Suspension in a Global-Games version of the Diamond-Dybvig model

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Abstract

This work builds on the model in Goldstein and Pauzner (GP) (2005), a global-games version of the Diamond-Dybvig (DD) (1983) model in which there is uncertainty about the long-term return and in which agents observe noisy signals about that return. GP limited their investigation to a banking contract that makes a noncontingent promised payoff to those who withdraw early until the bank's resources are exhausted. We amend the contract and permit suspension. As we show, there is a class of suspension policies that gives rise to uniqueness without requiring the new assumption introduced in a proof in GP; namely, the short-term return is also random. In general, both the GP policy and my generalization of it to allow suspension seem not to be the best banking contracts. However, if the return uncertainty is sufficiently small, then there are policies in the class we study that imply ex ante welfare close to the first-best outcome in DD, which itself is an upper bound on welfare in the model with return uncertainty.

1 Introduction

This papers builds on the model in Goldstein and Pauzner (GP) (2005), a global-games version of the Diamond-Dybvig (DD) (1983) model in which

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there is uncertainty about the long-term return and in which agents observe noisy signals about that return.

GP limited their investigation to a banking contract that makes a non-contingent promised payoff to those who withdraw early until the bank's resources are exhausted. Here, that contract is amended to permit suspension. There are two reasons for this. First, suspension works perfectly in the no aggregate-risk version of the DD model. (It uniquely implements the first-best outcome.) Because versions of GP are close to that model, it is plausible that suspension would also work well in such versions. Second, as shown below, there is a class of suspension policies that gives rise to uniqueness without requiring the new assumption introduced in a proof in GP—namely, that the short-term return is also random.

In general, both the GP policy and the generalization of it that allows suspension seem not to be the best banking contracts in a model with return uncertainty and signals about it. Viewed as mechanisms, such contracts make no attempt to elicit information about the signals that agents receive and to make suspension and payoffs to at least some depositors contingent on that information. However, if the return uncertainty is sufficiently small, then there are policies in the class I study that imply ex ante welfare close to the first-best outcome in DD, which itself is an upper bound on the welfare in the model with return uncertainty. Moreover, for one such policy, noisy signals are not necessary for uniqueness. In other words, the ex ante welfare properties of the DD suspension policy are robust to the kind of uncertainty introduced by GP, provided it is small.

The virtue of such uncertainty is that in a long sequence of i.i.d. realizations of the model, all of the following four outcomes occur with positive probability: $\{good\ long-term\ return, poor\ long-term\ return\} \times \{suspension\ not\ invoked, suspension\ invoked\}$. Put differently, the model with a small amount of GP uncertainty combined with a banking contract that permits suspension gives a unique equilibrium, does well in terms of ex ante welfare, and is able to account for a rich history of banking-system outcomes, including ones with a variety of banking-system difficulties—difficulties that are rare, but do occur.

The remainder of the paper is organized as follows. The model is set out in Section 2. Section 3 has the uniqueness results. Section 4 deals with ex ante welfare. Section 5 contains the proofs, while section 6 offers concluding remarks.

2 Model

Our model is essentially GP, but without uncertainty about the short-term return. There are three dates, one good and a nonatomic unit measure of agents. Each agent is endowed with 1 unit of good at date 0. The agent becomes impatient with probability λ and the agent becomes patient with probability $1-\lambda$. Impatient agents can consume only at date 1. They derive utility $u(c_1)$ from date 1 consumption c_1. Patient agents can consume at both date 1 and date 2. They derive utility $u(c_1+c_2)$ from consumption bundle (c_1, c_2). Type is i.i.d. across agents and there is no aggregate uncertainty about type. The function $u(.)$ satisfies the following properties: it is twice continuously differentiable, increasing, strictly concave, and satisfies $u(0) = 0$, $u'(0) = \infty$, and $-cu''(c)/u'(c) > 1$[1].

A risky production technology is available to everyone. One unit of date-0 investment yields $R > 1$ units of date-2 output with probability $p(\theta)$ and 0 with probability $1-p(\theta)$, where θ, labeled the fundamental, is uniformly distributed over $[0,1]$ and $p : [0,1] \to [0,1]$ and is continuous, strictly increasing, and satisfies $p(0) = 0$ and $p(1) = 1$. If liquidated at date 1, the investment pays return 1. The technology has a higher long-term return in the sense that $E_\theta(p(\theta))u(R) > u(1)$.

Figure 1 shows two examples of possible $p(\theta)$ functions. Limiting cases of each, indicated by the arrows, serves different purposes later.

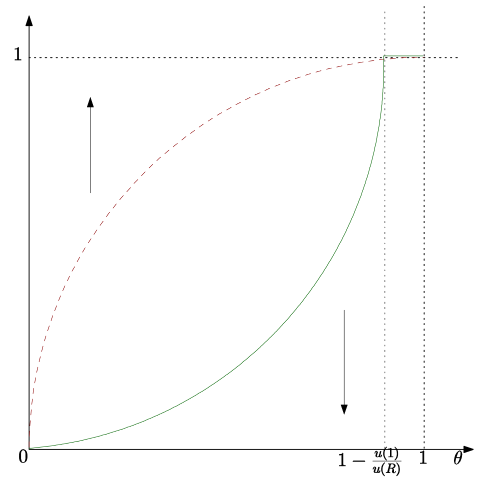

Figure 1: $p(\theta)$

[1] Although these assumptions, taken together, rule out CARA, they are innocuous in general.

The fundamental θ, which is not observed, determines a private signal received by each agent. The private signal is $\theta' = \theta + \varepsilon'$, where ε' is independently and uniformly distributed over $[-\varepsilon, \varepsilon]$. It follows that given signal θ', θ is uniformly distributed over $[\max(0, \theta' - \varepsilon), \min(1, \theta' + \varepsilon)]$[2].

The timeline is as follows. At date 0, the bank offers a deposit contract $(x, \delta) \in [1, +\infty) \times [\lambda, 1]$ with $x\delta \leq 1$. It gives each agent who is able to withdraw at date 1 x units of output per unit of deposit, while allowing no more than δ proportion of the agents to withdraw at date 1. For the moment, I assume that agents deposit in the bank and that all available resources are invested[3].

Just prior to date 1, the fundamental, the agent types and their signals are realized. Then agents simultaneously choose between withdraw and wait. The date-1 payoff from playing withdraw is

$$c_1 = \begin{cases} x \text{ with probability } \delta / \max(\delta, n) \\ 0 \text{ with probability } 1 - \delta / \max(\delta, n) \end{cases},$$

where n is the proportion who choose to withdraw. At date 2, the long-term return is realized. Each agent who played wait and each agent who played withdraw but received 0 at date 1 get the date 2 payoff

$$c_2 = \begin{cases} R[1 - x\min(\delta, n)]/[1 - \min(\delta, n)] \text{ if } R \text{ is realized} \\ 0 \text{ otherwise} \end{cases}.$$

Because each impatient agent necessarily plays withdraw, it is sufficient to focus on the play of the patient. I focus on symmetric strategies and allow randomization. Thus, a strategy is $y : [-\varepsilon, 1 + \varepsilon] \to [0, 1]$, where $y(\theta')$ is the probability that a patient agent with signal θ' plays withdraw. The solution concept is Nash equilibrium.

