tail distribution (Heikkinen 2002), a jump-diffusion model with the jump intensity as a time-dependent function was used by (Samuelides 2001), etc. We introduce here new stochastic model of the short rate based on the intrinsic properties of daily rate changes.

2. Analysis of the historical data

Statistics of overnight rates r_i is based on the nature of daily changes $\Delta r_i = r_{i+1} - r_i$. The stochastic characteristics of the OR changes Δr_i were derived from the available historical data (USD currency is used as an example). Typical pattern of OR daily changes is presented in Fig.1, and its probability distribution - in Fig.2.

¹ The views expressed in this paper are solely those of the authors and do not necessarily reflect the views of the Bank of Montreal or the Photonics Research Ontario.

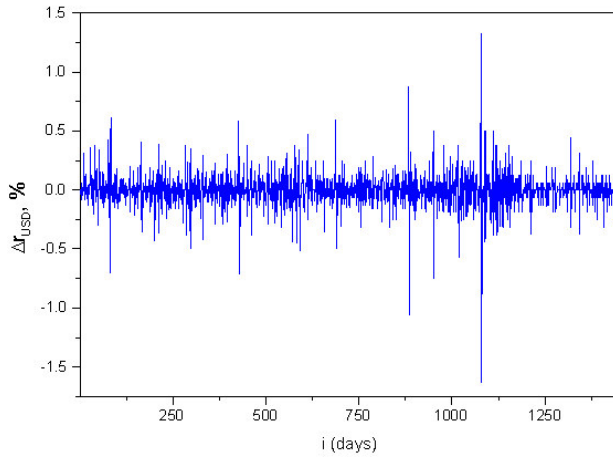


Fig.1.

Overnight rate daily changes (USD, August 1995 - August 2001)

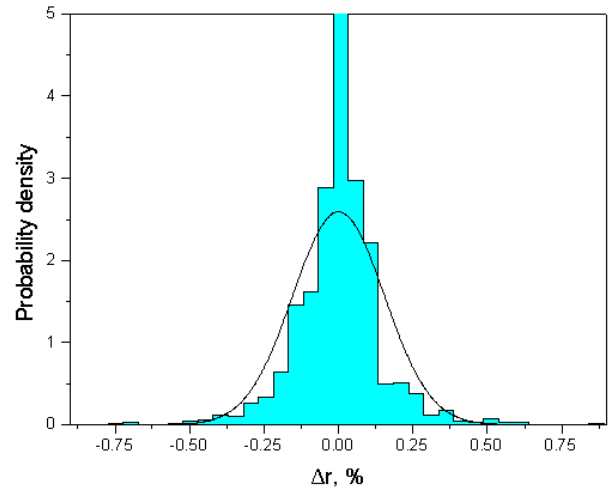


Fig.2.

Probability distribution of historical overnight rate daily changes (curve is the Gaussian function)

This distribution represents narrow central peak ($\pm 0.1\%$) and wide background (“fat tails”) expanding in the range of $\pm 0.5\%$. The first one reflects low volatility OR behavior, and the second one is due to relatively rare “hectic days” jumps in OR. For comparison, we plotted normal distribution function with the same standard deviation σ_0 as the historical data

probability distribution. The difference is obvious. Stochastic distribution of this type cannot be characterized by standard deviation only. There are additional parameters: upper and lower bounds: $\Delta r_u = \text{percentile}(\Delta r_i, p_u)$ and $\Delta r_L = \text{percentile}(\Delta r_i, 1 - p_u)$, upper and lower shortfall values: $\Delta r_{su} = \text{average}(\Delta r_i, \Delta r_i \geq \Delta r_u)$ and $\Delta r_{sL} = \text{average}(\Delta r_i, \Delta r_i \leq \Delta r_L)$. Shortfall value (or *tail conditional expectation* (Artzner, Delbaen, Eber and Heath 1999)) is used as a risk coherent measure. In our case the shortfall value is an essential parameter for calibration of our model. Confidence level is assumed as $p_u = 0.99$. Statistical analysis results are summarized in the following table.

