Preference-based Cooperation in a Prisoner's Dilemma Game: Whole Population Cooperation without Information Flow across Matches

Hanjoon Michael Jung

Lahore University of Management Sciences

2007

Preference-based Cooperation in a Prisoner's Dilemma Game: Whole Population Cooperation without Information Flow across Matches

Hanjoon Michael Jung[*,†]
Department of Economics, Pennsylvania State University
University Park, PA, 16802, USA

June 26, 2007

Abstract

This paper studies the possibility of cooperation based on players' preferences. Consider the following infinitely repeated game, similar to Ghosh and Ray (1996). At each stage, uncountable numbers of players are randomly matched without information about their partners' past actions and play a prisoner's dilemma game. The players have the option to continue their relationship, and they all have the same discount factor. Also, they have two possible types: high ability player (H) or low ability player (L). H can produce better outcomes for its partner as well as for itself than L can. I look for an equilibrium that is robust against both pair-wise deviation and individual deviation and call such equilibrium a social equilibrium. I show that in this setting, long term cooperative behavior can arise in a social equilibrium. H wants to match and play only with another H because an HH match produces better outcomes for H than an HL match. So H would break a match with L to increase the possibility of meeting another H, and thus H would not play any cooperative action with L. L knows this intention of H and realizes that L can only cooperate with another L. Consequently, both HH and LL matches are endowed with a scarcity value. This scarcity value is utilized by players to sustain cooperative relationships. Therefore, in a social equilibrium, whole players can play long term cooperative actions because of their preferences for their partners' types.

Journal of Economic Literature Classification Number: C72, C78

Keywords: Folk theorem, Random-matching, Social equilibrium, Type-based payoffs

[*]I am most grateful to Kalyan Chatterjee and James Jordan for their helpful comments and encouragement. I also wish to thank Gaurab Aryal for helpful comments.

[†]*Email address:* hxj124@psu.edu

1 Introduction

The motivation of the present paper comes from studies in the area of Folk Theorem. Classical literature in folk theorem, developed by Fudenberg and Maskin (1986), Kandori (1992), and Ellison (1994), showed that a long-term cooperative relationship in a prisoner's dilemma is possible without any legal enforcement, assuming that players' past actions affect their future payoffs. Based on a different assumption that players' past actions might not necessarily affect their future payoffs because they can change their partners in a large population, Ghosh and Ray (1996), hereinafter referred to as GR, maintained that a long-term cooperative relationship is still possible according to the structure of their model. However, GR's model is sound only when there exist a significantly large proportion of players who have zero discount factors and thus would not play any cooperative action. Therefore, GR's model can be considered as a partial population cooperation model. As a follow-up of GR, the present study is motivated to seek whole population cooperation possibilities in a prisoner's dilemma game assuming that players' past actions might not affect their future payoffs.

In this study, the whole population cooperation takes place based on two assumptions. First, every player is assumed to be either a high-ability player (H-player) or a low-ability player (L-player) according to her production ability. An H-player is defined as a player who can produce better outcomes for her partner as well as for herself than an L-player. Second, players in a common pair have the option to continue their relationships if they both wish. In this setting, I look for a long-term cooperative behavior that is robust against both pair-wise deviation and individual deviation as GR intended in their study.

The present study shows that such a cooperative behavior can happen in equilibrium because of players' preferences for their partners' types. An H-player wants to match and play only with another H-player because a high-ability partner produces better outcomes than a low-ability partner. So, when an H-player meets an L-player, the H-player would break the relationship with the L-player in order to increase the possibility to meet another H-player. Thus, an H-player would not play any cooperative action with an L-player. Since an L-player is aware of H-players' intentions, she realizes that she can only cooperate with another L-player. Consequently, two kinds of matches, the H-H match and the L-L match, are endowed with a *scarcity value*. Players can use this scarcity value to sustain their cooperative relationship. Therefore, the result shows that in equilibrium a long-term cooperative relationship among the whole population is possible based on players' preferences for their partners.

This paper is organized as follows. Section 2 is devoted to detailed description of the model. Section 3 introduces the concept of a social equilibrium. Section 4 presents the results of this study, including the folk theorem of this model.

2 The Model

The following setting of the model comes from GR. A continuum of players are randomly matched in pairs and bilaterally play an infinitely repeated stage game with an option to break up their relationship. Each stage of the game consists of two substages. At the first substage, players in a common pair play a prisoner's dilemma game with an action set $[0, \tilde{a}] \subset \mathbb{R}$. At the second substage, after watching the actions chosen before, players decide whether to break up their relationship. Only when both players in a common pair decide to maintain their relationship, can they play the stage game between themselves at the next stage. If one of the players in a common pair breaks up the relationship, then both in the pair go into *the pool of unmatched players* and would be randomly matched with other players in the pool. At the next stage, all players bilaterally repeat this stage game.

The present model introduces new features into the setting of GR. All players have the same discount factor δ, but they have their own types. Each player is either an H-player or an L-player. An H-player has higher abilities to produce an outcome than an L-player does. Based on this ability difference, the present model reflects the situation in which a partner of an H-player can benefit from the high ability of the H-player by sharing the produced outcome. Therefore, it is assumed that a player's payoff depends on her partner's type as well as on her own type and also depends on her partner's and her actions so that when other things being equal, **a player gets a better payoff when she cooperates with an H-player than when she cooperates with an L-player**.

