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Abstract

A pillage game is a coalitional game that is meant to be a model of *Hobbesian anarchy*. The *spatial pillage game* introduces a *spatial feature* into the pillage game by assuming that players are located in regions. Players can travel from one region to another in one move and can form a coalition and combine their power only with players in the same region. A coalition has power only within its region. Under this spatial restriction, some members of a coalition can pillage less powerful coalitions without any cost. The feasibility of pillages between coalitions determines the *dominance relation*. *Core, stable set, and farsighted core* are adopted as alternative solution concepts.

JEL Classification Numbers: C71, D74, R19

Keywords: allocation by force, coalitional games, pillage game, spatial restriction, stable set, farsighted core

1 Introduction

Hobbesian anarchy is a state of society before a government ensuring property rights is organized. Without such an organization, no individuals are safe to secure their wealth. Individuals could be tempted to pillage others whenever possible and beneficial. Although a coalition could be formed to secure their wealth, some members of the coalition may still be tempted to betray the others and to take their wealth. Consequently, in Hobbesian anarchy, the possibility of the stable distribution of wealth is questionable.

A substantial amount of literature on *allocation by force* has been devoted to this possibility. Skaperdas (1992) showed that a cooperative outcome is possible in equilibrium if the probability of winning in conflict is sufficiently robust against

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each individual's action. Hirshleifer (1995) found the conditions to make Hobbesian anarchy stable. Also, Hirshleifer (1991), Konrad and Skaperdas (1998), and Muthoo (1991) studied the situations in which property right is partially secured. These studies analyzed noncooperative models in which the formation of coalitions is limited or not allowed.

Different from the previous models, Piccione and Rubinstein (2006) and Jordan (2005) developed models of Hobbesian anarchy that allow the formation of coalitions. Piccione and Rubinstein introduced *the jungle* in which coercion governs economic transactions and they compared the equilibrium allocation of the jungle with the equilibrium allocation of an exchange economy. Jordan introduced *pillage games* and examined stable sets of allocations in which the power of pillaging balances endogenously.

The *spatial pillage game* is an extended version of a pillage game. In most literature on "allocation by force" including the papers reviewed above, there is no restriction on using power. Thus any individual or coalitions can pillage another individual or other coalitions if one is more powerful than others. However, the acts of pillaging and defending are inevitably under spatial restriction. Members of a coalition, if they move together, cannot simultaneously pillage two less powerful coalitions that are far apart from each other. Likewise, two coalitions cannot combine their power to defend themselves together against another powerful coalition unless they are close enough to each other. The spatial pillage game introduces a *space concept*, which conditions power usage based on location, into a Hobbesian anarchy model that allows the formation of coalitions, in the hope of understanding how spatial restriction affects stable distributions of wealth.

The assumptions about the space concept are as follows. There are regions and each player can stay in only one of the regions. Players can change their regions to pillage others. The regions are connected with one another, and thus players can travel from a region to another in one move. Players can form a coalition and combine their power only after getting together in a common region. If coalitions are in different regions, they cannot combine their power. The influence of the power of each coalition is limited within its region. Therefore, a coalition cannot pillage two other coalitions in different regions simultaneously.

The other assumptions of the spatial pillage game are the same as the original pillage games. A fixed amount of wealth is allocated among a finite number of players. Some players can form a coalition under the spatial restriction. A coalition can pillage less powerful coalitions within its region without any cost. An increase in the wealth of a coalition causes an increase in its power. Since the power of each coalition is endogenously determined, the spatial pillage game cannot have a characteristic function, which exogenously determines the power of each coalition.

The pillage games are characterized by *power functions* that determine the feasibility of pillages between coalitions. Jordan presented three power functions classified by the degree of their dependence on the sizes of coalitions. *Wealth is power* is one of the power functions and specifies the power of each coalition as its total wealth. Therefore, "wealth is power" is characterized as independent of the sizes of coalitions. Only the pillage game with this function has a stable set in every possible case. Therefore, the spatial pillage game adopts "wealth is power" as a power function.

As criteria for stable distributions of wealth and players, three solution concepts are explored, i.e., core, stable set, and farsighted core. First, the core is the collection of states at which pillage is not possible, thus it is one of the most persuasive solution concepts. However, because of its strong requirement, the core is too small to represent stable situations as shown in Theorem 6. Second, a stable set is much bigger than the core if it exists, as shown in Theorem 21. A stable set is a collection of states that is both internally stable and externally stable. Internal stability requires that pillage not be possible between states in the collection and external stability requires that pillage at a state outside the collection result in another state inside the collection. In most cases, however, there are no stable set. And even when they exist, they contain implausible states as shown in Theorems 21 and 28. Third, farsighted core, which was introduced by Jordan (2005), solves these problems in stable sets, as shown in Theorem 33 and Lemma 34. A farsighted core is a collection of states at which *pillage in expectation* is not possible in the sense that some members of the pillage would end up being worse off, and consequently they would not join the pillage.

In section 2, we search for the core and stable sets. First, the core is characterized. Then, since one-player models and two-player models are trivial, we start from three-player models and completely characterize stable sets in those models. Finally, we show that no stable set exists in a $I-$player and $N-$region model where $I = 4$ and $N = 2$ or $I \geq 4$ and $N \geq 3$. In section 3, we construct a consistent expectation, defined in Definition 31, to find a farsighted core. After confirming the existence of a farsighted core in a consistent expectation, we explore one of the properties that farsighted cores of consistent expectations have in common. Then, we show that in a $I-$player and $N-$region model where $1 \leq I \leq 3$ or $N = 1$, there is the unique farsighted core of consistent expectations. In section 4, a suggestion for further research is presented.

2 Core and stable set

The environment of a spatial pillage game is defined in Definitions 1 and 2. We normalize the total wealth to unity. Note that definitions in this section are applied throughout the whole paper.

Definition 1 [1] *The finite set I is the set of* ***players****. A* ***coalition*** *is a subset of I. The set $A = \{w \in \mathbb{R}^I : w_i \geq 0$ for all $i \in I$ and $\sum_{z\in I} w_z = 1\}$ is the set of* ***allocations****.*

[1] We follow notations in Jordan (2005).

The definitions below concern the spatial environment.

Definition 2 *The finite set R is the set of* ***regions*** *and the Cartesian product R^I is the set of* ***distributions****. Given a distribution $p \in R^I$, the coalition $p^r = \{i \in I : p_i = r\}$ is the* ***population*** *at region r.*

A distribution is short for a population distribution and denotes how players are distributed over the regions. For example, the distribution $p = (1, 1, 2)$ expresses that players 1 and 2 are at region 1 and player 3 is at region 2. Also, it means $p^1 = \{1, 2\}$ and $p^2 = \{3\}$.

A **state** denotes both the allocation and distribution of the status quo.

Definition 3 *The Cartesian product $X = A \times R^I$ is the set of* ***states****.*

For instance, the ordered pair $(w, p) = ((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 2))$ is a state in the three-player and two-region model. The state (w, p) expresses that player 1 has $\frac{1}{2}$ and player 2 has $\frac{1}{4}$ while staying at region 1 and player 3 has $\frac{1}{4}$ while staying at region 2.

The **dominance** relation between states is defined as follows.

Definition 4 *Given states (w, p) and (w', p'), define $W = \{i : w'_i > w_i\}$ and $L = \{i : w'_i < w_i\}$. Suppose for some $r, q \in R$, i) $\{i : w'_i \neq w_i\} \subset p'^r$; ii) $\{i : p_i \neq p'_i\} = \varnothing$ or $\{i : p_i \neq p'_i\} = W \subset p^q$; and iii) $\sum_{i \in W} w_i > \sum_{i \in L} w_i$. Then (w', p')* ***dominates*** *(w, p).*

The dominance relation shows which state the status quo can move to. It must satisfy both *physical* and *spatial conditions*. The physical condition requires that the winning coalition W must have enough power to pillage the losing coalition L. Definition 4 presents this condition at *iii*). Jordan (2005) introduced a variety of physical conditions. The condition *iii*) above accords with the physical condition of the *wealth is power* in Jordan (2005). The spatial condition requires that the act of pillaging must satisfy spatial restriction. This condition is expressed at *i*) and *ii*) in Definition 4. The condition *i*) means that transfers of wealth happen only in destination region r where the pillage happens. The condition *ii*) denotes that only the winners can travel and that they are all from the common region q.

In this section, we adopt the solution concepts of **core** and **stable set**. The definition stated below follows Lucas (1992) and Jordan (2005).

Definition 5 *The set of undominated states is the* ***core*** *C. For any set E of states, let the set $U(E)$ be the set of states that are not dominated by any state in E. A set S of states is a* ***stable set*** *if it satisfies both $S \subset U(S)$, which means internal stability, and $S \supset U(S)$, which means external stability.*

Therefore, a stable set S is defined by the set of states that satisfies $S = U(S)$.

Theorem 6 embodies the core. Note that this result is applied throughout section 2.

Theorem 6 *The set* $\{(w,p) \in X$: *for each* i, $w_i = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}, \frac{1}{2},$ *or* $0\}$ *is the core* C.

Proof. Suppose $(w,p) \in \{(w,p) \in X$: for each i, $w_i = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}, \frac{1}{2}$, or $0\}$. If $w_i > 0$, then $w_i \geq \min\{\frac{1}{2}, \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}\}$. If $\frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} \geq \frac{1}{2}$, then $\#\{r \in R : \Sigma_{j\in p^r} w_j > 0\} = 1$ or 2 and thus for each i, $w_i = 1, \frac{1}{2}$, or 0. In this case, any coalition W cannot pillage another coalition L such that $W \cap L = \varnothing$ because if $\sum_{i\in L} w_i > 0$, then $\sum_{i\in L} w_i \geq \frac{1}{2}$ and so $\frac{1}{2} \geq \sum_{i\notin L} w_i \geq \sum_{i\in W} w_i$. If $\frac{1}{2} > \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}$, then $\#\{i : w_i > 0\} = \#\{r \in R : \Sigma_{j\in p^r} w_j > 0\}$ since $\#\{i : w_i > 0\} \geq \#\{r \in R : \Sigma_{j\in p^r} w_j > 0\}$ and $\#\{i : w_i > 0\} \times \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} = \#\{i : w_i > 0\} \times \min\{\frac{1}{2}, \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}\} \leq \Sigma_{i\in I} w_i = 1$, and thus for each i, $w_i = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}$ or 0 since $w_i \geq \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}$ or 0. In this case, we have for each $r \in R$, $\Sigma_{j\in p^r} w_j = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}$ or 0, and so any coalition W such that $W \subset p^q$ for some $q \in R$ cannot pillage another coalition L such that $W \cap L = \varnothing$ because if $\sum_{i\in L} w_i > 0$, then $\sum_{i\in L} w_i \geq \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}$ and $\frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} \geq \sum_{i\in W} w_i$. Therefore, (w,p) is not dominated. Since (w,p) is arbitrary, every state in the set $\{(w,p) \in X$: for each i, $w_i = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}, \frac{1}{2}$, or $0\}$ is not dominated.

Suppose $(w,p) \notin \{(w,p) \in X$: for each i, $w_i = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}, \frac{1}{2}$, or $0\}$. Then there exists i such that $w_i \notin \{0, \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}, \frac{1}{2}, 1\}$. If $w_i > \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}$, then there exists $q \in R$ such that $\frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} > \Sigma_{j\in p^q} w_j > 0$ since $\Sigma_{j\in I} w_j = 1$, and thus player i can pillage another player j such that $w_j > 0$ and $p_j = q$ since $w_i > \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} > \Sigma_{k\in p^q} w_k \geq w_j > 0$. If $\#\{r \in R : \Sigma_{j\in p^r} w_j > 0\} = 1$ and $w_i < 1$, then either $1 > w_i > \frac{1}{2}$ or $\frac{1}{2} > w_i > 0$ since $w_i \notin \{\frac{1}{2}, 0\}$, and thus player i can pillage player j such that $w_j > 0$ or the coalition $W = \{k : k \neq i$ and $p_k = p_i\}$ can pillage player i. If $\#\{r \in R : \Sigma_{j\in p^r} w_j > 0\} \geq 2$ and $\frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} > w_i$, then $\frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}}$, which denotes the average wealth of regions except the region p_i, is well defined, and thus either $w_i \geq \frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}}$ or $\frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}} > w_i$. If $w_i \geq \frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}}$, then $\Sigma_{j\in p^{p_i}} w_j > \frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}}$, which means that the wealth of the region p_i is greater than the average wealth of regions except the region p_i, since $\frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}} > w_i \geq \frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}}$, and thus all players in the region p_i can pillage another region q such that $w_i \geq \Sigma_{j\in p^q} w_j > 0$. If $\frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}} > w_i$, then all players in q such that $\Sigma_{j\in p^q} w_j \geq \frac{\Sigma_{j\notin p^{p_i}} w_j}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0 \text{ and } r\neq p_i\}}$ can pillage the player i. This means that (w,p) is dominated by some state in X. Since (w,p) is arbitrary, every state in $X \setminus \{(w,p) \in X$: for each i, $w_i = \frac{1}{\#\{r\in R:\Sigma_{j\in p^r} w_j>0\}}, \frac{1}{2}$, or $0\}$ is dominated. ∎

2.1 Stable set in three-player models

To characterize stable sets, we divide states into four groups according to their distributions and allocations; group 1) all players are in one region; group 2) players have less than halves and occupy two regions; group 3) only one player has a half or more and the player stays alone in his region; and group 4) only one player has a half or more and the player is together with only another player in his region. It is easy to analyze the states in groups 1), 2), and 3) to find a stable set, it is not in group 4), however. Thus we would devote most this subsection to analyzing the states in group 4). For simplicity of expression, we call a state in group 4) a *basic state* and a set of basic states a *basic set.* Note that definitions and results in this subsection are applied to subsection 2.2 as well as subsection 2.1.

Definition 7 formalizes a **basic set** and a **basic state**.

Definition 7 *For any three distinct players $i, j,$ and k, define the set $B(i; j, k)$ of distributions by $B(i; j, k) = \{p \in R^I : \text{for some region } r \in R,\ p^r = \{i, j\} \text{ or } \{i, k\}\ \}$ and define the correspondence $B^i_{j,k} : [\frac{1}{2}, 1] \times R^I \longrightarrow\!\!\!\!\!\!\longrightarrow X$ by $B^i_{j,k}(a, \dot{p}) = \{(w, p) \in X : p = \dot{p},\ w_i \geq a, \text{ and } w_i + w_j + w_k = 1\}$. For each $p \in B(i; j, k)$, the set $B^i_{j,k}(\frac{1}{2}, p)$ of states is called a* ***basic set.*** *A state in a basic set is called a* ***basic state****.*

The set $B(i; j, k)$ denotes the set of distributions such that either player i and player j, or player i and player k constitute all population in some region. For example, let $p = (1, 1, 2)$ and $p' = (1, 2, 1)$, then $p, p' \in B(1; 2, 3)$ because player 1 shares region 1 only with player 2 at the distribution p and only with player 3 at the distribution p'. The basic sets are visualized on the hyperplane of states in Figure 1. The black area and the gray area denote the basic set $B^1_{2,3}(\frac{1}{2}, (1, 1, 2))$ and the basic set $B^2_{1,3}(\frac{1}{2}, (1, 1, 2))$, respectively. They are all possible basic sets under the distribution $(1, 1, 2)$.

In Figure 1, consider the basic state $(w, p) = ((\frac{7}{12}, \frac{3}{12}, \frac{1}{6}), (1, 1, 2))$ where players 1 has $\frac{7}{12}$ and player 2 has $\frac{3}{12}$ while staying at region 1 and player 3 has $\frac{1}{6}$ while staying at region 2. Note that player 1 cannot pillage players 2 and 3 simultaneously because players 2 and 3 are in different regions. If player 1 pillages player 3 at (w, p), then the allocation of the state is located on the left arrow in the figure, and the distribution changes from $(1, 1, 2)$ to $(2, 1, 2)$. If player 1 pillages player 2 at (w, p), then the state is located on the right arrow, and the distribution does not change.

For notational simplicity, we define the following set of states.

Definition 8 *For any three distinct players $i, j,$ and k, define the correspondence $H^i_{j,k} : [\frac{1}{2}, 1] \times R^I \longrightarrow\!\!\!\!\!\!\longrightarrow X$ by $H^i_{j,k}(a, \dot{p}) = \{(w, p) \in X : p = \dot{p},\ w_i = a, \text{ and } w_i + w_j + w_k = 1\}$.*

For each $(a, p) \in [\frac{1}{2}, 1] \times R^I$, the set $H^i_{j,k}(a, p)$ consists of the states such that $w_i = a$ in $B^i_{j,k}(\frac{1}{2}, p)$. In Figure 2, the bold horizontal line and the dot denote $H^i_{j,k}(\frac{7}{12}, (1, 1, 2))$ and $w = (\frac{7}{12}, \frac{3}{12}, \frac{1}{6})$, respectively.

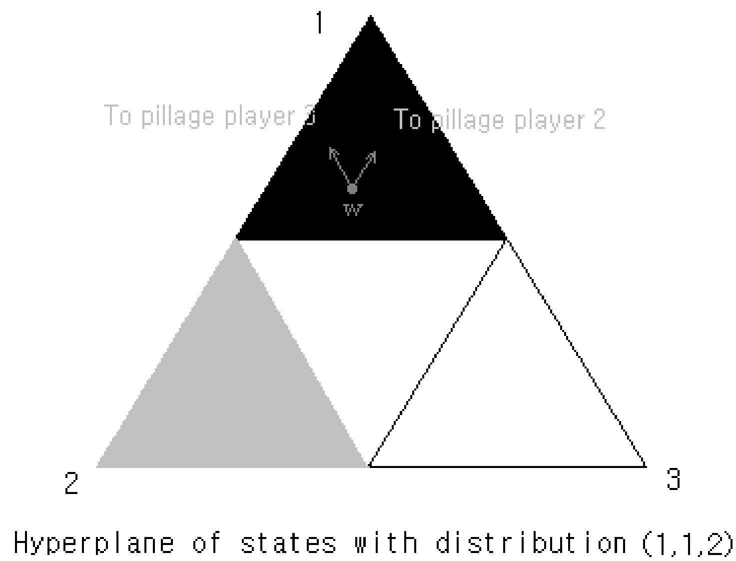

Figure 1: Basic Sets

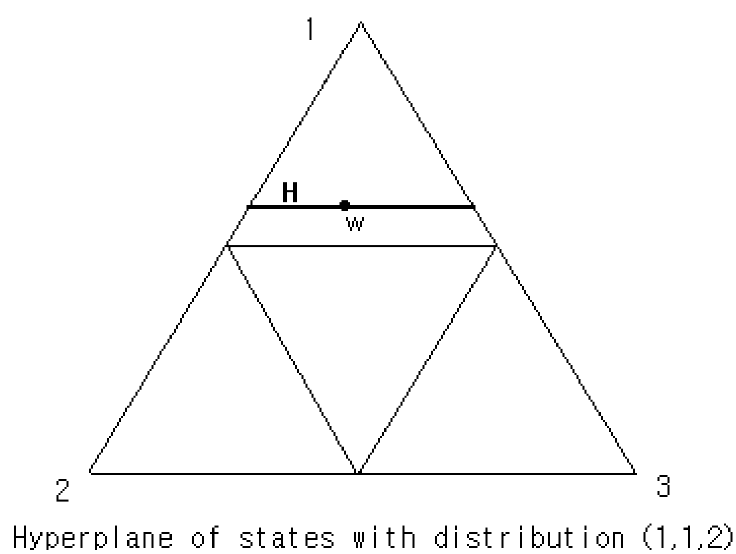

Figure 2: The Set $H^1_{2,3}(\frac{7}{12}, (1,1,2))$

Definition 9 introduces the condition that a stable set has to satisfy. The condition is related to basic sets, and thus we call this condition the **basic condition**. If a set S' of states lacks the basic condition, then S' cannot satisfy internal stability and external stability simultaneously.