Statistics on Δr (USD) at the confidence level 99%							
Δr_{sL}	Δr_L	Average	Median	σ_0	Δr_u	Δr_{su}	Max / Δr /
- 0.651 %	- 0.415 %	0.001 %	0.000 %	0.151 %	0.400 %	0.586 %	1.630 %

The average of the rate daily changes is close to zero (less than 0.002%), and skewness is very low (we assume it negligible). Based on the historical experience one can anticipate correlation between daily changes within several days interval. Direct calculations of auto-correlation parameter $z_k = \text{covar}(\Delta r_i, \Delta r_{i+k})$ of the daily changes revealed that the one-day auto-correlation (“memory”) works as a strong stabilizer: $z_1 = -0.29$ (USD). For most of other currencies parameter z_1 is also negative (for example, $z_1 = -0.25$ CAD, $z_1 = -0.30$ EUR). The influence of the negative correlation of this magnitude on the rate temporal behavior is obviously very significant (the “next day” rate change provides some compensation of the “previous day” rate change, thus keeping rate not too far from the initial value).

3. Overnight rate stochastic model

Based in the analysis of the historical stochastic characteristics of the OR we introduce the following model. The daily change (i^{th} day) with m days of "memory" is chosen as a weighted sum of independent random values ε_i (each associated with i^{th} day representing "non-correlated" daily changes)

$$\Delta r_i = \sum_{j=0}^m \alpha_j \varepsilon_{i-j} (\sigma_1, \sigma_2, l), \quad (1)$$

where vector α is normalized: $\sum_{j=0}^m \alpha_j^2 = 1$.

Auto-correlation vector \mathbf{z} is defined as

$$z_k = \frac{\overline{\Delta r_i \cdot \Delta r_{i+k}}}{(\overline{\Delta r_i})^2} \quad (2)$$

Here and below the ensemble averaging of a stochastic variable Ψ or averaging of the historical data set Ψ_i by i is denoted as $\overline{\Psi}$. Combining (1) and (2) we have

$$z_k = \frac{\overline{\sum_{j=0}^m \alpha_j \varepsilon_{i-j} \cdot \sum_{q=0}^m \alpha_q \varepsilon_{i+k-q}}}{\left(\overline{\sum_{j=0}^m \alpha_j \varepsilon_{i-j}} \right)^2} \quad (3)$$

Assume that ε_i are independent: $\overline{\varepsilon_i \cdot \varepsilon_j} = \sigma_i^2 \delta_{ij}$ (symbol δ_{ij} denotes Kroneker symbol). Then

$$z_k = \sum_{j=0}^{m-k} \alpha_j \cdot \alpha_{j+k}, \quad k = 0, \dots, m, \quad z_0 = 1 \quad (4)$$

Vector α can be derived using values of \mathbf{z} (from market data) by the following iterative procedure:

<i>Iteration procedure A</i>
1 st iteration: Choose initial iteration values $\alpha_0^{(1)} = \sqrt{1 - z_1^2}$, $\alpha_{j \geq 1}^{(1)} = 0$;
(*) s th iteration: calculate $\alpha_k^{(s+1)} = \frac{1}{\alpha_0^{(s)}} \left(z_k - \sum_{j=1}^{m-k} \alpha_j^{(s)} \cdot \alpha_{j+k}^{(s)} \right)$ in the reverse order ($k = m, \dots, 0$);
Repeat (*) until equation (4) is satisfied with required precision.

Convergence of this procedure is quite reliable (see Fig.3 for maximal error of the equation (4) versus number of iterations). Dependence of auto-correlation factors z_i and parameters α_i on the number of days i is presented in Fig.4.

The dominant feature of the auto-correlation pattern is the high negative ‘next-day’ auto-correlation (z_1 and α_1). Parameters σ_1 and σ_2 of the stochastic process $\varepsilon(\sigma_1, \sigma_2, l)$ with ‘fat tail’ probability distribution $P(x; \sigma_1, \sigma_2, l)$ represent the low-volatility and the high-volatility components, and parameter l is their relative contribution to the deviation.