The payoff functions of the players are as follows. For any $I, J \in \{H, L\}$, the function $\Pi_{IJ} : [0, \tilde{a}]^2 \longrightarrow \mathbb{R}$ denotes a payoff function of I-type when she works with J-type. For example, let $a, a' \in [0, \tilde{a}]$, then $\Pi_{HL}(a, a')$ denotes the payoff to an H-player when she works with an L-player under her action a and her partner's action a'. Here, the players' actions a and a' can be referred to as *cooperation levels*. Then, in order to reflect the prisoner's dilemma setting, it is assumed that for each $a, a' \in [0, \tilde{a}]$, if $a > 0$, then $\Pi_{IJ}(0, a') > \Pi_{IJ}(a, a')$. In addition, the payoff under zero actions, $\Pi_{IJ}(0, 0)$, is normalized to zero.

In this study, three assumptions about the payoff functions from GR's model are adopted and adapted. First, the payoff function Π_{IJ} is assumed to be continuous, and the function $\Pi_{JJ}(a, a)$ is assumed to be strictly increasing in a. This assumption is used for the sake of simplicity. Second, there exists $a \in (0, \tilde{a}]$ such that $\Pi_{JJ}(a, a) > (1 - \delta)\Pi_{JJ}(0, a)$. Third, given any $a_L \in [0, \tilde{a}]$, there exists $a \in (0, \tilde{a}]$ such that $\pi\Pi_{HH}(a, a) + (1 - \pi)\Pi_{HL}(a, a_L) > (1 - \pi)\Pi_{HL}(0, a_L)$ where π is the proportion of H-players in the pool of unmatched players. If the second or the third assumption does not hold, then players might not have any incentive to play a positive action. Therefore, the latter two assumptions are used to exclude a trivial case in which players have no incentive to cooperate with their partners and prefer to play zero actions.

Regarding information, a player has limited information about types and actions. A player is informed only of her own type. However, if her partner plays a positive action, she can figure out her partner's type by comparing the outcomes drawn from her action and her partner's action. This is because, other things being equal, the cooperative action performed by a high ability partner brings out a better outcome than the action performed by a low ability partner. Note that a player cannot figure out her partner's type if her partner plays a zero action because of the normalization of the payoffs. In addition, a player knows only her own actions and her partners' actions from the beginning, but they do not know the actions taken by others. A player's *personal history* is defined as the record of her type, the types of her partners who have played positive actions, and all the actions taken by her partners and her from the beginning. Therefore, a pure strategy of a player is a possible mapping from her personal histories either to the set of the actions $[0, \tilde{a}]$ for the first substages or to the set of the breakup decisions for the second substages.

3 Social Equilibrium

In this study, our interest is restricted to *social norms* and *steady states* like in the study of GR. A social norm is a profile of pure strategies such that players of the same type use the same pure strategy. A state is steady if the proportion of H-players in the pool of unmatched players, π, is constant over time[1]. Moreover, our study focuses on the cooperation possibility based on players' preferences for the high ability of an H-player. So, we rule out the cases in which a player prefers betraying an H-player partner rather than cooperating with the H-player partner because of a possible huge payoff when she betrays the H-player partner. In addition, our equilibrium is required to satisfy two criteria: "Individual incentive constraint" and "Bilateral rationality," which were proposed by GR. These two criteria require an equilibrium to be proof against *individual deviation* and *pair-wise deviation*, respectively[2].

These two criteria are applied to five possible phases. First, these are applied to the phase in which two H-players are matched into a pair and they are aware of their partners' types. In this phase, H-players solve the following optimization problem;

[1] For information about the feasibility of a constant π, please refer to GR. Here, another interpretation of a constant π is presented. If we assume that the relationship will be exogenously broken up with a probability $\theta > 0$ regardless of players' breakup decisions, then we can easily show that a constant π is feasible. In addition, given any $\pi > 0$ and any positive number $\varepsilon > 0$, we can find an exogenous breakup probability $\theta > 0$ such that $\varepsilon > \theta$ and θ makes π a constant proportion of H-players in the pool over time. Therefore, the steady state in which $\pi > 0$ and $\theta = 0$ can be interpreted as the limit of the exogenous breakup cases.

[2] In GR, *individual incentive constraint* is defined as a social norm under which, given that other players follow the norm, no player has an incentive to deviate from the norm. In addition, *bilateral rationality* is defined as a social norm under which, given that other players follow the norm, no matched pair of players who have followed the norm can improve their payoffs by making a joint change from the norm. For more information about these criteria, please refer to GR.

given $0 \leq x_H \leq \delta \max_{a\in[0,\tilde{a}]}\{\frac{\Pi_{HH}(a,a)}{1-\delta} - \Pi_{HH}(0,a)\}$,

$$V_H^F(x_H) \equiv \max_{a\in[0,\tilde{a}]} \frac{\Pi_{HH}(a,a)}{1-\delta} \tag{1}$$

$$s.t.\ \frac{\Pi_{HH}(a,a)}{1-\delta} \geq \Pi_{HH}(0,a) + \delta x_H \tag{2}$$

where x_H denotes a present value to an H-player when she is in the pool of unmatched players. Given a present value to an H-player, this optimization problem yields the highest possible cooperation level, which, therefore, satisfies bilateral rationality, among all the cooperation levels that satisfy individual incentive constraint.

