Wealth Martingale and Neighborhood Turnpike Property in Dynamically Complete Market with Heterogeneous Investors

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WEALTH MARTINGALE AND NEIGHBORHOOD TURNPIKE PROPERTY IN DYNAMICALLY COMPLETE MARKET WITH HETEROGENEOUS INVESTORS

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Abstract

In the current paper, a dynamically complete financial market with finite and countable heterogeneous investors has been constructed. Self-dynamic game is defined, that is, the investors determine the optimal bankruptcy time first, and then the optimal portfolio policy. Sub-game perfect Nash equilibrium bankruptcy time is derived and it is confirmed that there exists a unique value of efficient terminal wealth for each investor. The interesting theorem of the current paper proves that the vector of efficient terminal wealth exhibits neighborhood turnpike property if the corresponding optimal path of wealth accumulation is a martingale for each investor. This result would be regarded as an interesting neighborhood turnpike theorem in mathematical finance because it focuses on terminal wealth accumulation of the investors which indeed plays a crucial role in mathematical finance. And it also provides us with an internal\intrinsic and a simple relationship between fairness and efficiency characterizations of the modern financial-market institutions.

Keywords: Optimal portfolio policy, Sub-game perfect Nash equilibrium bankruptcy time, Heterogeneous investors, Wealth martingale, Neighborhood turnpike theorem

JEL classification: C61, G11, G30

1. Introduction

As is well-known, when discussing efficient capital accumulation (see, Gong and Zou, 2000, 2002; Ray, 2010, and among others) in macroeconomics and turnpike theorem (see, McKenzie, 1976, 1998, and among others) in mathematical economics, efficiency is usually defined with reference to the final state (see, Radner, 1961; Kurz, 1965; Dai, 2012, 2013, and among others) or the terminal stocks (e.g., McKenzie, 1963, 1976). Similarly, in most mathematical-finance models (see, Merton, 1971, 1973; Pliska, 1986; Kramkov and Schachermayer, 1999, 2003; Touzi, 2000; Cvitanić and Wang, 2001; Schachermayer, 2001; Pham and Quenez, 2001; Brennan and Xia, 2002; Owen, 2002; Hugonnier and Kramkov, 2004; Schied, 2005), expected utility maximization problem is often specified with respect to the terminal wealth of the corresponding investor in any given complete or incomplete market. Nevertheless, most of existing literatures (e.g., Pliska, 1986; Touzi, 2000; Pham and Quenez, 2001; Owen, 2002; Kramkov and Schachermayer, 1999, 2003; Schied, 2005, and among others) share the common setting of an agent who maximizes the expected utility of her *pre-specified* terminal wealth up to a *pre-specified* terminal time.

However, in reality, especially in financial market, both the terminal time and the terminal wealth are *endogenously* determined, which actually throws new insights into our understanding of the microeconomic behaviors of the investors relative to the case where both the terminal time and the corresponding terminal wealth are exogenously given. Therefore, provided the above considerations, the current paper, along the same line of Karatzas and Wang (2001), Jeanblanc et al (2004), Øksendal and Sulem (2005) and Dai (2012), considers the case with endogenously determined terminal time, which would be translated to the category of financial economics, i.e., an optimal bankruptcy time in certain sense. Moreover, the current paper defines a self-dynamic game of the investors, that is, investors determine the optimal bankruptcy time first, and then the optimal portfolio policy, which thus results in a sub-game perfect Nash equilibrium bankruptcy time by using the famous backward induction rationality principle that is widely employed in dynamic non-cooperative game theories, thereby extending the framework of Karatzas and Wang (2001), Jeanblanc et al (2004), Øksendal and Sulem (2005) and Dai (2012). This may be regarded as the first innovation of the current paper.

The major and also interesting finding of the current paper is that the *vector* of the endogenously determined terminal wealth level for all heterogeneous investors exhibits *neighborhood turnpike property* if the corresponding wealth dynamics is a *martingale* for each investor, respectively, which can be achieved by the well-known Girsanov Theorem in most cases. And this would be regarded as the second contribution of the present model. Noting that neighborhood turnpike theorem (see, Bewley, 1982; McKenzie, 1982; Yano, 1984a, 1984b, 1998, 1999; Marena and Montrucchio, 1999;

Guerrero-Luchtenberg, 2000; Sahashi, 2002; Kamihigashi and Roy, 2007; Kondo, 2008; Dai, 2012) as well as asymptotic turnpike theorem (e.g., Scheinkman, 1976; Brock and Scheinkman, 1976; Araujo and Scheinkman, 1977; Yano, 1984c, 1985, 1999; Sahashi, 2002; Dai, 2012, and among others) plays a crucial role in optimal growth theory, the neighborhood turnpike theorem proved in the current paper would be seen as an extension to some extent and thus a natural correspondence in financial economics compared to existing studies focused on macro-economics, in particular, optimal growth theory or capital accumulation theory in dynamic general equilibrium economy (e.g., Becker, 1980; Bewley, 1982; Yano, 1984a, 1984b, 1985; Coles, 1985; Nishimura and Shimomura, 2002, and among others).

Finally, noting that market selection theory (e.g., Blume and Easley, 1992, 2006; Luo, 1998; Sandroni, 2000; Chiarella and He, 2001; Chiarella et al, 2006, and among others), specifically wealth-driven selection theory (see, De Long et al, 1991; Anufriev, 2008; Anufriev and Dindo, 2010; Brianzoni et al, 2012, and among others), plays a more and more important role in both financial economics and evolutionary economics, the present investigation would in certain sense provides us with some new inspirations to existing approach from the viewpoint of wealth dynamics and its mathematical properties in a type of complete market with discretionary stopping time and heterogeneous investors as a whole.

In other words, when sub-game perfect Nash equilibrium bankruptcy time is defined for all heterogeneous investors, we then are led to the robust conclusion that the endogenously determined terminal wealth is indeed the *stationary state* of the optimal path of wealth accumulation for each investor if the optimal path of wealth accumulation exhibits martingale property, that is, the optimal path will spend the most time staying in the arbitrarily small neighborhood of the efficient terminal wealth, which is also a constant thanks to the optimal stopping theory of continuous time. Accordingly, wealth-driven selection theory can build on the above efficient terminal wealth, a constant, rather than the whole path, a stochastic differential equation (SDE). And we intuitively and heuristically argue that market selection theory based upon constant efficient terminal wealth will be a good approximation to that based on the whole path of optimal wealth accumulation by noting that the Lebesgue measure of possible loss or error would approach zero in infinity. Obviously, our approach makes things much easier. And we leave much deeper issues about market selection theory for future studies.

The present paper proceeds as follows. Section 2 presents the general model where the basic definitions and assumptions about the complete financial market facing heterogeneous investors are introduced; section 3 solves the casual individual optimization problem by using classical technique of dynamic programming and then optimal portfolio policy is derived; section 4 confirms the existence and uniqueness of the sub-game perfect Nash equilibrium bankruptcy time based upon the results given in section 3; section 5 proves the key theorem of the current paper, that is, neighborhood turnpike theorem of the vector of the efficient terminal wealth provided the results given in sections 3 and 4. There is a brief concluding section. All proofs, unless otherwise noted in the text, appear in the Appendix.

2. The general model

Suppose that there are I heterogeneous investors in the underlying market. We denote by $(\Omega^{(W^i)}, F^{(W^i)}, \{F_t^{(W^i)}\}_{0\le t\le \tau^i(\omega)}, P^{(W^i)})$ the filtered probability space with $\overline{F}^{(W^i)} \equiv \{F_t^{(W^i)}\}_{0\le t\le \tau^i(\omega)}$ the $P^{(W^i)}-$augmented filtration generated by the following $d-$dimensional standard Brownian motion $\{W^i(t)\}_{0\le t\le \tau^i(\omega)}$ with $F^{(W^i)} \equiv F^{(W^i)}_{\tau^i(\omega)}$, where $\tau^i(\omega)$ is a given stopping time for $\omega \in \Omega^{(W^i)}$. Moreover, we define

$$\widetilde{N}^i(dt,dz) = (\widetilde{N}_1^i(dt,dz_1),\ldots,\widetilde{N}_n^i(dt,dz_n))^T$$
$$= (N_1^i(dt,dz_1) - \nu_1^i(dz_1)dt,\ldots,N_n^i(dt,dz_n) - \nu_n^i(dz_n)dt)^T,$$

where $\{N_l^i\}_{l=1}^n$ are independent Poisson random measures with Lévy measures ν_l^i coming from n independent (1-dimensional) Lévy processes,

$$\eta_1^i(t) \equiv \int_0^t \int_{R_0} z_1 \widetilde{N}_1^i(ds,dz_1) \ , \ldots , \ \eta_n^i(t) \equiv \int_0^t \int_{R_0} z_n \widetilde{N}_n^i(ds,dz_n),$$