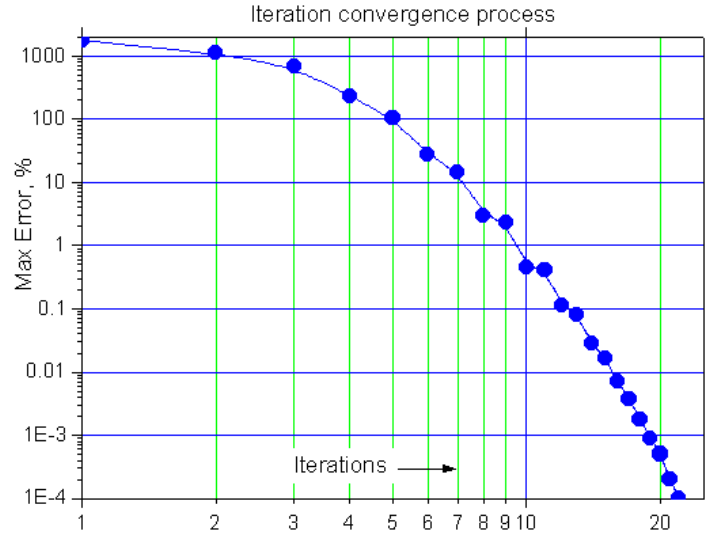


Fig.3. Convergence of the iterative procedure A

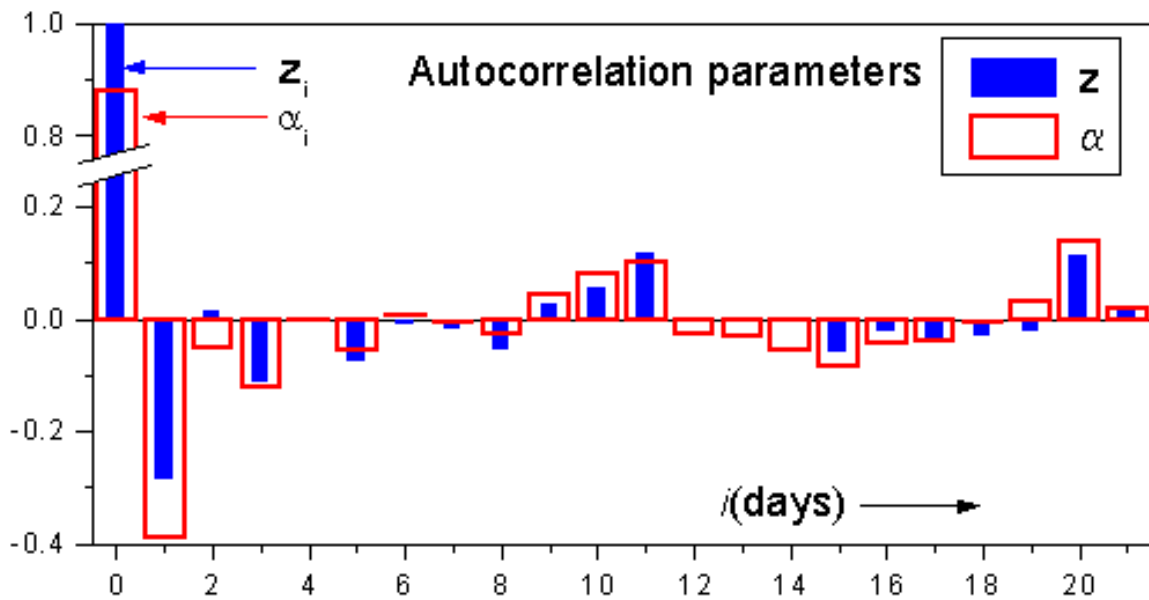


Fig.4.

The natural way to construct the probability distribution function for modelling daily changes of the overnight rate is to use a linear combination (weighted sum) of two normal distributions with different standard deviations.

$$P(x; \sigma_1, \sigma_2, l) = \frac{1}{\sqrt{2 \cdot \pi}} \left(\frac{1}{\sigma_1} e^{-\frac{1}{2} \left(\frac{x}{\sigma_1} \right)^2} + \frac{(1-l)}{\sigma_2} e^{-\frac{1}{2} \left(\frac{x}{\sigma_2} \right)^2} \right) \quad (5)$$

Similar approach (mixture of two lognormal densities as probability density function) to model the short-term dynamics of the euro/dollar was used by (Rzepowski 2002).

Parameters σ_1 , σ_2 and l can be found as a unique solution of the corresponding set of conditions: simultaneous fit of the model standard deviation, the upper bound and upper shortfall value to the historical ones. Calibration based on account of both the upper bound and shortfall provides efficient modelling of rare jump-like rate deviations. This solution can be found using iterative search of the minimum of the following objective function

$$F(\sigma_1, \sigma_2) = \left| p_u - \int_{-\infty}^{\Delta r_u} P(x; \sigma_1, \sigma_2, l) \cdot dx \right| + \left| \Delta r_{su} - \int_{\Delta r_u}^{\infty} x \cdot P(x; \sigma_1, \sigma_2, l) \cdot dx \right| \quad (6)$$

with $l = (\sigma_0^2 - \sigma_1^2) / (\sigma_2^2 - \sigma_1^2)$.