Second, based on the optimization problem above, the two criteria are applied to the phase for an H-player when she is newly matched and thus she does not know her partner's type; given x_H and $a_L^S \in [0, \tilde{a}]$,

$$V_H^S(x_H, a_L^S) \equiv \max_{a\in[0,\tilde{a}]} \pi\{\Pi_{HH}(a,a) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a, a_L^S) + \delta x_H\} \tag{3}$$

$$\begin{aligned} s.t.\ &\pi\{\Pi_{HH}(a,a) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a, a_L^S) + \delta x_H\} \\ &\geq \pi\{\Pi_{HH}(0,a) + \delta x_H\} + (1-\pi)\{\Pi_{HL}(0, a_L^S) + \delta x_H\} \end{aligned} \tag{4}$$

where a_L^S denotes an action of an L-player when she is newly matched.

Similarly, the two criteria are applied to the phases for L-players. Third, two L-players who are certain that their partners are L-players solve the following problem; given $0 \leq x_L \leq \delta \max_{a\in[0,\tilde{a}]}\{\frac{\Pi_{LL}(a,a)}{1-\delta} - \Pi_{LL}(0,a)\}$,

$$V_L^F(x_L) \equiv \max_{a\in[0,\tilde{a}]} \frac{\Pi_{LL}(a,a)}{1-\delta} \tag{5}$$

$$s.t.\ \frac{\Pi_{LL}(a,a)}{1-\delta} \geq \Pi_{LL}(0,a) + \delta x_L \tag{6}$$

where x_L denotes a present value to an L-player when she is in the pool of unmatched players.

Fourth, an L-player who is newly matched solves the following problem; given x_L and $a_H^S \in [0, \tilde{a}]$,

$$V_L^S(x_L, a_H^S) \equiv \max_{a\in[0,\tilde{a}]} \pi\{\Pi_{LH}(a, a_H^S) + \delta x_L\} + (1-\pi)\{\Pi_{LL}(a,a) + \delta V_L^F(x_L)\} \tag{7}$$

$$\begin{aligned} s.t.\ &\pi\{\Pi_{LH}(a, a_H^S) + \delta x_L\} + (1-\pi)\{\Pi_{LL}(a,a) + \delta V_L^F(x_L)\} \\ &\geq \pi\{\Pi_{LH}(0, a_H^S) + \delta x_L\} + (1-\pi)\{\Pi_{LL}(0,a) + \delta x_L\} \end{aligned} \tag{8}$$

where a_H^S denotes an action of an H-player when she is newly matched.

Finally, the two criteria are applied to the phase in which an H-player and an L-player are matched into a pair and they are aware of their partners' types. In equilibrium, players could have long-term cooperative relationships in the previous four phases only if they cannot achieve cooperation in this phase. So, given present values in the pool of unmatched players, we need to show that every cooperation level that satisfies individual incentive constraint does not give an H-player or an L-player a greater payoff than their present values in the pool. This condition is formalized at the condition 9 in Definition 1.

Now, we are ready to define our equilibrium, which we call a "Social Equilibrium." This social equilibrium is adopted and adapted from GR.

Definition 1 *A social equilibrium is a collection of actions* $(a_H^F, a_H^S, a_L^F, a_L^S)$ *and payoffs* $(V_H^F, V_H^S, V_L^F, V_L^S)$ *such that*

1. *given* V_H^S, a_H^F *solves (1) subject to (2);*
2. *given* V_H^S *and* a_L^S, a_H^S *solves (3) subject to (4);*
3. *given* V_L^S, a_L^F *solves (5) subject to (6);*
4. *given* V_L^S *and* a_H^S, a_L^S *solves (7) subject to (8);*
5. *the payoff* V_H^F *equals the maximum value* $V_H^F(V_H^S)$*;*
6. *the payoff* V_H^S *equals the maximum value* $V_H^S(V_H^S, a_L^S)$*;*
7. *the payoff* V_L^F *equals the maximum value* $V_L^F(V_L^S)$*;*
8. *the payoff* V_L^S *equals the maximum value* $V_L^S(V_L^S, a_H^S)$*;*

and for all $a', a'' \in [0, \tilde{a}]$,

$$\text{9. if } \frac{\Pi_{HL}(a', a'')}{1-\delta} \geq \Pi_{HL}(0, a'') + \delta V_H^S, \text{ then } V_H^S \geq \frac{\Pi_{HL}(a', a'')}{1-\delta} \text{ or} \tag{9}$$

$$\text{if } \frac{\Pi_{LH}(a'', a')}{1-\delta} \geq \Pi_{LH}(0, a') + \delta V_L^S, \text{ then } V_L^S \geq \frac{\Pi_{LH}(a'', a')}{1-\delta}. \tag{10}$$

4 Results

In this study, the results are similar to GR's in respect to the factors that can influence the level of cooperation in equilibrium. In both studies, cooperation is enhanced when players find their proper matches or when the discount factor goes up. However, while GR's results apply to partial population only, the following results show that a long-term cooperative relationship among the whole population is possible. The first result shows that there exists a social equilibrium. Like in GR, special assumptions on payoff functions are used for the existence of the equilibrium. Note that only Proposition 1 uses these special assumptions.

Assumption 1. For each $J \in \{H, L\}$, the payoff function $\Pi_{JJ}(a, a)$ is strictly concave, the function $\Pi_{JJ}(a, 0)$ is concave, and the function $\Pi_{JJ}(0, a)$ is convex[3].

[3] For the information about this assumption, please refer to GR.