where $R_0 \equiv R - \{0\}$ as the usual definition, and therefore the corresponding stochastic basis is given by $(\Omega^{(\widetilde{N}^i)}, F^{(\widetilde{N}^i)}, \{F_t^{(\widetilde{N}^i)}\}_{0\le t\le \tau^i(\omega)}, P^{(\widetilde{N}^i)})$ with $\overline{F}^{(\widetilde{N}^i)} \equiv \{F_t^{(\widetilde{N}^i)}\}_{0\le t\le \tau^i(\omega)}$ the $P^{(\widetilde{N}^i)}-$augmented filtration with $F^{(\widetilde{N}^i)} \equiv F^{(\widetilde{N}^i)}_{\tau^i(\omega)}$ for $\omega \in \Omega^{(\widetilde{N}^i)}$. Thus, we are provided with a new stochastic basis $(\Omega^i, F^i, \{F_t^i\}_{0\le t\le \tau^i(\omega)}, P^i)$, where $\Omega^i \equiv \Omega^{(W^i)} \times \Omega^{(\widetilde{N}^i)}$, $F^i \equiv F^{(W^i)} \otimes F^{(\widetilde{N}^i)}$, $F_t^i \equiv F_t^{(W^i)} \otimes F_t^{(\widetilde{N}^i)}$, $P^i \equiv P^{(W^i)} \otimes P^{(\widetilde{N}^i)}$ and $\overline{F}^i \equiv \{F_t^i\}_{0\le t\le \tau^i(\omega)}$, and also the underlying probability measure space is assumed to satisfy the so-called "usual conditions". Here, and throughout the current paper, E^i is used to denote the expectation operator with respect to (w. r. t.) the probability law P^i for $\forall i = 1, 2, ..., I$. Accordingly, we have the new stochastic basis $(\Omega, F, \{F_t\}_{0 \le t \le \tau(\omega)}, P)$ with $\Omega \equiv \Omega^1 \times ... \times \Omega^I$, $F \equiv F^1 \otimes ... \otimes F^I$, $F_t \equiv F_t^1 \otimes ... \otimes F_t^I$, $\tau(\omega) \equiv \tau^1(\omega) \vee ... \vee \tau^I(\omega)$ for $\forall \omega \in \Omega$, $P \equiv P^1 \otimes ... \otimes P^I$, $\overline{F} \equiv \{F_t\}_{0 \le t \le \tau(\omega)}$ denoting the corresponding filtration satisfying the usual conditions, and E denoting the expectation operator w. r. t. the probability law P .

We define the canonical Lebesgue measure μ on measure space $(R_+, \mathrm{B}(R_+))$ with $R_+ \equiv [0, \infty)$, $R_{++} \equiv (0, \infty)$ and $\mathrm{B}(R_+)$ the Borel sigma-algebra, and also the corresponding regular properties about Lebesgue measure are supposed to be fulfilled. Thus, we can define the following product measure space $(\Omega^i \times R_+, F^i \otimes \mathrm{B}(R_+))$ and $(\Omega \times R_+, F \otimes \mathrm{B}(R_+))$ with corresponding product measure $\mu \otimes P^i$ and $\mu \otimes P$, respectively, for $\forall i = 1, 2, ..., I$.

Now, based upon the probability space $(\Omega^i, F^i, \overline{F}^i, P^i)$, we define the dynamically complete market (see, Anderson and Raimondo, 2008, for example) as follows,

$$\begin{cases} dB(t) = r(t)B(t)dt, B(0) = 1. \\ dS_j^i(t) = S_j^i(t)\left[b_j^i(t)dt + \sum_{k=1}^{d} \sigma_{jk}(t)dW_k^i(t) + \sum_{l=1}^{n} \int_{R_0} \gamma_{jl}(t, z_l)\widetilde{N}_l^i(dt, dz_l) \right], S_j^i(0) > 0. \end{cases} \tag{1}$$

where $B(t)$ denotes the price process of a safe or riskless investment, i.e., bank account, and $S_j^i(t)$ denotes the price process of a risky investment, for instance, the stock, for $j = 1, 2, ..., m$ and $\forall i = 1, 2, ..., I$. And $r(t)$, $b_j^i(t)$, $\sigma_{jk}(t) \in R$ denote the riskless interest rate, the expectation return rate of the stock and the market volatility in period t , respectively, for $\forall i = 1, 2, ..., I$; $j = 1, 2, ..., m$ and $k = 1, 2, ..., d$. In particular, if we let $b_j(t)$ represent the true value of market mean return of stock j , then we get $E^i[b_j(t) \,|\, F_t^i] = b_j^i(t) >$ (res. $=$ or $<$) $b_j(t)$ if individual i

is an optimistic (res. a rational or a pessimistic) investor, which reflects heterogeneous beliefs in the underlying financial market for $\forall i=1,2,...,I$ and $\forall j=1,2,...,m$. With a little abuse of notations, we put $r \equiv r(0)$, $b_j^i \equiv b_j^i(0)$, $\sigma_{jk} \equiv \sigma_{jk}(0)$ and $\gamma_{jl}(z_l) \equiv \gamma_{jl}(0,z_l)$ for $\forall i=1,2,...,I$; $j=1,2,...,m$; $k=1,2,...,d$ and $l=1,2,...,n$. Moreover, all the above processes are supposed to be $F^i \otimes \mathrm{B}(R_+)-$progressively measurable. Then we have the following SDE of wealth accumulation,

$$dX^i(t) = X^i(t)\left[\sum_{j=1}^m \pi_j^i(t)\frac{dS_j^i(t)}{S_j^i(t)} + \left(1-\sum_{j=1}^m \pi_j^i(t)\right)r(t)dt\right]$$
$$= X^i(t)\left[\pi^i(t)^T(b^i(t)-r(t)\mathbf{1})+r(t)\right]dt + X^i(t)\pi^i(t)^T\sigma(t)dW^i(t)$$
$$+X^i(t)\pi^i(t)^T\int_{R_0^n}\gamma(t,z)\widetilde{N}^i(dt,dz), \tag{2}$$

subject to the initial conditions $X^i(0)=x^i \in R_{++}$, $W^i(0)=(0,...,0)^T$ P^i-a.s., and we denote by $\pi^i(t) \equiv (\pi_1^i(t),...,\pi_m^i(t))^T$ the portfolio policy. And as usual, we put $b^i(t) \equiv (b_1^i(t),...,b_m^i(t))^T$, $\mathbf{1} \equiv (1,1,...,1)^T$, $W^i(t) \equiv (W_1^i(t),...,W_d^i(t))^T$, where the superscript "T" denotes transpose, and $\sigma(t) \equiv (\sigma_{jk}(t)) \in R^{m\times d}$, $\gamma(t,z) \equiv (\gamma_{jl}(t,z_l)) \in R^{m\times n}$ denote bounded matrices. As before, we set up the initial conditions $\pi^i \equiv \pi^i(0) = (\pi_1^i(0),...,\pi_m^i(0))^T$, $b^i \equiv b^i(0) = (b_1^i(0),...,b_m^i(0))^T = (b_1^i,...,b_m^i)^T$, $\sigma \equiv \sigma(0) = (\sigma_{jk}(0)) = (\sigma_{jk})$ and also $\gamma \equiv \gamma(0,z) = (\gamma_{jl}(0,z_l)) = (\gamma_{jl}(z_l))$ for $\forall i=1,2,...,I$; $j=1,2,...,m$; $k=1,2,...,d$ and $l=1,2,...,n$. Furthermore, the wealth dynamics $X^i(t)$ is assumed to be $F^i \otimes \mathrm{B}(R_+)-$ adapted and all the remaining processes are $F^i \otimes \mathrm{B}(R_+)-$progressively measurable, for $\forall i=1,2,...,I$. When denoted in matrix form, we get,

$$dX(t) = f(X(t))dt + g(X(t))dW(t) + \int_{R_0^n} h(X(t),z)\widetilde{N}(dt,dz), \tag{3}$$

where $X(t) \equiv (X^1(t),...,X^I(t))^T$, $X(t)$ is assumed to be $F \otimes \mathrm{B}(R_+)-$adapted and $f(X(t))$,

$g(X(t))$, $h(X(t), z)$ are supposed to be $F \otimes \mathrm{B}(R_+)-$progressively measurable.

Now, as a preparation for proving our propositions and the key theorem, we, as usual, provide the following formal assumptions and definitions,

ASSUMPTION 1: *Here, and throughout the current paper, the real symmetric matrix* $\Sigma(t) \equiv \sigma(t)\sigma(t)^T$ *is assumed to be bounded and invertible,* $\gamma_{jl}(t, z_l) > -1 + \varepsilon \quad \mu \otimes \nu_l^i -$ *a.s. for* $\forall \varepsilon \in R_{++}$, $\forall j = 1, 2, ..., m$ *and* $l = 1, 2, ..., n$, *and we let* $\Theta^i(t) \equiv b^i(t) - r(t)\mathbf{1} \in R_{++} \quad \mu \otimes P^i -$ *a.e. for* $\forall i = 1, 2, ..., I$. *Moreover, with a little abuse of notations, we put* $\Sigma \equiv \Sigma(0) \equiv \sigma(0)\sigma(0)^T = \sigma\sigma^T$ *and* $\Theta^i \equiv \Theta^i(0) = b^i(0) - r(0)\mathbf{1} = b^i - r\mathbf{1}$ *for* $\forall i = 1, 2, ..., I$.

REMARK 1: It is worth emphasizing that $\Theta^i(t) \in R_{++} \quad \mu \otimes P^i -$ a.e. is just for the sake of simplicity and indeed without loss of any generality by noting that it does not essentially restrict our following turnpike theorem. Obviously, this assumption can be relaxed to imply much more cases at the cost of much more complicated computations.