Definition 9 *Given a set E of states, for any two distinct states $(w,p),(\dot{w},p) \in E \cap B^i_{j,k}(\frac{1}{2},p)$ such that $p \in B(i;j,k)$ and $1 > \dot{w}_i \geq w_i > \frac{1}{2}$, suppose that i) $0 < \dot{w}_j \leq w_j$ and $0 < \dot{w}_k \leq w_k$; and ii) $\dot{w}_k < w_k$ when $p_i = p_j$ and $\dot{w}_j < w_j$ when $p_i = p_k$. Then the set E of states is said to satisfy the **basic condition**.*

We can prove that a stable set satisfies the basic condition by way of contradiction. That is, if we assume that there is a stable set that lacks the basic condition, then we can show that the stable set cannot satisfy external stability and internal stability simultaneously.

Lemma 10 *A stable set satisfies the basic condition.*

Proof. By way of contradiction, suppose that there exists a stable set S that does not satisfies the basic condition. Then for some three distinct players i, j, and k, there exist two distinct states $(w,p),(\dot{w},p) \in S \cap B^i_{j,k}(\frac{1}{2},p)$ such that $p \in B(i;j,k)$; $1 > \dot{w}_i \geq w_i > \frac{1}{2}$; if $p_i = p_k$, then $\dot{w}_k > w_k$ or $\dot{w}_j \geq w_j$; and if $p_i = p_j$, then $\dot{w}_k \geq w_k$ or $\dot{w}_j > w_j$. Without loss of generality, we can assume that player i is together with player j in a common region, i.e., $p_i = p_j$. Then we must have that either $\dot{w}_k \geq w_k$ or $\dot{w}_j > w_j$. We can show that in each case, S cannot satisfy internal stability and external stability simultaneously.

Suppose that we have that $\dot{w}_k \geq w_k$. We first show that $\dot{w}_k > w_k$. Every state $(\ddot{w},p)$ in $B^i_{j,k}(\frac{1}{2},p)$ such that $\ddot{w}_k = w_k$ and $\ddot{w}_i = w_i$ has $\ddot{w}_j = w_j$ since $\ddot{w}_i + \ddot{w}_j + \ddot{w}_k = w_i + w_j + w_k = 1$. Thus we have that $(\ddot{w},p) = (w,p)$. Therefore, $(\dot{w},p)$ cannot have $\dot{w}_k = w_k$ and $\dot{w}_i = w_i$ since $(w,p) \neq (\dot{w},p)$. Every state $(\dddot{w},p)$ in $B^i_{j,k}(\frac{1}{2},p)$ such that $\dddot{w}_k = w_k$ and $\dddot{w}_i > w_i$ is the state that results from player i pillaging player j at the state (w,p); that is, such state $(\dddot{w},p)$ dominates (w,p). By internal stability, S cannot contain such state $(\dddot{w},p)$ and thus $(\dot{w},p)$ cannot be $\dot{w}_k = w_k$ and $\dot{w}_i > w_i$. Therefore, we must have that $\dot{w}_k \neq w_k$ and thus that $\dot{w}_k > w_k$.

Let the allocation w' be $w'_j = \dot{w}_j$, $w'_k = w_k$, and $w'_i = 1 - \dot{w}_j - w_k$. Since $\dot{w}_i \geq w_i$ and $\dot{w}_k > w_k$, we have $w_j = 1 - w_i - w_k > 1 - \dot{w}_i - \dot{w}_k = \dot{w}_j$. Thus we have $w'_i = 1 - \dot{w}_j - w_k > 1 - w_j - w_k = w_i$. Since $w'_k = w_k$, $w'_i > w_i$, and $w'_j = \dot{w}_j = 1 - w_k - w'_i = w_j - (w'_i - w_i)$, (w',p) dominates (w,p) by player i pillaging player j. Thus S cannot contain (w',p) according to internal stability. To satisfy external stability, S has to dominate (w',p).

However, we can show that S cannot dominate (w',p). The stable set S can dominate (w',p), only if S contains those states as follows; the states that result from player i pillaging player j at (w',p), the states that result from player i pillaging player k at (w',p), the states that result from players i and j pillaging player k at

(w',p), the states that result from player j pillaging player k at (w',p) when $w'_j > w'_k$, or the states that result from player k pillaging player j at (w',p) when $w'_k > w'_j$. Note that player j and player k are in different regions and so player i cannot pillage both of them simultaneously although player i has enough power to do it, i.e., $p_j \neq p_k$ and $w'_i > w'_j + w'_k$. We will show that S cannot contain any state above.

Every state that results from player i pillaging player j at (w',p) dominates (w,p), which is in S according to our assumption. By internal stability, S cannot contain those states. Every state that results from player i pillaging player k at (w',p) dominates $(\dot{w},p)$, which is in S according to our assumption. Similarly, S cannot contain those states. The states that result from players i and j pillaging player k at (w',p) are all dominated by $((0,...,w_i=1,...,0),(p_k,...,p_k))$, which is in the core. Thus S cannot contain those states. The states that result from player j pillaging player k at (w',p) or that result from player k pillaging player j at (w',p) are all dominated by either $((0,...,w_i=1,...,0),(p_k,...,p_k))$ or $((0,...,w_i=1,...,0),(p_j,...,p_j))$. Thus S cannot contain those states. Therefore, S cannot dominate (w',p) and thus cannot satisfy external stability. This contradiction shows that $\dot{w}_k \geq w_k$ is not possible.

Suppose that we have that $\dot{w}_j > w_j$. Then we can similarly show that S cannot dominate the state (w'',p) such that $w''_j = w_j$, $w''_k = \dot{w}_k$, and $w''_i = 1 - w_j - \dot{w}_k$. Consequently, the stable set S cannot satisfy internal stability and external stability simultaneously. This contradiction completes the proof. ■

Lemma 11 presents another condition that a stable set must follow. Lemma 10 examines the relation between two basic states in a stable set. Lemma 11 examines the relation between a basic state and another state whose distribution results from the move of the player who has a half or more at the basic state.

Lemma 11 *Suppose that $p \in B(i;j,k)$ and $(w,p) \in B^i_{j,k}(\frac{1}{2},p)$. Let a distribution $\dot{p}$ satisfy that $\dot{p}_z = p_z$ for each $z \neq i$ and $\dot{p}_i \in \{p_j,p_k\}$. Given a stable set S, if $(w,\dot{p}) \in S$, then $(w,p) \in S$.*

Proof. If $\dot{p}_i = p_i \in \{p_j,p_k\}$, then $\dot{p} = p$, and thus this result obviously follows. Now, we have to show that if $\dot{p}_i \neq p_i$ and $(w,\dot{p}) \in S$, then $(w,p) \in S$. Suppose by way of contradiction that $\dot{p}_i \neq p_i$ and $(w,\dot{p}) \in S$, but $(w,p) \notin S$. It suffices to show that S cannot contain any state that dominates (w,p).

Without loss of generality, we assume that $w_j \geq w_k$. Since $(w,\dot{p}) \in B^i_{j,k}(\frac{1}{2},\dot{p})$, we have that $w_i \geq \frac{1}{2}$ and $w_i + w_j + w_k = 1$. We first show that if $w_j > w_k$ then $w_j + w_k < \frac{1}{2}$. By way of contradiction, suppose not, that is, $w_j > w_k$ and $w_j + w_k = \frac{1}{2}$. Then $(w,\dot{p})$ is dominated by the state $(\ddot{w},\ddot{p})$ such that $\ddot{w}_i = \ddot{w}_j = \frac{1}{2}$, which is in the core, C, by player j pillaging all wealth of player k at $(w,\dot{p})$. This contradicts internal stability of S since $(w,\dot{p}) \in S$ and $C \subset S$.

Let (w',p') result from player j or players i and j pillaging player k at (w,p). Then we have $p'_j = p'_k$. If players i and j pillage player k at $(w,\dot{p})$, then $w'_i > w_i \geq \frac{1}{2}$ and $w'_j > 0$, and thus player i can deprive the other players of their all wealth by pillage since $p'_j = p'_k$ and $w'_j + w'_k < w'_i$. If player j alone pillages player k at $(w,\dot{p})$, then $w_j > w_k$ and thus $w'_i = w_i > \frac{1}{2}$ since $w_j + w_k < \frac{1}{2}$. Thus player i can also deprive the other players of their all wealth in one move since $p'_j = p'_k$ and $w'_j + w'_k < w'_i$. Therefore, (w', p') is dominated by some state $(\dot{w}', \dot{p}')$ in the core such that $\dot{w}'_i = 1$, and thus S cannot contain (w', p'). Similarly, we can show that S cannot contain any state that results from players i and k pillaging player j at (w, p).

Let (w'', p'') result from player i pillaging player j at (w, p). Then we have that $w''_i > w_i$, $w''_j < w_j$, and $w''_z = w_z$ for each $z \in I \setminus \{i, j\}$. Note that $\{z : p''_z \neq p_z\} \subset \{i\}$ and thus $\{z : p''_z \neq \dot{p}_z\} \subset \{i\}$ since $\dot{p}_z = p_z$ for each $z \neq i$. Therefore, (w'', p'') dominates $(w, \dot{p})$ by player i pillaging player j. Thus S cannot contain (w'', p''). Similarly, we can show that S cannot contain any state that results from player i pillaging player k at (w, p).

Consequently, S cannot contain any state that dominates $(\dot{w}, p)$ and thus cannot satisfy external stability. This contradiction completes the proof. ■

Lemma 12 shows another implication of Lemma 10 and Lemma 11. We will express basic states in a stable set with a function. Lemma 12 provides a basis to define the function that characterizes a stable set.

Lemma 12 *Given a stable set S, $S \cap H^i_{j,k}(a, p)$ has a single element for each $1 \geq a > \frac{1}{2}$ and $p \in B(i; j, k)$.*

Proof. It suffices to show that for each $1 > a > \frac{1}{2}$ and $p \in B(i; j, k)$, $S \cap H^i_{j,k}(a, p)$ has a single element because $H^i_{j,k}(1, p)$ has only one state regardless of p, which is in the core and so in a stable set. Suppose that $(w', p), (w, p) \in S \cap H^i_{j,k}(a, p)$ such that $1 > a > \frac{1}{2}$ and $p \in B(i; j, k)$. Then we have that $(w', p), (w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ and $1 > w'_i = w_i > \frac{1}{2}$. Suppose by way of contradiction that $w' \neq w$. By the basic condition of S, we have that either $w'_j \leq w_j$ and $w'_k < w_k$, or $w'_j < w_j$ and $w'_k \leq w_k$ since $1 > w'_i \geq w_i > \frac{1}{2}$. However, neither case is possible since $w'_i + w'_j + w'_k = w_i + w_j + w_k$. Therefore, we must have that $w' = w$, and thus $S \cap H^i_{j,k}(a, p)$ has at most one state for each $1 > a > \frac{1}{2}$ and $p \in B(i; j, k)$.

We need to show that $S \cap H^i_{j,k}(a, p) \neq \varnothing$ for each $1 > a > \frac{1}{2}$ and $p \in B(i; j, k)$ to complete the proof. By way of contradiction, suppose that there exists a stable set S such that $S \cap H^i_{j,k}(a, p) = \varnothing$ for some $1 > a > \frac{1}{2}$ and $p \in B(i; j, k)$. Without loss of generality, we can assume that player i is together with player j in a common region, i.e., $p_i = p_j$. Let $\bar{w}_j = \sup\{w_j : (w, p) \in S \cap B^i_{j,k}(a, p)\}$ and $\bar{w}_k = \sup\{w_k : (w, p) \in S \cap B^i_{j,k}(a, p)\}$.

We first show that $a + \bar{w}_j + \bar{w}_k \leq 1$. Suppose by way of contradiction that $a + \bar{w}_j + \bar{w}_k > 1$. Then by the definitions of $\bar{w}_j$ and $\bar{w}_k$, there exist states $(\dot{w}, p)$ and $(\ddot{w}, p)$ such that $(\dot{w}, p), (\ddot{w}, p) \in S \cap B^i_{j,k}(a, p)$ and $a + \dot{w}_j + \ddot{w}_k > 1$. Since $(\dot{w}, p), (\ddot{w}, p) \in B^i_{j,k}(a, p)$, we have that $\dot{w}_i, \ddot{w}_i \geq a$ and thus that $\dot{w}_i + \dot{w}_j + \ddot{w}_k > 1$ and $\ddot{w}_i + \dot{w}_j + \ddot{w}_k > 1$. Thus we have that $\dot{w}_j > \ddot{w}_j$ and $\dot{w}_k < \ddot{w}_k$ since $\dot{w}_i + \dot{w}_j + \dot{w}_k = \ddot{w}_i + \ddot{w}_j + \ddot{w}_k = 1$. However, the basic condition of S implies that both $\dot{w}_j \geq \ddot{w}_j$ and $\dot{w}_k \geq \ddot{w}_k$ if $\dot{w}_i \leq \ddot{w}_i$ and both $\dot{w}_j \leq \ddot{w}_j$ and $\dot{w}_k \leq \ddot{w}_k$ if $\dot{w}_i \geq \ddot{w}_i$. This contradiction guarantees that $a + \bar{w}_j + \bar{w}_k \leq 1$.

Define the allocation w to be $w_i = a$, $w_j = \bar{w}_j + \frac{1-(a+\bar{w}_j+\bar{w}_k)}{2}$, and $w_k = \bar{w}_k + \frac{1-(a+\bar{w}_j+\bar{w}_k)}{2}$. Then S cannot contain (w, p) since $(w, p) \in H^i_{j,k}(a, p)$ and $S \cap H^i_{j,k}(a, p) = \varnothing$. To prove that the assertion, $S \cap H^i_{j,k}(a, p) = \varnothing$, is impossible, it suffices to show that S cannot dominate (w, p).

First, we show that every state that results from player i pillaging player j at (w, p) cannot be in S. Let the state (w', p) result from player i pillaging player j at (w, p). Then we have that $w'_i > a$, $w'_j < w_j$, $w'_k = w_k$, and $(w', p) \in B^i_{j,k}(a, p)$; that is, player i increases its wealth through pillaging player j at the state (w, p) and player k maintains its wealth because the pillage does not affect player $k's$ wealth. If $1 > (\bar{w}_j + \bar{w}_k + a)$ then $w'_k > \bar{w}_k$, and thus $(w', p) \notin S$ because $\bar{w}_k$ is the supremum of the wealth that player k can have at states in $S \cap B^i_{j,k}(a, p)$ and $(w', p) \in B^i_{j,k}(a, p)$. If $1 = \bar{w}_j + \bar{w}_k + a$, then $w'_j < w_j = \bar{w}_j$ and $w'_k = w_k = \bar{w}_k$. Thus there exists a state $(w'', p) \in S \cap B^i_{j,k}(a, p)$ such that $w'_j < w''_j \le \bar{w}_j$ and $w''_k \le \bar{w}_k$ by the definitions of $\bar{w}_j$ and $\bar{w}_k$. Thus if $(w', p) \in S$, then the basic condition of S means that $w''_i < w'_i$ since $w'_j < w''_j$ and so that $w'_k < w''_k$. Since $w''_k \le \bar{w}_k = w'_k$, we have that $(w', p) \notin S$. Note that (w', p) is arbitrary such that (w', p) results from player i pillaging player j at (w, p). Therefore, S cannot contain the states that result from player i pillaging player j at (w, p).

Second, we prove that every state that results from player i pillaging player k at (w, p) cannot be in S. Suppose by way of contradiction that S contains a state (w''', p') that results from player i pillaging player k at (w, p). Then we have that $w'''_i > w_i$, $w'''_k < w_k$, and $w'''_j = w_j$. Consider the state (w''', p). Then we have that $p \in B(i; j, k)$ and $p'_z = p_z$ for each $z \neq i$ and $p'_i = p_k$. Lemma 11 means that $(w''', p) \in S \cap B^i_{j,k}(a, p)$ since $(w''', p') \in S$. Then we have that $\bar{w}_j \ge w'''_j$ according to the definition of $\bar{w}_j$. Since $w'''_j = w_j = \bar{w}_j + \frac{1-(a+\bar{w}_j+\bar{w}_k)}{2} \ge \bar{w}_j$, we have that $w'''_j = \bar{w}_j$ and thus that $1 = a + \bar{w}_j + \bar{w}_k$. By the definition of $\bar{w}_k$, there exists $(w^{(4)}, p)$ such that $(w^{(4)}, p) \in S \cap B^i_{j,k}(a, p)$ and $w'''_k < w^{(4)}_k < w_k = \bar{w}_k$. The basic condition of S implies that $w'''_j \le w^{(4)}_j$ since $w'''_k < w^{(4)}_k$. Since $w^{(4)}_j \le \bar{w}_j = w'''_j$ according to the definition of $\bar{w}_j$, we have that $w'''_j = w^{(4)}_j$. Therefore, we have that $w'''_i > w^{(4)}_i$, $w'''_k < w^{(4)}_k$, and $w'''_j = w^{(4)}_j$. This means that (w''', p') dominates $(w^{(4)}, p)$ by player i pillaging player k at $(w^{(4)}, p)$. This contradiction assures that $(w''', p') \notin S$. Since (w''', p') is arbitrary, S cannot contain the states that result from player i pillaging player k at (w, p).

Finally, we demonstrate that every state that dominates (w, p) and that is not covered by the two cases above is not in S. Note that these states result from either player j or player k moving to the other regardless of the move of player i. Consequently, player j and player k are in a common region at these states. Therefore, all such states are dominated by some state in the core such that player i has all of the wealth because player i, who has a majority of the power, $w_i \ge a > \frac{1}{2}$, can pillage both players in one move. Therefore, S cannot contain these states.

Consequently, S cannot dominate (w, p). This means that S cannot satisfy internal stability and external stability simultaneously. This contradiction guarantees that we must have that $S \cap H^i_{j,k}(a,p) \neq \varnothing$ for each $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$. ■

Definition 13 presents the conditions for the function that characterizes a stable set and names the function a **basic function**.

Definition 13 *For any three distinct players i, j, and k, let a function $\beta^i_{j,k} : [0,\frac{1}{2}] \times B(i;j,k) \longrightarrow [0,\frac{1}{2}]$ satisfy that $\beta^i_{j,k}(\lambda,p) \leq \lambda$ for each $(\lambda,p) \in [0,\frac{1}{2}] \times B(i;j,k)$. Define the set $B(\beta^i_{j,k})$ of states by $B(\beta^i_{j,k}) = \{(w,p) \in \bigcup_{p \in B(i;j,k)} B^i_{j,k}(\frac{1}{2},p) :$ for some $(\lambda,p) \in [0,\frac{1}{2}] \times B(i;j,k)$, $w_j = \beta^i_{j,k}(\lambda,p)$ and $w_k = \lambda - \beta^i_{j,k}(\lambda,p)\}$. Suppose that $\beta^i_{j,k}$ satisfies three conditions as follows; i) $B(\beta^i_{j,k})$ satisfies the basic condition; ii) if $p, \dot{p} \in B(i;j,k)$ satisfy that $\dot{p}_z = p_z$ for each $z \neq i$ and $p_i \neq \dot{p}_i$, then for each $\lambda \in [0,\frac{1}{2}]$, $\beta^i_{j,k}(\lambda,p) = \beta^i_{j,k}(\lambda,\dot{p})$; and for each $p \in B(i;j,k)$, iii) if $\lim\limits_{\lambda \to 1/2} \beta^i_{j,k}(\lambda,p) = \frac{1}{4}$, then $\beta^i_{j,k}(\frac{1}{2},p) = \frac{1}{4}$, otherwise $\beta^i_{j,k}(\frac{1}{2},p) = \frac{1}{2}$. Then $\beta^i_{j,k}$ is called a **basic function**.*

Lemma 14 characterizes the functions that generate the set satisfying the basic condition.