Assumption 2. The left-hand partial derivatives of $\Pi_{HL}(a_1, a_2)$ and $\Pi_{LH}(a_1, a_2)$ with respect to the first argument a_1 are continuous in the second argument a_2.

Assumptions 1 and 2 guarantee that the optimization functions $V_H^S(\cdot,\cdot)$ and $V_L^S(\cdot,\cdot)$ and the optimizers in these functions are continuous in their arguments. This property of continuity serves as a stepping-stone for the existence of a fixed point in the optimization problems above.

Assumption 3. For each $IJ \in \{HL, LH\}$, the payoff function $\Pi_{IJ}(a_1, a_2)$ is concave in a_1 and convex in a_2, and for $a_1 > 0$, $\Pi_{IJ}(0, a_2) - \Pi_{IJ}(a_1, a_2) \leq \Pi_{IJ}(0, a_2') - \Pi_{IJ}(a_1, a_2')$ if $a_2 > a_2'$.

Assumption 3 implies that in the different-type matches, *i.e.*, the H-L matches, the payoff $\Pi_{IJ}(a_1, a_2)$ decreases with her own action a_1 at an increasing rate and increases with her partner's action a_2 at an increasing rate. In addition, when a_1 is positive, the payoff difference $\Pi_{IJ}(0, a_2) - \Pi_{IJ}(a_1, a_2)$ decreases in a_2. This assumption is used for the sake of simplicity.

Under Assumptions 1, 2, and 3, Proposition 1 presents a sufficient condition for the existence of a social equilibrium. Like in GR, the notations below are used to simplify the sufficient condition. First, denote by a_H^1 and a_L^1 the maximizers of the functions $\Pi_{HH}(a, a) - (1-\delta)\Pi_{HH}(0, a)$ and $\Pi_{LL}(a, a) - (1-\delta)\Pi_{LL}(0, a)$, respectively. Next, let a_H^2 and a_L^2 denote the maximum values of a s.t.

$$\begin{aligned} & \pi\{\Pi_{HH}(0, a) - \Pi_{HH}(a, a)\} + (1-\pi)\{\Pi_{HL}(0, \tilde{a}) - \Pi_{HL}(a, \tilde{a})\} \\ \leq\ & \pi\{\Pi_{HH}(0, a_H^1) - \Pi_{HH}(a_H^1, a_H^1)\} \text{ and} \end{aligned}$$

$$\begin{aligned} & \pi\{\Pi_{LH}(0, \tilde{a}) - \Pi_{LH}(a, \tilde{a})\} + (1-\pi)\{\Pi_{LL}(0, a) - \Pi_{LL}(a, a)\} \\ \leq\ & (1-\pi)\{\Pi_{LL}(0, a_L^1) - \Pi_{LL}(a_L^1, a_L^1)\}, \text{ respectively.} \end{aligned}$$

Finally, let a_H^3 and a_L^3 denote the maximizers of the strictly concave functions $\pi\Pi_{HH}(a, a) + (1-\pi)\Pi_{HL}(a, \tilde{a})$ and $\pi\Pi_{LH}(a, \tilde{a}) + (1-\pi)\Pi_{LL}(a, a)$, respectively.

Here is a sufficient condition[4] for the existence of a fixed point in the aforementioned optimization problems.

Condition E If $a_H^3 \leq a_H^2$, then

$$\begin{aligned} & \pi\Pi_{HH}(a_H^3, a_H^3) + (1-\pi)\Pi_{HL}(a_H^3, \tilde{a}) \\ \leq\ & (\pi + \frac{1}{\delta})\Pi_{HH}(a_H^1, a_H^1) + (1 - \pi - \frac{1}{\delta})\Pi_{HH}(0, a_H^1)\}. \end{aligned} \tag{11}$$

If $a_H^3 > a_H^2$, then

$$\delta\pi\Pi_{HH}(0, a_H^2) + \delta(1-\pi)\Pi_{HL}(0, \tilde{a}) \leq \Pi_{HH}(a_H^1, a_H^1) - (1-\delta)\Pi_{HH}(0, a_H^1). \tag{12}$$

[4]For an intuitive description of this condition, please refer to GR.

If $a_L^3 \leq a_L^2$, then

$$\begin{aligned} & \pi\Pi_{LH}(a_L^3, \tilde{a}) + (1-\pi)\Pi_{LL}(a_L^3, a_L^3) \\ \leq \; & (1 - \pi + \frac{1}{\delta})\Pi_{LL}(a_L^1, a_L^1) + (\pi - \frac{1}{\delta})\Pi_{LL}(0, a_L^1)\}. \end{aligned} \tag{13}$$

If $a_L^3 > a_L^2$, then

$$\delta\pi\Pi_{LH}(0, \tilde{a}) + \delta(1-\pi)\Pi_{LL}(0, a_L^2) \leq \Pi_{LL}(a_L^1, a_L^1) - (1-\delta)\Pi_{LL}(0, a_L^1). \tag{14}$$

To sustain a social equilibrium, a fixed point in the optimization problems above has to satisfy the condition 9 in Definition 1 in which one of the types has no incentive to cooperate with the other type. If the ability difference between an H-player and an L-player is *wide enough*, then the H-player would have no incentive to cooperate with the L-player, and therefore, the fixed point would satisfy the condition 9 in Definition 1. Definition 2 below provides the level of the ability difference in which an H-player has no incentive to cooperate with an L-player.