Definition 1 *The function y is a symmetric equilibrium if it is a best response to the play of y by all other agents.*

Following the Global Games literature, a threshold equilibrium plays an important role in what follows.

[2] Agents utilize the information $\theta \in [0, 1]$.

[3] There exists a contract that is consistent with a unique equilibrium and under which agents want to deposit rather than remain in autarky.

Definition 2 *The function y is a threshold equilibrium if there exists $\widehat{\theta}$ such that $y(\theta') = 1$ for $\theta' < \widehat{\theta}$ and $y(\theta') = 0$ for $\theta' > \widehat{\theta}$.*

In what follows, I distinguish between two kinds of threshold equilibrium. Following most of the literature, I reserve the term run for a threshold equilibrium in which $\widehat{\theta} \geq 1 + \varepsilon$ and in which, therefore, $y(\theta) \equiv 1$. Any other threshold equilibrium is called interior.

I suspect that condition 1 can be relaxed so that equilibrium $\widehat{\theta} \in [-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$ exists. But the probability of a run of such equilibrium is either ε or $1-\varepsilon$ and ε is assumed to be negligible.

3 Conditions for uniqueness and for multiplicity

Before discussing about uniquenss, conditions on (x, δ) are established so that an interior threshold equilibrium exists.

Proposition 1 (Existence of threshold) *If (x, δ) satisfies*

$$\begin{aligned} 0 \;<\; & \int_{\lambda}^{\delta} [u(\frac{1-xn}{1-n}R) - u(x)]dn \\ & + \int_{\delta}^{1} [u(\frac{1-x\delta}{1-\delta}R) - u(x)]\frac{\delta}{n}dn, \end{aligned} \tag{1}$$

there exists $\widetilde{\varepsilon}$, which can depend on (x, δ), such that if $\varepsilon \in (0, \widetilde{\varepsilon}]$, then an interior equilibrium with cut-off value $\widehat{\theta} \in [\varepsilon, 1-\varepsilon]$ exists.

In fact, condition (1) is also a necessary condition on (x, δ) for the existence of such equilibrium. I next turn to uniqueness. Several properties are necessary for Global Games uniqueness to hold in this model. The somewhat problematic property is an interval of the fundamental for which wait is a dominant strategy. GP get this interval by making the short-term return depend on the fundamental as follows: they assume that there exists $\hat{\theta} \in (0,1)$ such that if $\theta \geq \hat{\theta}$, then the date 1 return is R. I get this property from suspension contracts that reserve sufficient resources for date 2 payoffs.

Let

$$A = \{(x, \delta) | \frac{(1-x\delta)R}{1-\delta} > x\}, \tag{2}$$

which is displayed in Figure 2.

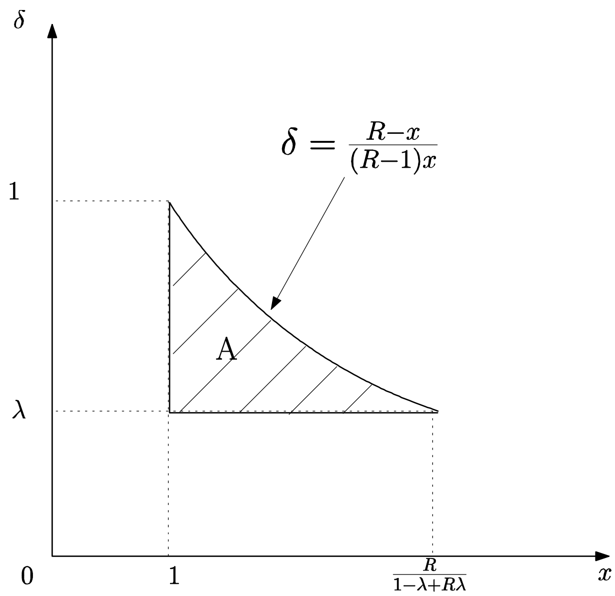

Figure 2: Set A

If the return R is realized, then the date-2 payoff is no less than $(1-x\delta)R/(1-\delta)$. Therefore, if the contract is in set A, then choosing wait is a dominant strategy for an agent who realizes a sufficiently high signal. This is sufficient to assure that the Global Games uniqueness result holds.

Proposition 2 (Uniqueness) *Let $(x, \delta) \in A$. There exists $\overline{\varepsilon}$, which can depend on (x, δ), such that if $\varepsilon \in (0, \overline{\varepsilon}]$, then there is a unique symmetric equilibrium and it is an interior threshold equilibrium.*

To avoid the dependence of $\overline{\varepsilon}$ on (x, δ) (in fact, $\overline{\varepsilon} \to 0$ as (x, δ) approaches the upper boundary of A), I can restrict contracts to a slightly smaller set.

Corollary 1 (ε independent of (x, δ)) *For any $k \in (0, R-1)$, let*

$$A_k = \{(x, \delta) | \frac{1 - x\delta}{1 - \delta} R > x + k\}. \tag{3}$$

There exists $\overline{\varepsilon}_k > 0$ such that for any $(x, \delta) \in A_k$, if $\varepsilon \in (0, \overline{\varepsilon}_k]$, then there is a unique equilibrium and it is an interior threshold equilibrium.

Contracts within A give rise to a unique equilibrium. Outside of A, the model always has a run equilibrium.

Proposition 3 (Existence of run) *If $(x, \delta) \notin A$, then a run equilibrium ($y(\theta') \equiv 1$) exists.*

In general, I am not sure about uniqueness outside of set A. However, if (x, δ) is near the upper boundary of A, then condition (1) is satisfied. That and Proposition 3 together imply multiplicity.

Corollary 2 (Existence of multiple equilibrium) *If $x > 1$, $\delta > \lambda$ and $(x, \delta) \notin A$ but near A, then multiple equilibria exist for a sufficiently small $\varepsilon > 0$.*

Contracts in A with $\delta = \lambda$ have a unique equilibrium even without noise; namely, even if $\varepsilon = 0$. All other contracts in A require noise for uniqueness.

Proposition 4 (Necessity of noise) *Suppose $\varepsilon = 0$. If $(x, \delta) \in A$ and $\delta > \lambda$, then multiple equilibria exist.*

4 Optimality

Although contracts in A have a unique equilibrium, in general, our contracts—and, hence, the GP contracts—seem not to be the best implementable arrangements. They preclude giving almost everyone some date 1 payoff when signals imply that 0 is a very likely date 2 return. In particular, our contract does not permit the bank to elicit information about the signals from agents and to make the suspension parameter and the date-1 payoff of at least some agents contingent on such information. Despite that, a bit can be said about ex ante (representative-agent) welfare under our class of contracts.