Lemma 14 *Let a function $\beta^i_{j,k}$ be a function from $[0,\frac{1}{2}] \times B(i;j,k)$ to $[0,\frac{1}{2}]$ such that $\beta^i_{j,k}(\lambda,p) \leq \lambda$ for each $(\lambda,p) \in [0,\frac{1}{2}] \times B(i;j,k)$. If $B(\beta^i_{j,k})$ satisfies the basic condition, then $\beta^i_{j,k}(\cdot,p)$ is uniformly continuous and non-decreasing on $[0,\frac{1}{2})$.*

Proof. If $B(\beta^i_{j,k})$ satisfies the basic condition, then for each $\frac{1}{2} > \lambda > \lambda' \geq 0$ and $p \in B(i;j,k)$, we have that $\lambda - \beta^i_{j,k}(\lambda,p) \geq \lambda' - \beta^i_{j,k}(\lambda',p)$ and $\beta^i_{j,k}(\lambda,p) \geq \beta^i_{j,k}(\lambda',p)$. Therefore, Given any $\varepsilon > 0$, we must have that $\varepsilon > \beta^i_{j,k}(\lambda,p) - \beta^i_{j,k}(\lambda',p) \geq 0$ for all $\lambda, \lambda' \in [0,\frac{1}{2})$ and $p \in B(i;j,k)$ such that $\varepsilon > \lambda - \lambda' \geq 0$. This shows that the function $\beta^i_{j,k}(\cdot,p)$ is uniformly continuous and non-decreasing on $[0,\frac{1}{2})$. ■

Corollary 15 shows properties of a basic function.

Corollary 15 *For each $p \in B(i;j,k)$, a basic function $\beta^i_{j,k}(\cdot,p) : [0,\frac{1}{2}] \to [0,\frac{1}{2}]$ is uniformly continuous and non-decreasing on $[0,\frac{1}{2})$.*

Proof. According to Lemma 14, this result follows. ■

Lemma 16 strengthens Lemma 10. More concretely, Lemma 16 shows that a stable set must satisfy three conditions that are reflected on a basic function.

Lemma 16 *Given a stable set S, for any three distinct players i, j, and k, there exists a unique basic function $\beta^i_{j,k}$ such that $(B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2},p) = S \cap B^i_{j,k}(\frac{1}{2},p)$ for each $p \in B(i;j,k)$.*

Proof. According to Lemma 12, $S \cap H^i_{j,k}(a,p)$ has a single state for each $1 \geq a > \frac{1}{2}$ and $p \in B(i;j,k)$. In addition, we have that $S \cap H^i_{j,k}(\frac{1}{2},p) \neq \varnothing$ for each $p \in B(i;j,k)$ since such a set contains some states in C, at which two players have halves. Therefore, we can define the function $\alpha : [\frac{1}{2},1] \times B(i;j,k) \to [0,\frac{1}{2}]$ as follows;

$i)$ $\alpha(w_i, p) = w_j$ such that $(w, p) \in S \cap \bigcup_{p \in B(i;j,k)} B^i_{j,k}(\frac{1}{2}, p)$; and for each $p \in B(i;j,k)$, $ii)$ if there exists $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ such that $w_i = \frac{1}{2}$ and $w_j = \frac{1}{4}$, then $\alpha(\frac{1}{2}, p) = \frac{1}{4}$, otherwise $\alpha(\frac{1}{2}, p) = \frac{1}{2}$. That is, the function α assigns each (w_i, p) the player $j's$ allocation according to $(w, p) \in S \cap \bigcup_{p \in B(i;j,k)} B^i_{j,k}(\frac{1}{2}, p)$.

Define the function $\beta^i_{j,k} : [0, \frac{1}{2}] \times B(i;j,k) \rightarrow [0, \frac{1}{2}]$ by $\beta^i_{j,k}(\lambda, p) = \alpha(1 - \lambda, p)$. Then it is easily seen that for each $(\lambda, p) \in [0, \frac{1}{2}] \times B(i;j,k)$, $\beta^i_{j,k}(\lambda, p) \leq \lambda$ and $((B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, p)) \setminus H^i_{j,k}(\frac{1}{2}, p) = (S \cap B^i_{j,k}(\frac{1}{2}, p)) \setminus H^i_{j,k}(\frac{1}{2}, p)$. Next, we will show that for each $p \in B(i;j,k)$, $(B(\beta^i_{j,k}) \cup C) \cap H^i_{j,k}(\frac{1}{2}, p) = S \cap H^i_{j,k}(\frac{1}{2}, p)$.

By the definition of $\beta^i_{j,k}$, we have that $B(\beta^i_{j,k}) \subset S$ and thus that $(B(\beta^i_{j,k}) \cup C) \cap H^i_{j,k}(\frac{1}{2}, p) \subset S \cap H^i_{j,k}(\frac{1}{2}, p)$ for each $p \in B(i;j,k)$. Note that if $(w', p) \in H^i_{j,k}(\frac{1}{2}, p)$ with $w'_j \notin \{0, \frac{1}{4}, \frac{1}{2}\}$, then (w', p) is dominated by some state in C such that two players have halves. Therefore, if $(w, p) \in S \cap H^i_{j,k}(\frac{1}{2}, p)$ for some $p \in B(i;j,k)$, then $w_j = 0$, $\frac{1}{4}$, or $\frac{1}{2}$. And thus $(w, p) \in (B(\beta^i_{j,k}) \cup C) \cap H^i_{j,k}(\frac{1}{2}, p)$ because if $w_j = 0$ or $\frac{1}{2}$ then $(w, p) \in C$ and if $w_j = \frac{1}{4}$ then $(w, p) \in B(\beta^i_{j,k})$.

To complete the proof, we must show that the function $\beta^i_{j,k}$ is a basic function. Since $(B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, p) = S \cap B^i_{j,k}(\frac{1}{2}, p)$ for each $p \in B(i;j,k)$, the set $B(\beta^i_{j,k})$ satisfies the basic condition as S does by Lemma 10. If $p, \dot{p} \in B(i;j,k)$ such that $\dot{p}_z = p_z$ for each $z \neq i$ and $p_i \neq \dot{p}_i$, then $S \cap H^i_{j,k}(w_i, p) = S \cap H^i_{j,k}(w_i, \dot{p})$ for each $1 \geq w_i \geq \frac{1}{2}$ by Lemma 11, and thus $\beta^i_{j,k}(\lambda, p) = \beta^i_{j,k}(\lambda, \dot{p})$ for each $\lambda \in [0, \frac{1}{2}]$. Now, we only need to prove that $\lim_{\lambda \rightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$ if and only if $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$ because if $\beta^i_{j,k}(\frac{1}{2}, p) \neq \frac{1}{4}$, then $S \cap H^i_{j,k}(\frac{1}{2}, p)$ has two elements at which player j has either 0 or $\frac{1}{2}$.

First, we prove that for some $p \in B(i;j,k)$, if $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$ then $\lim_{w_i \longrightarrow 1/2} \beta^i_{j,k}(w_i, p) = \frac{1}{4}$. Suppose that for some $p \in B(i;j,k)$, there exists $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $w_i = \frac{1}{2}$ and $w_j = \frac{1}{4}$. Without loss of generality, we assume that player i is together with player j in a common region, i.e., $p_i = p_j$. Since $\beta^i_{j,k}(\cdot, p)$ is uniformly continuous on $[0, \frac{1}{2})$ by Lemma 14, $\lim_{w_i \longrightarrow 1/2} \beta^i_{j,k}(w_i, p)$ always exists. Let $b = \lim_{w_i \longrightarrow 1/2} \beta^i_{j,k}(w_i, p)$. Suppose by way of contradiction that $b \neq \frac{1}{4}$.

Let $b > \frac{1}{4}$ first. Then there exists $\dot{w}_i$ such that $\beta^i_{j,k}(1 - \dot{w}_i, p) = \frac{1}{4}$ by the continuity of $\beta^i_{j,k}(\cdot, p)$ on $[0, \frac{1}{2})$, and thus there exists the allocation $\dot{w}$ such that $\dot{w}_j = \frac{1}{4}$ and $(\dot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$. Let $\dot{p} \in R^I$ such that $\dot{p}_z = p_z$ for each $z \neq i$ and $\dot{p}_i = p_k$. Then the state $(\dot{w}, \dot{p})$ dominates (w, p) by player i pillaging player k at (w, p) since $\dot{w}_i > w_i$, $\dot{w}_k < w_k$, and $\dot{w}_j = w_j$. Thus we have that $(\dot{w}, \dot{p}) \notin S$. However, every state that results from player i pillaging either player j or player k at $(\dot{w}, \dot{p})$ dominates $(\dot{w}, p) \in S$ as well. Every state that results from either player j or player k moving his region to pillage, regardless of the movement of player i, is dominated by some state in the core such that player i has the entire wealth. Therefore, S cannot dominate $(\dot{w}, \dot{p})$, and thus S lacks external stability.

Let $b < \frac{1}{4}$ next. Then there exists $\lambda \in [0, \frac{1}{2})$ with $\lambda - \beta^i_{j,k}(\lambda, p) > \frac{1}{4}$, and thus there exists λ'' such that $\lambda'' - \beta^i_{j,k}(\lambda'', p) = \frac{1}{4}$ since $\beta^i_{j,k}(\cdot, p)$ is continuous on $[0, \frac{1}{2})$. Then there exists $(\ddot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $\ddot{w}_k = \frac{1}{4}$ and $\ddot{w}_j = \beta^i_{j,k}(\lambda'', p)$, and $(\ddot{w}, p)$ dominates $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ by player i pillaging player j at (w, p). This shows that S lacks internal stability. Consequently, these contradictions ensure that $b = \frac{1}{4}$.

Lastly, we prove that for some $p \in B(i; j, k)$, if $\lim_{\lambda \longrightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$, then $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$. Suppose that for some $p \in B(i; j, k)$, $\lim_{\lambda \longrightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$. Note that $(w, p) \in B^i_{j,k}(\frac{1}{2}, p)$ with $w_i = \frac{1}{2}$, $w_j = \frac{1}{4}$, and $p_i = p_j$ is dominated only either by player i pillaging another player at (w, p), or by players i and j pillaging player k at (w, p). Note that $\beta^i_{j,k}(\lambda, p)$ and $\lambda - \beta^i_{j,k}(\lambda, p)$ denote player $j's$ allocation and player $k's$ allocation, respectively, when player i has $1 - \lambda$ at the distribution p in the stable set S. Because $\lim_{\lambda \longrightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$, we have that $\lim_{\lambda \longrightarrow 1/2} (\lambda - \beta^i_{j,k}(\lambda, p)) = \frac{1}{4}$. Therefore, the basic condition implies that a state (w'', p) such that $w''_j < \frac{1}{4}$ and $w''_k = \frac{1}{4}$ is not in S. Such a state (w'', p) results from player i pillaging player j at (w, p). Furthermore, the basic condition implies that a state (w''', p) such that $w'''_j = \frac{1}{4}$ and $w'''_k < \frac{1}{4}$ is not in S. By Lemma 11, S cannot contain such a state $(w''', \dot{p})$ that results from player i pillaging player k at (w, p). Finally, every state that results from players i and j pillaging player k at (w, p) is dominated by some state in the core such that player i has all of the wealth. Therefore, S must contain (w, p) to satisfy external stability, and thus we have that $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$. ∎

Jordan (2005) studied the pillage game of "wealth is power" power function without spatial restriction and found the unique stable set, the set of dyadic allocations. Definition 17 introduces a **dyadic state** that satisfies three conditions below and the set of dyadic states D. Theorem 18 establishes that the set D is the unique stable set in a one-region model. Note that Definition 17 and Theorem 18 are adapted from Jordan (2005) for the spatial pillage game.

Definition 17 *An allocation $w \in A$ is dyadic if for each i, $w_i = 0$ or $(\frac{1}{2})^{k_i}$ for some nonnegative integer k_i. A state (w, p) is **dyadic** if it satisfies that i) w is dyadic; ii) for each $r \in R$, $\sum_{i \in p^r} w_i = 0, \frac{1}{2}$, or 1; and iii) if there exists a region $r' \in R$ with $\sum_{i \in p^{r'}} w_i = \frac{1}{2}$, then there exists player z with $w_z = \frac{1}{2}$. The set D denotes the set of dyadic states.*

Theorem 18 (Theorem 3.3 in Jordan, 1999) *In a one-region model, the unique stable set is D.*

Lemma 19 reveals another implication of Theorem 18. It applies Theorem 18 to a general model, which possibly can have more than one region.

Lemma 19 *Define the set $\bar{X}$ of states by $\bar{X} = \{(w, p) \in X :$ for each region $r \in R$, $\sum_{i \in p^r} w_i = 0, \frac{1}{2}$, or 1 and if for some region $r' \in R$, $\sum_{i \in p^{r'}} w_i = \frac{1}{2}$, then for some* *player z, $w_z = \frac{1}{2}\}$. Then D is the unique set that satisfies both internal stability and external stability with respect to $\bar{X}$. In addition, a stable set includes D.*

Proof. For any region $r \in R$ and any distribution $\dot{p} \in R^I$, define the set $X(r;\dot{p})$ of states by $X(r;\dot{p}) = \{(w,p) \in X : p = \dot{p} \text{ and } \sum_{i \in \dot{p}^r} w_i = 1\}$. We first show that $D \cap X(r;p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(r;p)$. By Theorem 18, the unique stable set in a one-region model is the set of dyadic states. Given a region $r \in R$ and a distribution $p \in R^I$, define the function $w^{r,p} : X \longrightarrow [0,1]^{\#p^r}$ by $w^{r,p}(w,p)_1 = w_{\min p^r}, ..., w^{r,p}(w,p)_{\#p^r} = w_{\max p^r}$; that is, $w^{r,p}$ projects from X onto allocations of players in the region r of the distribution p. Then $\{w^{r,p}(w,p) : (w,p) \in D \cap X(r;p)\}$ is the set of allocations of dyadic states in the $\#p^r$-player one-region model, and thus it is the unique stable set by Theorem 18 in this one-region model. Note that in a one-region model, dominance relation between states is well defined without distributions. Thus it is easily seen that $(w',p) \in X(r;p)$ dominates $(w,p) \in X(r;p)$ if and only if $w^{r,p}(w',p)$ dominates $w^{r,p}(w,p)$ in the $\#p^r$-player one-region model; both mean that $\sum_{z \in \{i : w'_i > w_i\}} w_z > \sum_{z \in \{i : w'_i < w_i\}} w_z$. Therefore, $D \cap X(r;p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(r;p)$ because $\{w^{r,p}(w,p) : (w,p) \in D \cap X(r;p)\}$ is the unique stable set of allocations in the $\#p^r$-player one-region model.

For any region $r \in R$, any distribution $\dot{p} \in R^I$, and any player z with $\dot{p}_z \notin \dot{p}^r$, define the set $X(z,r;\dot{p})$ of states by $X(z,r;\dot{p}) = \{(w,p) \in X : p = \dot{p}, \sum_{i \in \dot{p}^r} w_i = \frac{1}{2}$, and $w_z = \frac{1}{2}\}$. We second prove that $D \cap X(z,r;p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(z,r;p)$. Note that $(w',p) \in X(z,r;p)$ dominates $(w,p) \in X(z,r;p)$ if and only if $2w^{r,p}(w',p)$ dominates $2w^{r,p}(w,p)$ in the $\#p^r$-player one-region model. It is easily seen that $\{2w^{r,p}(w,p) : (w,p) \in D \cap X(z,r;p)\}$ is the set of allocations of dyadic states in the $\#p^r$-player one-region model, and thus by Theorem 18, $\{2w^{r,p}(w,p) : (w,p) \in D \cap X(z,r;p)\}$ is the unique stable set. Therefore, $D \cap X(z,r;p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(z,r;p)$.

Third, we check that a state in $X(r;p)$ can be dominated only by another state in $X(r;p)$. If $(w,p) \in X(r;p)$ is dominated by another state (w',p'), then because the coalition $\{i : w'_i > w_i\} \subset p^r$ pillages the coalition $\{i : w'_i < w_i\} \subset p^r$ within region r, we have that $p_i = p'_i = r$ for any $i \in \{i : w'_i \neq w_i\}$. Since the pillage does not affect the coalition $\{i : w'_i = w_i\}$, we have that $p_i = p'_i$ for any $i \in \{i : w'_i = w_i\}$. Since $p' = p$ and $p'_i = r$ for each $i \in \{i : w'_i > 0\}$, we have that $(w',p') \in X(r;p)$.

Suppose that $\bar{S} \subset \bar{X}$ is a set that satisfies both internal stability and external stability with respect to $\bar{X}$. We next demonstrate that $\bar{S} = D$. The set $\bar{S}$ must dominate every state in $X(r,p) \setminus \bar{S}$. However, $X(r,p) \setminus \bar{S}$ can be dominated only by some state in $X(r,p)$, and thus $\bar{S} \cap X(r,p)$ dominates every state in $X(r,p) \setminus \bar{S}$. Since $\bar{S} \cap X(r,p)$ is internally stable, $\bar{S} \cap X(r,p)$ is a set that satisfies both internal stability and external stability with respect to $X(r,p)$. Therefore, we have that $\bar{S} \cap X(r,p) = D \cap X(r;p)$. Since r and p are arbitrary, we have that $\bar{S} \cap \bigcup_{(r,p) \in R \times R^I} X(r,p) =$

$D\cap\bigcup_{(r,p)\in R\times R^I} X(r;p)$. Note that a state in $X(z,r;p)$ can be dominated only by some state in $X(z,r;p)\cup\bigcup_{(r,p)\in R\times R^I} X(r;p)$. Any state in $D\cap\bigcup_{(r,p)\in R\times R^I} X(r;p)$ cannot dominate another state in $X(z,r;p)$ because a state (w',p') that results from player z with $w_z=\frac{1}{2}$ and $p_z\neq r$ pillaging other players at region r at (w,p) in $X(z,r;p)$ has $1>w'_z>\frac{1}{2}$, and thus $(w',p')\notin D$. Therefore, $\bar{S}\cap X(z,r;p)$ must dominate every state in $X(z,r;p)\setminus\bar{S}$ because $\bar{S}$ dominates every state in $X(z,r;p)\setminus\bar{S}$. Since $\bar{S}\cap X(z,r;p)$ is internally stable, $\bar{S}\cap X(z,r;p)$ satisfies both internal stability and external stability with respect to $X(z,r;p)$. Thus we have that $\bar{S}\cap X(z,r;p)=D\cap X(z,r;p)$ because $D\cap X(z,r;p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(z,r;p)$. Since r, p, and z with $\dot{p}_z\notin\dot{p}^r$ are arbitrary, we have that $\bar{S}\cap\bigcup_{(r,p)\in R\times R^I}(\bigcup_{z\notin p^r} X(z,r;p))=D\cap\bigcup_{(r,p)\in R\times R^I}(\bigcup_{z\notin p^r} X(z,r;p))$. Since $\bigcup_{(r,p)\in R\times R^I}(X(r;p)\cup\bigcup_{z\notin p^r} X(z,r;p))=\bar{X}$ and $\bar{S},D\subset\bar{X}$, we have that $\bar{S}=D$.

Finally, we complete the proof that D is the unique set that satisfies both internal stability and external stability with respect to $\bar{X}$. We have proven that if a set satisfies both internal stability and external stability with respect to $\bar{X}$, then it must be D. Therefore, we need to show that D satisfies both internal stability and external stability with respect to $\bar{X}$. Because for any states $(w,p),(w',p')\in D$, we have that $\sum_{z\in\{i:w'_i>w_i\}} w_z\leq\sum_{z\in\{i:w'_i<w_i\}} w_z$ or $\sum_{z\in\{i:w_i>w'_i\}} w'_z\leq\sum_{z\in\{i:w_i<w'_i\}} w'_z$, the set D is internally stable. Note that for each r, p, and z with $p_z\notin p^r$, $D\cap X(r;p)$ and $D\cap X(z,r;p)$ satisfy external stability with respect to $X(r;p)$ and $X(z,r;p)$, respectively. Therefore, D is externally stable with respect to $\bar{X}$. Consequently, D satisfies both internal stability and external stability with respect to $\bar{X}$.