Definition 2 *Define a_H^4 as the value of a such that $(1-\delta\pi)\Pi_{LH}(0, a) = \delta(1-\pi)\Pi_{LL}(a_L^1, a_L^1)$. The ability difference between an H-player and an L-player is said to be **wide enough** if $\delta\pi\Pi_{HH}(a_H^1, a_H^1) \geq (1-\delta+\delta\pi)\Pi_{HL}(a_H^4, \tilde{a})$ whenever a_H^4 exists.*

Proposition 1 *Under Assumptions 1,2, and 3, a social equilibrium exists if Condition E holds and the ability difference between an H-player and an L-player is wide enough.*

Proof. See Appendix. ■

Examples with specific payoff functions can be found in GR. The payoff functions from GR, however, have to be adapted for the L-players. In GR, one of two types has no incentive to play any positive action. The zero action by this type lowers a present value to the other type in the pool of unmatched players, and this lowered present value in turn makes an ongoing cooperative relationship more valuable. As a result, although an one-period payoff from betrayal is high, the type who has an incentive to play a positive action can sustain a long-term cooperative relationship among themselves. In the present model, on the other hand, when H-players are newly matched with L-players, they play positive cooperative actions a_H^S. Since the H-players' actions a_H^S significantly improve present values to L-players in the pool, if one-period payoffs to L-players when they betray other L-players are as high as those in GR, then L-players would prefer betraying their partners more than cooperating with them. Therefore, the payoff functions from GR need to be modified so that L-players can sustain long-term cooperative relationships among themselves.

The second result describes cooperation levels in each phase in equilibrium. There are two possible phases in which a player can play different levels of cooperation.

First, a player can play a cooperative action when she knows that her partner is of the same type as herself. Second, a player can play another cooperative action when she is newly matched, and thus, does not know her partner's type. Proposition 2 below shows that a player plays a higher cooperative action in the first phase than in the second phase except that she achieves the same level of cooperation when she plays full cooperative actions in both phases. According to the interpretation of GR, Proposition 2 characterizes a social equilibrium into a "testing phase" and a "cooperation phase." In the testing phase, the players are "cautious," and as a result, they have less to achieve. If they are confirmed that they are matched with the same type of players as themselves, then they move into the cooperation phase where they can play at greater cooperation levels.

Proposition 2 *In a social equilibrium, $a_J^F \geq a_J^S$ where $J \in \{H, L\}$ with strict inequality holding whenever $a_J^F < \tilde{a}$.*

Proof. Consider an H-player case. If $a_H^F = \tilde{a}$, then it is trivial. Let $a_H^F < \tilde{a}$ in equilibrium. By way of contradiction, suppose that $a_H^F \leq a_H^S$. Then, we have that $\Pi_{HH}(0, a_H^S) + \delta V_H^S \geq \frac{\Pi_{HH}(a_H^S, a_H^S)}{1-\delta}$ by the constraint (2). Then,

$$
\begin{aligned}
(1-\delta)\{\Pi_{HH}(0, a_H^S) - \Pi_{HH}(a_H^S, a_H^S)\} &\geq \delta\{\Pi_{HH}(a_H^S, a_H^S) - (1-\delta)V_H^S\} \\
&\geq \delta\{\Pi_{HH}(a_H^F, a_H^F) - (1-\delta)V_H^S\} = \delta(1-\delta)(V_H^F - V_H^S) \\
&> \delta(1-\delta)(V_H^F - V_H^S) + \frac{1-\pi}{\pi}(1-\delta)\{\Pi_{HL}(a_H^S, a_L^S) - \Pi_{HL}(0, a_L^S)\}
\end{aligned}
$$

where the fact $\Pi_{HL}(a_H^S, a_L^S) - \Pi_{HL}(0, a_L^S) < 0$ is used at the last inequality. This contradicts (4). Therefore, we have $a_H^F > a_H^S$. Similarly, we can show $a_L^F \geq a_L^S$ with strict inequality holding whenever $a_L^F < \tilde{a}$. ■

The final result goes one step further from GR's. In their paper, as players become infinitely patient, the cooperation level in equilibrium approaches full cooperation once players find their proper matches. In the present model, Proposition 3 below states that *when players are sufficiently patient, they play the maximal cooperation level in equilibrium right after they check that they are matched with the same type partners as themselves*, which is the folk theorem of this study.

Proposition 3 (Folk Theorem) *There exists a discount factor $\delta^* < 1$ such that for any $\delta \in [\delta^*, 1)$, $a_H^F = a_L^F = \tilde{a}$ in a social equilibrium under δ, whenever the social equilibrium exists.*

Proof. By way of contradiction, suppose not. Then, for any $\delta < 1$, there exists $1 > \delta' \geq \delta$ such that under δ', there exists a social equilibrium with $a_H^F < \tilde{a}$ or $a_L^F < \tilde{a}$. First, consider the case in which for any $\delta < 1$, there exists $1 > \delta' \geq \delta$ such that under δ', there exists a social equilibrium with $a_H^F < \tilde{a}$. In the social equilibrium under the discount factor δ', let V_H^S be a present value to an H-player in the pool of unmatched players. Then, according to the constraint (2), we have that

$$\frac{\Pi_{HH}(\tilde{a},\tilde{a})}{1-\delta'} < \Pi_{HH}(0,\tilde{a}) + \delta' V_H^S. \tag{15}$$