ASSUMPTION 2: *The initial conditions of the wealth processes* $X^i(0) = x^i \in R_{++}$ $(\forall i = 1, 2, ..., I)$ *and* $X(0) = x \in R_{++}$ *are all supposed to be deterministic and bounded.*

ASSUMPTION 3 (Preference): *It is assumed that all the investors exhibit log preference* (*special CRRA preference*) *w. r. t. terminal wealth.*

REMARK 2: First, it is well-known that constant relative risk aversion (CRRA) utility function such as the well-known log preference has been broadly employed in both macroeconomic and financial models (see, among others, Dai, 2012, 2013; Dai et al., 2013; He and Krishnamurthy, 2012; Goll and Kallsen, 2000; Pang, 2006) owing to the tractability of the corresponding optimal solutions and also without loss of any generality in most cases. Thus, the current paper also employs CRRA preference but not other types of utility function, say, constant absolute risk aversion (CARA) preference. Second, empirical studies (see, Brunnermeier and Nagel, 2008) have confirmed that CRRA model provides us with a good approach to study microeconomic behaviors in reality. Third, noting that the key issue of the present paper is turnpike property of wealth dynamics, the proof of our major conclusion, indeed, is robust to the specification of utility functions. To summarize, Assumption 3 just plays a role of technical condition to make things much easier, in other words, the current model can be naturally extended to include very general preference functions while the major claim still keeping invariant.

Now, we need the following definitions,

DEFINITION 1 (Self-Dynamic Game): *It is supposed that the order of action of investors proceeds as follows*:

Step 1: *They choose an optimal bankruptcy time, i.e., an optimal stopping time, given any optimal portfolio policy. That is, they first determine the time-dimensional control variable.*

Step 2: *Given the optimal bankruptcy time derived in Step* 1, *they choose an optimal portfolio policy, that is, determine the space-dimensional control variable.*

And we denote by $\Gamma^i(X^i(t))$ *the* self-dynamic game *defined above for* $\forall i = 1, 2, ..., I$.

Why will we especially impose the self-dynamic game structure on the investors in the present financial market? On the one hand, notice that we will employ the concept of sequential rationality to derive the optimal bankruptcy time in the following discussion and also as you can see below that our major result mainly depends on the optimal bankruptcy time rather than the optimal portfolio policy, we give the special self-dynamic game in Definition 1 to naturally establish the sub-game perfect Nash equilibrium bankruptcy time in the following definition; on the other hand, as clearly stated in Introduction, many studies derive optimal portfolio policy in models where the terminal time or the bankruptcy time is already pre-specified, that is, they have in fact implicitly assumed that the terminal time or the bankruptcy time is determined (although there it is exogenous) before the determination of optimal portfolio policy, and it is in this sense that we argue that the present model is indeed compatible with existing work and it further extends the existing literatures by introducing the interesting self-dynamic game in Definition 1 into the decision process of the investors. What's more, it is particularly worthwhile mentioning that such kind of self-dynamic game structure is not at all new in existing economics literatures. For example, similar specification has been sufficiently employed in modern consumption theory (e.g., Amador et al., 2006, and references therein). Nevertheless, it is the first time, to the best of our knowledge, for us to incorporate such game structure into mathematical finance to capture much more interesting background stories. Naturally, we then formulate the corresponding game equilibrium,

DEFINITION 2 (Sub-Game Perfect Nash Equilibrium Bankruptcy Time): *Given the self-dynamic game* $\Gamma^i(X^i(t))$ ($\forall i = 1, 2, ..., I$) *defined above, the following algorithm of computation is employed with the help of backward induction rationality principle, which is widely used in dynamic non-cooperative game theories,* *Step* 1: *Investors determine their optimal portfolio policy for any given bankruptcy time.*

Step 2: *Based upon the results given in Step* 1*, optimal bankruptcy time is derived and hence we name it* sub-game perfect Nash equilibrium bankruptcy time*, which is denoted by* $\tau^{i*}(\omega)$ *for* $\forall \omega \in \Omega^i$ *and* $\forall i = 1, 2, ..., I$.

REMARK 3: It is obvious that one can determine the optimal time-space policy simultaneously from the viewpoint of pure mathematical technique (see, Øksendal and Sulem, 2005). It is, nonetheless, widely noticed that investors in reality generally do not determine their optimal bankruptcy time and optimal portfolio policy simultaneously. This phenomenon may be partially explained from the following viewpoints: first, people are usually not used to the way of determining time-space control variables simultaneously because the optimal decisions of time dimension and that of the space dimension in general share totally different properties or characteristics, most importantly, as already noticed by the literatures of endogenous longevity (see, Chakraborty, 2004; de la Croix and Ponthiere, 2010, and among others), it is reasonable to argue that the longevity of the time dimension and the consumption or investment of the space dimension are closely related to each other, that is, individuals' optimal consumption or investment will be correspondingly changed or modified as long as they are definitely informed that their longevity has been increased or decreased, and you can easily see that this basic logic also applies here and our approach relying on the above self-dynamic game to some extent successfully avoids such kind of complex interactions between the control variable of time dimension and that of the space dimension; second, acting like this will definitely bring us high cost of computation when noting that people prefer simple rules-of-thumb strategies (e.g., Ellison and Fudenberg, 1993) in reality. Most importantly, our definition actually implies much richer background stories and economic logics.

DEFINITION 3 (Admissible Strategy): *We call the control* $\pi^i(t) \in [0,1]^m$ Markov admissible strategy *if it is time-consistent and also the corresponding wealth process* $X^i(t,\omega) \in R_+$ $\mu \otimes P^i -$ *a.e. and then we define the set of* Markov admissible strategy *as* A^i *for* $\forall i = 1, 2, ..., I$. *Moreover, we define* $\mathrm{T}^i \equiv \{$ $F^i -$ *stopping times* $\}$ *with the corresponding element* $\tau^i(\omega)$, $F^i -$ *predictable for* $\forall \omega \in \Omega^i$ *and* $\forall i = 1, 2, ..., I$ *, representing the* admissible bankruptcy time.

Finally, we give some mathematical notations that will be used in the following proofs. For any vector $x \in R^I$, we give the norms, $\| x \|_1 \equiv | x_1 | + | x_2 | + ... + | x_I |$, $\| x \|_2 \equiv \sqrt{| x_1 |^2 + | x_2 |^2 + ... | x_I |^2}$, $\| x \|_\infty \equiv$

$\{\max_i |x_i|; i=1,2,...,I\}$, where $|x_i|$ denotes the absolute value of x_i for $\forall i=1,2,...,I$.

3. Optimal portfolio policy

In particular, in the present paper, we just consider the case where the consumption process is omitted. This is to make things easier on the one hand, and also to focus on the optimal wealth dynamics on the other hand. Therefore, by Assumptions 1 and 3, the individual optimization problem facing investor i reads as follows,

$$\max_{\pi^i(t)\in A^i} J^i \equiv E^i_{(s,x^i)}\left[e^{-\rho^i(\tau^i-s)}\log X^i(\tau^i)\right], \tag{4}$$

s.t.

$$dX^i(t) = X^i(t)\left[\pi^i(t)^T\Theta^i(t)+r(t)\right]dt + X^i(t)\pi^i(t)^T\sigma(t)dW^i(t)$$

$$+X^i(t)\pi^i(t)^T\int_{R_0^n}\gamma(t,z)\widetilde{N}^i(dt,dz)$$

$$X^i(0)=x^i>0,\quad P^i-\text{a.s.} \tag{5}$$

where $E^i_{(s,x^i)}$ denotes the expectation operator, defined in the previous section, that depends on initial conditions $(s,x^i)\in R_+\times R_{++}$, $\rho^i\in R_{++}$ denotes the subjective discount factor and $\tau^i\equiv\tau^i(\omega)\in \mathrm{T}^i$, which is given in Definition 3, for $\forall i=1,2,...,I$.

Now, employing the classical technique of dynamic programming, the following proposition is thus established,

PROPOSITION 1: *Given the optimization problem constructed in* (4) *and* (5)*, and based upon the above assumptions, we get if,*

$$\beta^i\Theta^i(t)+\beta^i(\beta^i-1)\Sigma(t)\mathbf{1}$$

$$+\sum_{l=1}^n\int_{R_0}\beta^i\left[\left(1+\mathbf{1}^T\gamma^{(l)}(t,z)\right)^{\beta^i-1}-1\right]\gamma^{(l)}(t,z)\nu_l^i(dz_l)\le 0, \tag{6}$$

in which $\gamma^{(l)}(t,z)$ *represents the* l*-th column of matrix* $\gamma(t,z)$. *Then, there exists an optimal* $\pi^{i*}(t)\in A^i$ *such that,*

$$\beta^i\Theta^i(t)+\beta^i(\beta^i-1)\Sigma(t)\pi^{i*}(t)$$

$$+\sum_{l=1}^{n}\int_{R_0}\beta^i\left[\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)^{\beta^i-1}-1\right]\gamma^{(l)}(t,z)\nu_l^i(dz_l)=0, \tag{7}$$

with $\beta^i\in R_{++}$ *a solution of the following equation,*

$$\rho^i=\beta^i\left[\pi^{i*}(t)^T\Theta^i(t)+r(t)\right]+\frac{1}{2}\beta^i(\beta^i-1)\pi^{i*}(t)^T\Sigma(t)\pi^{i*}(t)$$

$$+\sum_{l=1}^{n}\int_{R_0}\left[\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)^{\beta^i}-1-\beta^i\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right]\nu_l^i(dz_l), \tag{8}$$

for $\forall i=1,2,\ldots,I$.