In addition, It is easily seen that a stable set S includes D. Note that a state in $\bar{X}$ can be dominated only by another state in $\bar{X}$. Every state (w',p') that results from player z with $w_z=\frac{1}{2}$ being involved in pillaging other players at (w,p) in $\bar{X}$ satisfies that $\sum_{i\in p'^r} w'_i=1$ for some $r\in R$ and thus that $(w',p')\in\bar{X}$. Every state (w'',p'') that results from players in some region r pillaging other players in the same region r at (w,p) in $\bar{X}$ satisfies that for each $r\in R$, $\sum_{i\in p''^r} w''_i=0,\frac{1}{2}$, or 1 and that if $\sum_{i\in p''^r} w''_i=\frac{1}{2}$ for some region $r\in R$, then $w''_z=\frac{1}{2}$ for some player z. Thus we have that $(w'',p'')\in\bar{X}$. Since a stable set S dominates every state in $\bar{X}\setminus S$, $S\cap\bar{X}$ dominates every state in $\bar{X}\setminus S$. Since $S\cap\bar{X}$ is internally stable, $S\cap\bar{X}$ satisfies both internal stability and external stability with respect to $\bar{X}$. Therefore, we have that $S\cap\bar{X}=D$ and thus that $D\subset S$. Since S is an arbitrary stable set, a stable set includes D. ■

Proposition 20 completely characterizes stable sets in the three-player and two-region model.

Proposition 20 *In the three-player and two-region model, a set S is a stable set if and only if $S=B(\beta^1_{2,3})\cup B(\beta^2_{3,1})\cup B(\beta^3_{1,2})\cup D$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$.*

Proof. We prove the necessary condition first. Suppose that S is a stable set. By Lemma 16, there exist basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ such that $B(\beta^1_{2,3})\cup B(\beta^2_{3,1})\cup$

$B(\beta^3_{1,2}) \subset S$. By Lemma 19, we have that $D \subset S$. Therefore, we must have that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D \subset S$. To show that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D \supset S$, it suffices to show that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$ is externally stable.

Let $\dot{S} = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$. If $(\mathring{w}, p) \in X \setminus \dot{S}$ with $\mathring{w}_z < \frac{1}{2}$ for each $z \in I$, then $(\mathring{w}, p)$ is dominated by some state in D such that two players have halves. Let $\bar{X} = \{(w, p) \in X :$ for each region $r \in R$, $\sum_{i \in p^r} w_i = 0, \frac{1}{2}$, or 1 and if for some region $r' \in R$, $\sum_{i \in p^{r'}} w_i = \frac{1}{2}$, then for some player z, $w_z = \frac{1}{2}\}$. If $(\mathring{w}, p) \in X \setminus \dot{S}$ and $(\mathring{w}, p) \in \bar{X}$, then by Lemma 19, $(\mathring{w}, p)$ is dominated by some state in D such that either two players have halves, or one player has all of the wealth. If $(\mathring{w}, p) \in X \setminus \dot{S}$ with $\mathring{w}_i > \frac{1}{2}$ and $p_j = p_k$, then $(\mathring{w}, p)$ is dominated by some state in D such that player i has all of the wealth.

Let $(\mathring{w}, p) \in X \setminus \dot{S}$ satisfy that $p \in B(i; j, k)$ and $(\mathring{w}, p) \in B^i_{j,k}(\frac{1}{2}, p)$. Without loss of generality, we assume that $p_i = p_j$. Since $1 > \mathring{w}_i \geq \frac{1}{2}$ and a basic function $\beta^i_{j,k}(\cdot, p) : [0, \frac{1}{2}] \longrightarrow [0, \frac{1}{2}]$ is uniformly continuous on $[0, \frac{1}{2})$, $\lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$ is well defined. If $\mathring{w}_i > \frac{1}{2}$ and $\mathring{w}_j > \beta^i_{j,k}(1 - \mathring{w}_i, p)$ or $\mathring{w}_i = \frac{1}{2}$ and $\mathring{w}_j > \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, both of which mean that $\mathring{w}_j > \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, then $\mathring{w}_k < 1 - \mathring{w}_i - \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$. Thus there exists a state $(w, p) \in B(\beta^i_{j,k})$ with $w_i > \mathring{w}_i$ and $w_k = \mathring{w}_k$. In this case, (w, p) dominates $(\mathring{w}, p)$ by player i pillaging player j. If $\mathring{w}_i > \frac{1}{2}$ and $\mathring{w}_j < \beta^i_{j,k}(1 - \mathring{w}_i, p)$ or $\mathring{w}_i = \frac{1}{2}$ and $\mathring{w}_j < \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, both of which mean that $\mathring{w}_j < \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, then $(w', \dot{p}) \in B(\beta^i_{j,k})$ such that $w'_j = \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, $\dot{p}_z = p_z$ for all $z \neq i$, and $\dot{p}_i \neq p_i$ dominates $(\mathring{w}, p)$ by player i pillaging player k. If $\mathring{w}_i = \frac{1}{2}$ and $\mathring{w}_j = \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p) \neq \frac{1}{4}$, then some state in the core such that two players have halves dominates $(\mathring{w}, p)$. Therefore, $\dot{S}$ is externally stable, and thus $\dot{S} = S$.

Next, we prove the sufficient condition, that is, if functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ are basic functions, then the set $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$ is a stable set. Suppose that $S' = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$. Then S' is externally stable as shown above. Now, we need to show that S' is internally stable.

Notice that each $B(i; j, k)$ has four elements and each element p in $B(i; j, k)$ has its counterpart distribution $\dot{p}$ such that $\dot{p} \in B(i; j, k)$, $p_i \neq \dot{p}_i$, $\dot{p}_j = p_j$, and $\dot{p}_k = p_k$. For example, $B(1; 2, 3) = \{(1, 1, 2), (1, 2, 1), (2, 1, 2), (2, 2, 1)\}$ and $(1, 1, 2)$ and $(1, 2, 1)$ are counterpart distributions to $(2, 1, 2)$ and $(2, 2, 1)$, respectively. Therefore, by the second condition of a basic function, if $(w, p), (w', p) \in B(\beta^i_{j,k})$ with $w_i > w'_i > \frac{1}{2}$ and $p_i = p_j$, then $(w, \dot{p}), (w', \dot{p}) \in B(\beta^i_{j,k})$ with $\dot{p}_i = \dot{p}_k$, and vice versa. Then we have that $w_j \leq w'_j$ and $w_k < w'_k$ by the first condition of a basic function since $p_i = p_j$ and $\dot{p}_i = \dot{p}_k$. Similarly, we have that $w_j < w'_j$ and $w_k \leq w'_k$. Consequently, we have that $w_j < w'_j$ and $w_k < w'_k$.

First, we prove that each $B(\beta^1_{2,3})$, $B(\beta^2_{3,1})$, and $B(\beta^3_{1,2})$ is internally stable. Let $(w, p), (w', p') \in B(\beta^i_{j,k})$ such that $(w, p) \neq (w', p')$ and $(w, p) \notin C$, which is the core.

Since $\{p_j, p_k\} = \{p'_j, p'_k\} = R$, i.e., players j and k are distributed all over regions at p and p', we have that either $p'_j \neq p_j$ and $p'_k \neq p_k$, or $p'_j = p_j$ and $p'_k = p_k$. Thus if $p'_j \neq p_j$, then $\{z : p_z \neq p'_z\} \nsubseteq p^r$ for each $r \in R$, and so (w', p') does not dominate (w, p).

Suppose that p and p' satisfies that $p'_j = p_j$ and $p'_k = p_k$. If $w_i, w'_i > \frac{1}{2}$, then *i)* $w_j < w'_j$ and $w_k < w'_k$; *ii)* $w_j > w'_j$ and $w_k > w'_k$; or *iii)* $w = w'$. If either *i)* $w_j < w'_j$ and $w_k < w'_k$, or *ii)* $w_j > w'_j$ and $w_k > w'_k$, then $\{z : w'_z \neq w_z\} \nsubseteq p^r$ for each $r \in R$. If $w = w'$ then $\sum_{z\in\{i:w'_i>w_i\}} w_z = \sum_{z\in\{i:w'_i<w_i\}} w_z = 0$. If $w_i > \frac{1}{2}$ and $w'_i = \frac{1}{2}$, then $\sum_{z\in\{i:w'_i>w_i\}} w_z \leq \frac{1}{2} < \sum_{z\in\{i:w'_i<w_i\}} w_z$. If $w_i = \frac{1}{2}$, $w_j = \frac{1}{4}$, and $w'_i > \frac{1}{2}$, then since $\lim_{\lambda\longrightarrow\frac{1}{2}} \beta^i_{j,k}(\lambda) = \frac{1}{4}$, $w_j = \frac{1}{4} > w'_j$ and $w_k = \frac{1}{4} > w'_k$. Thus we have that $\{i : w'_i \neq w_i\} \nsubseteq p^r$ for each $r \in R$. If $w_i = \frac{1}{2}$, $w_j = \frac{1}{4}$, and $w'_i = \frac{1}{2}$, then since $w'_j = \frac{1}{4}$ or $\frac{1}{2}$, we have that $\sum_{z\in\{i:w'_i>w_i\}} w_z = \sum_{z\in\{i:w'_i<w_i\}} w_z = 0$ or $\frac{1}{4}$. Therefore, in these cases, (w', p') does not dominate (w, p). Since $(w, p), (w', p') \in B(\beta^i_{j,k})$ with $(w, p) \neq (w', p')$ and $(w, p) \notin C$ are arbitrary, each set $B(\beta^1_{2,3})$, $B(\beta^2_{3,1})$, and $B(\beta^3_{1,2})$ is internally stable.

Second, we check internal stability of the set $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$. Let $(w, p) \in B(\beta^i_{j,k})$ and $(w'', p'') \in B(\beta^j_{k,i})$. Then $w_i \geq \frac{1}{2}$ and $w''_j \geq \frac{1}{2}$, and thus $w_j \leq \frac{1}{2}$ and $w''_i \leq \frac{1}{2}$. If $w_i > w''_i$ then $\sum_{z\in\{y:w''_y>w_y\}} w_z \leq \frac{1}{2} \leq w_i \leq \sum_{z\in\{y:w''_y<w_y\}} w_z$. If $w_i = w''_i$ then $w_i = w''_i = \frac{1}{2}$. Since $w_j \in \{\frac{1}{4}, \frac{1}{2}\}$ by the third condition of a basic function, $\sum_{z\in\{i:w''_i>w_i\}} w_z = \sum_{z\in\{i:w''_i<w_i\}} w_z = 0$ or $\frac{1}{4}$. Therefore, (w, p) does not dominate (w'', p''). Similarly, we can prove that (w'', p'') does not dominate (w, p). Consequently, $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$ is internally stable.

Finally, we examine internal stability of $S' = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$. Note that $(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})) \cap \bar{X} \subset C$ and that $C \subset D$. Since a state in D can be dominated only by another state in $\bar{X}$ and D is internally stable, any state in D is not dominated by another state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$. Therefore, it suffices to show that any state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$ is not dominated by another state in D. Let $(w, p) \in B(\beta^i_{j,k})$. Then we have that $w_i \geq \frac{1}{2}$. If $w_i = 1$ or $w_i = \frac{1}{2}$ and $w_j = \frac{1}{2}$, then $(w, p) \in C$, and thus (w, p) is not dominated by any state in D. If $1 > w_i > \frac{1}{2}$, then by the basic condition of $B(\beta^i_{j,k})$, we have that $w_j > 0$ and $w_k > 0$. Also, if $w_i = \frac{1}{2}$ and $w_j = \frac{1}{4}$, then $w_k = \frac{1}{4}$. In these cases, player i cannot pillage both players j and k simultaneously since $p_j \neq p_k$ and cannot be pillaged by another player since $w_i > \frac{1}{2} > \max\{w_j, w_k\}$. Thus if a state (w''', p''') dominates (w, p), then (w''', p''') satisfies that for some player z, $w_z \notin \{0, \frac{1}{4}, \frac{1}{2}, 1\}$, that is, $(w''', p''') \notin D$. Since (w, p) is arbitrary, any state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$ is not dominated by another state in D.

Therefore, S' is a stable set. Since functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ are arbitrary basic functions, $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$ for any basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ is a stable set. ■

Figures 3 and 4 show one possible stable set S on the hyperplanes. Dots and bold

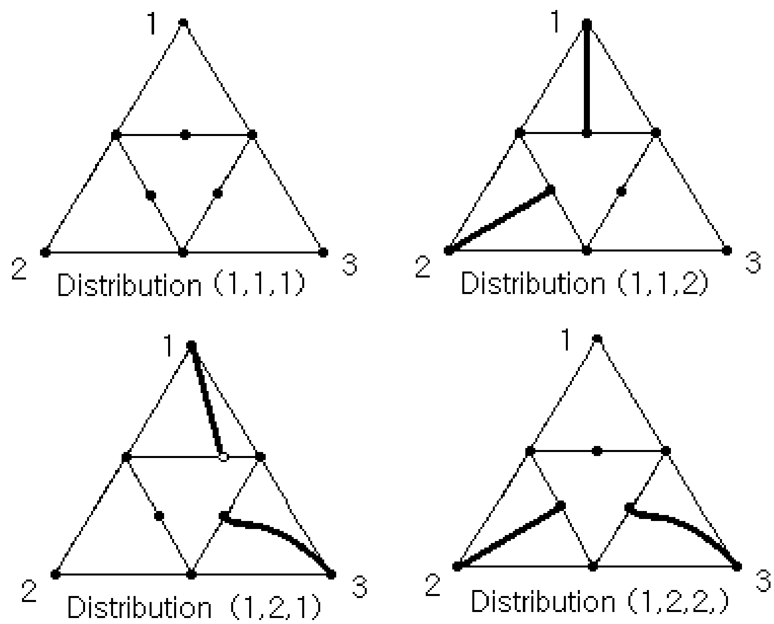

Figure 3: Stable set in the hyperplanes of states with $p_1 = 1$

curves in the figures denote states in S at each distribution. As shown in the proof of Proposition 20, the bold curves in the figures can be expressed with basic functions.

Theorem 21 generalizes Proposition 20 to the three-player $N-$region models where $N \geq 2$.

Theorem 21 *In a three-player $N-$region model where $N \geq 2$, a set S is a stable set if and only if $S = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D \cup U(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C)$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ where the set $U(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C)$ is the set of states that are not dominated by any state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C$.*

Proof. For any two distinct regions $q, r \in R$, define the set $X^{q,r}$ of states by $X^{q,r} = \{(w, p) \in X : \text{for each } i,\ p_i = q \text{ or } r\}$. Then, it is easily seen that a state in $X^{r,q}$ can be dominated only by some state in $X^{r,q}$ because the act of the pillage does not disperse players. If there are more than two regions, then define the set $X^{indiv.}$ of states by $X^{indiv.} = \{(w, p) \in X : \text{for any three distinct regions } o, q, \text{ and } r,\ \{p_1, p_2, p_3\} = \{o, q, r\}\}$. That is, $X^{indiv.}$ is the set of states at which each player occupies its own region alone, i.e., individual region distribution. Note that any state in $X^{indiv.}$ does not dominate any other state in X, however, it can be dominated by some state at which only one region contains two players, whose distribution results from one player pillaging another player. Therefore, a set S is a stable set if and only if $i)$ for any two distinct regions q and r, $S \cap X^{r,q}$ is both internally stable and

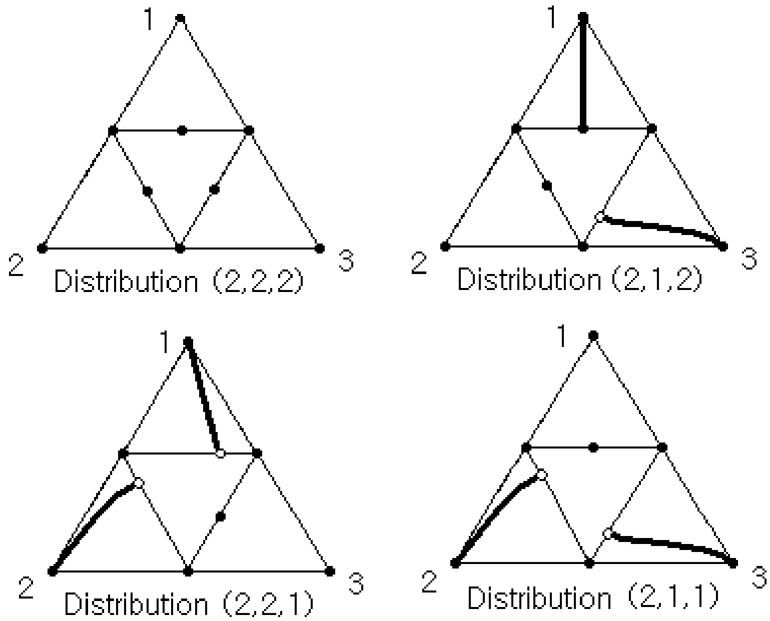

Figure 4: Stable set in the hyperplanes of states with $p_1 = 2$

externally stable with respect to $X^{r,q}$; and ii) S dominates all states in $X^{indiv.}$ except itself $X^{indiv.} \cap S$.

In the three-player and two-region model, by Proposition 20, a set S is a stable set if and only if $S = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$. Without loss of generality, given any two distinct regions q and r, we can regard a state (w, p) in $X^{r,q}$ as the state (w, p) in the two-region model, and vice versa. Then, it is easily seen that $(w', p') \in X^{r,q}$ dominates $(w, p) \in X^{r,q}$ if and only if (w', p') dominates (w, p) in the two-region model. Therefore, for any two distinct regions q and r, $S \cap X^{r,q}$ is both internally stable and externally stable with respect to $X^{r,q}$ if and only if $S \cap X^{r,q} = (B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup D) \cap X^{r,q}$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$. The observation that for any basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$, the set $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C$ dominates all states in $X^{indiv.}$ except $U(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C)$ completes the proof. ■

Figures 5 and 6 show one possible stable set S on the hyperplanes. Figure 5 covers distributions where at least two players are in a common region and Figure 6 covers the other distributions, where each player occupies its own region alone. In the figures, dots, bold lines, and the gray area denote states in the stable set S. Note that except for the three corner points and three middle points, the gray area does not contain boundary lines.

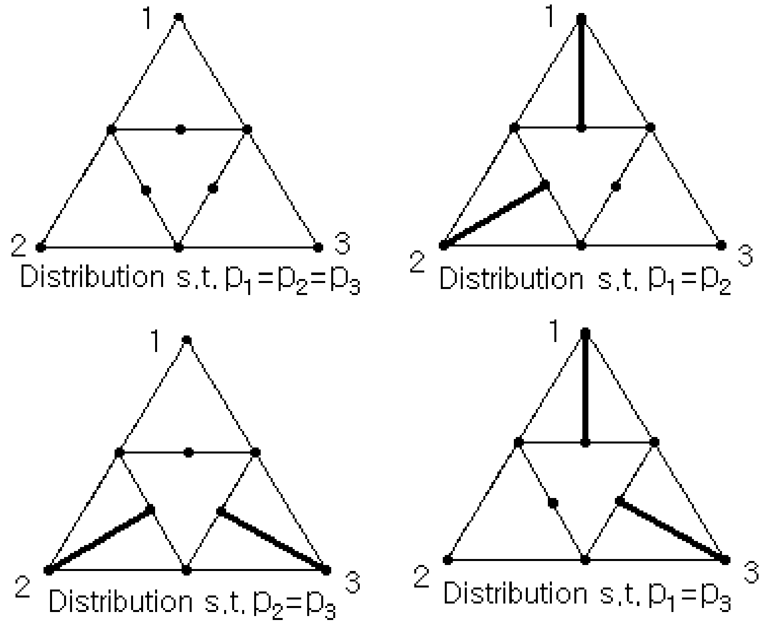

Figure 5: Stable set in the hyperplanes of states such that $p_j = p_k$ for two distinct players j and k.