In addition, we have that

$$V_H^S \leq \frac{1}{\delta'}\{\frac{\Pi_{HH}(a_H^1,a_H^1)}{1-\delta'} - \Pi_{HH}(0,a_H^1)\} \tag{16}$$

where a_H^1 is a maximizer of $\Pi_{HH}(a,a)-(1-\delta)\Pi_{HH}(0,a)$. Note that since $\frac{1}{\delta'}\{\frac{\Pi_{HH}(\tilde{a},\tilde{a})}{1-\delta'}$ $-\Pi_{HH}(0,\tilde{a})\} < V_H^S \leq \frac{1}{\delta'}\{\frac{\Pi_{HH}(a_H^1,a_H^1)}{1-\delta'}$ $-\Pi_{HH}(0,a_H^1)\}$, we have that $a_H^1 < \tilde{a}$. By combining (15) with (16), we have that

$$\begin{aligned} & \frac{\Pi_{HH}(\tilde{a},\tilde{a})}{1-\delta'} < \Pi_{HH}(0,\tilde{a}) + \frac{\Pi_{HH}(a_H^1,a_H^1)}{1-\delta'} - \Pi_{HH}(0,a_H^1) \\ \Longleftrightarrow \quad & \frac{1}{1-\delta'}\{\Pi_{HH}(\tilde{a},\tilde{a}) - \Pi_{HH}(a_H^1,a_H^1)\} < \Pi_{HH}(0,\tilde{a}) - \Pi_{HH}(0,a_H^1). \end{aligned} \tag{17}$$

However, since the inequality (17) holds for any $\delta < 1$ and thus for any $1 > \delta' \geq \delta$, (17) is contradiction. Similarly, we can show that it is contradiction that for any $\delta < 1$, there exists $1 > \delta' \geq \delta$ such that under δ', there exists a social equilibrium with $a_L^F < \tilde{a}$. This completes the proof. ■

Distinguished from GR's model, the present model is significant in two aspects. First, in GR, if there exists a social equilibrium, it must be unique, because one type always prefers to play a zero action and the other type has only one best response to the zero action. In the present model, however, there could be multiple social equilibria, because players can play different levels of initial actions (a_H^S, a_L^S). Since each type has the best response to the initial action of the other type, there could be multiple social equilibria elicited from multiple initial actions. Second, in GR, a change in π, the proportion of non-myopic players of unmatched players, directly influences the payoffs. An increase in π results in an increase in the present values when players are in the unmatched pool and also results in a non-increase in the payoffs to non-myopic players when they find non-myopic partners. In the present model, however, due to the possible existence of multiple equilibria, a change in π, the proportion of H-players in the pool of unmatched players, does not have a clear effect on the payoffs $(V_H^F, V_H^S, V_L^F, V_L^S)$. This is because the impact from a change in π could be diluted with the influence from a change in equilibria.

In conclusion, a long-term cooperative behavior that is robust to both pair-wise deviation and individual deviation is possible among the whole population in equilibrium. Regarding the cooperative behavior, after players play a lower cooperation in the testing phase, they move on to higher cooperation in the cooperation phase. If players are patient enough, both H-players and L-players can achieve full cooperation once they find their proper matches. Therefore, based on players' preferences for their partners' types, a long-term cooperative relationship among the whole population is possible in equilibrium.

Appendix

Proof of Proposition 1. Consider (1) subject to (2). Note that $V_H^F(x_H)$ is continuous and non-increasing in x_H. Also, we have that $V_H^F(x_H) - x_H > 0$ because of (2). Given $x_H \in [0, \frac{1}{\delta}\{\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} - \Pi_{HH}(0, a_H^1)\}]$ and $a_L^S \in [0, \tilde{a}]$, define

$$a_H^S(x_H, a_L^S) \in \arg\max_{a \in [0,\tilde{a}]} \pi\{\Pi_{HH}(a,a) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a, a_L^S) + \delta x_H\}$$
$$s.t.\ \pi\{\Pi_{HH}(0,a) + \delta x_H\} + (1-\pi)\{\Pi_{HL}(0, a_L^S) + \delta x_H\}$$
$$\leq \pi\{\Pi_{HH}(a,a) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a, a_L^S) + \delta x_H\}.$$

Let $a_H^2(x_H, a_L^S)$ be the maximum value of a such that

$$\begin{aligned} & \pi\{\Pi_{HH}(0,a) + \delta x_H\} + (1-\pi)\{\Pi_{HL}(0, a_L^S) + \delta x_H\} \\ \leq \quad & \pi\{\Pi_{HH}(a,a) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a, a_L^S) + \delta x_H\} \\ \Longleftrightarrow \quad & \pi\{\Pi_{HH}(0,a) - \Pi_{HH}(a,a)\} + (1-\pi)\{\Pi_{HL}(0, a_L^S) - \Pi_{HL}(a, a_L^S)\} \\ \leq \quad & \delta\pi\{V_H^F(x_H) - x_H\}. \end{aligned}$$