A special random number generator is required to produce series of values $\varepsilon(\sigma_1, \sigma_2, l)$ with probability distribution (5). For simulation experiments it was used random number generator based on rejection method (Press1992). Typically, volatility σ_1 (quiet periods) is much lower than σ_2 (hectic periods) by 3 to 6 times, and weight l of the σ_2 -part of the distribution is less than 0.2. For example, results of the calibration procedure for USD OR are: $\sigma_1=0.12\%$, $\sigma_2=0.38\%$ and $l=0.06$. Convergence of the minimization procedure of the object function (6) is shown in Fig. 5.

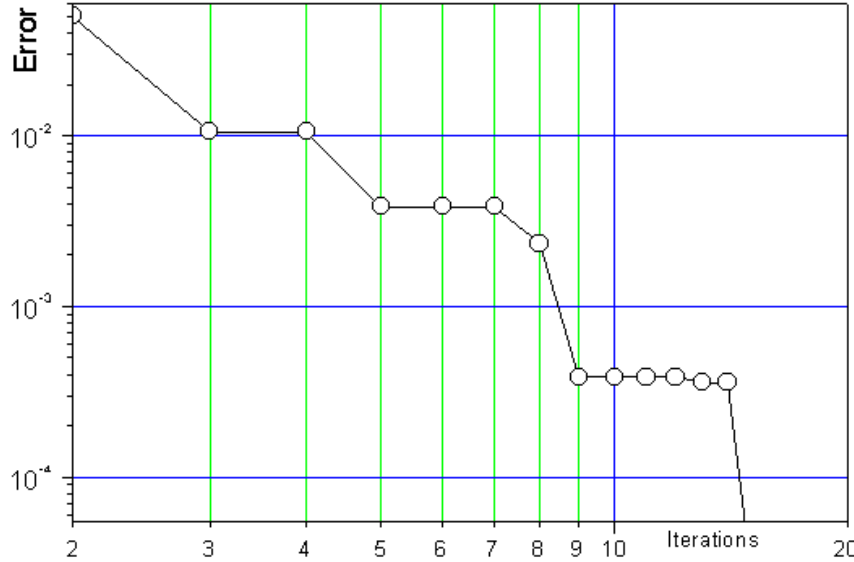


Fig.5
Objective function (6) versus iteration number

Calibration algorithm (resulting in identification of parameters $\alpha_i (i=1..m)$, l , σ_1 and σ_2) satisfies conditions of a low risk model (precision fit is better than 10^{-4}). Fig.6 illustrates best fit of the model probability distribution function (5) to historical distribution. For comparison there is the Gaussian function plotted with the historical standard deviation.