Contracts in A with $\delta = \lambda$ always have a unique equilibrium and it is tempting to focus on such contracts. The first result says that limiting the contracts further to those with $\delta = \lambda$ can be costly.

Proposition 5 (Desirability of $\delta > \lambda$) *There exist economies $(p(.), R)$ for which the best contract in A has $\delta > \lambda$.*

As shown in the proof, one way to get such economies is by using a limiting case of the $p(\theta)$ function depicted by the solid curve in Figure 1. For such $p(\theta)$ functions, the optimal contract within set A has $\delta > \lambda$ and, by way of proposition 4, has a unique equilibrium only in the presence of noise.

The economies in the last proposition have a lot of uncertainty and, therefore, are ones for which the gain from enriching the class of contracts is presumably large. If I interpret the DD model to have $p(\theta) \equiv 1$, then they are also economies that are far from the DD economy. In such economies, there seems to be a big gain from eliciting information from agents. I next turn to economies where such gain could be negligible. These economies are close to the DD economy in the sense that they have $p(\theta)$ functions that are close to $p(\theta) \equiv 1$ (see the dotted $p(\theta)$ function in Figure 1). Note that DD first-best welfare is an upper bound on ex ante welfare in our environment. The next result says that for economies close to DD in that sense, our contract has ex ante welfare close to that upper bound and, hence, is close to being ex ante optimal.

Let x^* be the first-best date-1 consumption in DD. The assumption $-cu''(c)/u'(c) > 1$ implies that $(1 - \lambda x^*)R/(1 - \lambda) - x^* > 0$. Hence, there exists $k \in (0, (1 - \lambda x^*)R/(1 - \lambda) - x^*)$. Consider a sequence of economies $\{p_l(.), \varepsilon_l\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} p_l(\theta) = 1$ for $\theta > 0$ and $\lim_{l\to\infty} \varepsilon_l \to 0$, where ε_l satisfies Corollary 1 for the function $p_l(\theta)$.

Proposition 6 (Perturbation of DD) *Let $k \in (0, (1 - \lambda x^*)R/(1 - \lambda) - x^*)$. For any $(x^*, \delta) \in A_k$ (see (3)), the limit of ex ante welfare is the first-best welfare in DD.*

Note that A_k contains the DD suspension contract (x^*, λ). Under this contract, I obtain uniqueness even if there is no noise—even if agents observe the fundamental directly. It follows that the DD suspension contract is robust to the introduction of a small amount of date-2 return uncertainty without the introduction of noisy signals.

5 Proofs

Before presenting the proofs, it is helpful to introduce additional notation and some preliminary lemmas. Lemmas 1-2 show that (x, δ) gives rise to the properties necessary for the Global Games results to hold in our environment. Lemma 3 applies the GP Global Games technique.

Following the Global Games literature, I consider a threshold strategy, denoted by its cut-off value $\widehat{\theta}$. At a threshold equilibrium, the proportion of agents who choose to withdraw depends on the fundamental and the cut-off value. The function of withdrawal is:

$$n(\theta,\widehat{\theta})=\begin{cases}1 & \text{if} \quad \theta\leq\widehat{\theta}-\varepsilon\\ \lambda & \text{if} \quad \theta>\widehat{\theta}+\varepsilon\\ 1-\frac{1-\lambda}{2\varepsilon}(\theta-\widehat{\theta}+\varepsilon) & \text{if} \quad \text{otherwise}\end{cases}. \tag{4}$$

Suppose the proportion of agents who choose withdraw is n and the fundamental is θ. If an agent plays wait, she receives expected utility $u([1-x\min(\delta,n)]/[1-\min(\delta,n)]R)p(\theta)$. If she plays withdraw, she receives x units of good at date 1 with probability $\delta/\max(\delta,n)$; otherwise she has to wait and withdraw at date 2. The function of the expected gain from choosing wait conditional on θ is

$$v(\theta,n,x,\delta)=\{u[\frac{1-x\min(\delta,n)}{1-\min(\delta,n)}R]p(\theta)-u(x)\}\frac{\delta}{\max(\delta,n)}. \tag{5}$$

The conditional distribution of θ, given $\theta'\in[\varepsilon,1-\varepsilon]$, is uniform over $[\theta'-\varepsilon,\theta'+\varepsilon]$, and thus the function of the conditional expected gain from playing wait is

$$U(\theta',\widehat{\theta},x,\delta)=\frac{1}{2\varepsilon}\int_{\theta'-\varepsilon}^{\theta'+\varepsilon}v(\theta,n(\theta,\widehat{\theta}),x,\delta)d\theta. \tag{6}$$

Global Games literature often makes use of monotonicity. But the function $v(\cdot,n(\cdot,\widehat{\theta}),x,\delta)$ here is not increasing in θ. Lemma 1 shows that this function crosses the horizontal axis at most once, and this is enough for uniqueness. Lemma 1 also gives a necessary and sufficient condition for $\widehat{\theta}$ to be a threshold equilibrium.

Lemma 1 *For any threshold $\widehat{\theta}$, $v(\theta,n(\theta,\widehat{\theta}),x,\delta)$ satisfies the single crossing property: there exists at most one θ such that $v(\theta,n(\theta,\widehat{\theta}),x,\delta)=0$.*

$U(\widehat{\theta},\widehat{\theta},x,\delta)$ is strictly increasing and continuous in $\widehat{\theta}\in[\varepsilon,1-\varepsilon]$.

If $\widehat{\theta}\in[\varepsilon,1-\varepsilon]$ satisies $U(\widehat{\theta},\widehat{\theta},x,\delta)=0$, then $\widehat{\theta}$ is a threshold equilibrium.

Proof. For any threshold strategy $\widehat{\theta}$, $n(\theta,\widehat{\theta})$ is decreasing in θ and p is strictly increasing in θ. Therefore

$$u[\frac{1-x\min(\delta,n(\theta,\widehat{\theta}))}{1-\min(\delta,n(\theta,\widehat{\theta}))}R]p(\theta)-u(x) \tag{7}$$

is strictly increasing in θ and is equal to 0 for at most one θ. Then $v(\theta,n(\theta,\widehat{\theta}),x,\delta)$ is equal to 0 for at most one θ.

$U(\widehat{\theta},\widehat{\theta},x,\delta)$ is is strictly increasing in $\widehat{\theta}\in[\varepsilon,1-\varepsilon]$, because when both the private signal and the threshold strategy increase by the same amount, an agent's belief about how many other agents withdraw is unchanged, but the return from waiting is higher.