Then, $a_H^2(x_H, a_L^S) > 0$ since $V_H^F(x_H) - x_H > 0$. Also, $a_H^2(x_H, a_L^S)$ is continuous in x_H and a_L^S because 1) $\pi\{\Pi_{HH}(0,a) - \Pi_{HH}(a,a)\} + (1-\pi)\{\Pi_{HL}(0, a_L^S) - \Pi_{HL}(a, a_L^S)\}$ is strictly increasing in a and continuous in a and a_L^S; and 2) $\delta\pi\{V_H^F(x_H) - x_H\}$ is continuous in x_H. In addition, let $a_H^3(a_L^S)$ denote the maximizer of the strictly concave function $\pi\Pi_{HH}(a,a) + (1-\pi)\Pi_{HL}(a, a_L^S)$. Then $a_H^3(a_L^S) > 0$ because given any $a_L^S \in [0, \tilde{a}]$, there exists $a > 0$ s.t. $\pi\Pi_{HH}(a,a) + (1-\pi)\Pi_{HL}(a, a_L^S) > (1-\pi)\Pi_{HL}(0, a_L^S)$. The function $\Pi_{HL}(a_1, a_2)$ is concave in a_1, and therefore, its left-hand partial derivative with respect to a_1, $\frac{\partial \Pi_{HL}(a_1 - 0, a_2)}{\partial a_1}$, is well-defined on $(0, \tilde{a}]^2$. According to Assumption 2, $\frac{\partial \Pi_{HL}(a_1 - 0, a_2)}{\partial a_1}$ is continuous in a_2. Also, the function $\Pi_{HH}(a,a)$ is strictly concave in a. Therefore, $a_H^3(a_L^S)$ is continuous in a_L^S. Note that $a_H^S(x_H, a_L^S) = \min\{a_H^2(x_H, a_L^S), a_H^3(a_L^S)\}$. Since $a_H^2(x_H, a_L^S)$ and $a_H^3(a_L^S)$ are positive and continuous in x_H and a_L^S, so is $a_H^S(x_H, a_L^S)$. Define

$$\Phi_H(x_H, a_L^S) \equiv \max_{a \in [0,\tilde{a}]} \pi\{\Pi_{HH}(a,a) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a, a_L^S) + \delta x_H\}$$
$$s.t.\ \pi\{\Pi_{HH}(0,a) - \Pi_{HH}(a,a)\} + (1-\pi)\{\Pi_{HL}(0, a_L^S) - \Pi_{HL}(a, a_L^S)\}$$
$$\leq \delta\pi\{V_H^F(x_H) - x_H\}.$$

Then $\Phi_H(x_H, a_L^S) = \pi\{\Pi_{HH}(a_H^S(x_H, a_L^S), a_H^S(x_H, a_L^S)) + \delta V_H^F(x_H)\} + (1-\pi)\{\Pi_{HL}(a_H^S(x_H, a_L^S), a_L^S) + \delta x_H\}$, and $\Phi_H(x_H, a_L^S)$ is continuous in x_H and a_L^S. Similarly, we can define

$a_L^S(x_L, a_H^S)$ and $\Phi_L(x_L, a_H^S)$ for an L-player and show that $a_L^S(x_L, a_H^S)$ and $\Phi_L(x_L, a_H^S)$ are continuous in x_L and a_H^S.

Let

$$\begin{aligned}\hat{V}_H^S &\equiv \max_{a\in(0,\tilde{a}]}\{\frac{\Pi_{HH}(a,a)}{1-\delta} - \Pi_{HH}(0,a)\} \text{ and} \\ \hat{V}_L^S &\equiv \max_{a\in(0,\tilde{a}]}\{\frac{\Pi_{LL}(a,a)}{1-\delta} - \Pi_{LL}(0,a)\}.\end{aligned}$$

Then $\hat{V}_H^S$ and $\hat{V}_L^S$ exist and are positive because $\Pi_{JJ}(a,a) > (1-\delta)\Pi_{JJ}(0,a)$ for some $a > 0$ where $J \in \{H, L\}$. If

$$\max_{a_L^S}\{\Phi_H(\hat{V}_H^S, a_L^S)\} \leq \hat{V}_H^S \text{ and} \tag{18}$$

$$\max_{a_H^S}\{\Phi_L(\hat{V}_L^S, a_H^S)\} \leq \hat{V}_L^S, \tag{19}$$

then by using the values $a_H^S(x_H, a_L^S)$, $a_L^S(x_L, a_H^S)$, $\min\{\Phi_H(x_H, a_L^S), \hat{V}_H^S\}$, and $\min\{\Phi_L(x_L, a_H^S), \hat{V}_L^S\}$, we can construct a continuous function from $[0,\tilde{a}]^2 \times [0, \hat{V}_H^S] \times [0, \hat{V}_L^S]$ to $[0,\tilde{a}]^2 \times [0, \hat{V}_H^S] \times [0, \hat{V}_L^S]$ such that a fixed point of the function, whose existence is guaranteed by Brouwer's Fixed Point Theorem, solves (3) subject to (4) and solves (7) subject to (8). Therefore, to complete the proof, we should check that (18) and (19) are equivalent to Condition E and also should show that if the ability difference between an H-player and an L-player is wide enough, then the fixed point satisfies the condition 9 in Definition 1.