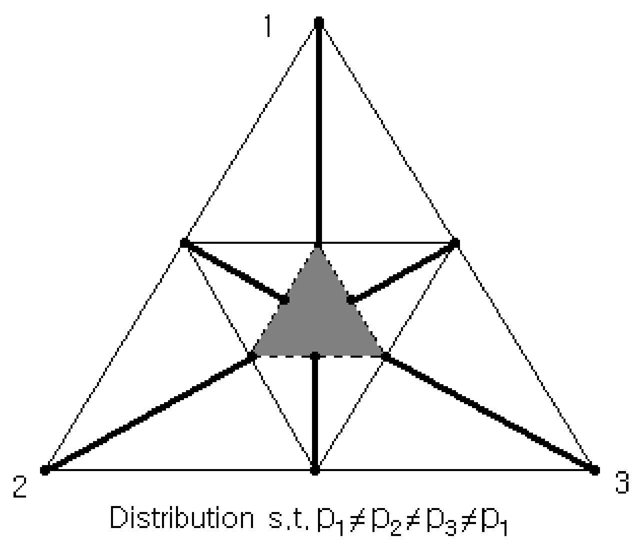

Figure 6: Stable set in the hyperplane of states such that $p_1 \neq p_2 \neq p_3 \neq p_1$

2.2 Stable set in $I-$player and $N-$region models where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$

A stable set does not exist in a $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$. First, we prove that in the four-player and two-region model, a stable set must contain a group of states out of basic sets. Second, we discover some properties of a group of states that are not in basic sets, but are in a stable set. Next, we show that in the four-player and two-region model, if there exists a stable set S, then we can find a state (w, p) such that (w, p) cannot be in S and S cannot dominate (w, p). It is because S contains four states and the properties of these states assure that S dominates every state that dominates (w, p). Finally, we generalize the result in the four-player and two-region model and verify the nonexistence of a stable set in a $I-$player and $N-$region model where $I \geq 4$ and $N \geq 3$.

Lemma 22 shows another property of a stable set. To satisfy both internal and external stabilities, a stable set must contain some states outside the basic sets as well as some basic states. Lemma 22 reveals relation among states that are outside the basic states and belong to some stable set.

Lemma 22 *In the four-player and two-region model, for any player j, let a distribution p satisfy $p^1 = \{j\}$ or $p^2 = \{j\}$. Then given a stable set S, there exists a positive real number a_p such that $[0, a_p] \subset \{w_j : (w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. In particular, if $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, then $[0, w'_j] \subset \{w_j : (w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.*

Proof. Let a distribution $\dot{p} \in B(i; j, k)$ with $\dot{p}_z = p_z$ for each $z \neq i$. Note that $B(i; j, k)$ is the set of distributions at which player i is together with only either player j or player k; that is, at the distribution $\dot{p}$, there exists a region $r \in R$ such that $\dot{p}^r = \{i, j\}$ or $\{i, k\}$. At the distribution p, player j is alone in a region and player k is together with the other players including player i. Therefore, we must have that $\dot{p}_i = \dot{p}_j$ so that player i is together with only one player in a common region. By Lemma 16, there exists a basic function $\beta^i_{j,k}$ with $(B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, \dot{p}) = S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. By the first condition of a basic function, there exists $\dot{\lambda} \in (0, \frac{1}{2})$ with $\beta^i_{j,k}(\dot{\lambda}, \dot{p}) > 0$. According to Corollary 15, $\beta^i_{j,k}(\cdot, \dot{p})$ is uniformly continuous on $[0, \frac{1}{2})$. Since $\beta^i_{j,k}(0, \dot{p}) = 0$, the intermediate value theorem implies that $[0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})] \subset \beta^i_{j,k}([0, \dot{\lambda}], \dot{p})$.

To prove the first assertion, it suffices to show that $[0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})] \subset \{w_j : (w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. For any $\delta \in [0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})]$, let $\lambda_\delta = \min\{\lambda \in [0, \dot{\lambda}] : \beta^i_{j,k}(\lambda, \dot{p}) = \delta\}$, which is well defined by the uniform continuity of $\beta^i_{j,k}$ on $[0, \dot{\lambda}]$ because if a function is continuous, then the inverse image of a closed set under the function is a closed set. Given any $\lambda_\delta \in [0, \dot{\lambda}]$, let the allocation w^δ satisfy that $w^\delta_i = 1 - \lambda_\delta$, $w^\delta_j = \beta^i_{j,k}(\lambda_\delta, \dot{p})$, and $w^\delta_k = \lambda_\delta - \beta^i_{j,k}(\lambda_\delta, \dot{p})$. Then we have that $(w^\delta, \dot{p}) \in B(\beta^i_{j,k}) \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$ and thus that $(w^\delta, \dot{p}) \in S$. Suppose by way of contradiction that $(w^\delta, p) \notin S$. Every state that results from player i pillaging either player j or player k at (w^δ, p) dominates

$(w^\delta, \dot{p}) \in S$ as well as (w^δ, p). Every state that results from either player j or player k engaging in pillage at (w^δ, p), regardless of player $i's$ participation, is dominated by some state in the core such that player i has the total wealth because players j and k get together in a common region and player i has greater than a half. Therefore, S cannot dominate (w^δ, p), and thus S lacks external stability. This contradiction guarantees that $(w^\delta, p) \in S$ and thus that $\delta = w_j^\delta \in \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$ since $(w^\delta, p) \in B^i_{j,k}(\frac{1}{2}, p)$. Since $\delta \in [0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})]$ is arbitrary, we have that $[0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.

In particular, if $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, then by Lemma 11, $(w', \dot{p}) \in S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. By the same way as shown above, we can show that for any $\delta \in [0, w'_j]$, $\delta \in \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$ and thus that $[0, w'_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. ∎

Lemmas 23, 24, and 25 strengthen Lemma 22 by revealing relations between states that are outside the basic states and belong to some stable set.

Lemma 23 *In the four-player and two-region model, for any player j, let a distribution p satisfy either $p^1 = \{j\}$ or $p^2 = \{j\}$. Then given a stable set S, for any $a \in (0, \frac{1}{2})$, there exists a state $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ such that $a > w_j + w_k > w_j > 0$ and $[0, w_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. In addition, if $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $w_j > w'_j > 0$, then we have that $w_k \geq w'_k > 0$.*

Proof. By Lemma 16, there exists a basic function $\beta^i_{j,k}$ with $B(\beta^i_{j,k}) \subset S$. Let the distribution $\dot{p} \in B(i; j, k)$ satisfy $\dot{p}_z = p_z$ for each $z \neq i$. Then since $\beta^i_{j,k}(\cdot, \dot{p})$ is defined on $[0, \frac{1}{2}]$, for any $a \in (0, \frac{1}{2})$, we can find the allocation w^a with $w^a_j = \beta^i_{j,k}(a, \dot{p})$, $w^a_k = a - \beta^i_{j,k}(a, \dot{p})$, and $w^a_i = 1 - a$. Then we have that $(w^a, \dot{p}) \in B(\beta^i_{j,k})$. By Lemma 22, there exists a state $(\dot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $\dot{w}_j > 0$ and $[0, \dot{w}_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. Let a state $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ satisfy that $0 < w_j < \min\{\dot{w}_j, w^a_j\}$. We will show that (w,p) satisfies all required conditions, that is, $a > w_j + w_k > w_j > 0$ and $[0, w_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.

Since $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, by Lemma 11, we have that $(w, \dot{p}) \in S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. Since $w^a_j > w_j > 0$ and $w^a_i > \frac{1}{2}$, the basic condition means that $w_i > w^a_i$ and $w_k > 0$, and thus we have that $a > 1 - w_i = w_j + w_k > w_j > 0$. Since $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, Lemma 22 assures the second condition, $[0, w_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.

In addition, let $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $w'_j \in (0, w_j)$. Then by Lemma 11, we have that $(w', \dot{p}) \in S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. Since $w_j > w'_j > 0$ and $w_i > 1 - a > \frac{1}{2}$, the basic condition of S implies that $w_i < w'_i < 1$ and thus that $w_k \geq w'_k > 0$. ∎

Lemma 24 *In the four-player and two-region model, for the distinct four players i, j, k, and y, let distributions p and p' satisfy either $p^1 = \{j\}$ and $p'^1 = \{i, j, y\}$, or $p^2 = \{j\}$ and $p'^2 = \{i, j, y\}$. Given a stable set S, suppose that $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ and $(w', p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')$. If $w_i, w'_i > \frac{1}{2}$ and $w_j = w'_k$, then $w_k \geq w_j$ or $w'_y \geq w'_k$.*

Proof. By way of contradiction, suppose that $w_i, w'_i > \frac{1}{2}$, $w_j = w'_k$, $w_j > w_k$, and $w'_k > w'_y$. Lemma 22 implies that $[0, w'_k] \subset \{w_k : (w, p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')\}$. Since $w'_k = w_j > w_k$, there exists a state $(\dot{w}', p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')$ with $\dot{w}'_k = w_k$. Let $\dot{p}' \in B(i; k, y)$ satisfy $\dot{p}'_i \neq p'_i$ and $\dot{p}'_z = p'_z$ for each $z \neq i$. Since $(w', p'), (\dot{w}', p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')$, we have that $(w', \dot{p}'), (\dot{w}', \dot{p}') \in S \cap B^i_{k,y}(\frac{1}{2}, \dot{p}')$ by Lemma 11. Since $w_i, w'_i > \frac{1}{2}$ and $w'_k > w_k = \dot{w}'_k$, the basic condition of S means that $w'_i < \dot{w}'_i$ and thus that $w'_y \geq \dot{w}'_y$. Since $w'_k > w'_y$, we have that $w_j = w'_k > w'_y \geq \dot{w}'_y$ and thus that $\dot{w}'_i = 1 - \dot{w}'_k - \dot{w}'_y > 1 - \dot{w}'_k - w_j = 1 - w_k - w_j = w_i$. Since $w_i + w_j + w_k = 1$, we have that $w_y = 0$. Therefore, we have that $\dot{w}'_i > w_i > \frac{1}{2}$, $\dot{w}'_k = w_k$, and $\dot{w}'_y > w_y = 0$. Note that the distribution p' results from players i and y moving to the region of player j at the distribution p. Therefore, $(\dot{w}', p') \in S$ dominates $(w, p) \in S$ by players i and y pillaging player j at (w, p). This contradiction guarantees that if $w_i, w'_i > \frac{1}{2}$ and $w_j = w'_k$, then $w_k \geq w_j$ or $w'_y \geq w'_k$. ■

Lemma 25 *In the four-player and two-region model, for any player j, let a distribution p satisfy either $p^1 = \{j\}$ or $p^2 = \{j\}$. Given a stable set S, if $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ and $(w', p) \in S \cap B^i_{j,y}(\frac{1}{2}, p)$ with $w_j = w'_j > 0$, then $w_i = w'_i$.*

Proof. Suppose by way of contradiction that $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, $(w', p) \in S \cap B^i_{j,y}(\frac{1}{2}, p)$, $w_j = w'_j > 0$ and $w_i \neq w'_i$. Since $(w, p) \in B^i_{j,k}(\frac{1}{2}, p)$ and $(w', p) \in B^i_{j,y}(\frac{1}{2}, p)$, we have that $w_i \geq \frac{1}{2} \geq w_j + w_k$ and $w'_i \geq \frac{1}{2} \geq w'_j + w'_y$. Since $w_j, w'_j > 0$, we have that $w_i > w_k$ and $w'_i > w'_y$. Therefore, if $w_i > w'_i$ then $(w, p) \in S$ dominates $(w', p) \in S$ by either player i or players i and k pillaging player y at (w', p). Similarly, if $w_i < w'_i$ then $(w', p) \in S$ dominates $(w, p) \in S$. This contradiction completes the proof. ■

Lemma 26 synthesizes the previous results in this subsection and shows that in the four-player and two-region model, a stable set S contains four distinct states that satisfy six conditions introduced in this lemma. We can use these four states to show nonexistence of stable set. The six conditions guarantee that there exists a state (w, p) that S cannot contain or dominate.

Lemma 26 *In the four-player and two-region model, a stable set S contains four states $(\dot{w}, p')$, $(\ddot{w}, p'')$, $(\dddot{w}, p''')$, and $(\mathring{w}, p''')$ such that for some four distinct players i, j, k, and y, i) distributions p', p'', and p''' satisfy either $p'^1 = \{j\}$, $p''^1 = \{i, j, y\}$, and $p'''^1 = \{k\}$, or $p'^2 = \{j\}$, $p''^2 = \{i, j, y\}$, and $p'''^2 = \{k\}$; ii) $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$, $(\ddot{w}, p'') \in S \cap B^i_{k,j}(\frac{1}{2}, p'')$, and $(\dddot{w}, p'''), (\mathring{w}, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p''')$; iii) $\ddot{w}_k > \dot{w}_k \geq \dot{w}_j > 0$; iv) $\ddot{w}_j, \dddot{w}_j < \frac{1}{4}$; v) $\frac{1}{4} > \dddot{w}_k > \dot{w}_k + \dot{w}_j$; and vi) $\mathring{w}_j \geq \mathring{w}_k = \dot{w}_j$.*

Proof. Let distributions p, $p(1)$, $p(2)$, $p(3)$, $p(4)$, and $p(5)$ satisfy that either $p^1 = \{i, j, y\}$, $p(1)^1 = \{j\}$, $p(2)^1 = \{i, j, k\}$, $p(3)^1 = \{k\}$, $p(4)^1 = \{i, k, y\}$, and $p(5)^1 = \{y\}$; or $p^2 = \{i, j, y\}$, $p(1)^2 = \{j\}$, $p(2)^2 = \{i, j, k\}$, $p(3)^2 = \{k\}$, $p(4)^2 = \{i, k, y\}$, and $p(5)^2 = \{y\}$. By Lemma 23, there exist states $(\ddot{w}, p) \in S \cap B^i_{k,j}(\frac{1}{2}, p)$, $(\ddot{w}^{(1)}, p(1)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(1))$, $(\ddot{w}^{(2)}, p(2)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(2))$, $(\ddot{w}^{(3)}, p(3)) \in S \cap$

$B^i_{k,j}(\frac{1}{2}, p(3))$, $(\dddot{w}^{(4)}, p(4)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(4))$, and $(\ddot{w}^{(5)}, p(5)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(5))$ with $0 < \ddot{w}_j, \ddot{w}_k, \ddot{w}^{(1)}_j, \ddot{w}^{(1)}_y, \dddot{w}^{(2)}_k, \dddot{w}^{(2)}_y, \dddot{w}^{(3)}_j, \dddot{w}^{(3)}_k, \dddot{w}^{(4)}_j, \dddot{w}^{(4)}_y, \ddot{w}^{(5)}_k, \ddot{w}^{(5)}_y < \frac{1}{4}$. Lemma 23 also implies that there exist states $(\dot{w}, p) \in S \cap B^i_{k,y}(\frac{1}{2}, p)$, $(\mathring{w}, p) \in S \cap B^i_{k,j}(\frac{1}{2}, p)$, $(\dot{w}^{(1)}, p(1)) \in S \cap B^i_{j,k}(\frac{1}{2}, p(1))$, $(\mathring{w}^{(1)}, p(1)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(1))$, $(\dot{w}^{(2)}, p(2)) \in S \cap B^i_{y,j}(\frac{1}{2}, p(2))$, $(\mathring{w}^{(2)}, p(2)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(2))$, $(\mathring{w}^{(3)}, p(3)) \in S \cap B^i_{k,j}(\frac{1}{2}, p(3))$, and $(\mathring{w}^{(4)}, p(4)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(4))$ such that $0 < \dot{w}_k + \dot{w}_y, \dot{w}^{(1)}_j + \dot{w}^{(1)}_k, \dot{w}^{(2)}_j + \dot{w}^{(2)}_y < \min\{\ddot{w}_k, \ddot{w}^{(1)}_j, \ddot{w}^{(5)}_y, \dddot{w}^{(2)}_y, \dddot{w}^{(3)}_k, \dddot{w}^{(4)}_j\}$ and $\dot{w}_k = \mathring{w}_k = \dot{w}^{(1)}_j = \mathring{w}^{(1)}_j = \dot{w}^{(2)}_y = \mathring{w}^{(2)}_y = \mathring{w}^{(3)}_k = \mathring{w}^{(4)}_j > 0$.

Since $\dot{w}^{(1)}_j = \mathring{w}^{(1)}_j > 0$, Lemma 25 means that $\dot{w}^{(1)}_i = \mathring{w}^{(1)}_i$ and thus that $\dot{w}^{(1)}_k = \mathring{w}^{(1)}_y$. If $\mathring{w}^{(1)}_y \geq \mathring{w}^{(1)}_j$ then $\dot{w}^{(1)}_k = \mathring{w}^{(1)}_y \geq \mathring{w}^{(1)}_j = \dot{w}^{(1)}_j$. Therefore, if $\mathring{w}^{(1)}_y \geq \mathring{w}^{(1)}_j$ and $\mathring{w}^{(3)}_j \geq \mathring{w}^{(3)}_k$, then the states $(\dot{w}^{(1)}, p(1))$, $(\ddot{w}, p)$, $(\dddot{w}^{(3)}, p(3))$, and $(\mathring{w}^{(3)}, p(3))$ satisfy all required conditions; that is, for some four distinct players i, j, k, and y, *i)* distributions $p(1)$, p, and $p(3)$ satisfy either $p(1)^1 = \{j\}$, $p^1 = \{i, j, y\}$, and $p(3)^1 = \{k\}$, or $p(1)^2 = \{j\}$, $p^2 = \{i, j, y\}$, and $p(3)^2 = \{k\}$; *ii)* $(\dot{w}^{(1)}, p(1)) \in S \cap B^i_{j,k}(\frac{1}{2}, p(1))$, $(\ddot{w}, p) \in S \cap B^i_{k,j}(\frac{1}{2}, p)$, and $(\dddot{w}^{(3)}, p(3)), (\mathring{w}^{(3)}, p(3)) \in S \cap B^i_{k,j}(\frac{1}{2}, p(3))$; *iii)* $\ddot{w}_k > \dot{w}^{(1)}_k \geq \dot{w}^{(1)}_j > 0$; *iv)* $\ddot{w}_j, \dddot{w}^{(3)}_j < \frac{1}{4}$; *v)* $\frac{1}{4} > \dddot{w}^{(3)}_k > \dot{w}^{(1)}_j + \dot{w}^{(1)}_k$; and *vi)* $\mathring{w}^{(3)}_j \geq \mathring{w}^{(3)}_k = \dot{w}^{(1)}_j$.