PROOF: See Appendix A. ■

REMARK 4: Proposition 1 proves the existence of optimal portfolio policy for each investor under our assumptions and constructions while we cannot express the optimal solution explicitly in the present circumstance, which however is not important. Actually, we only need to prove the *existence* of the optimal portfolio policy for the purpose of the current paper. And also optimal consumption-portfolio policy can be derived by using the classical martingale duality approach (see, Karatzas et al, 1987, 1991) for very general preference functions. Therefore, this is why we argue that employing log preference is just for the sake of simplicity and also without loss of any generality for the purpose of the present paper.

4. Sub-game perfect Nash equilibrium bankruptcy time

By Definition 2, Step 1 has been completed in section 3, and then the major goal of the present section is to complete Step 2, that is, to determine the sub-game perfect Nash equilibrium bankruptcy time corresponding to the case without consumption. It follows from Proposition 1 that optimal wealth accumulation amounts to,

$$dX^i(t) = X^i(t)\left[\pi^{i*}(t)^T\Theta^i(t) + r(t)\right]dt + X^i(t)\pi^{i*}(t)^T\sigma(t)dW^i(t)$$

$$+X^i(t)\pi^{i*}(t)^T\int_{R_0^n}\gamma(t,z)\widetilde{N}^i(dt,dz), \qquad (9)$$

subject to $X^i(0) = x^i \in R_{++}$, $P^i -$a.s.. Now, we need the following assumption,

ASSUMPTION 4: *In what follows, suppose that there exist processes* $\theta^i(s,z) \equiv (\theta_1^i(s,z_1), \theta_2^i(s,z_2), \ldots,$ $\theta_n^i(s,z_n))^T \in R^n$ *with* $\theta_l^i(s,z_l) \le 1$ *and* $\alpha^i(s) \in R^d$ *that are* $\overline{F}^i -$*predictable such that,*

- $\pi^{i*}(s)^T\sigma(s)\alpha^i(s) + \sum_{l=1}^{n}\int_{R_0}\pi^{i*}(s)^T\gamma^{(l)}(s,z)\theta_l^i(s,z_l)\nu_l^i(dz_l)$

$$= \pi^{i*}(s)^T\Theta^i(s) + r(s), \quad \mu\otimes P^i - a.e. \qquad (10)$$

- $\int_0^{\tau^i(\omega)}\|\alpha^i(s)\|_2^2\, ds < \infty, \quad P^i - a.s. \qquad (11)$

- $\sum_{l=1}^{n}\int_0^{\tau^i(\omega)}\int_{R_0}\left[|\log(1+\theta_l^i(s,z_l))| + \|\theta^i(s,z)\|_2^2\right]\nu_l^i(dz_l) < \infty, \quad P^i - a.s. \qquad (12)$

for a.a. $(s,\omega) \in [0,\tau^i(\omega)]\times\Omega^i$, $z_l \in R_0$, *and* $\gamma^{(l)}(t,z)$ *represents the* l*-th column of matrix* $\gamma(t,z)$, $\forall i = 1,2,\ldots,I$ *and* $\forall l = 1,2,\ldots,n$.

Now, letting,

$$Z^i(t) \equiv \exp\left\{\sum_{l=1}^{n}\int_0^t\int_{R_0}\left[\log(1-\theta_l^i(s,z_l)) + \theta_l^i(s,z_l)\right]\nu_l^i(dz_l)ds\right.$$

$$-\int_0^t\alpha^i(s)^T dW^i(s) - \int_0^t\|\alpha^i(s)\|_2^2\, ds$$

$$\left.+\sum_{l=1}^{n}\int_0^t\int_{R_0}\log\left(1-\theta_l^i(s,z_l)\right)\widetilde{N}_l^i(ds,dz_l)\right\}, \qquad (13)$$

And define a new measure Q^i on $F_{\tau^i}^i$ by,

$$dQ^i(\omega) = Z^i(\omega,\tau^i(\omega))dP^i(\omega), \qquad (14)$$

i.e., $Z^i(\omega,\tau^i(\omega))$ is the well-known Radon-Nikodym derivative. By Assumption 4, $Z^i(\omega,\tau^i(\omega))$ satisfies the Novikov condition, that is,

$$E^i\left[\exp\left(\frac{1}{2}\int_0^{\tau^i(\omega)}\|\alpha^i(s)\|_2^2\, ds\right.\right.$$

$$\left.\left.+\sum_{l=1}^{n}\int_0^{\tau^i}\int_{R_0}\left\{\left(1-\theta_l^i(s,z_l)\right)\log\left(1-\theta_l^i(s,z_l)\right)+\theta_l^i(s,z_l)\right\}\nu_l^i(dz_l)ds\right)\right]<\infty, \tag{15}$$

Then $E^i[Z^i(\omega,\tau^i(\omega))]=1$. And hence, according to the well-known Girsanov Theorem for Lévy processes, Q^i is a new probability measure on $F^i_{\tau^i}$ and $X^i(t)$ will be a (local) martingale w. r. t. the probability law Q^i. Define,

$$\widetilde{N}^i_{Q^i l}(ds,dz_l)\equiv\theta_l^i(ds,dz_l)\nu_l^i(dz_l)ds+\widetilde{N}^i_l(ds,dz_l), \tag{16}$$

$$dW^i_{Q^i}(s)\equiv\alpha^i(s)ds+dW^i(s), \tag{17}$$

Then $\widetilde{N}^i_{Q^i l}(\cdot,\cdot)$ and $W^i_{Q^i}(\cdot)$ are compensated Poisson random measure of $\widetilde{N}^i_l(\cdot,\cdot)$ and Brownian motion under Q^i, respectively, for $(s,\omega)\in[0,\tau^i(\omega)]\times\Omega^i$, $\forall i=1,2,...,I$ and $\forall l=1,2,...,n$. And we will denote by $E^i_{Q^i}$ the expectation operator w. r. t. the probability law Q^i, $\forall i=1,2,...,I$, and E_Q the expectation operator w. r. t. the probability law $Q\equiv Q^1\otimes...\otimes Q^I$. Now, the optimal wealth accumulation given in (9) can be rewritten as follows,

$$dX^i(t)=X^i(t)\pi^{i*}(t)^T\sigma(t)dW^i_{Q^i}(t)+X^i(t)\pi^{i*}(t)^T\int_{R_0^n}\gamma(t,z)\widetilde{N}^i_{Q^i}(dt,dz), \tag{18}$$

subject to $X^i(0)=x^i\in R_{++}$ Q^i-a.s.. And hence the optimal stopping problem facing investor i can be expressed as follows,

$$\sup_{\tau^i(\omega)\in\mathrm{T}^i} E^i_{Q^i,(s,x^i)}\left[e^{-\rho^i(\tau^i-s)}\log X^i(\tau^i)\right], \tag{19}$$

subject to the new Lévy SDE given by (18), for $\forall i=1,2,...,I$. Let $Y^i(t)\equiv(s+t,X^i(t))^T$ and $Y^i(0)\equiv(s,x^i)^T$, then the differential generator of $Y^i(t)$ reads as follows,

$$L^i\phi^i(s,x^i)=\frac{\partial\phi^i}{\partial s}+\frac{1}{2}(\pi^{i*})^T\Sigma\pi^{i*}(x^i)^2\frac{\partial^2\phi^i}{\partial(x^i)^2}+\sum_{l=1}^{n}\int_{R_0}\Big\{\phi^i\left(s,x^i+x^i(\pi^{i*})^T\gamma^{(l)}(z)\right)$$

$$-\phi^i(s,x^i)-(\pi^{i*})^T\gamma^{(l)}(z)x^i\frac{\partial\phi^i}{\partial x^i}\Big\}\nu_l^i(dz_l), \tag{20}$$

If we try a function ϕ^i of the form,

$\phi^i(s,x^i)=e^{-\rho^i s}(x^i)^{\lambda^i}$ for some constant $\lambda^i\in R$,

Then we get,

$$L^i\phi^i(s,x^i)=e^{-\rho^i s}\Big[-\rho^i(x^i)^{\lambda^i}+\tfrac{1}{2}(\pi^{i*})^T\Sigma\pi^{i*}(x^i)^2\lambda^i(\lambda^i-1)(x^i)^{\lambda^i-2}$$

$$+\sum_{l=1}^{n}\int_{R_0}\Big\{\left(x^i+x^i(\pi^{i*})^T\gamma^{(l)}(z)\right)^{\lambda^i}-(x^i)^{\lambda^i}$$

$$-(\pi^{i*})^T\gamma^{(l)}(z)x^i\lambda^i(x^i)^{\lambda^i-1}\Big\}\nu_l^i(dz_l)\Big]$$

$$=e^{-\rho^i s}(x^i)^{\lambda^i}\psi^i(\lambda^i),$$

where,

$$\psi^i(\lambda^i)\equiv-\rho^i+\frac{1}{2}(\pi^{i*})^T\Sigma\pi^{i*}\lambda^i(\lambda^i-1)$$

$$+\sum_{l=1}^{n}\int_{R_0}\Big\{\left(1+(\pi^{i*})^T\gamma^{(l)}(z)\right)^{\lambda^i}-1-(\pi^{i*})^T\gamma^{(l)}(z)\lambda^i\Big\}\nu_l^i(dz_l),$$

Notice that,

$\psi^i(1)=-\rho^i<0$ and $\lim_{\lambda^i\to\infty}\psi^i(\lambda^i)=\infty$.