First, check that (18) and (19) are equivalent to Condition E. Note that

$$\hat{V}_H^S = \frac{1}{\delta}\{\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} - \Pi_{HH}(0, a_H^1)\}.$$

In addition, note that $\Phi_H(x_H, \tilde{a}) \geq \Phi_H(x_H, a_L^S)$ for every $a_L^S \in [0, \tilde{a}]$ since $\Pi_{HL}(0, a_2)$ $-\Pi_{HL}(a_1, a_2) \leq \Pi_{HL}(0, a_2') - \Pi_{HL}(a_1, a_2')$ if $a_2 > a_2'$. Therefore, if $a_H^3(\tilde{a}) \leq a_H^2(\hat{V}_H^S, \tilde{a})$, *i.e.*, $a_H^3 \leq a_H^2$, then

$$\begin{aligned}\max_{a_L^S}\{\Phi_H(\hat{V}_H^S, a_L^S)\} \leq \hat{V}_H^S &\iff \Phi(\hat{V}_H^S, \tilde{a}) \leq \hat{V}_H^S \\ &\iff \pi\Pi_{HH}(a_H^3, a_H^3) + (1-\pi)\Pi_{HL}(a_H^3, \tilde{a}) \\ +\delta\pi\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} &+ \delta\frac{(1-\pi)}{\delta}\{\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} - \Pi_{HH}(0, a_H^1)\} \\ &\leq \frac{1}{\delta}\{\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} - \Pi_{HH}(0, a_H^1)\},\end{aligned}$$

which is equivalent to (11). If $a_H^3 > a_H^2$, then by the definition of a_H^2,

$$\begin{aligned}
&\Phi_H(\hat{V}_H^S, \tilde{a}) \leq \hat{V}_H^S \\
&\iff \pi\Pi_{HH}(0, a_H^2) + (1-\pi)\Pi_{HL}(0, \tilde{a}) \\
&+\frac{1-\pi+\delta\pi}{1-\delta}\Pi_{HH}(a_H^1, a_H^1) - (1-\pi)\Pi_{HH}(0, a_H^1) \\
&\leq \frac{1}{\delta}\{\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} - \Pi_{HH}(0, a_H^1)\},
\end{aligned}$$

which is equivalent to (12). Similarly, we can show that (19) is equivalent to (13) and (14). Therefore, (11), (12), (13), and (14) are a sufficient condition for the existence of a fixed point.

Finally, we need to show that the fixed point satisfies (9) and (10). Let V_H^S and V_L^S be parts of the fixed point such that V_H^S and V_L^S satisfy the respective conditions 6 and 8 in Definition 1. Then, we have $V_H^S(V_H^S, a_L^S) = V_H^S$ and $V_L^S(V_L^S, a_H^S) = V_L^S$ for some a_L^S and a_H^S. Note that $V_H^F(x_H)$ and $V_L^F(x_L)$ are non-increasing in x_H and x_L, respectively, and thus, $V_H^F(x'_H) \geq V_H^F(\hat{V}_H^S)$ and $V_L^F(x'_L) \geq V_L^F(\hat{V}_L^S)$ for any x'_H and x'_L. By (3) and (7), we have that

$$V_H^S \geq \delta\pi V_H^F(\hat{V}_H^S) + \delta(1-\pi)V_H^S \iff V_H^S \geq \frac{\delta\pi}{1-\delta+\delta\pi}\frac{\Pi_{HH}(a_H^1, a_H^1)}{1-\delta} \tag{20}$$

$$\text{and } V_L^S \geq \delta\pi V_L^S + \delta(1-\pi)V_L^F(\hat{V}_L^S) \iff V_L^S \geq \frac{\delta(1-\pi)}{1-\delta\pi}\frac{\Pi_{LL}(a_L^1, a_L^1)}{1-\delta}. \tag{21}$$

Suppose that $a', a'' \in [0, \tilde{a}]$ satisfy the premises of (9) and (10). Then,

$$\begin{aligned}
&\frac{\Pi_{LH}(0, a')}{1-\delta} \geq \frac{\Pi_{LH}(a'', a')}{1-\delta} \geq \Pi_{LH}(0, a') + \delta V_L^S \\
&\implies \delta\frac{\Pi_{LH}(0, a')}{1-\delta} \geq \delta\frac{\delta(1-\pi)}{1-\delta\pi}\frac{\Pi_{LL}(a_L^1, a_L^1)}{1-\delta}
\end{aligned} \tag{22}$$

where (21) is used at the inequality (22). Since the function $\Pi_{LH}(0, a)$ is convex, from (22), we can find that there exists a_H^4 and that $a' \geq a_H^4$. Since $\tilde{a} \geq a''$, we have that $\frac{\Pi_{HL}(a_H^4, \tilde{a})}{1-\delta} \geq \frac{\Pi_{HL}(a', a'')}{1-\delta}$. Since the ability difference between an H-player and an L-player is wide enough,

$$\begin{aligned}
&\frac{\delta\pi}{1-\delta+\delta\pi}\Pi_{HH}(a_H^1, a_H^1) \geq \Pi_{HL}(a_H^4, \tilde{a}) \\
&\implies V_H^S \geq \frac{\Pi_{HL}(a_H^4, \tilde{a})}{1-\delta} \geq \frac{\Pi_{HL}(a', a'')}{1-\delta}
\end{aligned}$$

where the second inequality follows from (20). Since $a', a'' \in [0, \tilde{a}]$ are arbitrary, the fixed point satisfies the condition 9 in Definition 1. This completes the proof.///

References

[1] Ellison, G (1994), "Cooperation in the prisoner's dilemma with anonymous random matching," *Review of Economic Studies*, 61, 567-588.

[2] Fudenberg, D and Maskin, E (1986), "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54, 533-556.

[3] Ghosh, P and Ray, D (1996), "Cooperation in Community Interaction without Information flows," *Review of Economic Studies*, 63, 491-519.

[4] Kandori, M (1992), "Social norms and community enforcement," *Review of Economic Studies*, 59, 63-80.