Note that if $\mathring{w}_i = \frac{1}{2}$, then since $\mathring{w}_j + \mathring{w}_k = \frac{1}{2}$ and $\mathring{w}_k < \frac{1}{4}$, we have that $\mathring{w}_j > \frac{1}{4} > \mathring{w}_k$, and thus some state in the core such that players i and j have halves dominates $(\mathring{w}, p) \in S$. This contradiction shows that we must have that $\mathring{w}_i > \frac{1}{2}$. Similarly, we can show that we have that $\mathring{w}^{(1)}_i$, $\mathring{w}^{(2)}_i$, $\mathring{w}^{(3)}_i$, and $\mathring{w}^{(4)}_i > \frac{1}{2}$. Therefore, if $\mathring{w}^{(1)}_y < \mathring{w}^{(1)}_j$, then since $\mathring{w}_i$, $\mathring{w}^{(1)}_i$, and $\mathring{w}^{(2)}_i > \frac{1}{2}$, Lemma 24 implies that $\mathring{w}_j \geq \mathring{w}_k$ and $\mathring{w}^{(2)}_k \geq \mathring{w}^{(2)}_y$. Since $\dot{w}_k = \mathring{w}_k$, by Lemma 25, we have that $\dot{w}_i = \mathring{w}_i$ and thus that $\dot{w}_y = \mathring{w}_j \geq \mathring{w}_k = \dot{w}_k$. In this case, the states $(\dot{w}, p)$, $(\ddot{w}^{(5)}, p(5))$, $(\dddot{w}^{(2)}, p(2))$, and $(\mathring{w}^{(2)}, p(2))$ satisfy all six conditions; that is, for some four distinct players i, k, y, and j, *i)* distributions p, $p(5)$, and $p(2)$ satisfy $p^1 = \{k\}$, $p(5)^1 = \{i, j, k\}$, and $p(2)^1 = \{y\}$, or $p^2 = \{k\}$, $p(5)^2 = \{i, j, k\}$, and $p(2)^2 = \{y\}$; *ii)* $(\dot{w}, p) \in S \cap B^i_{k,y}(\frac{1}{2}, p)$, $(\ddot{w}^{(5)}, p(5)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(5))$, and $(\dddot{w}^{(2)}, p(2)), (\mathring{w}^{(2)}, p(2)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(2))$; *iii)* $\ddot{w}^{(5)}_y > \dot{w}_y \geq \dot{w}_k > 0$, *iv)* $\ddot{w}^{(5)}_k, \dddot{w}^{(2)}_k < \frac{1}{4}$, *v)* $\frac{1}{4} > \dddot{w}^{(2)}_y > \dot{w}_k + \dot{w}_y$, and *vi)* $\mathring{w}^{(2)}_k \geq \mathring{w}^{(2)}_y = \dot{w}_k$. Similarly, if $\mathring{w}^{(3)}_j < \mathring{w}^{(3)}_k$, then $(\dot{w}^{(2)}, p(2))$, $(\ddot{w}^{(1)}, p(1))$, $(\dddot{w}^{(4)}, p(4))$, and $(\mathring{w}^{(4)}, p(4))$ satisfy all six conditions. ■

Proposition 27 proves nonexistence of stable set in the four-player and two-region model.

Proposition 27 *No stable set exists in the four-player and two-region model.*

Proof. By way of contradiction, suppose that there exists a stable set S in the four-player and two-region model. Then by Lemma 26, there exist four states $(\dot{w}, p')$, $(\ddot{w}, p'')$, $(\dddot{w}, p''')$, and $(\mathring{w}, p''')$ such that for some four distinct players i, j, k, and y, *i)* distributions p', p'', and p''' satisfy either $p'^1 = \{j\}$, $p''^1 = \{i, j, y\}$, and $p'''^1 = \{k\}$, or $p'^2 = \{j\}$, $p''^2 = \{i, j, y\}$, and $p'''^2 = \{k\}$; *ii)* $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$, $(\ddot{w}, p'') \in S \cap B^i_{k,j}(\frac{1}{2}, p'')$, and $(\dddot{w}, p''')$, $(\mathring{w}, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p''')$; *iii)* $\ddot{w}_k > \dot{w}_k \geq \dot{w}_j > 0$; *iv)* $\ddot{w}_j, \dddot{w}_j < \frac{1}{4}$; *v)* $\frac{1}{4} > \dddot{w}_k > \dot{w}_k + \dot{w}_j$; and *vi)* $\mathring{w}_j \geq \mathring{w}_k = \dot{w}_j$.

Define the set of states $T(\dot{w}, \ddot{w}, \dddot{w}; p') = \{(w, p) : w_i = \frac{1-\dot{w}_j}{2}, w_j = \dot{w}_j, \min\{\ddot{w}_k, \dddot{w}_k - \dot{w}_j\} \geq w_k > \dot{w}_k, \text{ and } p = p'\}$. Then, we can show that $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ has uncountably many elements. Note that by conditions *iii)* and *v)*, $\min\{\ddot{w}_k, \dddot{w}_k - \dot{w}_j\} > \dot{w}_k$. Let $a \in \mathbb{R}^4$ satisfy that $a_i = \frac{1-\dot{w}_j}{2}$, $a_j = \dot{w}_j$, $\min\{\ddot{w}_k, \dddot{w}_k - \dot{w}_j\} \geq a_k > \dot{w}_k$, and $a_y = 1 - a_i - a_j - a_k$. Since $1 > \dot{w}_j > 0$ and $\dot{w}_k > 0$, we have that $a_i, a_j, a_k \in (0, 1)$. Note that $a_y = 1 - \frac{1-\dot{w}_j}{2} - \dot{w}_j - a_k > \frac{1}{2} - \frac{\dot{w}_j}{2} - \dot{w}_k > \frac{1}{4}$, that is, $a_y \in (\frac{1}{4}, 1)$. Since $a_i + a_j + a_k + a_y = 1$, a is an allocation in the four-player model. Since a satisfies all requirements to be an allocation in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$, we have that $(a, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$. It is easily seen that for any $\varepsilon \in (0, a_k - \dot{w}_k)$, $((a_i, a_j, a_k - \varepsilon, a_y + \varepsilon), p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$.

We will prove that $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ contains some state that S cannot contain or dominate. First, we show that there exists a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ that S cannot contain. Note that for any distinct states (w, p'), $(w', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$, we have that $w_i = w'_i$, $w_j = w'_j$, and either $w'_y > w_y > w_k > w'_k$ or $w_y > w'_y > w'_k > w_k$. If $w'_y > w_y > w_k > w'_k$, then (w', p') dominates (w', p') by player y pillaging player k. Similarly, if $w_y > w'_y > w'_k > w_k$, then (w, p') dominates (w', p'). Therefore, internal stability of S means that $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$ has at most one element. In addition, for any $(w, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$, there exists a state $(w'', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ such that $w''_i = w_i$, $w''_j = w_j$, and $w''_y > w_y > w_k > w''_k$; that is, every state in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ is dominated by another state in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$. Therefore, there exists a state $(w^T, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ such that (w^T, p') dominates a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$, which can be empty. Then we have that $(w^T, p') \notin S$. Next, we show that S cannot dominate (w^T, p').

Let the set of states $T_1(w^T, p') = \{(w, p) : (w, p) \text{ results from player } i \text{ pillaging player } j \text{ at } (w^T, p')\}$. Note that $w^T_k + w^T_y = 1 - w^T_i - w^T_j = 1 - \frac{1-\dot{w}_j}{2} - \dot{w}_j = \frac{1-\dot{w}_j}{2} = w^T_i$. Let $(w^{T_1}, p) \in T_1(w^T, p')$. Since $w^T_i = w^T_k + w^T_y$, $w^T_j = \dot{w}_j$, and $w^T_k > \dot{w}_k$, we have that $w^{T_1}_i > w^{T_1}_k + w^{T_1}_y$, $w^{T_1}_j < w^T_j = \dot{w}_j$, and $w^{T_1}_k = w^T_k > \dot{w}_k$. Since $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$, Lemma 22 implies that $[0, \dot{w}_j] \subset \{w_j : (w, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')\}$. Therefore, $S \cap B^i_{j,k}(\frac{1}{2}, p')$ contains $(\dot{w}^{T_1}, p')$ with $\dot{w}^{T_1}_j = w^{T_1}_j$. Since $\dot{w}_j > \dot{w}^{T_1}_j$, by Lemma 23, we have that $\dot{w}_k \geq \dot{w}^{T_1}_k$ and thus that $w^{T_1}_k > \dot{w}^{T_1}_k$. Therefore, (w^{T_1}, p) in $T_1(w^T, p')$ is dominated by $(\dot{w}^{T_1}, p')$ in $S \cap B^i_{j,k}(\frac{1}{2}, p')$ by player i pillaging players k and y. Since $(w^{T_1}, p) \in T_1(w^T, p')$ is arbitrary, every state in $T_1(w^T, p')$ is dominated by some state in $S \cap B^i_{j,k}(\frac{1}{2}, p')$ through player i pillaging players k and y. Therefore, we must have that $T_1(w^T, p') \cap S = \varnothing$.

Let the set of states $T_2(w^T, p') = \{(w, p) : (w, p) \text{ results from player } i \text{ pillaging player } k \text{ at } (w^T, p')\}$. Then for each $(w^{T_2}, p') \in T_2(w^T, p')$, we have that $w^{T_2}_i > w^{T_2}_k + w^{T_2}_y$, $w^{T_2}_j = \dot{w}_j$, and $w^{T_2}_y = w^T_y > w^T_k > \dot{w}_k$ since $w^T_i = w^T_k + w^T_y$, $w^T_j = \dot{w}_j$, and $w^T_y > \frac{1}{4} > w^T_k$. Therefore, every state in $T_2(w^T, p')$ is dominated by $(\dot{w}, p') \in$

$S \cap B^i_{j,k}(\frac{1}{2}, p')$ through players i and k pillaging player y when $\dot{w}_k > w_k^{T_2}$, through player i pillaging player y when $\dot{w}_k = w_k^{T_2}$, or through player i pillaging players k and y when $\dot{w}_k < w_k^{T_2}$. Therefore, we have that $T_2(w^T, p') \cap S = \varnothing$.

Let the set of states $T_3(w^T, p') = \{(w, p) : (w, p)$ results from player i pillaging player y at $(w^T, p')\}$. Then for each $(w^{T_3}, p') \in T_3(w^T, p')$, we have that $w_i^{T_3} > w_k^{T_3} + w_y^{T_3}$, $w_j^{T_3} = \dot{w}_j$, and $w_k^{T_3} = w_k^T > \dot{w}_k$ since $w_i^T = w_k^T + w_y^T$, $w_j^T = \dot{w}_j$, and $w_k^T > \dot{w}_k$. Therefore, every state in $T_3(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$ through player i either pillaging players k and y when $w_y^{T_3} > 0$, or pillaging player k when $w_y^{T_3} = 0$. Therefore, we have that $T_3(w^T, p') \cap S = \varnothing$.

Let the set of states $T_4(w^T, p') = \{(w, p) : (w, p)$ results from player y pillaging player k at $(w^T, p')\}$. Let $(w^{T_4}, p') \in T_4(w^T, p')$. Since $w_i^T = w_k^T + w_y^T$ and $w_y^T > \frac{1}{4} > \dot{w}_k$, we have that $w_i^{T_4} \geq w_y^{T_4} > \dot{w}_k$ and $w_j^{T_4} = \dot{w}_j$. Note that dominance relation in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ is transitive; that is, if $(w, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ is dominated by $(w', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ and (w', p') is dominated by $(w'', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$, then because $w_i = w_i' = w_i''$, $w_j = w_j' = w_j''$, and $w_y'' > w_y' > w_y > w_k > w_k' > w_k''$, (w, p') is dominated by (w'', p') through player y pillaging player k at (w, p'). Therefore, if $w_k^{T_4} > \dot{w}_k$, then (w^{T_4}, p) is still in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ and thus since (w^T, p') dominates a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$, which can be empty, (w^{T_4}, p') dominates a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$. This means that $(w^{T_4}, p') \notin S$. If $\dot{w}_k \geq w_k^{T_4} > 0$, then (w^{T_4}, p') is dominated by $(\dot{w}, p')$ in $S \cap B^i_{j,k}(\frac{1}{2}, p')$ either through player i pillaging player y when $\dot{w}_k = w_k^{T_4}$, or through players i and k pillaging player y when $\dot{w}_k > w_k^{T_4}$. If $w_k^{T_4} = 0$, then $w_i^{T_4} = w_y^{T_4} > w_j^{T_4}$, and thus (w^{T_4}, p') is dominated by some state in the core such that players i and y have halves. Since $(w^{T_4}, p') \in T_4(w^T, p')$ is arbitrary, we have that $T_4(w^T, p') \cap S = \varnothing$.

Let the set of states $T_5(w^T, p') = \{(w, p) : (w, p)$ results from player y pillaging player j at $(w^T, p_1)\}$. Then for each $(w^{T_5}, p) \in T_5(w^T, p')$, we have that $w_i^{T_5} + w_k^{T_5} > w_j^{T_5} + w_y^{T_5}$ and $\frac{1}{2} > w_i^{T_5} > w_k^{T_5}$ since $w_i^T > w_y^T$, $w_k^T > w_j^T$ and $\frac{1}{2} > w_i^T > w_k^T$. Therefore, every state in $T_5(w^T, p')$ is dominated by some state in the core such that players i and k have halves. Therefore, we have that $T_5(w^T, p') \cap S = \varnothing$.

Let the set of states $T_6(w^T, p') = \{(w, p) : (w, p)$ results from player k pillaging player j at $(w^T, p')\}$. Then for each $(w^{T_6}, p) \in T_6(w^T, p')$, we have that $w_i^{T_6} + w_y^{T_6} > w_j^{T_6} + w_k^{T_6}$ and $\frac{1}{2} > w_i^{T_6} > w_y^{T_6}$ since $\frac{1}{2} > w_i^T > w_y^T > w_k^T > w_j^T$. Therefore, every state in $T_6(w^T, p')$ is dominated by some state in the core such that players i and y have halves. Therefore, we have that $T_6(w^T, p') \cap S = \varnothing$.

Let the set of states $T_7(w^T, p') = \{(w, p) : (w, p)$ results from players i and y pillaging player k at $(w^T, p')\}$. Then for each $(w^{T_7}, p) \in T_7(w^T, p')$, we have that $w_i^{T_7} > w_k^{T_7} + w_y^{T_7}$, $w_j^T = \dot{w}_j$, and $w_y^{T_7} > \dot{w}_k$ since $w_i^T = w_k^T + w_y^T$ and $w_y^T > \dot{w}_k$. Therefore, every state in $T_7(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$. Therefore, we have that $T_7(w^T, p') \cap S = \varnothing$.

Let the set of states $T_8(w^T, p') = \{(w, p) : (w, p)$ results from players i and y pillaging player j at $(w^T, p')\}$. Let $(w^{T_8}, p'') \in T_8(w^T, p')$. Since $w_i^T > w_j^T + w_y^T$,

$\ddot{w}_k \geq w_k^T$, and $w_y^T > \frac{1}{4} > \ddot{w}_j$, we have that $w_i^{T_8} > w_j^{T_8} + w_y^{T_8}$, $\ddot{w}_k \geq w_k^T = w_k^{T_8}$, and $w_y^{T_8} > w_y^T > \ddot{w}_j$. When $w_k^{T_8} = \ddot{w}_k$, (w^{T_8}, p'') is dominated by $(\ddot{w}, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')$ through player i pillaging players j and y when $\ddot{w}_j < w_j^{T_8}$, through player i pillaging player y when $\ddot{w}_j = w_j^{T_8}$, or through players i and j pillaging player y when $\ddot{w}_j > w_j^{T_8}$. Now, we check the case that $\ddot{w}_k > w_k^{T_8}$. Since $(\ddot{w}, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')$, Lemma 22 implies that $[0, \ddot{w}_k] \subset \{w_k : (w, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')\}$. Therefore, $S \cap B_{k,j}^i(\frac{1}{2}, p'')$ contains the state $(\ddot{w}^{T_8}, p'')$ such that $\ddot{w}_k^{T_8} = w_k^{T_8}$. Since $\ddot{w}_k > \ddot{w}_k^{T_8}$, Lemma 23 implies that $\ddot{w}_j \geq \ddot{w}_j^{T_8}$ and thus that $w_y^{T_8} > w_y^T > \frac{1}{4} > \ddot{w}_j \geq \ddot{w}_j^{T_8}$. Therefore, $(w^{T_8}, p'') \in T_8(w^T, p')$ is dominated by the state $(\ddot{w}^{T_8}, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')$ through player i pillaging players j and y when $\ddot{w}_j^{T_8} < w_j^{T_8}$, through player i pillaging player y when $\ddot{w}_j^{T_8} = w_j^{T_8}$, or through players i and j pillaging player y when $\ddot{w}_j^{T_8} > w_j^{T_8}$. Since $(w^{T_8}, p'') \in T_8(w^T, p')$ is arbitrary, we have that $T_8(w^T, p') \cap S = \varnothing$.

Let the set of states $T_9(w^T, p') = \{(w, p) : (w, p)$ results from players i and k pillaging player y at $(w^T, p')\}$. Then for each $(w^{T_9}, p') \in T_9(w^T, p')$, we have that $w_i^{T_9} > w_k^{T_9} + w_y^{T_9}$, $w_j^{T_9} = \dot{w}_j$, and $w_k^{T_9} > \dot{w}_k$ since $w_i^T = w_k^T + w_y^T$ and $w_k^T > \dot{w}_k$. Therefore, every state in $T_9(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B_{j,k}^i(\frac{1}{2}, p')$. Therefore, we have that $T_9(w^T, p') \cap S = \varnothing$.

Let the set of states $T_{10}(w^T, p') = \{(w, p) : (w, p)$ results from players i and k pillaging player j at $(w^T, p')\}$. Let $(w^{T_{10}}, p) \in T_{10}(w^T, p')$. Since $w_i^T > w_y^T > \frac{1}{4}$, $\mathring{w}_j \geq \mathring{w}_k = \dot{w}_j = w_j^T$, and $\dddot{w}_k - \dot{w}_j \geq w_k^T > \dot{w}_j$, we have that $w_i^{T_{10}} > w_y^{T_{10}} > \frac{1}{4}$, $\mathring{w}_j \geq w_j^T > w_j^{T_{10}}$, and $\dddot{w}_k > w_k^{T_{10}} > \mathring{w}_k$. Since $(\dddot{w}, p''') \in S \cap B_{k,j}^i(\frac{1}{2}, p''')$, Lemma 22 implies that $[0, \dddot{w}_k] \subset \{w_k : (w, p''') \in S \cap B_{k,j}^i(\frac{1}{2}, p''')\}$. Therefore, $S \cap B_{k,j}^i(\frac{1}{2}, p'')$ contains the state $(\dddot{w}^{T_{10}}, p''')$ with $\dddot{w}_k^{T_{10}} = w_k^{T_{10}}$. Since $\dddot{w}_k > w_k^{T_{10}} > \mathring{w}_k$, Lemma 23 means that $\dddot{w}_j \geq \dddot{w}_j^{T_{10}} \geq \mathring{w}_j$. Since $w_y^{T_{10}} > \frac{1}{4} > \dddot{w}_j \geq \dddot{w}_j^{T_{10}} > w_j^{T_{10}}$, $(\dddot{w}^{T_{10}}, p''') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')$ dominates the state $(w^{T_{10}}, p)$ by players i and j pillaging player y. Since $(w^{T_{10}}, p) \in T_{10}(w^T, p')$ is arbitrary, every state in $T_{10}(w^T, p')$ is dominated by some state in $S \cap B_{k,j}^i(\frac{1}{2}, p''')$. Therefore, we have that $T_{10}(w^T, p') \cap S = \varnothing$.

Let the set of states $T_{11}(w^T, p') = \{(w, p) : (w, p)$ results from players k and y pillaging player j at $(w^T, p')\}$. Then for each $(w^{T_{11}}, p) \in T_{11}(w^T, p')$, we have that $w_i^{T_{11}} < w_j^{T_{11}} + w_k^{T_{11}} + w_y^{T_{11}}$, $\frac{1}{2} > w_y^{T_{11}}$, and $\frac{1}{4} > w_k^{T_{11}} > w_j^{T_{11}}$ since $w_i^T < w_j^T + w_k^T + w_y^T$, $\frac{1}{2} > w_y^T + w_j^T$, and $\frac{1}{4} > \dddot{w}_k \geq w_k^T + w_j^T$. Note that by Lemma 19, $D \subset S$. Therefore, every state in $T_{11}(w^T, p')$ is dominated by some state (w, p) in D such that $w_y = \frac{1}{2}$, $w_j = w_k = \frac{1}{4}$, and $p^{p'_i} = I$ through players j, k, and y pillaging player i. Therefore, we have that $T_{11}(w^T, p') \cap S = \varnothing$.