Therefore, there exists $\lambda^i>1$ such that $\psi^i(\lambda^i)=0$. And with this value of λ^i we put,

$$\phi^i(s,x^i)=\begin{cases}e^{-\rho^i s}C^i(x^i)^{\lambda^i}, & (s,x^i)\in D^i\\ e^{-\rho^i s}\log x^i, & (s,x^i)\notin D^i\end{cases} \tag{21}$$

for some constant $C^i>0$, remains to be determined. If we let,

$$g^i(s,x^i)\equiv e^{-\rho^i s}\log x^i, \tag{22}$$

Hence, applying (20) to (22) leads us to,

$$L^i g^i(s,x^i) = e^{-\rho^i s}\Bigg[-\rho^i \log x^i - \frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*}$$

$$+\sum_{l=1}^{n}\int_{R_0}\left\{\log\left(1+(\pi^{i*})^T\gamma^{(l)}(z)\right)-(\pi^{i*})^T\gamma^{(l)}(z)\right\}\nu_l^i(dz_l)\Bigg] > 0$$

$$\Leftrightarrow x^i < \exp\Bigg\{\frac{1}{\rho^i}\Bigg[-\frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*}$$

$$+\sum_{l=1}^{n}\int_{R_0}\left\{\log\left(1+(\pi^{i*})^T\gamma^{(l)}(z)\right)-(\pi^{i*})^T\gamma^{(l)}(z)\right\}\nu_l^i(dz_l)\Bigg]\Bigg\},$$

Therefore,

$$U^i = \Bigg\{(s,x^i); x^i < \exp\Bigg(\frac{1}{\rho^i}\Bigg[-\frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*}$$

$$+\sum_{l=1}^{n}\int_{R_0}\left\{\log\left(1+(\pi^{i*})^T\gamma^{(l)}(z)\right)-(\pi^{i*})^T\gamma^{(l)}(z)\right\}\nu_l^i(dz_l)\Bigg]\Bigg)\Bigg\}, \tag{23}$$

Thus, we guess that the continuation region D^i has the form,

$$D^i \equiv \{(s,x^i); 0 < x^i < x^{i*}\}, \tag{24}$$

For some x^{i*} such that $U^i \subseteq D^i$, i.e.,

$$x^{i*} \geq \exp\Bigg\{\frac{1}{\rho^i}\Bigg[-\frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*}$$

$$+\sum_{l=1}^{n}\int_{R_0}\left\{\log\left(1+(\pi^{i*})^T\gamma^{(l)}(z)\right)-(\pi^{i*})^T\gamma^{(l)}(z)\right\}\nu_l^i(dz_l)\Bigg]\Bigg\}, \tag{25}$$

Hence, with (24) we can rewrite (21) as follows,

$$\phi^i(s,x^i) = \begin{cases} e^{-\rho^i s} C^i (x^i)^{\lambda^i}, & 0 < x^i < x^{i*} \\ e^{-\rho^i s}\log x^i, & x^i \geq x^{i*} \end{cases} \tag{26}$$

for some constant $C^i > 0$, remains to be determined. We guess that the value function ϕ^i is C^1 at $x^i = x^{i*}$ and this implies the following "smooth fit" conditions,

$$C^i (x^{i*})^{\lambda^i} = \log x^{i*} \quad \text{(continuity at } x^i = x^{i*}) \tag{27}$$

$$C^i \lambda^i (x^{i*})^{\lambda^i - 1} = (x^{i*})^{-1} \quad \text{(differentiability at } x^i = x^{i*}) \tag{28}$$

Combining equation (27) with (28) shows that,

$$\frac{C^i (x^{i*})^{\lambda^i}}{C^i \lambda^i (x^{i*})^{\lambda^i - 1}} = \frac{\log x^{i*}}{(x^{i*})^{-1}} \Leftrightarrow x^{i*} = \exp\left(\frac{1}{\lambda^i}\right), \tag{29}$$

and,

$$C^i = \frac{1}{\lambda^i} (x^{i*})^{-\lambda^i} = \frac{1}{\lambda^i} \exp(-1), \tag{30}$$

To summarize, then we have,

PROPOSITION 2: *Based upon the above assumptions and constructions, if,*

$$\int_0^\infty \left\{ -\pi^{i*}(t)^T \Sigma(t) \pi^{i*}(t) + \sum_{l=1}^n \int_{R_0} \left[\pi^{i*}(t)^T \gamma^{(l)}(t,z)) \right]^2 \nu_l^i (dz_l) - 2\rho^i \right\} dt < \infty, \tag{31}$$

$Q^i -$ *a.s., where* $\gamma^{(l)}(t,z)$ *represents the* l *-th column of matrix* $\gamma(t,z)$. *Then we obtain the optimal discretionary stopping time* $\tau^{i*}(\omega) \equiv \inf\{t \geq 0; X^i(t) = x^{i*}\} \in \mathrm{T}^i$. *In particular, if* $\pi^{i*}(t)$, $\Sigma(t)$, $\gamma(t,z)$ *and* $\sigma(t)$ *are all constants, and*

$$\sum_{l=1}^n \int_{R_0} \left\{ \log\left(1 + (\pi^{i*})^T \gamma^{(l)}(z)\right) - (\pi^{i*})^T \gamma^{(l)}(z) \right\} \nu_l^i (dz_l) > \frac{1}{2} (\pi^{i*})^T \Sigma \pi^{i*}, \tag{32}$$

with $z \geq 0$ $\nu^i -$ *a.s., then we get* $\tau^{i*}(\omega) < \infty$ $Q^i -$ *a.s.. Moreover,*

$$g^{i*}(s, x^i) = e^{-\rho^i s} \frac{1}{\lambda^i} (x^{i*})^{-\lambda^i} (x^i)^{\lambda^i},$$

which is a supermeanvalued majorant of $g^i(s, x^i)$ *with* x^{i*} *and* λ^i *given by* (29) *and* $\psi^i(\lambda^i) = 0$, *respectively, for* $\forall i = 1, 2, ..., I$.

PROOF: See Appendix B. ■

REMARK 5: First, it is worth emphasizing that it follows from Proposition 2 that the efficient terminal wealth is *endogenously* determined. The efficient terminal wealth, rather, plays the role of free boundary in the corresponding optimal stopping problem of continuous time. Existing literatures usually refer to the optimal stopping time as an optimal bankruptcy time. Nonetheless, this is far from being the only economic story in reality. For example, one may think of the following scenario: all investors try their best to invest as skillfully as possible before "retiring" from the stock market and putting all their holdings in the bank, that is, they need to determine an optimal "retiring time" from the risky stock market, as is argued by Karatzas and Wang (2001). And finally, it is also worth noting that one just need to demonstrate the *existence* of the optimal bankruptcy time or sub-game perfect Nash equilibrium bankruptcy time for the major goal of the current paper. So, employing log preference just makes things much easier from the view of point of mathematical computations without losing any necessary economic logics.

5. Wealth martingale and neighborhood turnpike theorem

According to Proposition 2, we get, based upon the optimal portfolio given in Proposition 1 and the well-known Girsanov Theorem for Lévy diffusions, that the optimal wealth accumulation reads as follows,

$$dX^i(t) = X^i(t)\pi^{i*}(t)^T \sigma(t) dW^i_{Q^i}(t) + X^i(t)\pi^{i*}(t)^T \int_{R_0^n} \gamma(t,z)\widetilde{N}^i_{Q^i}(dt,dz), \tag{18'}$$

subject to $X^i(0) = x^i \in R_{++}$ $Q^i -$a.s. for $\forall i = 1,2,...,I$. And also it follows from Proposition 2 that the sub-game perfect Nash equilibrium bankruptcy time reads as follows,

$$\tau^{i*}(\omega) \equiv \inf\{t \geq 0; X^i(t) = x^{i*}\}, \tag{33}$$

with x^{i*} given by (29). It is easily seen from (18') that $X^i(t)$ is a martingale w. r. t. the probability law Q^i for $\forall i = 1,2,...,I$. And the major goal of the present section is to explore the turnpike property of the vector of efficient terminal wealth $x^* \equiv (x^{1*},...,x^{I*})^T$. Before giving the turnpike theorem, we first introduce the following lemma,

LEMMA 1 (Norm Equivalence): *For any real vector* $x \in R^I$*, and based on the mathematical notations given at the end of section* 2*, then we have* $\| x \|_2 \leq I \| x \|_\infty$.