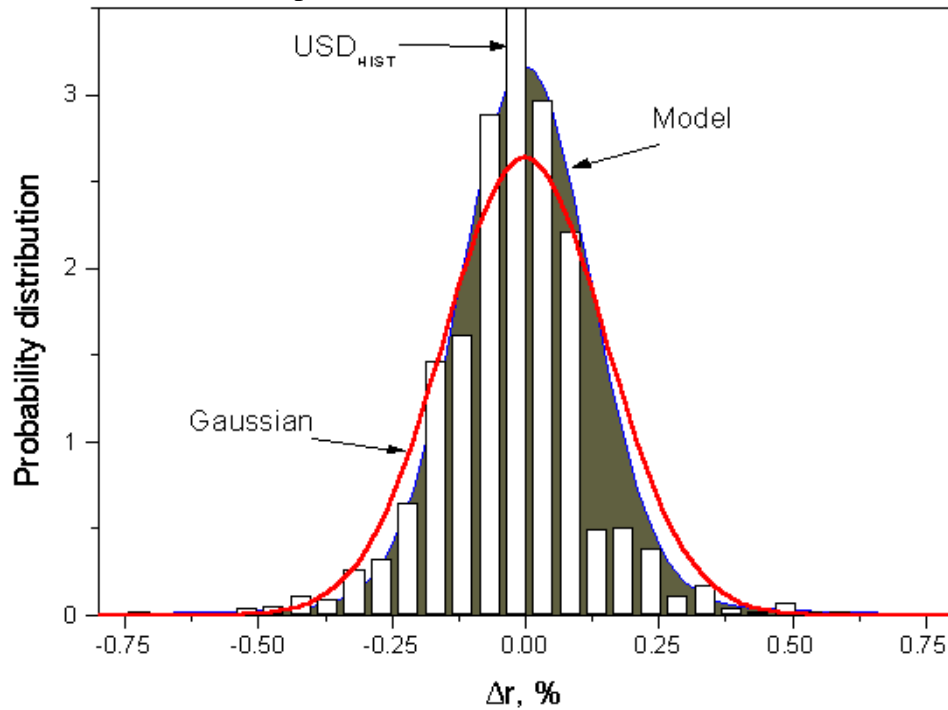


Fig.6

Model ‘fat-tail’ function fits much better the historical distribution than a Gaussian. It is even more persuasive in Fig.7 for the tail area of distributions.

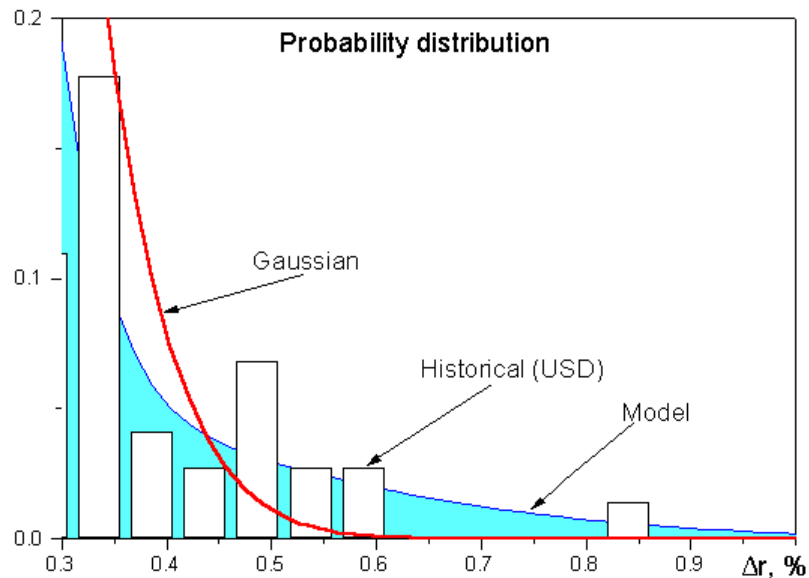


Fig.7.

4. Overnight rate stochastic simulation

An example of the historical rate path and two simulated scenarios for USD OR (starting from August 1, 1995) is presented in Fig.8. The simulated path of the OR (curve

"Autocorrelated") well illustrates two main features of the OR: periods with low volatility and periods with rapid fall or rise of the rate. The simulation path ("Non-correlated") illustrates importance of the auto-correlation account (simulation with zero correlation shows unstable behavior: rapid long-range deviations that are not observed historically). This illustration cannot obviously be used for statistical risk estimations: it is merely visual aid in the model examination.