$U(\widehat{\theta},\widehat{\theta},x,\delta)=0$ implies that $v(\theta,n(\theta,\widehat{\theta}),x,\delta)$ crosses zero at most once. Observing signal θ' below $\widehat{\theta}$ shifts the probability from positive values of v to negative values. Thus $U(\theta',\widehat{\theta},x,\delta)<U(\widehat{\theta},\widehat{\theta},x,\delta)=0$. Similarly, θ' above $\widehat{\theta}$ imply $U(\theta',\widehat{\theta},x,\delta)>U(\widehat{\theta},\widehat{\theta},x,\delta)=0$. ■

With the strict monotonicity of (7), it is not hard to see that $v(\cdot,n(\cdot,\widehat{\theta}),x,\delta)$ is equal to zero at most once. Also a higher signal θ' shifts the integral of U to the right. Thus $U(\widehat{\theta},\widehat{\theta},x,\delta)=0$ is enough to ensure that $\widehat{\theta}$ is an equilibrium. Then the only task is to guarantee the existence of such $\widehat{\theta}$. Lemma 2 provides a sufficient condition for that, under which wait is the dominant action for a large θ'. I obtain such property as a natural result of incorporating suspension into their contracts, while [4] assumes that the short-run return is also high with positive probability.

Lemma 2 *Define* $\underline{\theta}:A\to[0,1]$ *and* $\overline{\theta}:A\to[0,1]$*:*

$$\underline{\theta}(x,\delta) = p^{-1}\left(\frac{u(x)}{u(\frac{1-x\lambda}{1-\lambda}R)}\right) \tag{8}$$

$$\overline{\theta}(x,\delta) = p^{-1}\left(\frac{u(x)}{u(\frac{1-x\delta}{1-\delta}R)}\right), \tag{9}$$

Let $(x,\delta)\in A$*. The function* U *has two dominance intervals: if* $\theta'>\overline{\theta}(x,\delta)+\varepsilon$*, then wait is the dominant action; If* $\theta'<\underline{\theta}(x,\delta)-\varepsilon$*, then withdraw is the dominant action.*

Proof. Note that $\frac{1-x\delta}{1-\delta}R>x$; then $\underline{\theta}$ and $\overline{\theta}$ are well-defined. If $\theta'<\underline{\theta}(x,\delta)-\varepsilon$, then the fundamental $\theta<\underline{\theta}(x,\delta)$. Note that $u[\frac{1-xn}{1-n}R]p(\theta)-u(x)$ is strictly decreasing in n and strictly increasing in θ. Then we have

$$\begin{aligned} & u[\frac{1-x\min(\delta,n)}{1-\min(\delta,n)}R]p(\theta)-u(x) \\ < \ & u[\frac{1-x\lambda}{1-\lambda}R]p(\underline{\theta})-u(x)=0, \end{aligned} \tag{10}$$

which implies $v(\theta,n,x,\delta)<0$ independent of $n\in[\lambda,1]$. Hence if $\theta'\leq\underline{\theta}(x,\delta)-\varepsilon$, wait is the dominant action. The rest of the lemma follows similarly. ■

Contracts in Set A reserve enough resources for future withdrawals even when the others choose withdraw. Hence the upper dominance interval is obtained. Then equilibrium $\widehat{\theta}$ can found by the intermediate value theorem. In fact, the restriction on set A is strong so that $\widehat{\theta}$ is the unique equilibrium. Lemma 3 applies the GP technique and precludes other equilibrium possibilities.

Lemma 3 *Let* $(x, \delta) \in A$*. If* $\varepsilon \in (0, \overline{\varepsilon}(x, \delta)]$ *with*

$$\overline{\varepsilon}(x, \delta) = \min\{(1 - \overline{\theta}(x, \delta))/3, \underline{\theta}(x, \delta)/3, 1/4\}, \tag{11}$$

then any equilibrium is a threshold equilibrium.

Proof. Note that $\overline{\theta}(x, \delta)$, $\underline{\theta}(x, \delta) \in (0, 1)$; then, $\overline{\varepsilon}(x, \delta) > 0$ and $(0, \overline{\varepsilon}(x, \delta)]$ is well-defined. Suppose by contradiction that y is a non-threshold equilibrium. Since y is fixed throughout this proof, it is helpful to suppress the dependence of U and n on strategy y. Let $U(\theta', x, \delta)$ be the function of expected gain from wait and contract (x, δ). Let $n(\theta)$ be the proportion of those who choose withdraw. θ denotes the fundamental and θ' is the private signal.

Define

$$\theta'_B = \sup\{\theta' : U(\theta', x, \delta) < 0\}. \tag{12}$$

By Lemma 2, $[-\varepsilon, \underline{\theta}(x, \delta) - \varepsilon] \subseteq \{\theta' : U(\theta', x, \delta) < 0\}$ and hence θ'_B is well-defined. Furthermore, $[\overline{\theta}(x, \delta) + \varepsilon, 1 + \varepsilon] \cap \{\theta' : U(\theta', x, \delta) < 0\} = \emptyset$, which implies $\theta'_B \leq \overline{\theta}(x, \delta) + \varepsilon \leq 1 - 2\varepsilon$.

Define

$$\theta'_A = \sup\{\theta' < \theta'_B : U(\theta', x, \delta) > 0\}. \tag{13}$$

Since y is not a threshold equilibrium, there must exist $\theta' < \theta'_B$ such that $U(\theta', x, \delta) > 0$, and then θ'_A is well defined. By Lemma 2, $[\overline{\theta}(x, \delta) + \varepsilon, 1 + \varepsilon] \cap \{\theta' < \theta'_B : U(\theta', x, \delta) > 0\} = \emptyset$ and hence $\theta'_A \geq \underline{\theta}(x, \delta) - \varepsilon \geq 2\varepsilon$.

Continuity of U implies

$$U(\theta'_A, x, \delta) = U(\theta'_B, x, \delta) = 0. \tag{14}$$

Conditional on signal θ'_A, which belongs to $[2\varepsilon, 1 - 2\varepsilon]$, the fundamental θ is uniformly distributed over $[\theta'_A - \varepsilon, \theta'_A + \varepsilon]$. Similarly, conditional on θ'_B, θ is uniform over $[\theta'_B - \varepsilon, \theta'_B + \varepsilon]$. Define $\theta'_1 = \max\{\theta'_B - \varepsilon, \theta'_A + \varepsilon\}$, $\overleftarrow{\theta'_1} = \min\{\theta'_B - \varepsilon, \theta'_A + \varepsilon\}$, $\overleftarrow{\theta'_2} = \theta'_A - \varepsilon$ and $\theta'_2 = \theta'_B + \varepsilon$. Then $[\theta'_A - \varepsilon, \theta'_A + \varepsilon]$

and $[\theta'_B - \varepsilon, \theta'_B + \varepsilon]$ can be decomposed into three intervals: $[\theta'_1, \overleftarrow{\theta'_1}]$, $[\theta'_1, \theta'_2]$ and $[\overleftarrow{\theta'_2}, \overleftarrow{\theta'_1}]$. Define the mirror image transformation, $\overleftarrow{\theta}:[\theta'_1, \theta'_2] \to [\overleftarrow{\theta'_2}, \overleftarrow{\theta'_1}]$:

$$\overleftarrow{\theta} = \theta'_A + \theta'_B - \theta. \tag{15}$$

Then (14) implies

$$\int_{\theta'_1}^{\theta'_2} v(\theta, n(\theta), x, \delta) d\theta = \int_{\theta'_1}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) d\theta^4. \tag{16}$$

In what follows, I only consider the situation with $\theta'_B - \theta'_A \in (\varepsilon, 2\varepsilon)$. However the proof can be generalized without difficulty.