PROOF: Noting that,

$$|x_1|^2+|x_2|^2+\ldots+|x_I|^2 \le (|x_1|+|x_2|+\ldots+|x_I|)^2$$

$$\Leftrightarrow (|x_1|^2+|x_2|^2+\ldots+|x_I|^2)^{\frac{1}{2}} \le |x_1|+|x_2|+\ldots+|x_I| \Leftrightarrow \|x\|_2 \le \|x\|_1, \tag{34}$$

And,

$$|x_1|+|x_2|+\ldots+|x_I| \le \max_i |x_i| + \max_i |x_i| + \ldots + \max_i |x_i| = I \max_i |x_i| \Leftrightarrow \|x\|_1 \le I\|x\|_\infty, \tag{35}$$

So combining (34) with (35) yields the desired result. ■

Now, we state and prove the key theorem of the present investigation,

THEOREM 1 (Neighborhood Turnpike Theorem): *Given* $X^i(t)$ *and* x^{i*} *determined by* (18') *and* (33), *respectively, for* $\forall i = 1,2,\ldots,I$. *If we define* $X(t) \equiv (X^1(t),\ldots,X^I(t))^T$, *which is given by* (3), *and* $x^* \equiv (x^{1*},\ldots,x^{I*})^T$, *then we get* $\|X(t)-x^*\|_2 \le \varepsilon$, $Q-a.s.$ *for* $t \in [0,\tau^*]$ *with* $\tau^* \equiv \tau^{1*} \vee \ldots \vee \tau^{I*} \le \infty$, *for* $\forall \varepsilon > 0$, *which may depend on* τ^*, *and for* $Q \triangleq Q^1 \otimes \ldots \otimes Q^I$.

PROOF: By the Doob's Martingale Inequality,

$$Q^i\left\{\sup_{0\le t\le T}|X^i(t)| \ge \xi^i\right\} \le \frac{1}{\xi^i}E^i_{Q^i}\left[|X^i(T)|\right] = \frac{x^i}{\xi^i},\ \ \forall \xi^i > 0,\ \ \forall 0 < T < \tau^{i*}(\omega).$$

Without loss of generality, we put $\xi^i = 2^k$ for $\forall k \in N$, then we get,

$$Q^i\left\{\sup_{0\le t\le T}|X^i(t)| \ge 2^k\right\} \le \frac{1}{2^k}x^i,\ \ \forall k \in N$$

By the well-known Borel-Cantelli Lemma,

$$Q^i\left\{\sup_{0\le t\le T}|X^i(t)| \ge 2^k\, i.m.k\right\} = 0.$$

where $i.m.k$ denotes "infinitely many k". So for a.a. $\omega \in \Omega^i$, there exists $\bar{k}(\omega)$ such that,

$$\sup_{0\le t\le T}|X^i(t)| < 2^k, \text{ a.s. for } k \ge \bar{k}(\omega).$$

Thus, we see that,

$$\lim_{T\to\tau^{i*}(\omega)}\sup_{0\le t\le T}|X^i(t)|<2^k, \text{ a.s. for } k\ge\bar{k}(\omega). \tag{36}$$

Consequently, $X^i(t)=X^i(t,\omega)$ is uniformly bounded for $\forall t\in[0,T]$ for $\forall 0<T<\tau^{i*}(\omega)$ and for a.a. $\omega\in\Omega^i$. Furthermore, it is easily seen from Proposition 2 that $X^i(t)-x^{i*}$ is also an F_t^i-martingale w. r. t. Q^i. So, applying the Doob's Martingale Inequality again implies that,

$$Q^i\left\{\sup_{0\le t\le T}|X^i(t)-x^{i*}|\ge\frac{\varepsilon^i}{I}\right\}\le\frac{I}{\varepsilon^i}E^i_{Q^i}\left[|X^i(T)-x^{i*}|\right],\ \forall\varepsilon^i>0,\ \forall 0<T<\tau^{i*}(\omega).$$

Provided the definition of $\tau^{i*}(\omega)$ given by (33), we see that there exists $\delta^i>0$ such that the above martingale inequality still holds for $\forall t\in B^i_{\delta^i}(\tau^{i*}(\omega))\equiv\{t\ge 0;|t-\tau^{i*}|<\delta^i\}$, for $\forall i=1,2,\ldots,I$ by applying Doob's Optional Sampling Theorem. Without loss of any generality, we set $\delta^i=2^{-k}$ for $\forall k\in N$. Hence, for $\forall T^i_k\in B^i_{\delta^i}(\tau^{i*}(\omega))$ and according to the continuity of martingale w. r. t. t for any given $\omega\in\Omega^i$, condition (36) and the well-known Lebesgue Dominated Convergence Theorem, we obtain,

$$\limsup_{k\to\infty}Q^i\left\{\sup_{0\le t\le T^i_k}|X^i(t)-x^{i*}|\ge\frac{\varepsilon^i}{I}\right\}\le\frac{I}{\varepsilon^i}\limsup_{k\to\infty}E^i_{Q^i}\left[|X^i(T^i_k)-x^{i*}|\right]=0.$$

which yields,

$$\limsup_{k\to\infty}Q^i\left\{\sup_{0\le t\le T^i_k}|X^i(t)-x^{i*}|<\frac{\varepsilon^i}{I}\right\}\ge 1.$$

Letting $\varepsilon^i=2^{-k}I$, $\forall k\in N$, we get,

$$\limsup_{k\to\infty}Q^i\left\{\sup_{0\le t\le T^i_k}|X^i(t)-x^{i*}|<2^{-k}\right\}=1,\ \forall k\in N.$$

It follows from the well-known Fatou's Lemma that,

$$Q^i\left\{\sup_{0\le t\le\tau^{i*}(\omega)}|X^i(t)-x^{i*}|<2^{-k}\right\}=1,\ \forall k\in N.$$

Then applying the Borel-Cantelli Lemma again implies that,

$$Q^i\left\{\sup_{0\le t\le \tau^{i*}(\omega)} |X^i(t)-x^{i*}|<2^{-k}\, i.m.k\right\}=1.$$

in which $i.m.k$ denotes "infinitely many k". So for a.a. $\omega\in\Omega^i$, there exists $\tilde{k}(\omega)$ such that,

$$\sup_{0\le t\le \tau^{i*}(\omega)} |X^i(t)-x^{i*}|<2^{-k}, \text{ for } k\ge \tilde{k}(\omega).$$

That is,

$$\sup_{0\le t\le \tau^{i*}(\omega)} |X^i(t)-x^{i*}|<\frac{\varepsilon^i}{I},\ \ Q^i-\text{a.s., for } \forall i=1,2,...,I.$$

Now, using Lemma 1 reveals that,

$$\|X(t)-x^*\|_2\le I\,\|X(t)-x^*\|_\infty = I\max_i |X^i(t)-x^{i*}|$$

$$\le I\max_i \sup_{0\le t\le \tau^{i*}(\omega)} |X^i(t)-x^{i*}|\le I\max_i \frac{\varepsilon^i}{I}\le \max_i \varepsilon^i\equiv\varepsilon,\ \ Q-\text{a.s.}$$

which gives the desired result. ■

REMARK 6: Theorem 1 would be regarded as a *stability characterization* of optimal wealth dynamics, that is, the optimal wealth path will always stay in the $\varepsilon-$neighborhood of endogenously determined efficient terminal wealth for each investor. And our stability theorem can be, in some sense, seen as a natural correspondence to Liapounov stability theorem or dual Liapounov stability theorem confirmed in macroeconomic models with an integration of competitive equilibrium theory and optimal growth theory, see Yano (1999) and references therein for more details. Moreover, it is easily seen from the proof of Theorem 1 that it is the martingale property but not the explicit forms of optimal portfolio policy and sub-game perfect Nash equilibrium bankruptcy time that plays the key role in demonstrating the neighborhood turnpike property of the vector of efficient terminal wealth for all heterogeneous investors as a whole. And one, if motivated, can employ very general utility functions and take the consumption strategy into account without changing the neighborhood turnpike theorem proved in the present paper if the martingale property of wealth dynamics still holds. That is to say, we have confirmed, in certain sense, the *equivalence* between the wealth martingale and the neighborhood turnpike property of endogenously determined efficient terminal wealth with being robust to preference specifications. This is, obviously, an interesting finding of the current paper.

Finally, one may easily check that the neighborhood turnpike theorem proved in Theorem 1 is a little *stronger* than existing ones (see, Bewley, 1982; McKenzie, 1982; Yano, 1984a, 1984b, 1998, 1999; Marena and Montrucchio, 1999; Sahashi, 2002; Kamihigashi and Roy, 2007; Kondo, 2008, and among others) by noting that the optimal wealth path will *always* stay in the arbitrarily small neighborhood of the efficient terminal wealth rather than *eventually* lie in the arbitrarily small neighborhood of the efficient terminal wealth. Moreover, our stability theorem does not depend on the choice of preference function, elasticity of intertemporal substitution and time discount factor.

6. Concluding remarks

This study explores the turnpike properties of wealth dynamics in a dynamically complete market (see, Anderson and Raimondo, 2008) with heterogeneous investors. Notice that most of existing literatures consider the case where the expected utility is maximized with respect to the pre-specified terminal wealth, the present study extends the traditional approach and efficient terminal wealth is indeed endogenously derived following the optimal bankruptcy time or the sub-game perfect Nash equilibrium bankruptcy time based upon the present study's background.

Rather, the result that *wealth martingale implies neighborhood turnpike property* of the above efficient terminal wealth as a whole for all investors has been demonstrated, which would in some sense shed some lights on our understanding of both wealth-driven selection theory when *discretionary* stopping time is taken into account and neighborhood turnpike theorem of optimal wealth accumulation in financial economics.