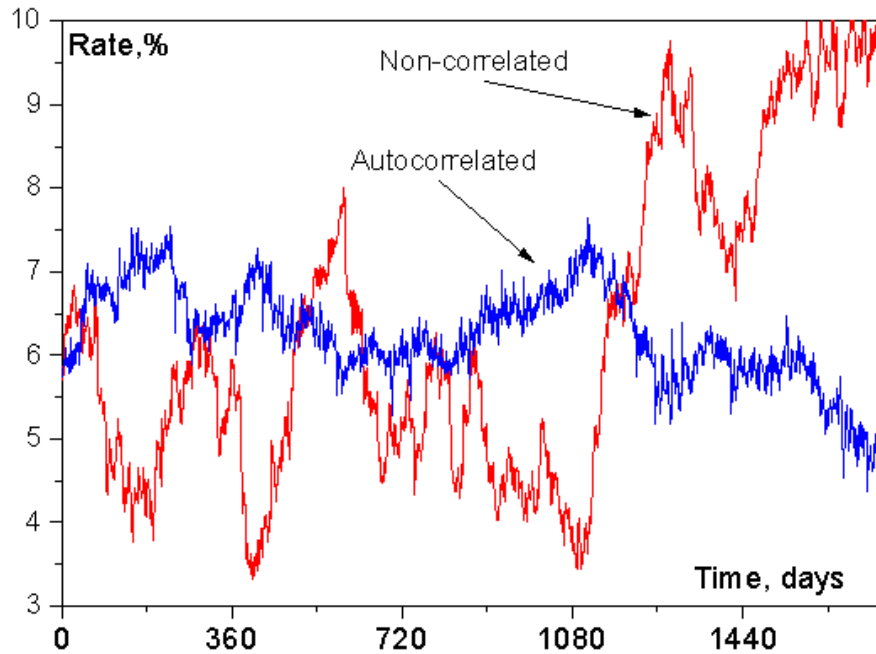


Fig.8

Statistical characteristics of the model and its ability to give “statistical forecast” are presented in Fig.9. Based on model parameters identified above using historical data from August 1, 1995 to August 1, 2001 (1450 days) we applied Monte Carlo simulation method to derive lower and upper shortfall values as well as lower and upper percentiles (with confidence level 99%) of the overnight rate for extended time horizon of 1740 days (to May 18, 2002). For comparison, the historical data are plotted on the same graph.

One can see that historical OR curve is locked inside limits of the 1st and 99th percentiles (open circles) during the whole time span used for model parameters estimation (1450 days). It is important to note that even unusually steep downturn of the OR beyond this time period ($1450 < t < 1740$) is still inside the limits of shortfall curve (solid circles). It means that the model is able to “predict” (at given confidence level) the limits of the OR path.

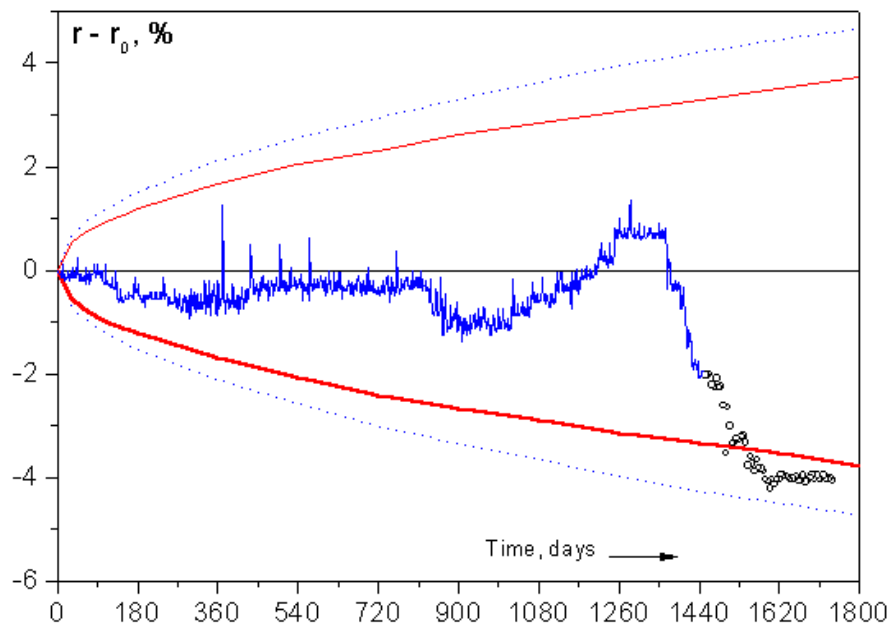


Fig.9

Tendency of the OR probability distribution changes as the time period (term) increases is illustrated in Fig.10 and 11. It is important to note that standard deviation of simulated rates increases as time term increases at much lower rate than it could in case of the normally (Gaussian) distributed stochastic process.