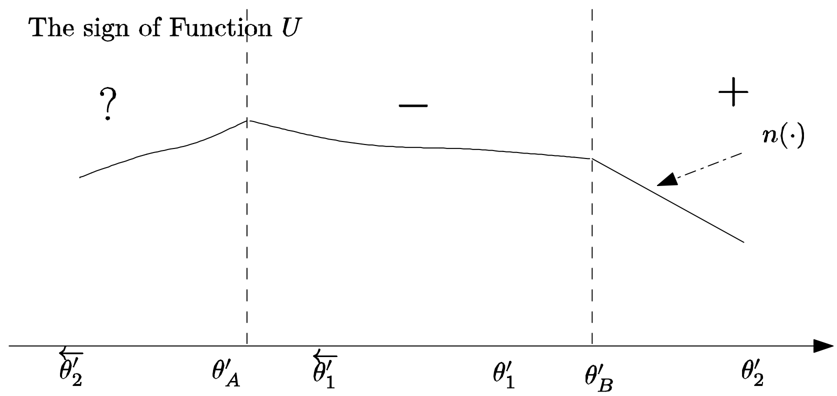

Figure 3

Let $\theta(n)$ be the inverse of the restriction of $n(\theta)$ to $[\theta'_1, \theta'_2]$. Define the "inverse" of $n(\overleftarrow{\theta})$: $\overrightarrow{\theta}(n) = \inf\{\theta : n(\overleftarrow{\theta}) < n \text{ or } \theta = \theta'_2\}$. The following claim provides the properties of these functions.

Claim 1 *$n(\theta)$ decreases over $[\theta'_1, \theta'_2]$ at the rate of $(1-\lambda)/2\varepsilon$; $n(\theta)$ increases over $[\overleftarrow{\theta'_2}, \overleftarrow{\theta'_1}]$ at a rate less than $(1-\lambda)/2\varepsilon$; For any $\theta' \in [\overleftarrow{\theta'_1}, \theta'_1]$, $n(\theta') \geq n(\theta'_1)$; $n(\theta'_2) \leq n(\overleftarrow{\theta'_2}) \leq n(\theta'_1)$; For all $\theta' \in [\theta'_1, \theta'_2]$, $\overleftarrow{\theta'} < \theta'_1$ and $n(\overleftarrow{\theta'}) \geq n(\overleftarrow{\theta'_2})$; $\theta(n)$ is strictly decreasing; $n(\overleftrightarrow{\theta}(n)) = n$; $\overrightarrow{\theta}(n)$ is strictly decreasing.*

Proof of Claim 1. *(12) and (13) imply that $U(\theta', x, \delta) < 0$ for $\theta' \in (\theta'_A, \theta'_B)$ and that $U(\theta', x, \delta) > 0$ for $\theta' > \theta'_B$. As the fundamental θ increases from θ'_1 to θ'_2, patient agents who observe signals $\theta' \in [\theta'_1 + \varepsilon, \theta'_2 + \varepsilon]$*

[4] I suppress the dependence of n on strategy y in function $n(.,.)$.

gradually replace patient agents who observe signals $\theta' \in [\theta'_A, \theta'_B]$. Hence $n(.)$ decreases in $[\theta'_1, \theta'_2]$ at the rate of $(1-\lambda)/2\varepsilon$. Note that the sign of $U(\theta', x, \delta)$ in $[\theta'_A - \varepsilon, \theta'_A]$ is undetermined, as indicated in Figure 3. Then, as the fundamental θ decreases from $\overleftarrow{\theta'_1}$ to $\overleftarrow{\theta'_2}$, patient agents who observe signals $\theta' \in [\theta'_A, \theta'_B]$ and who thus choose withdraw, are gradually replaced by patient agents who observe $[\theta'_A - \varepsilon, \theta'_A]$ and who may or may not choose wait. Hence $n(.)$ increases in $[\overleftarrow{\theta'_2}, \overleftarrow{\theta'_1}]$ at a rate less than $(1-\lambda)/2\varepsilon$. As θ moves from $\overleftarrow{\theta'_1}$ to θ'_1, patient agents, who observe signals $\theta' \in [\overleftarrow{\theta'_1} - \varepsilon, \theta'_A]$ and who thus may or may not choose wait, are gradually replaced by patient agents who observe $[\theta'_B, \theta'_1 + \varepsilon]$ and choose wait. Then for any $\theta' \in [\overleftarrow{\theta'_1}, \theta'_1]$, $n(\theta') \geq n(\theta'_1)$. Suppose by contradiction $n(\overleftarrow{\theta'_2}) > n(\theta'_1)$; then, I have $n(\theta') > n(\theta'_1)$ for all $\theta' \in [\overleftarrow{\theta'_2}, \overleftarrow{\theta'_1}]$, and hence (16) does not hold, a contradiction. Because $n(\theta)$ is strictly decreasing in $[\theta'_1, \theta'_2]$, $\theta(n)$ is strictly decreasing. Note that $n(\overleftarrow{\theta})$ is decreasing in $[\theta'_1, \theta'_2]$. Then, $\overrightarrow{\theta}(n)$ is strictly decreasing by its definition. ■

The LHS of (16) can be rewritten as

$$\int_{\theta'_1}^{\theta'_2} v(\theta, n(\theta), x, \delta) d\theta = \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\theta(n), n, x, \delta) d\theta(n) + \int_{n(\overleftarrow{\theta'_2})}^{n(\theta'_2)} v(\theta(n), n, x, \delta) d\theta(n). \tag{17}$$

Because $\overleftarrow{\overrightarrow{\theta}(n)} \leq \theta'_1 < \theta(n)$ for all $n \in [n(\overleftarrow{\theta'_2}), n(\theta'_1))$ and $v(\theta, n, x, \delta)$ is strictly increasing in θ, I have

$$\int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\theta(n), n, x, \delta) d\theta(n) > \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\overleftarrow{\overrightarrow{\theta}(n)}, n, x, \delta) d\theta(n). \tag{18}$$

The RHS of (16) can be rewritten as

$$\begin{aligned}
&\int_{\theta'_1}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) d\theta \\
= &\int_{\theta'_1}^{\overrightarrow{\theta}(n(\theta'_1))} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) d\theta + \int_{\overrightarrow{\theta}(n(\theta'_1))}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) A(\theta) d\theta \\
&+ \int_{\overrightarrow{\theta}(n(\theta'_1))}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta)(1 - A(\theta)) d\theta,
\end{aligned} \tag{19}$$

where

$$A(\theta) = \begin{cases} 1 \quad \text{if} \quad \frac{\partial n(\overleftarrow{\theta})}{\partial \theta} \ \exists \text{ and } < 0 \\ 0 \qquad\qquad \text{otherwise} \end{cases} . \tag{20}$$