Furthermore, it would be necessary for us to give some intuitive explanations regarding the major concern and also contribution of the present limited investigation. As you can see, the issue of optimal bankruptcy time plays at least the same, if not more important, role as that of optimal portfolio policy in mathematical finance as well as in real-world financial markets. In fact, the above two dimensions of optimal decisions facing the investors in financial markets are intimately correlated with each other and hence not at all independent of each other as implicitly assumed in many existing literatures. For example, if we suppose that the time is discrete and the classical sequential rationality is fulfilled, then any given economic agent will stop her investment in the current period when she was informed at the last period that she will definitely die in the next period. Naturally, if there is an initial investment period and based upon the well-known backward induction principle, what will be the investor's optimal investment arrangement when she is informed about her exact longevity at the initial investment period? Or, if she is informed about her exact longevity at any given indeterminate or intermediate investment period, then what will be her optimal investment decision now. So, the paper emphasizes that both the optimal portfolio policy and the optimal bankruptcy time should be endogenously determined to capture the above complex decision circumstance facing any investor in the financial market. Interestingly, the advantage of the optimal stopping theory employed here is that we can also endogenously determine the efficient terminal wealth of the investors. Noting that it is the terminal wealth that plays a crucial role in establishing optimal portfolio policy for investors, the endogenous constant terminal wealth rather than the exogenously given terminal wealth widely used in existing studies indeed makes things much easier in explicit computation and also implies much richer economic implications, i.e., as in reality, the endogenous terminal wealth captures the fact that it indeed depends on many relevant environment parameters such as discount factor, portfolio policy and the underlying stochastic fluctuation.

Last but not least, one can interpret the major result that *wealth martingale implies neighborhood turnpike property* from the following two perspectives: on the one hand, if we focus on the financial institutional arrangement for some developing economies, then the condition wealth martingale implies the underlying requirement that the financial market is perfect and no arbitrage opportunities exist, which of course is the baseline characteristic of advanced or developed financial markets nowadays, that is, investors accumulate financial wealth in a fair market environment; on the other hand, neighborhood turnpike property pictures the efficient level of the underlying financial market, i.e., all investors accumulate their financial wealth to maximize their utilities, respectively. To summarize, the major contribution *wealth martingale implies neighborhood turnpike property* provides us with an internal\intrinsic relationship between the issue of *fairness* and that of *efficiency* of the financial-market institutions in real-world economies. Accordingly, the basic or possible lesson, as mentioned by the reviewer, derived from our exploration is that financial institutional arrangements in reality should impose some exogenous constraints on the heterogeneous investors so that they have to agree to accumulate their financial wealth following the martingale path needed.

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Appendix

A. Proof of Proposition 1.

The corresponding stochastic Bellman partial differential equation (SBPDE) reads as follows,

$$-J_t^i(t,X^i(t))$$

$$= \max_{\pi^i(t)\in A^i}\left\{\begin{aligned}&J_{X^i}^i(t,X^i(t))X^i(t)\left[\pi^i(t)^T\Theta^i(t)+r(t)\right]+\tfrac{1}{2}J_{X^iX^i}^i(t,X^i(t))X^i(t)^2\\&\times\pi^i(t)^T\Sigma(t)\pi^i(t)+\sum_{l=1}^n\int_{R_0}\left[J^i(t,X^i(t)+X^i(t)\pi^i(t)^T\gamma^{(l)}(t,z))\right.\\&\left.-J^i(t,X^i(t))-J_{X^i}^i(t,X^i(t))X^i(t)\pi^i(t)^T\gamma^{(l)}(t,z)\right]\nu_l^i(dz_l)\end{aligned}\right\}, \tag{A.1}$$

where $\gamma^{(l)}(t,z)$ represents the l-th column of matrix $\gamma(t,z)$, and the boundary value condition is given by,

$$J^i(\tau^i,X^i(\tau^i))=e^{-\rho^i(\tau^i-s)}\log X^i(\tau^i), \tag{A.2}$$

If we try,

$$J^i(t,X^i(t))=e^{-\rho^i(t-s)}C^iX^i(t)^{\beta^i}, \tag{A.3}$$

for some C^i, $\beta^i>0$, to be determined. Then we get,

$$J_t^i(t,X^i(t))=-\rho^ie^{-\rho^i(t-s)}C^iX^i(t)^{\beta^i}, \tag{A.4}$$

$$J_{X^i}^i(t,X^i(t))=e^{-\rho^i(t-s)}C^i\beta^iX^i(t)^{\beta^i-1}, \tag{A.5}$$

$$J_{X^iX^i}^i(t,X^i(t))=e^{-\rho^i(t-s)}C^i\beta^i(\beta^i-1)X^i(t)^{\beta^i-2}, \tag{A.6}$$

Substituting (A.3)-(A.6) into (A.1) gives,

$$\rho^i=\max_{\pi^i(t)\in A^i}\left\{\begin{aligned}&\beta^i\left[\pi^i(t)^T\Theta^i(t)+r(t)\right]+\tfrac{1}{2}\beta^i(\beta^i-1)\pi^i(t)^T\Sigma(t)\pi^i(t)+\\&\sum_{l=1}^n\int_{R_0}\left[\left(1+\pi^i(t)^T\gamma^{(l)}(t,z)\right)^{\beta^i}-1-\beta^i\pi^i(t)^T\gamma^{(l)}(t,z)\right]\nu_l^i(dz_l)\end{aligned}\right\}, \tag{A.7}$$

Performing the maximization produces,

$$\Lambda^i(\pi^i(t))\equiv\beta^i\Theta^i(t)+\beta^i(\beta^i-1)\Sigma(t)\pi^i(t)$$

$$+\sum_{l=1}^{n}\int_{R_0}\beta^i\left[\left(1+\pi^i(t)^T\gamma^{(l)}(t,z)\right)^{\beta^i-1}-1\right]\gamma^{(l)}(t,z)\nu_l^i(dz_l)=0\,, \tag{A.8}$$

It follows from Assumption 1 that,

$$\Lambda^i(0)=\beta^i\Theta^i(t)=\beta^i[b^i(t)-r(t)\mathbf{1}]>0\,, \tag{A.9}$$

Thus, we see that if,

$$\Lambda^i(\mathbf{1})\equiv\beta^i\Theta^i(t)+\beta^i(\beta^i-1)\Sigma(t)\mathbf{1}$$

$$+\sum_{l=1}^{n}\int_{R_0}\beta^i\left[\left(1+\mathbf{1}^T\gamma^{(l)}(t,z)\right)^{\beta^i-1}-1\right]\gamma^{(l)}(t,z)\nu_l^i(dz_l)\le 0\,, \tag{A.10}$$

Then there exists an optimal $\pi^{i*}(t)\in(0,1]^m$ for any $\beta^i\in R_{++}$. Now, inserting this $\pi^{i*}(t)$ into (A.7) reveals that,

$$\varphi^i(\beta^i)\equiv-\rho^i+\beta^i\left[\pi^{i*}(t)^T\Theta^i(t)+r(t)\right]+\frac{1}{2}\beta^i(\beta^i-1)\pi^{i*}(t)^T\Sigma(t)\pi^{i*}(t)$$

$$+\sum_{l=1}^{n}\int_{R_0}\left[\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)^{\beta^i}-1-\beta^i\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right]\nu_l^i(dz_l)=0\,, \tag{A.11}$$

Noting that,

$$\varphi^i(0)=-\rho^i<0 \quad\text{and}\quad \lim_{\beta^i\to\infty}\varphi^i(\beta^i)=\infty\,, \tag{A.12}$$

Consequently, there exists $\beta^i\in R_{++}$ such that $\varphi^i(\beta^i)=0$. Moreover, one can derive C^i from using the boundary value condition given in (A.2). And this completes the proof. ■

B. Proof of Proposition 2.

It follows from the "Integro-variational inequalities for optimal stopping" (see, Theorem 2.2, pp. 29) of Øksendal and Sulem (2005), we need to prove the following cases,

(i) We need to prove that $\phi^i\ge g^i$ on D^i, i.e.,

$$C^i(x^i)^{\lambda^i}\ge\log x^i \quad\text{for}\quad 0<x^i<x^{i*}.$$

Define $\varsigma(x^i)\equiv C^i(x^i)^{\lambda^i}-\log x^i$. By our chosen values of C^i and x^{i*}, we get $\varsigma(x^{i*})=$

$\varsigma'(x^{i*}) = 0$. Moreover, since $\varsigma''(x^i) = C^i \lambda^i (\lambda^i - 1)(x^i)^{\lambda^i - 2} + (x^i)^{-2} > 0$ with our chosen value of λ^i, then we get $\varsigma(x^i) > 0$ for all $0 < x^i < x^{i*}$. And this completes the proof.

(ii) Outside D^i we have $\phi^i(s, x^i) = e^{-\rho^i s} \log x^i$ and so,

$$L^i \phi^i(s, x^i) = e^{-\rho^i s} \Bigg[-\rho^i \log x^i - \frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*}$$

$$+ \sum_{l=1}^{n} \int_{R_0} \left\{ \log\left(1 + (\pi^{i*})^T \gamma^{(l)}(z)\right) - (\pi^{i*})^T \gamma^{(l)}(z) \right\} \nu_l^i(dz_l) \Bigg] \le 0 \quad \text{for } \forall x^i \ge x^{i*}$$

$$\Leftrightarrow x^i \ge \exp\left\{ \frac{1}{\rho^i} \Bigg[-\frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*} \right.$$

$$\left. + \sum_{l=1}^{n} \int_{R_0} \left\{ \log\left(1 + (\pi^{i*})^T \gamma^{(l)}(z)\right) - (\pi^{i*})^T \gamma^{(l)}(z) \right\} \nu_l^i(dz_l) \Bigg] \right\}, \quad \forall x^i \ge x^{i*}$$

$$\Leftrightarrow x^{i*} \ge \exp\left\{ \frac{1}{\rho^i} \Bigg[-\frac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*} \right.$$

$$\left. + \sum_{l=1}^{n} \int_{R_0} \left\{ \log\left(1 + (\pi^{i*})^T \gamma^{(l)}(z)\right) - (\pi^{i*})^T \gamma^{(l)}(z) \right\} \nu_l^i(dz_l) \Bigg] \right\}.$$

which holds by (25).