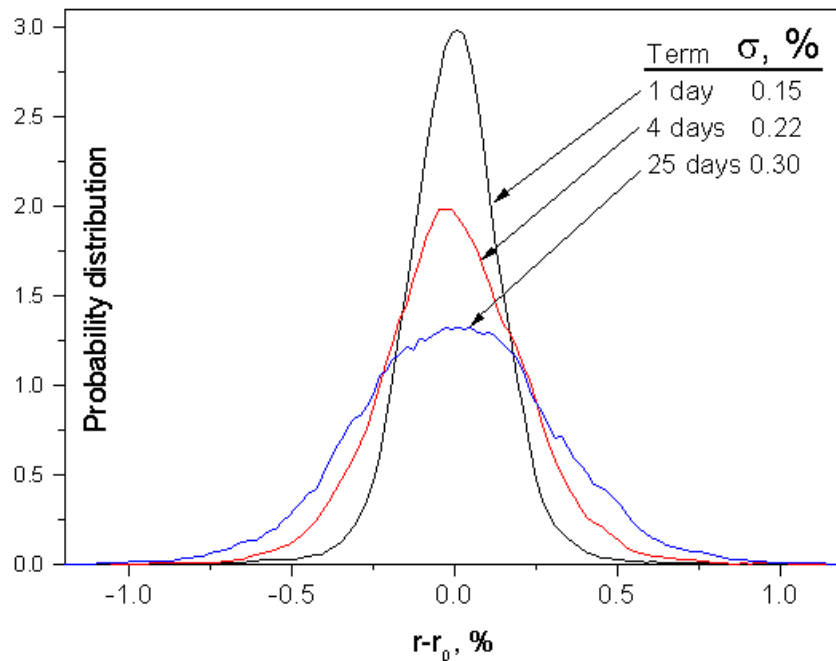


Fig.10

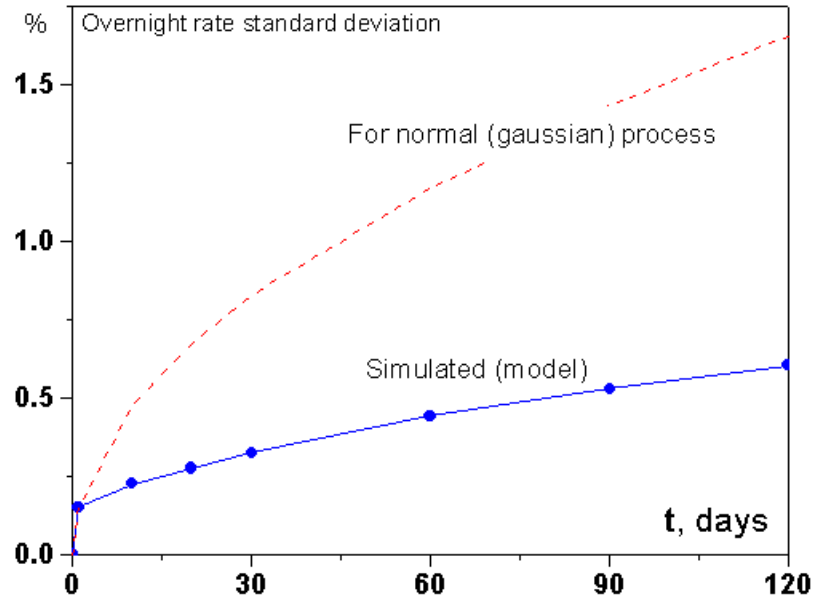


Fig.11

In case of a Gaussian process standard deviation increases as

$$\sigma_t^G = \sigma_0^{Hist} \cdot \sqrt{t}$$

where $\sigma_0^{Hist} = 0.151$ corresponds to the historically identified standard deviation of rate daily changes (see dotted curve in Fig.11). The simulated rates have much lower standard deviation (see ‘simulated’ curve) because of the negative auto -correlation influence.

5. Overnight indexed swap modeling

As an example of the OR model application we used the exposure valuation of the overnight indexed swaps (OIS) in the Monte Carlo methodology framework. Value V_M of the OIS (long position) was calculated as the sum of differential payments $P \cdot (r_0 - r_i)$ at moments i compounded from i to the swap maturity M (commonly used formula):

$$V_M = P \sum_{i=1}^M (r_0 - r_i) \Delta t \prod_{j=i}^M (1 + r_j \cdot \Delta t)^{\Delta t} \quad (7)$$

Here $P=100\%$ is the notional, rate r_0 is fixed, and r_i is the modeled floating rates, payment period is one day $\Delta t = 1/360$.

The Monte Carlo set $V_M^{(k)}$ was calculated (k is the index of the k^{th} scenario for the contract maturity M) using the OR model. In Fig.12 the 95th percentile of the OIS exposure as a function of the maturity M at various model parameters is plotted.