If $\overrightarrow{\theta}(n)$ is differentiable with $\frac{\partial \overrightarrow{\theta}(n)}{\partial n} < 0$ at n, then, by definition of $\overrightarrow{\theta}(n)$, $n(\overleftarrow{\theta})$ is also differentiable with $\frac{\partial n(\overleftarrow{\theta})}{\partial \theta} < 0$ at $\overrightarrow{\theta}(n)$. Note that $\overrightarrow{\theta}(n)$ is strictly decreasing and hence is almost everywhere differentiable with $\frac{\partial \overrightarrow{\theta}(n)}{\partial n} < 0$. Thus I have $A(\overrightarrow{\theta}(n)) = 1$ almost everywhere. Then the second integral in (19) becomes

$$\begin{aligned} & \int_{\overrightarrow{\theta}(n(\theta'_1))}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) A(\theta) d\theta \\ = & \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\overleftarrow{\overrightarrow{\theta}(n)}, n, x, \delta) A(\overrightarrow{\theta}(n)) d[\overrightarrow{\theta}(n) - \theta(n)] \\ & + \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\overleftarrow{\overrightarrow{\theta}(n)}, n, x, \delta) d\theta(n). \end{aligned} \tag{21}$$

Combining (18), (17), (19), and (21), I have

$$\begin{aligned} & \int_{n(\overleftarrow{\theta'_2})}^{n(\theta'_2)} v(\theta(n), n, x, \delta) d\theta(n) < \int_{\theta'_1}^{\overrightarrow{\theta}(n(\theta'_1))} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) d\theta \\ & + \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\overleftarrow{\overrightarrow{\theta}(n)}, n, x, \delta) A(\overrightarrow{\theta}(n)) d[\overrightarrow{\theta}(n) - \theta(n)] \\ & + \int_{\overrightarrow{\theta}(n(\theta'_1))}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta)(1 - A(\theta)) d\theta. \end{aligned} \tag{22}$$

I will reach a contradiction against (22) in what follows.

The integrals on the RHS of (22) have the same length as the integral on the LHS of (22):

$$\begin{aligned}
&\int_{\theta_1'}^{\overrightarrow{\theta}(n(\theta_1'))} d\theta + \int_{n(\theta_1')}^{n(\overleftarrow{\theta_2'})} A(\overrightarrow{\theta}(n))d[\overrightarrow{\theta}(n) - \theta(n)] + \int_{\overrightarrow{\theta}(n(\theta_1'))}^{\theta_2'} (1 - A(\theta))d\theta \\
= &\int_{\theta_1'}^{\theta_2'} d\theta - \int_{\overrightarrow{\theta}(n(\theta_1'))}^{\theta_2'} A(\theta)d\theta \\
&+ \int_{n(\theta_1')}^{n(\overleftarrow{\theta_2'})} A(\overrightarrow{\theta}(n))d\overrightarrow{\theta}(n) - \int_{n(\theta_1')}^{n(\overleftarrow{\theta_2'})} A(\overrightarrow{\theta}(n))d\theta(n) \\
= &\int_{\theta_1'}^{\theta_2'} d\theta - \int_{n(\theta_1')}^{n(\overleftarrow{\theta_2'})} A(\overrightarrow{\theta}(n))d\theta(n) \\
= &\int_{n(\overleftarrow{\theta_2'})}^{n(\theta_2')} d\theta(n).
\end{aligned} \tag{23}$$

Integration by substitution and $\overrightarrow{\theta}(n(\overleftarrow{\theta_2'})) = \theta_2'$ imply

$$\int_{\overrightarrow{\theta}(n(\theta_1))}^{\theta_2'} A(\theta)d\theta = \int_{n(\theta_1)}^{n(\overleftarrow{\theta_2'})} A(\overrightarrow{\theta}(n))d\overrightarrow{\theta}(n),$$

and hence the second equality of (23). $\int_{\theta_1'}^{\theta_2'} d\theta = \int_{n(\theta_1')}^{n(\theta_2')} d\theta(n)$ and $A(\overrightarrow{\theta}(n)) = 1$ almost everywhere imply the third equality of (23).

Because, by claim 1, $n(\overleftarrow{\theta_2'}) \geq n(\theta_2')$ and $\theta(n)$ is decreasing in n, the LHS of (22) can be treated as a regular integral with nonnegative weight on value $v(\theta(n), n, x, \delta)$. Because, by claim 1, $n(\theta_1') \geq n(\overleftarrow{\theta_2'})$ and $\overrightarrow{\theta}(n) - \theta(n)$ is decreasing in n, the second integral on the RHS of (22) is a regular integral with nonnegative weights on value $v(\overleftarrow{\overrightarrow{\theta}(n), n, x, \delta})$. $n(\theta_1') \geq n(\overleftarrow{\theta_2'})$ implies $\overrightarrow{\theta}(n(\theta_1')) < \theta_2'$. Hence the third integral on the RHS of (22) is a regular integral with nonnegative weight on value $v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta)$. A similar statement holds for the first integral. Therefore, the weights on the value of v on both sides of (22) are nonnegative.

The rest of the argument is divided into two cases: $v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta) \geq 0$ and $v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta) < 0$. Suppose $v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta) \geq 0$. Because $v(\theta, n, x, \delta)$ is strictly increasing in θ and strictly decreasing in n when $v(\theta, n, x, \delta) \geq 0$, we have condition 1: $v(\theta(n), n, x, \delta) > v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta)$

for all $n \in [n(\theta'_2), n(\overleftarrow{\theta'_2}))$[5]. In other words, the function v within the integral on the LHS of (22) is bounded below by $v(\theta(n(\overleftarrow{\theta'_2})), n(\overleftarrow{\theta'_2}), x, \delta)$.

Note that $\overleftarrow{\theta'} < \theta'_1 < \theta(n)$ and $n(\overleftarrow{\theta'}) \geq n(\overleftarrow{\theta'_2})$ for all $\theta' \in [\theta'_1, \theta'_2]$ and $n \in (n(\theta'_2), n(\theta'_1)]$. Therefore we have condition 2: $v(\overleftarrow{\theta'}, n(\overleftarrow{\theta'}), x, \delta) < v(\theta(n(\overleftarrow{\theta'_2})), n(\overleftarrow{\theta'_2}), x, \delta)$ for $\theta' \in [\theta'_1, \theta'_2)$, and $v(\overleftarrow{\overleftrightarrow{\theta}(n)}, n, x, \delta) < v(\theta(n(\overleftarrow{\theta'_2})), n(\overleftarrow{\theta'_2}), x, \delta)$ for $n \in (n(\overleftarrow{\theta'_2}), n(\theta'_1)]$. The function v within the integral on the RHS of (22) is bounded above by $v(\theta(n(\overleftarrow{\theta'_2})), n(\overleftarrow{\theta'_2}), x, \delta)$.