Finally, let the set of states $T_{12}(w^T, p') = \{(w, p) : (w, p)$ results from players i, k, and y pillaging player j at $(w^T, p')\}$. Let $(w^{T_{12}}, p) \in T_{12}(w^T, p')$. Since $w_i^T > w_y^T + w_j^T$ and $w_i^T > w_y^T > \frac{1}{4} > w_k^T + w_j^T$, we have that $w_i^{T_{12}} > w_y^{T_{12}} > \frac{1}{4} > w_j^{T_{12}} + w_k^{T_{12}}$. If $w_i^{T_{12}} > \frac{1}{2}$, then $(w^{T_{12}}, p)$ is dominated by some state in the core such that player i has the total wealth. If $w_i^{T_{12}} \leq \frac{1}{2}$, then $(w^{T_{12}}, p)$ is dominated by some state in the core at which players i and j have halves. Since $(w^{T_{12}}, p') \in T_{12}(w^T, p')$ is arbitrary, we have that $T_{12}(w^T, p') \cap S = \varnothing$.

Therefore, $(w^T, p) \notin S$ cannot be dominated by any state in S. This contradiction shows that there is no stable set in the four-player and two-region model. ■

Theorem 28 generalizes Proposition 27 to a $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$.

Theorem 28 *No stable set exists in a $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$.*

Proof. Suppose by way of contradiction that there exists a stable set S. For any four distinct players i, j, k, and y, define the set $F(i,j,k,y)$ of states by $F(i,j,k,y) = \{(w,p) : w_i + w_j + w_k + w_y = 1,\ p^1 \cup p^2 = \{i,j,k,y\}$, and $p^3 = I \setminus \{i,j,k,y\}$ if $N \geq 3\}$. Then any state in $F(i,j,k,y)$ cannot be dominated by another state in $X \setminus F(i,j,k,y)$. Thus $S \cap F(i,j,k,y)$ is externally stable with respect to $F(i,j,k,y)$. Obviously $S \cap F(i,j,k,y)$ is internally stable. Therefore, $S \cap F(i,j,k,y)$ is both internal stable and external stable with respect to $F(i,j,k,y)$. It is easily seen that $S \cap F(i,j,k,y)$ of states can be adapted for a stable set in the four-player and two-region model. This contradicts Proposition 27, which shows nonexistence of stable set in the four-player and two-region model. This contradiction completes the proof. ■

3 Core in expectation

As has been shown in section 2, stable set is not appropriate for a solution to the spatial pillage game. In a $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$, no stable set exists. In three-player models, there exist stable sets. However, they contain implausible states, such as some states in the set of states $X_{\#I} = \{(w,p) : \text{for some play } i,\ 1 > w_i > \frac{1}{2}\}$. Every state in $X_{\#I}$ is naturally thought not to be stationary because one of the players has enough power to pillage the others, so the player would achieve all of the wealth.

These problems with the solution concept of stable set are caused by weakness of external stability, which requires that any state outside a stable set be directly dominated by some state in the stable set. Some states in $X_{\#I}$ are directly dominated only by other states in $X_{\#I}$, thus a stable set must contain some states in $X_{\#I}$ to satisfy external stability. This shows why stable sets in three-player models contain some states in $X_{\#I}$. In $I-$player and $N-$region models where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$, if an internally stable set S' includes a set of states that dominates every state in $X_{\#I} \setminus S'$, then there exists some state $(w,p) \notin S'$ that S' cannot dominate. It is because the internally stable set S' inevitably dominates every state that dominates (w,p). This shows why no stable set exists in these models.

Jordan (2005) introduced the solution concept of **farsighted core** of **consistent expectation** that is based on the concept of **dominance in expectation**. We can settle the problem with the solution concept of stable set by adopting the solution concept of farsighted core, as shown in Theorem 33 and Lemma 34. Theorem 33 assures that a farsighted core of a consistent expectation exists. Lemma 34 guarantees that a farsighted core of a consistent expectation does not contains any state in $X_{\#I}$. Note that the definitions of expectation and dominance in expectation are adapted for a spatial pillage game.

An **expectation** is a belief that all players have in common and indicates how each state proceeds.

Definition 29 *An **expectation** is a function $f : X \longrightarrow X$ satisfying, for some integer $k \geq 2$, $f^k = f^{k-1}$ where $f^k = f \circ f^{k-1}$. Let $f_w(w,p)$ and $f_p(w,p)$ denote the allocation and the distribution at $f(w,p)$, respectively.*

Dominance in Expectation between states indicates the possible state that the present state can change to provided that players follow the expectation after a change. Both *physical* and *spatial conditions* should be satisfied in order for a *winning coalition in expectation*, who end up being better off, to change its present state through defeating a *losing coalition in expectation*, who end up being worse off. Physical and spatial conditions are reflected on condition *iii*) and conditions *i*) and *ii*) in Definition 30, respectively.

Definition 30 *Let an expectation f satisfy that $f^k = f^{k+1}$. Given states (w,p) and (w',p'), define $W_f = \{i : f^k_w(w',p')_i > w_i\}$ and $L_f = \{i : f^k_w(w',p')_i < w_i\}$. Suppose that for some $r \in R$, i) $\{i : w'_i \neq w_i\} \subset p'^r$; ii) for all $q \neq r$, $p'^q = p^q \setminus (W_f \cap p'^r)$; and iii) $\sum_{i \in W_f \cap p'^r} w_i > \sum_{i \in L_f \cap p'^r} w_i$. Then (w',p') **dominates** (w,p) **in expectation**.*

An expectation is **consistent** if it is organized in accord with the relation of dominance in expectation. If a state (w,p) proceeds to another state (w',p') in expectation, then (w',p') dominates (w,p) in expectation. If a state (w'',p'') is stationary in expectation, then no state dominates (w'',p'') in expectation.

Definition 31 *An expectation f is **consistent** if $f(w,p)$ dominates (w,p) in expectation when $f(w,p) \neq (w,p)$ and (w,p) is undominated in expectation when $f(w,p) = (w,p)$.*

Farsighted core and **farsighted supercore**[1] are defined as follows.

Definition 32 *Given an expectation f, the **farsighted core** under the expectation f is the set of states $K_f = \{(w,p) \in X :$ under the expectation f, no state in X dominates (w,p) in expectation$\}$. The **farsighted supercore** C_S is the intersection of all farsighted cores of consistent expectations.*

[1] *Farsighted supercore* is named after Roth's (1976) *supercore*.

Theorem 33 assures that there exists a consistent expectation that has the set D of states as farsighted core.

Theorem 33 (Existence of a consistent expectation) *There exists a consistent expectation f such that $K_f = D$.*

Proof. Let $\bar{X} = \{(w,p) \in X :$ for each region $r \in R$, $\sum_{i \in p^r} w_i = 0, \frac{1}{2}$, or 1 and for some region $q \in R$, if $\sum_{i \in p^q} w_i = \frac{1}{2}$, then for some player z, $w_z = \frac{1}{2}\}$. Then by Lemma 19, D is the unique set that satisfies both internal stability and external stability with respect to $\bar{X}$. Therefore, it suffices to construct a consistent expectation f such that for some positive integer k, $f^k(X \setminus \bar{X}) \subset D$.

For any positive integer $n \geq 2$, define $X^n = \{(w,p) :$ there exist n distinct regions $r_1, ..., r_n$ such that *i)* $\sum_{i \in p^{r_1} \cup \cdots \cup p^{r_n}} w_i = 1$; *ii)* $\sum_{i \in p^{r_1}} w_i = \max_{r' \in R}\{\sum_{i \in p^{r'}} w_i\} > \sum_{i \in p^{r_n}} w_i > 0$; and *iii)* for each $j \in p^{r_1}$ and some nonnegative integer k_j, $w_j = (\sum_{z \in p^{r_1}} w_z) \times (\frac{1}{2})^{k_j}\}$. Then any state in X^2 is dominated by some state in D through players in a wealthier region pillaging other players in another region. Therefore, we construct f such that every state in X^2 is dominated in expectation by some state in D. Similarly, we can construct f such that for any integer $k \geq 2$, any state in $X^{k+1} \setminus X^k$ is dominated in expectation by some state in X^k. Note that for any state $(w,p) \in X^n$, if $f_w(w,p)_i > w_i$, then for any positive integer k, $f_w^k(w,p)_i \geq f_w(w,p)_i$ and that during the change from (w,p) to $f^{n-1}(w,p)$, pillage happens in each region at most once.

Now, we design the expectation f to satisfy that any state in $X \setminus (X^{\#R} \cup \bar{X})$ is dominated in expectation by some state in $X^{\#R}$. For some regions q and q', let $(w,p) \in X \setminus X^{\#R}$ satisfy that $\sum_{i \in p^q} w_i > \sum_{i \in p^{q'}} w_i > 0$. Then we have that $(w,p) \notin \bar{X}$. Since $(w,p) \notin X^{\#R}$, there exist region r and player i such that $i \in p^r$, $\sum_{z \in p^r} w_z = \max_{r' \in R}\{\sum_{z \in p^{r'}} w_z\}$, and for any nonnegative integer k_i, $w_i \neq (\sum_{z \in p^r} w_z) \times (\frac{1}{2})^{k_i}$. Theorem 18 assures that there exists the state $(w',p) \in X^{\#R}$ such that *i)* $w'_i = w_i$ when $i \notin p^r$; *ii)* for some nonnegative integer k_i, $w'_i = (\sum_{z \in p^r} w_z) \times (\frac{1}{2})^{k_i}$ when $i \in p^r$; and *iii)* $\sum_{z \in \{y : w'_y > w_y\}} w_z > \sum_{z \in \{y : w'_y < w_y\}} w_z$. Then $(w',p) \in X^{\#R}$ dominates (w,p), and thus we can make (w',p) dominate (w,p) in expectation.

Let $(\dot{w},p) \in X \setminus \bar{X}$ satisfy that for any two regions r and q, if $\sum_{i \in p^r} \dot{w}_i > 0$ and $\sum_{i \in p^q} \dot{w}_i > 0$, then $\sum_{i \in p^r} \dot{w}_i = \sum_{i \in p^q} \dot{w}_i$. Note that $(\dot{w},p) \notin X^{\#R}$. For some distinct regions r and q, if $\sum_{i \in p^r} \dot{w}_i = \sum_{i \in p^q} \dot{w}_i = \frac{1}{2}$ and $\dot{w}$ is dyadic, then since $(\dot{w},p) \notin \bar{X}$, for each i, $\dot{w}_i < \frac{1}{2}$. In this case, a coalition E such that $\sum_{i \in E} \dot{w}_i = \frac{1}{2}$, $p^r \nsubseteq E$ (or $p^q \nsubseteq E$), and $E \subset \{i : w_i > 0\}$ can pillage all players in one of the regions and divide their booty in proportion to their wealth. For each region r, if $\sum_{i \in p^r} \dot{w}_i = 0$ or $\frac{1}{2}$ and there exists player i such that for any nonnegative integer k_i, $\dot{w}_i \neq (\frac{1}{2})^{k_i}$, then by Theorem 18, there exists an allocation $\dot{w}'$ such that *i)* when $z \notin p^{p_i}$, $\dot{w}'_z = \dot{w}_z$; *ii)* when $z \in p^{p_i}$, for some nonnegative integer k_z, $\dot{w}'_z = (\sum_{y \in p^{p_i}} \dot{w}_y) \times (\frac{1}{2})^{k_z}$; and *iii)* $\sum_{z \in \{y : \dot{w}'_y > \dot{w}_y\}} \dot{w}_z > \sum_{z \in \{y : \dot{w}'_y < \dot{w}_y\}} \dot{w}_z$. Then a coalition $\dot{E}$ that consists of $\{z : \dot{w}'_z > \dot{w}_z\}$

and player j such that $p_j \neq p_i$ and $\dot{w}_j > 0$ can pillage the other players at region p_i and proportion their wealth to $\dot{w}'$ while giving $\frac{1}{2}(\frac{1}{2} + \dot{w}_j)$ to player j. Note that players in the coalition $\dot{E}$ who earn nothing or even lose their wealth at $(\dot{w}', p')$ participate in $\dot{E}$ because they expect that their wealth will increase in the future movement. Similarly, in case that for each region r, $\sum_{i \in p^r} \dot{w}_i < \frac{1}{2}$, we can construct f so that $(\dot{w}, p)$ is dominated in expectation by some state in $X^{\#R}$. Note that for any $(w, p) \in X \setminus (X^{\#R} \cup \bar{X})$, if $f_w(w, p)_i > w_i$, then for any positive integer k, $f_w^k(w, p)_i \geq f_w(w, p)_i$ and that during the change from (w, p) to $f^{\#R}(w, p)$, pillage happens in each region at most once.

Now, we only need to check if the expectation f is consistent. The expectation f satisfies that $i)$ for any $(w, p) \in X$, if $f_w(w, p)_i > w_i$, then for any positive integer k, $f_w^k(w, p)_i \geq f_w(w, p)_i$; and $ii)$ during the change from (w, p) to $f^{\#R}(w, p)$, pillage can happen in each region at most once. If a player experiences pillage in his region, then he will never be pillaged during the rest of the process of f; that is, for any $(w, p) \in X \setminus K_f$ and any $i \in I$, if p_i satisfies that $\{z : f_w(w, p)_z \neq w_z\} \subset p^{p_i}$, then any positive integer k, $f_w^k(w, p)_i \geq f_w(w, p)_i$. It is easily seen that for any state $(w, p) \in X$ and any positive integer k, if $f^k(w, p) \neq f^{k-1}(w, p)$, then $i)$ there exists a region $r \in R$ such that $\{z : f_w^k(w, p)_z \neq f_w^{k-1}(w, p)_z\} \subset f_p^k(w, p)^r$; $ii)$ for all $q \neq r$, $f_p^k(w, p)^q = f_p^{k-1}(w, p)^q \setminus \{z : f_w^k(w, p)_z > f_w^{k-1}(w, p)_z\}$; and $iii)$ $\sum_{y \in \{z : f_w^k(w,p)_z > f_w^{k-1}(w,p)_z\}} f_w^{k-1}(w, p)_y > \sum_{y \in \{z : f_w^k(w,p)_z > f_w^{k-1}(w,p)_z\}} f_w^{k-1}(w, p)_y$. These show that $f(w, p)$ dominates (w, p) in expectation when $f(w, p) \neq (w, p)$ and (w, p) is undominated in expectation when $f(w, p) = (w, p)$. Therefore, f is consistent. ■

The expectation f constructed above shows how the neutrality assumption is modified in a spatial pillage game. If some players expect that they would be pillaged during the process of f, then they combine their power under spatial restriction to protect themselves although some of them are not pillaged immediately. For example, the state $((\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), (1, 2, 2))$ is not dominated in expectation by $((\frac{2}{3}, 0, \frac{1}{3}), (2, 2, 2))$ because player 3 will be against player 1 to protect player 2 in the expectation that after player 1 pillaging player 2, player 1 would pillage player 3. This shows that neutrality is modified. However, $((\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), (1, 2, 2))$ is dominated in expectation by $((\frac{1}{2}, 0, \frac{1}{2}), (1, 2, 2))$ because player 1 keeps neutral. Also, the state $((\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), (1, 1, 2))$ is dominated in expectation by $((\frac{2}{3}, 0, \frac{1}{3}), (1, 1, 2))$ because player 3 cannot protect player 2 because of spatial restriction. In these cases, neutrality is not modified. Therefore, as Jordan (2005) said, "the concept of domination in expectation constitutes an endogenous modification of the neutrality assumption" in a spatial pillage game.

Lemma 34 shows that in a consistent expectation, if one of players has a majority of the wealth, then the player would finally have all of the wealth. Lemmas 34 and 35 are used to prove Proposition 36.

Lemma 34 *Let the set of states $X_{\#I} = \{(w, p) :$ for some play i, $1 > w_i > \frac{1}{2}\}$ and the set of states $D_0 = \{(w, p) :$ for each i, $w_i = 0$ or $1\}$. Then for any consistent* *expectation f, there exists a positive integer k such that $f^k(X_{\#I}) = D_0$.*

Proof. For any integer $n \geq 2$, define $X_n = \{(w,p) :$ there exist n distinct players $i_1, ..., i_n$ such that $1 > w_{i_1} > \frac{1}{2}$ and $w_{i_1} + \cdots + w_{i_n} = 1\}$. Let f be a consistent expectation such that $f^k = f^{k+1}$. Then we have that $X_2 \cap K_f = \varnothing$ because every state in X_2 is dominated in expectation by some state in $D_0 \subset K_f$. Suppose that for some integer $n \geq 2$, $X_n \cap K_f = \varnothing$, then we can show that $X_{n+1} \cap K_f = \varnothing$. For two distinct players i and j, let $(w', p') \in X_{n+1}$ and (w,p) satisfy that $w'_i > \frac{1}{2}$, $w'_j > 0$, $w_i = w'_i + w'_j$, $w_z = w'_z$ and $p_z = p'_z$ for each $z \notin \{i,j\}$, and $p_i = p_j = p'_j$. Then the state (w,p) is in X_n, and thus K_f does not contain (w,p). Since $w_i > \frac{1}{2}$, $f^k(w,p)$ satisfies that $f^k_w(w,p)_i \geq w_i$. Otherwise any change in (w,p) is not possible. Since $w_i > w'_i$, the state (w,p) dominates (w',p') in expectation, and so K_f does not contain (w',p'). Since (w',p') is arbitrary, we have that $X_{n+1} \cap K_f = \varnothing$. Consequently, we have that $X_{\#I} \cap K_f = \varnothing$. It is easily seen that if $(w'', p'') \in X_{\#I}$ and $w''_i > \frac{1}{2}$, then $f^k_w(w'', p'')_i \geq w''_i$. Therefore, we have that $f^k(X_{\#I}) = D_0$. ■

Lemma 35 (Lemma 3.10 in Jordan, 1999) *For some positive integer k, let w be a dyadic allocation such that for each i, if $w_i > 0$ then $w_i \geq 2^{-(k+1)}$. If an allocation w' satisfies that $\sum_{z \in \{i : w'_i > w_i\}} w_z > \sum_{z \in \{i : w'_i < w_i\}} w_z$, then there exists a dyadic allocation w'' such that $\sum_{z \in \{i : w''_i > w'_i\}} w'_z > \sum_{z \in \{i : w''_i < w'_i\}} w'_z$ and for each i, if $w''_i > 0$ then $w''_i \geq 2^k$.*

Proposition 36 shows that every farsighted core of a consistent expectation includes the set D^* of dyadic states at which one of players has a half of the wealth or the total wealth.