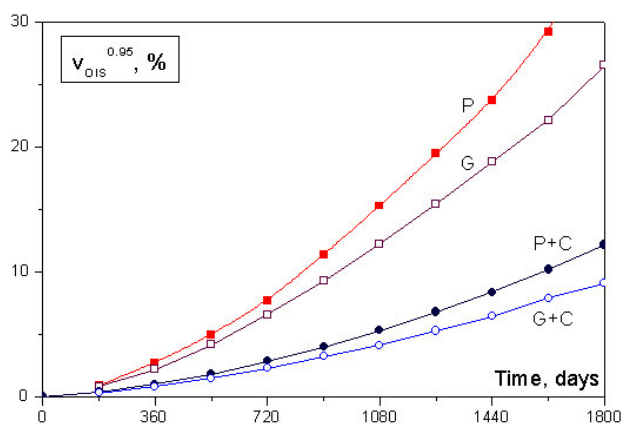


Fig.12

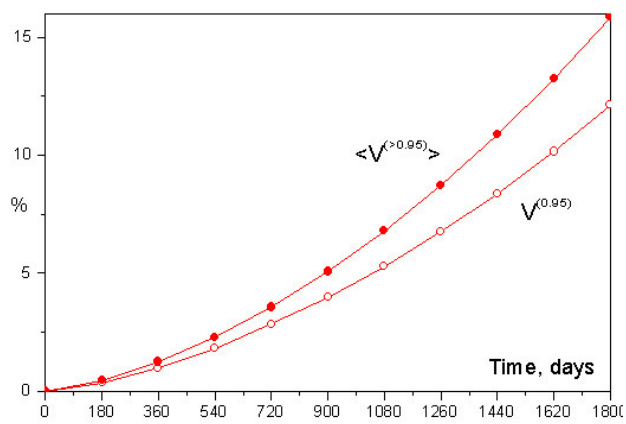


Fig.13

The curve P+C corresponds to the complete (correct) OR model with account of auto-correlations (C) and the fat-tail daily changes distribution (P). Modelling with a gaussian (G) distribution causes underestimation (curve G+C), lack of auto-correlations in the model causes significant overestimation (curve G is for normally distributed daily changes, and curve P is for non-correlated daily changes fat-tail distributed). The whole term profile of the "worst case" (0.95th percentile $V^{(0.95)}$) value and corresponding shortfall $\langle V^{(>0.95)} \rangle$ of the OIS exposures is presented in Fig.13.

6. Conclusion

The paper presents a new model for the stochastic dynamics of the short rates that accounts both for fat-tail distribution and auto-correlation vector. The model originates from direct statistical analysis of market data. This analysis shows that probability distribution of rate daily changes is far from gaussian and that next-day rate changes are anti-correlated. This fat-tail probability distribution is modelled by weighted sum of two gaussian functions. The auto-correlation function derived directly from market data is incorporated into the model. Calibration of the stochastic model is based on percentile measures and shortfall values of historical data. Numerical simulations of the overnight rate temporal behavior and its application to the exposure estimations of the overnight indexed swap illustrate that there is significant influence of the non-gaussian probability distribution and auto-correlations.

References

- Artzner P, Delbaen F, Eber J.-M and Heath D 1999 Coherent measures of Risk *Math. Finance* **9** (3) 203-228
- Embrechts P, McNeil A and Straumann D 1999 Correlation: pitfalls and alternatives *RISK Magazine, May*, 69-71
- Heikkinen V.-P and Kanto A (2002) Value-at-risk estimation using non-integer degrees of freedom of Student's distribution, *Journal of Risk*, **4**(4), 77 – 84
- James J and Webber N 2000 Interest rate modelling *Wiley*
- Kim J and Finger C C 2000 A stress test to incorporate correlation breakdown *RiskMetrics Journal* **1**

- Lee Y 1999 The Federal funds market and the overnight Eurodollar market *Deutsche Bank AG, DB Research, Research Note RN-99-2*
- Loretan M and English W B 2000 Evaluating “correlation breakdowns” during periods of market volatility *Board of Governors of the Federal Reserve System, February*
- Press W H, Flannery B P, Teukolsky S A and Vetterling W A 1992 Numerical Recipes: The Art of Scientific Computing *Cambridge University Press, Cambridge*
- Rzepowski B (2002) Heterogeneous expectations, currency options and the Euro/dollar, *Quantitative Finance* **2** 147 – 157
- Samuelides Y and Nahum E 2001 A tractable market model with jumps for pricing short-term interest rate derivatives *Quantitative Finance* **1** (2) 270-283