Then conditions 1-2 imply

$$
\begin{aligned}
& \int_{n(\overleftarrow{\theta'_2})}^{n(\theta'_2)} v(\theta(n), n, x, \delta) d\theta(n) \\
> \; & v(\theta(n(\overleftarrow{\theta'_2})), n(\overleftarrow{\theta'_2}), x, \delta) \int_{n(\overleftarrow{\theta'_2})}^{n(\theta'_2)} d\theta(n) \\
= \; & v(\theta(n(\overleftarrow{\theta'_2})), n(\overleftarrow{\theta'_2}), x, \delta) \{ \int_{\theta'_1}^{\overrightarrow{\theta}(n(\theta'_1))} d\theta \\
& + \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} A(\overrightarrow{\theta}(n)) d[\overrightarrow{\theta}(n) - \theta(n)] \\
& + \int_{\overrightarrow{\theta}(n(\theta'_1))}^{\theta'_2} (1 - A(\theta)) d\theta \} \\
> \; & \int_{\theta'_1}^{\overrightarrow{\theta}(n(\theta'_1))} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta) d\theta \\
& + \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\overleftarrow{\overleftrightarrow{\theta}(n)}, n, x, \delta) A(\overrightarrow{\theta}(n)) d[\overrightarrow{\theta}(n) - \theta(n)] \\
& + \int_{\overrightarrow{\theta}(n(\theta'_1))}^{\theta'_2} v(\overleftarrow{\theta}, n(\overleftarrow{\theta}), x, \delta)(1 - A(\theta)) d\theta, \qquad (24)
\end{aligned}
$$

which contradicts (22).

[5] If $n(\overleftarrow{\theta'_2}) = n(\theta'_2)$, both sides of inequality (22) are equal to zero, a contradiction. Therefore $n(\overleftarrow{\theta'_2}) > n(\theta'_2)$, which implies that $[n(\theta'_2), n(\overleftarrow{\theta'_2}))$ is well-defined.

Suppose $v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta) < 0$, which implies

$$u[\frac{1 - x\min(\delta, n(\overleftarrow{\theta_2'}))}{1 - \min(\delta, n(\overleftarrow{\theta_2'}))}R]p(\theta(n(\overleftarrow{\theta_2'}))) - u(x) < 0.$$

Because $\overleftarrow{\theta'} < \theta_1' < \theta(n)$ and $n(\overleftarrow{\theta'}) \geq n(\overleftarrow{\theta_2'})$ for all $\theta' \in [\theta_1', \theta_2']$ and $n \in (n(\theta_2'), n(\theta_1')]$, I have

$$\begin{aligned} & u[\frac{1 - x\min(\delta, n(\overleftarrow{\theta'}))}{1 - \min(\delta, n(\overleftarrow{\theta'}))}R]p(\overleftarrow{\theta'}) - u(x) \\ < \; & u[\frac{1 - x\min(\delta, n(\overleftarrow{\theta_2'}))}{1 - \min(\delta, n(\overleftarrow{\theta_2'}))}R]p(\theta(n(\overleftarrow{\theta_2'}))) - u(x), \end{aligned}$$

which implies $v(\overleftarrow{\theta'}, n(\overleftarrow{\theta'}), x, \delta) < 0$ for $\theta' \in [\theta_1', \theta_2']$ and hence $v(\overleftarrow{\overrightarrow{\theta}(n)}, n, x, \delta) < 0$ for $n \in [n(\overleftarrow{\theta_2'}), n(\theta_1')]$. Then the RHS of (22) is strictly negative, which implies

$$\int_{n(\overleftarrow{\theta_2'})}^{n(\theta_2')} v(\theta(n), n, x, \delta) d\theta(n) < 0. \tag{25}$$

Also, $v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta) < 0$ implies

$$\int_{n(\theta_1')}^{n(\overleftarrow{\theta_2'})} v(\theta(n), n, x, \delta) d\theta(n) < 0. \tag{26}$$

If $\overleftarrow{\theta_1'} \leq \theta_1'$, by claim 1, I have for $\theta \in [\overleftarrow{\theta_1'}, \theta_1']$, $n(\theta) \geq n(\theta_1') \geq n(\overleftarrow{\theta_2'})$ and $\theta < \theta(n(\overleftarrow{\theta_2'}))$, which implies

$$\int_c v(\theta, n(\theta), x, \delta) d\theta \leq \int_c v(\theta(n(\overleftarrow{\theta_2'})), n(\overleftarrow{\theta_2'}), x, \delta) d\theta \leq 0. \tag{27}$$

If $\overleftarrow{\theta'_1} > \theta'_1$, (27) also holds. Then (25), (26) and (27) imply

$$\begin{aligned}
& U(\theta_B, x, \delta) \\
= & \int_c v(\theta, n(\theta), x, \delta) d\theta \\
& + \int_{n(\theta'_1)}^{n(\overleftarrow{\theta'_2})} v(\theta(n), n, x, \delta) d\theta(n) \\
& + \int_{n(\overleftarrow{\theta'_2})}^{n(\theta'_2)} v(\theta(n), n, x, \delta) d\theta(n) \\
< & \ 0,
\end{aligned} \tag{28}$$

which contradicts (14). ■

Lemma 3 gives an upper bound on ε so that only threshold equilibria exist. The following proof finds a different upper bound so that in a set (see (1)) larger than A, there exists a threshold equilibrium. Both conclusions hold for a common ε interval.

Proof of Prop 1. Since p is continuous at 1, $\widetilde{\varepsilon}$ can be found so that if $\varepsilon \in (0, \widetilde{\varepsilon}]$, the following is satisfied:

$$\begin{aligned}
0 < & \int_\lambda^\delta \{u[\frac{1 - xn}{1 - n} R] p(1 - 2\varepsilon) - u(x)\} dn \\
+ & \int_\delta^1 \{u[\frac{1 - x\delta}{1 - \delta} R] p(1 - 2\varepsilon) - u(x)\} \frac{\delta}{n} dn.
\end{aligned} \tag{29}$$