Proposition 36 *Let $D^* = \{(w,p) \in D :$ there exists player i with $w_i = 1$ or $\frac{1}{2}\}$. Then the farsighted supercore C_s includes D^*.*

Proof. [2]For any player i, define the set $X(i)$ of states by $X(i) = \{(w,p) \in X : w_i = \frac{1}{2}$ and for some region r, $\sum_{z \in p^r \setminus \{i\}} w_z = \frac{1}{2}\}$. Let f be a consistent expectation with $f^k = f^{k+1}$. We first show that $f^k(X(i)) \subset X(i)$. Let $(\dot{w}, \dot{p}) \in X(i)$. Note that only a winning coalition in expectation, $\{z : f^k_w(\dot{w}, \dot{p})_z > \dot{w}_z\}$, can emigrate to another region, that is, for some player z, if $f^k_p(\dot{w}, \dot{p})_z \neq \dot{p}_z$, then $f^k_w(\dot{w}, \dot{p})_z > \dot{w}_z$. By way of contradiction, suppose that $f^k_w(\dot{w}, \dot{p})_i > \frac{1}{2}$. Then by Lemma 34, $f^k(\dot{w}; \dot{p})_i = 1$. In this case, we have that $\sum_{z \in \{y : f^k_w(\dot{w}, \dot{p})_y > \dot{w}_y\}} \dot{w}_z = \sum_{z \in \{y : f^k_w(\dot{w}, \dot{p})_y < \dot{w}_y\}} \dot{w}_z$. Thus $f^k(\dot{w}, \dot{p})$ cannot dominate $(\dot{w}, \dot{p})$ in expectation because all players in the losing coalition in expectation, $\{z : f^k_w(\dot{w}, \dot{p})_z < \dot{w}_z\}$, are in a common region. Since $f^k_w(\dot{w}, \dot{p})_i < \frac{1}{2}$ is not possible, we must have that $f^k_w(\dot{w}, \dot{p})_i = \frac{1}{2}$. If there exists a region r such that $\sum_{z \in \dot{p}^r} f^k_w(\dot{w}, \dot{p})_z < \frac{1}{2}$, then $f^k_w(\dot{w}, \dot{p})$ is dominated in expectation by some state in $X_{\#I} = \{(w,p) :$ for some play z, $1 > w_z > \frac{1}{2}\}$ through player i pillaging players at

[2]The proof of Proposition 36 is similar to the proof of Theorem in Jordan (2005).

region r. Therefore, for some region r, we have that $\sum_{z\in f_p^k(\dot{w},\dot{p})^r\setminus\{i\}} f_w^k(\dot{w},\dot{p})_z = \frac{1}{2}$ and thus that $f^k(\dot{w},\dot{p}) \in X(i)$. Since $(\dot{w},\dot{p}) \in X(i)$ is arbitrary, we have that $f^k(X(i)) \subset X(i)$.

Next, we show that $D^* \subset K_f$ and thus that $D^* \subset C_s$. Suppose by way of contradiction that $D^* \cap X(i) \nsubseteq f^k(X(i))$. For any nonnegative integer n, define $D_n = \{(w,p) \in D : \text{for each } i,\ w_i = 0 \text{ or } w_i \geq 2^{-n}\}$. Note that $\bigcup_{n\in\mathbb{N}} D_n = D^*$. Thus there exists a positive integer m with $m = \min\{n : D_n \cap X(i) \nsubseteq f^k(X(i))\}$. Since $D_1 \subset C_S$, m is greater than 1. Let $(\dot{w},\dot{p})$ be in $(D_m \cap X(i)) \setminus f^k(X(i))$. Since $(\dot{w},\dot{p}) \notin f^k(X(i))$, we have that $(\dot{w},\dot{p}) \neq f^k(\dot{w},\dot{p})$. Since f is consistent and $(\dot{w},\dot{p}), f^k(\dot{w},\dot{p}) \in X(i)$, we have that $\sum_{z\in\{y\neq i: f_w^k(\dot{w},\dot{p})_y > \dot{w}_y\}} \dot{w}_z > \sum_{z\in\{y\neq i: f_w^k(\dot{w},\dot{p})_y < \dot{w}_y\}} \dot{w}_z$ and thus that $\sum_{z\in\{y\neq i: 2f_w^k(\dot{w},\dot{p})_y > 2\dot{w}_y\}} 2\dot{w}_z > \sum_{z\in\{y\neq i: 2f_w^k(\dot{w},\dot{p})_y < 2\dot{w}_y\}} 2\dot{w}_z$. Note that the allocation $\mathring{w}$ that consists of players $I \setminus \{i\}$ such that for each $z \neq i$, $\mathring{w}_z = 2\dot{w}_z$ is a dyadic allocation in $(\#I-1)-$player model. Therefore, by Lemma 35, there exists a dyadic allocation $\dot{w}'$ in $\#I-$player model such that $\dot{w}'_i = \frac{1}{2}$, $\sum_{z\in\{y\neq i: 2\dot{w}'_y > 2f_w^k(\dot{w},\dot{p})_y\}} 2f_w^k(\dot{w},\dot{p})_z > \sum_{z\in\{y\neq i: 2\dot{w}'_y < 2f_w^k(\dot{w},\dot{p})_y\}} 2f_w^k(\dot{w},\dot{p})_z$, and for each z, if $\dot{w}'_z > 0$ then $\dot{w}'_z \geq 2^{-(m-1)}$. Since $f^k(\dot{w},\dot{p}) \in X(i)$, there exists some region r such that $\sum_{z\in f_p^k(\dot{w},\dot{p})^r\setminus\{i\}} f_w^k(\dot{w},\dot{p})_z = \frac{1}{2}$. Let the distribution $\dot{p}'$ satisfy that for each z, if $\dot{w}'_z \neq f_w^k(\dot{w},\dot{p})_z$ then $\dot{p}'_z = r$, otherwise $\dot{p}'_z = f_p^k(\dot{w},\dot{p})_z$. If $(\dot{w}',\dot{p}') \in f^k(X(i))$, then $(\dot{w}',\dot{p}')$ dominates $f^k(\dot{w},\dot{p})$ in expectation. Therefore, we have that $(\dot{w}',\dot{p}') \notin f^k(X(i))$ and $(\dot{w}',\dot{p}') \in D_{m-1} \cap X(i)$. This contradicts the definition of m. Consequently, we have that for each i, $D^* \cap X(i) \subset f^k(X(i))$. Since $D_0 = (D^* \setminus X(i)) \subset K_f$, we have that $D^* \subset K_f$. Since f is an arbitrary consistent expectation, we have that $D^* \subset C_s$. ■

Theorem 37 shows that D is the unique farsighted core in a $I-$player and $N-$region model where $1 \leq I \leq 3$ or $N = 1$. Note that Jordan (2005) used *one-step expectation,* where every state reaches its stationary state within one step, and had the same result as Theorem 37 in one-region models. Therefore, the definition of expectation in Jordan (2005) can be generalized to *finite-step expectation,* where some states take finite steps, possibly more than one step, to reach their stationary states.

Theorem 37 *In a $I-$player and $N-$region model where $1 \leq I \leq 3$ or $N = 1$, D is the unique farsighted core of consistent expectations, and thus the farsighted supercore is D.*

Proof. By Theorem 33, D is a farsighted core of some consistent expectation. To show uniqueness, we assume that f is a consistent expectation. We first prove that $D = K_f$ in one-region models. For any state $(w,p) \notin K_f$, we have that $\sum_{z\in\{i: f_w^k(w,p)_i > w_i\}} w_z > \sum_{z\in\{i: f_w^k(w,p)_i < w_i\}} w_z$. For any states $(w',p'), (w'',p'') \in K_f$, we have that $\sum_{z\in\{i: w''_i > w'_i\}} w'_z \leq \sum_{z\in\{i: w''_i < w'_i\}} w'_z$. Therefore, K_f satisfies external stability and internal stability, that is, K_f is a stable set. Theorem 18 implies that $K_f = D$.

Let $1 \leq I \leq 3$. Then for any $(w,p) \in D$, there exists player i with $w_i = 1$ or $\frac{1}{2}$. Thus we have that $(w,p) \in D^* = \{(w,p) \in D : \text{there exists player } i \text{ with } w_i = 1 \text{ or}$

$\frac{1}{2}\}$. By Lemma 34, we have that $D = D^* \subset C_s \subset K_f$. To show that $K_f \subset D$, let $(w,p) \notin D$. If for some player i, $w_i > \frac{1}{2}$, then by Lemma 34, $f^k(w,p) \in D_0 \subset D$, and thus $(w,p) \notin K_f$. If for each player i, $w_i \leq \frac{1}{2}$, then (w,p) is dominated in expectation by some state in $D \subset K_f$ such that two players have halves through two players or one player pillaging another player. Thus we have that $(w,p) \notin K_f$ since f is consistent. Therefore, we have that $K_f = D$.

Since f is an arbitrary consistent expectation, D is the unique farsighted core. Therefore, the farsighted supercore is D in a $I-$player and $N-$region model where $1 \leq I \leq 3$ or $N = 1$. ■

Example 38 provides a consistent expectation f with $D \nsubseteq K_f$ and $K_f \nsubseteq D$, that is, a farsighted core of a consistent expectation might contain nondyadic states and rule out dyadic states. Therefore, Example 38 shows that Theorem 37 cannot be generalized.

Example 38 *In the five-player three-region model, there exists a consistent expectation f such that for some $\varepsilon \in (0, \frac{1}{12})$, $K_f = D \cup \{((\frac{1}{4}+\varepsilon, \frac{1}{4}+\varepsilon, \frac{1}{4}+\varepsilon, 0, \frac{1}{4}-3\varepsilon), (1,1,2,1,3))\} \setminus \{((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0), (1,1,1,1,2))\}$.*

Proof. Let $(w', p') = ((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0), (1,1,1,1,2))$. First, construct f such that $(w', p') \longrightarrow f(w', p') = ((3\varepsilon, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}-3\varepsilon), (3,1,1,1,3)) \longrightarrow f^2(w', p') = ((2\varepsilon, \frac{1}{4}, \frac{1}{4}+\varepsilon, \frac{1}{4}, \frac{1}{4}-3\varepsilon), (2,1,2,1,3)) \longrightarrow f^3(w', p') = ((\frac{1}{4}+\varepsilon, \frac{1}{4}+\varepsilon, \frac{1}{4}+\varepsilon, 0, \frac{1}{4}-3\varepsilon), (1,1,2,1,3))$. If $f^3(w', p') \in K_f$, then for each $k \in \{1,2,3\}$, $f^k(w', p')$ dominates $f^{k-1}(w', p')$ in expectation, and thus we have that $(w', p') \notin K_f$.

Next, we make the expectation f consistent such that $f^3(w', p') \in K_f$. Note that only a winning coalition in expectation who would be better off in the farsighted core can move to other regions. Therefore, it suffices to construct the consistent expectation f such that some of players who change their region at $f^3(w', p')$ wind up being worse off at the final state in K_f.

For three distinct players $i, j, k \in \{1,2,3\}$, let $X(i;5) = \{(w,p) : w_j = w_k = \frac{1}{4}+\varepsilon$, $p_j = f_p^3(w', p')_j$, $p_k = f_p^3(w', p')_k$, $p_i = p_5 = 3$, $p_4 = 3$ or 1, and $\sum_{z \in p^3} w_z = \frac{2}{4} - 2\varepsilon\}$. Construct the expectation f such that every state in $X(i;5)$ is dominated in expectation by the state in D^* at which players j and k have halves. Then we have made the expectation f such that no state in $X(i;5)$ dominates $f^3(w', p')$ in expectation because any state that results from only player i changing her region winds up the state at which player i has zero.

For three distinct players $i, j, k \in \{1,2,3\}$, let $X(i,j;5) = \{(w,p) : w_k = \frac{1}{4} + \varepsilon$, $p_k = f_p^3(w', p')_k$, $p_i = p_j = p_5 = 3$, $p_4 = 3$ or 1, and $\sum_{z \in p^3} w_z = \frac{3}{4} - \varepsilon\}$. If $(w,p) \in X(1,2;5)$ satisfies that for some player z, $w_z > \frac{1}{2}(\frac{3}{4} - \varepsilon)$, then we make (w,p) dominated in expectation by the state $(\dot{w}, p)$ with $\dot{w}_z = \frac{3}{4} - \varepsilon$ and $\dot{w}_3 = \frac{1}{4} + \varepsilon$. Then $(\dot{w}, p)$ is dominated in expectation by some state $(\dot{w}', p') \in D^*$ with $\dot{w}'_z = 1$. If $(w,p) \in X(1,2;5)$ satisfies that $\frac{1}{2}(\frac{3}{4} - \varepsilon) \geq \max\{w_z : z \neq 3\} > \frac{1}{4} - \varepsilon$, then we make (w,p) dominated in expectation by the state in D^* at which players 3 and $\min\{z : w_z > \frac{1}{4} - \varepsilon\}$ have halves. If $(w,p) \in X(1,2;5)$ satisfies that $\max\{w_z : z \neq 3\} \leq \frac{1}{4} - \varepsilon$, then we make (w,p) dominated in expectation by the state in D^* at which player 3 has a half and players z and y such that $z,y \neq 3$ and $w_z + w_y > \frac{1}{4} - \varepsilon$ have quarters. To embody the expectation f, choose players z and y such that $5z + y = \min\{5l + m :$ for two distinct players $l,m \neq 3$, $w_l + w_m > \frac{1}{4} - \varepsilon\}$, which is the first by lexicographic ordering. Similarly, we construct f such that no state in $X(1,3;5)$ or $X(2,3;5)$ dominates $f^3(w',p')$ in expectation.

For three distinct players $i,j,k \in \{1,2,3\}$, let $X(i,j;k) = \{(w,p) : w_5 = \frac{1}{4} - 3\varepsilon$, $p_5 = 3$, $p_i = p_j = p_k = f^3_p(w',p')_k$, $p_4 = 1$ or $f^3_p(w',p')_k$, and $\sum_{z \in p^{f^3_p(w',p')_k}} w_z = \frac{3}{4} + 3\varepsilon\}$. If $(w,p) \in X(1,2;3)$ satisfies that for some player z, $w_z > \frac{1}{2}(\frac{3}{4} + 3\varepsilon)$, then we make (w,p) dominated in expectation by the state $(\dot{w},\dot{p})$ with $\dot{w}_z = \frac{3}{4} + 3\varepsilon$. Then the state $(\dot{w}',\dot{p}') \in D^*$ with $\dot{w}'_z = 1$ dominates $(\dot{w},\dot{p})$ in expectation. If $(w,p) \in X(1,2;3)$ satisfies that $\frac{1}{2}(\frac{3}{4} + 3\varepsilon) \geq \max\{w_z : z \neq 5\} > \frac{1}{4} + 3\varepsilon$, then we make (w,p) dominated in expectation by the state in D^* at which players 5 and $\min\{z : w_z > \frac{1}{4} + 3\varepsilon\}$ have halves. If $(w,p) \in X(1,2;3)$ satisfies that $\#\{z : z \neq 5$ and $\frac{1}{4} + 3\varepsilon \geq w_z > \frac{1}{4}(\frac{3}{4} + 3\varepsilon)\} = 3$, then we make (w,p) dominated in expectation by the state $(\dot{w},\dot{p})$ such that players z and y who satisfy $5z + y = \min\{5l + m :$ for two distinct players $l,m \in \{1,3,4\}$, $w_l + w_m > \frac{1}{2}(\frac{3}{4} + 3\varepsilon)\}$ have $\frac{1}{2}(\frac{3}{4} + 3\varepsilon)s$ and make $(\dot{w},\dot{p})$ dominated in expectation by the state in D^* such that players z and y have halves. In this case, one of players 1 and 2 ends up being worse off. If $(w,p) \in X(1,2;3)$ satisfies that $\max\{w_z : z \neq 5\} \leq \frac{1}{4} + 3\varepsilon$ and $\#\{z : w_z > \frac{1}{4}(\frac{3}{4} + 3\varepsilon)\} \leq 2$, then we make (w,p) dominated in expectation by the state $(\dot{w},\dot{p})$ such that player i who has the least number among the wealthiest players at region 2, that is, $i = \min\{z \in p^2 :$ for any player $y \in p^2$, $w_z \geq w_y\}$, has $\frac{1}{2}(\frac{3}{4} + 3\varepsilon)$ and other two players z and y that satisfy $5z + y = \min\{5l + m : l,m \notin \{i,5\}$, $l \neq m$, and $w_l, w_m \leq \frac{1}{4}(\frac{3}{4} + 3\varepsilon)\}$ have $\frac{1}{4}(\frac{3}{4} + 3\varepsilon)s$. Then we make $(\dot{w},\dot{p})$ dominated in expectation by $(\dot{w}',\dot{p}')$ in D^* such that player i has a half and players z and y have quarters. Similarly, we construct f such that no state in $X(1,3;2)$ or $X(2,3;1)$ dominates $f^3(w',p')$ in expectation.

Finally, we construct the rest of the expectation f according to the way introduced in Theorem 33. Then we have that $K_f = D \cup \{f^3(w',p')\} \setminus \{(w',p')\}$, $(w',p') \in D$, and $f^3(w',p') \notin D$. Now, we only have to examine if f is consistent.

The expectation f is designed to satisfy that during the process of the expectation f, pillage can happen in each region at most once. If a player experiences pillage in his region, then he will never be pillaged during the rest of the process of f; that is, for any $(w,p) \in X \setminus K_f$ and any $i \in I$, if p_i satisfies that $\{z : f_w(w,p)_z \neq w_z\} \subset p^{p_i}$, then any positive integer k, $f^k_w(w,p)_i \geq f_w(w,p)_i$. It is easily seen that for any state $(w,p) \in X$ and any positive integer k, if $f^k(w,p) \neq f^{k-1}(w,p)$, then $i)$ there exists a region $r \in R$ such that $\{z : f^k_w(w,p)_z \neq f^{k-1}_w(w,p)_z\} \subset f^k_p(w,p)^r$; $ii)$ for all $q \neq r$, $f^k_p(w,p)^q = f^{k-1}_p(w,p)^q \setminus \{z : f^k_w(w,p)_z > f^{k-1}_w(w,p)_z\}$; and $iii)$ $\sum_{y \in \{z : f^k_w(w,p)_z > f^{k-1}_w(w,p)_z\}} f^{k-1}_w(w,p)_y > \sum_{y \in \{z : f^k_w(w,p)_z > f^{k-1}_w(w,p)_z\}} f^{k-1}_w(w,p)_y$. These show that $f(w,p)$ dominates (w,p) in expectation when $f(w,p) \neq (w,p)$ and (w,p) is undominated in expectation when $f(w,p) = (w,p)$. Therefore, f is consistent. ■

4 Suggestion for further research

Throughout this paper, we have assumed that regions are connected with one another and thus players can travel from one region to another in one move. The results based on this assumption are meaningful in that they give general understanding of how spatial restriction affects stable distribution of wealth. Also, for applications, when we consider that many countries, which could be regarded as regions, are surrounded by the sea and we can travel from one country to another through the sea, the assumption seems to be an approximation to reality.

However, in order to describe real situations more exactly, we can generate a general model where some regions are not connected and thus players cannot travel between these regions in one move. A **geography correspondence** G embodies the general models as follows.

Definition 39 *A **geography correspondence** is a correspondence $G : R \longrightarrow\!\!\longrightarrow R$ satisfying for any $r \in R$, i) $r \in G(r)$; ii) if $r' \in G(r)$ then $r \in G(r')$; and iii) there exists a positive integer k such that $G^k(r) = R$ where $G^k(r) = G^{k-1}(G(r))$.*

For any $r \in R$, $G(r)$ denotes the regions that players at region r can go to in one move. Condition *i*) means that players can stay in their regions. Condition *ii*) means that connections between two regions are bilateral. And condition *iii*) means that there is no separated region where players cannot travel. For example, we can define G as $G(1) = \{1, 2\}$, $G(2) = \{1, 2, 3\}$, and $G(3) = \{2, 3\}$, then G describes that three regions are located along a line.

The general model characterized by a geography correspondence is different from the previous model presented above in terms that weak players may be able to change states to defend themselves and thus some coalitions may not pillage less powerful coalitions. The following example shows how it works. Suppose that there are three players and five regions. Let G describe that five regions are located along a line, that is, $G(1) = \{1, 2\}, G(2) = \{1, 2, 3\}, ...,$ and $G(5) = \{4, 5\}$. Consider the state $(w, p) = ((\frac{4}{9}, \frac{1}{9}, \frac{4}{9}), (1, 3, 5))$, which expresses that players 1 and 3 have $\frac{4}{9}s$ while staying at region 1 and at region 5, respectively, and player 2 has $\frac{1}{9}$ while staying at region 3. At the state (w, p), player 2 can change the state to discourage another player who tried to pillage player 2. In cases that player 1 or players 1 and 3 approach player 2 to pillage, player 2 can move to region 4 to change the state so that player 3 can pillage player 2 alone. If player 3 alone pillages player 2, then player 3 has enough power to pillage player 1. Thus player 1 would not try to pillage player 2. In case that player 3 alone approaches player 2 to pillage, we can apply the same logic.

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