(iii) Noting from (29) that $x^{i*} < \infty$, thus $[0, x^{i*}]$ is compact set by Heine-Borel Theorem. Accordingly, ϕ^i is bounded on $[0, x^{i*}]$ via applying the fact that $\phi^i \in C^2(R^2)$ and the well-known Weierstrass Theorem. So, it suffices to check that,

$$\left\{ e^{-\rho^i \tau^i} \log X^i(\tau^i) \right\}_{\tau^i \in \mathrm{T}^i} \text{ is uniformly integrable on } [x^{i*}, \infty).$$

where T^i denotes the set of admissible stopping time and the uniform topology is naturally induced by the norm, which is induced by inner product, of Hilbert space $L^2(\Omega^i, F^i, Q^i)$. For this to hold, it suffices to show that there exists a constant $M^i < \infty$ such that,

$$E_{Q^i}^i\left[e^{-2\rho^i\tau^i}\left(\log X^i(\tau^i)\right)^2\right] \le M^i \quad \text{for all} \quad \tau^i(\omega)\in \mathrm{T}^i \quad \text{and} \quad X^i(\tau^i)\ge x^{i*}.$$

Since,

$$0<\log X^i(\tau^i)<X^i(\tau^i) \quad \text{on} \quad [x^{i*},\infty).$$

Hence, we get,

$$E_{Q^i}^i\left[e^{-2\rho^i\tau^i}\left(\log X^i(\tau^i)\right)^2\right] \le E_{Q^i}^i\left[e^{-2\rho^i\tau^i}\left(X^i(\tau^i)\right)^2\right]$$

$$= E_{Q^i}^i\Bigg[(x^i)^2\exp\Bigg(2\int_0^{\tau^i}\Big\{-\tfrac{1}{2}\pi^{i*}(t)^T\Sigma(t)\pi^{i*}(t)+\sum_{l=1}^n\int_{R_0}\Big[\log\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)$$

$$-\pi^{i*}(t)^T\gamma^{(l)}(t,z)\Big]\nu_l^i(dz_l)\Big\}dt-2\rho^i\tau^i+2\int_0^{\tau^i}\pi^{i*}(t)^T\sigma(t)dW_{Q^i}^i(t)$$

$$+2\sum_{l=1}^n\int_0^{\tau^i}\int_{R_0}\log\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)\widetilde{N}_{Q^il}^i(dt,dz_l)\Bigg)\Bigg], \tag{B.1}$$

$$=(x^i)^2E_{Q^i}^i\Bigg[\exp\Bigg(\int_0^{\tau^i}\Big\{-\pi^{i*}(t)^T\Sigma(t)\pi^{i*}(t)+2\sum_{l=1}^n\int_{R_0}\Big[\log\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)$$

$$-\pi^{i*}(t)^T\gamma^{(l)}(t,z)\Big]\nu_l^i(dz_l)\Big\}dt-2\rho^i\tau^i+2\int_0^{\tau^i}\pi^{i*}(t)^T\sigma(t)dW_{Q^i}^i(t)$$

$$+\sum_{l=1}^n\int_0^{\tau^i}\int_{R_0}\Big\{\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)^2-1-2\log\left(1+\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right)\Big\}\nu_l^i(dz_l)dt\Bigg)\Bigg] \tag{B.2}$$

$$=(x^i)^2E_{Q^i}^i\Bigg[\exp\Bigg(\int_0^{\tau^i}\Big\{-\pi^{i*}(t)^T\Sigma(t)\pi^{i*}(t)+\sum_{l=1}^n\int_{\mathbb{R}_0}\left[\pi^{i*}(t)^T\gamma^{(l)}(t,z)\right]^2\nu_l^i(dz_l)\Big\}dt$$

$$-2\rho^i\tau^i+2\int_0^{\tau^i}\pi^{i*}(t)^T\sigma(t)dW_{Q^i}^i(t)\Bigg)\Bigg], \tag{B.3}$$

where we have used Assumption 2 and $\gamma^{(l)}(t,z)$ denotes the l-th column of matrix $\gamma(t,z)$. Moreover, from (B.1) to (B.2) we have used the following fact, i.e., for the following equation,

$$d\Psi(t)=\Psi(t^-)\int_{R_0}\left(e^{\zeta(t,z)}-1\right)\widetilde{N}(dt,dz),\quad \Psi(0)=1. \tag{B.4}$$

which has the solution,

$$\Psi(t)=\exp\left\{\int_0^t\int_{R_0}\zeta(s,z)N(ds,dz)-\int_0^t\int_{R_0}\left(e^{\zeta(s,z)}-1\right)\nu(dz)ds\right\}$$

$$= \exp\left\{\int_0^t \int_{R_0} \zeta(s,z)\widetilde{N}(ds,dz) - \int_0^t \int_{R_0} \left[e^{\zeta(s,z)} - 1 - \zeta(s,z)\right]\nu(dz)ds\right\}. \tag{B.5}$$

Suppose,

$$\int_0^t \int_{R_0} \left(e^{\zeta(s,z)} - 1\right)^2 \nu(dz)ds < \infty,$$

Then by (B.4) we see that $E^i_{Q^i}\left[\Psi(t)\right] = 1$ and hence by (B.5) we obtain,

$$E^i_{Q^i}\left[\exp\left(\int_0^t \int_{R_0} \zeta(s,z)\widetilde{N}(ds,dz)\right)\right] = \exp\left\{\int_0^t \int_{R_0} \left[e^{\zeta(s,z)} - 1 - \zeta(s,z)\right]\nu(dz)ds\right\}.$$

If we put $\zeta(s,z) = \log(1 + \pi^{i*}(s)^T \gamma^{(l)}(s,z))^2$, then (B.2) follows. Thus, we conclude from (B.3) that if,

$$\int_0^\infty \left\{-\pi^{i*}(t)^T \Sigma(t)\pi^{i*}(t) + \sum_{l=1}^n \int_{\mathbb{R}_0} \left[\pi^{i*}(t)^T \gamma^{(l)}(t,z))\right]^2 \nu^i_l(dz_l) - 2\rho^i\right\}dt < \infty, \quad Q^i - \text{a.s.}$$

Then the desired result follows.

(iv) By using Itô's rule and (18), we get,

$$X^i(t) = x^i \exp\left[\int_0^t \left\{-\tfrac{1}{2}\pi^{i*}(s)^T \Sigma(s)\pi^{i*}(s) + \sum_{l=1}^n \int_{R_0} \left[\log\left(1 + \pi^{i*}(s)^T \gamma^{(l)}(s,z)\right)\right.\right.\right.$$

$$\left. - \pi^{i*}(s)^T \gamma^{(l)}(s,z)\right]\nu^i_l(dz_l)\Big\} ds + \int_0^t \pi^{i*}(s)^T \sigma(s) dW^i_{Q^i}(s)$$

$$\left. + \sum_{l=1}^n \int_0^t \int_{R_0} \log\left(1 + \pi^{i*}(s)^T \gamma^{(l)}(s,z)\right)\widetilde{N}^i_{Q^i l}(ds,dz_l)\right],$$

In particular, if $\pi^{i*}(t)$, $\Sigma(t)$, $\gamma(t,z)$ and $\sigma(t)$ are all constants, then we get,

$$X^i(t) = x^i \exp\left[\left\{-\tfrac{1}{2}(\pi^{i*})^T \Sigma \pi^{i*} + \sum_{l=1}^n \int_{R_0} \left[\log\left(1 + (\pi^{i*})^T \gamma^{(l)}(z)\right)\right.\right.\right.$$

$$\left. - (\pi^{i*})^T \gamma^{(l)}(z)\right]\nu^i_l(dz_l)\Big\} t + \pi^{i*}(t)^T \sigma(t) W^i_{Q^i}(t)$$

$$\left. + \sum_{l=1}^n \int_0^t \int_{R_0} \log\left(1 + (\pi^{i*})^\top \gamma^{(l)}(z)\right)\widetilde{N}^i_{Q^i l}(ds,dz_l)\right],$$

By the law of iterated logarithm for Brownian motion, we see that if,

$$\sum_{l=1}^{n}\int_{R_0}\left\{\log\left(1+(\pi^{i*})^T\gamma^{(l)}(z)\right)-(\pi^{i*})^T\gamma^{(l)}(z)\right\}\nu_l^i(dz_l)>\frac{1}{2}(\pi^{i*})^T\Sigma\pi^{i*},$$

with $z\geq 0$ ν^i $-$a.s., then we get $\lim_{t\to\infty}X^i(t)=\infty$ Q^i $-$ a.s. and particularly,

$$\tau^{i*}(\omega)<\infty \quad Q^i - \text{a.s.}$$

■