It is not hard to see

$$\begin{aligned}
U(\widehat{\theta}, \widehat{\theta}, x, \delta) & = \frac{1}{2\varepsilon} \int_{\overline{\theta} - \varepsilon}^{\overline{\theta} + \varepsilon} v(\theta, n(\theta, \overline{\theta}), x, \delta) d\theta \\
& = \int_\lambda^\delta \{u[\frac{1 - xn}{1 - n} R] p(\theta(n)) - u(x)\} dn \\
& \quad + \int_\delta^1 \{u[\frac{1 - x\delta}{1 - \delta} R] p(\theta(n)) - u(x)\} \frac{\delta}{n} dn \\
& > \int_\lambda^\delta \{u[\frac{1 - xn}{1 - n} R] p(\widehat{\theta} - \varepsilon) - u(x)\} dn \\
& \quad + \int_\delta^1 \{u[\frac{1 - x\delta}{1 - \delta} R] p(\widehat{\theta} - \varepsilon) - u(x)\} \frac{\delta}{n} dn,
\end{aligned} \tag{30}$$

where $\theta(n)$ is the inverse of the restriction of $n(.)$ to $[\widehat{\theta}-\varepsilon, \widehat{\theta}+\varepsilon]$. Then, I have $U(1-\varepsilon, 1-\varepsilon, x, \delta) > 0$, combining (29) and the inequality in (30). I consider a contract with $x \geq 1$. Then $U(\varepsilon, \varepsilon, x, \delta) < 0$ for a sufficiently small ε. Then, intermediate value theorem implies the existence of $\widehat{\theta} \in [\varepsilon, 1-\varepsilon]$ so that $U(\widehat{\theta}, \widehat{\theta}, x, \delta) = 0$. Then by lemma 1, $\widehat{\theta}$ is an equilibrium. ■

For a sufficiently large θ', the above proof only makes sure that wait is the best response when half of the impatient agents choose withdraw, while wait is the dominant action in the Proof of Prop 2. This is consistent with the fact that (1) is less restrictive than Set A.

Proof of Prop 2. By lemma 2, $U(\widehat{\theta}, \widehat{\theta}, x, \delta) > 0$ if $\widehat{\theta} = \overline{\theta}(x, \delta) + \varepsilon$; $U(\widehat{\theta}, \widehat{\theta}, x, \delta) < 0$ if $\widehat{\theta} = \underline{\theta}(x, \delta) - \varepsilon$. By lemma 1, $U(\widehat{\theta}, \widehat{\theta}, x, \delta)$ is continuous and strictly increasing in $\widehat{\theta} \in [\varepsilon, 1-\varepsilon]$. Let $\overline{\varepsilon}$ satisfy (11) and thus $[\underline{\theta}(x, \delta) - \varepsilon, \overline{\theta}(x, \delta) + \varepsilon]$ is a subset of $[\varepsilon, 1-\varepsilon]$. Then, by the intermediate value theorem, there exists a unique $\widehat{\theta} \in [\underline{\theta}(x, \delta) - \varepsilon, \overline{\theta}(x, \delta) + \varepsilon]$ such that $U(\widehat{\theta}, \widehat{\theta}, x, \delta) = 0$. Lemma 1 implies that $\widehat{\theta}$ is a threshold equilibrium. The model does not have other threshold equilibria because $\widehat{\theta}$ such that $U(\widehat{\theta}, \widehat{\theta}, x, \delta) = 0$ is unique. Lemma 3 implies that there are no other types of equilibrium. ■

In the above, $\overline{\varepsilon}$ approaches 0 as (x, δ) approaches the upper boundary of A. When I fix the contract (x, δ) and consider perturbation on the model parameters including ε, it will be convenient to have $\overline{\varepsilon}$ independent of (x, δ). Corollary 1 serves this purpose. It finds a uniform bound by keeping (x, δ) away from the upper boundary.

Proof of Coro 1. Let $\underline{\theta}$ and $\overline{\theta}$ satisfy (8) and (9); then, we have

$$\underline{\theta}(x, \delta) \equiv p^{-1}\left(\frac{u(x)}{u(\frac{1-\lambda x}{1-\lambda}R)}\right) \geq p^{-1}\left(\frac{u(1)}{u(R)}\right) > 0 \tag{31}$$

$$\overline{\theta}(x, \delta) \equiv p^{-1}\left(\frac{u(x)}{u(\frac{1-\delta x}{1-\delta}R)}\right) \leq p^{-1}\left(\frac{u(x)}{u(x+k)}\right) \leq p^{-1}\left(\frac{u(R-k)}{u(R)}\right) < 1. \tag{32}$$

The first inequality of (31) holds because $x \geq 1$. Because $(x, \delta) \in A_k$, and hence $\frac{1-\delta x}{1-\delta}R > x+k$, the first inequality of (32) holds. The second inequality holds because $x + k \leq R$.

Let

$$\overline{\varepsilon}_k^* = \min\{(1 - p^{-1}(\frac{u(R-k)}{u(R)}))/3, p^{-1}(\frac{u(1)}{u(R)})/3, 1/4\}, \tag{33}$$

(31)-(33) imply

$$0 < \overline{\varepsilon}_k^* \leq \overline{\varepsilon}(x, \delta) \tag{34}$$

for all $(x, \delta) \in A_k$. By Proposition 2, when $0 < \varepsilon < \overline{\varepsilon}_k^*$, any $(x, \delta) \in A_k$ gives rise to uniqueness. ■

So far, suspensions in A appear important in that they are the only contracts generating the upper dominance interval. If the upper dominance interval is eliminated, ignoring signals becomes possible and a run always occurs in one equilibrium as it appears in [1]. In this sense, contracts outside of A are vulnerable to run, as shown in the following Proof.

Proof of Prop 3. Consider any $(x, \delta) \notin A$ so that

$$\frac{1 - x\delta}{1 - \delta} R \leq x. \tag{35}$$

If $y \equiv 1$ is adopted, then $n(\theta, 1 + \varepsilon) \equiv 1$ and

$$v(\theta, n(\theta, 1 + \varepsilon), x, \delta) \equiv \{u[\frac{1 - x\delta}{1 - \delta} R]p(\theta) - u(x)\}\delta < 0. \tag{36}$$

Then wait is the best response to $y \equiv 1$. Thus $y \equiv 1$ is an equilibrium. ■

The following is another example in the global game literature, where noisy signal is necessary for uniqueness. Contracts in A with $\delta > \lambda$ are vulnerable to run in the sense that all agents could coordinate perfectly to choose withdraw whenever the signal is outside the upper dominance interval. In this case, coordination failure leads to efficiency loss (note that the inequality in (37) still holds for θ less than but sufficiently close to $\overline{\theta}$).

Proof of Prop 4. Note that $v(1, 1, x, \delta) > 0$ and $v(0, 1, x, \delta) < 0$. By the intermediate value theorem, there exists $\overline{\theta} \in (0, 1)$ so that $v(\overline{\theta}, 1, x, \delta) = 0$. Similarly, $\underline{\theta} \in (0, 1)$ exists so that $v(\underline{\theta}, \lambda, x, \delta) = 0$.

$\delta > \lambda$ implies

$$0 = v(\underline{\theta}, \lambda, x, \delta) = v(\overline{\theta}, 1, x, \delta) < v(\overline{\theta}, \lambda, x, \delta), \tag{37}$$

where the inequality holds because $v(\theta, n, x, \delta)$ is strictly decreasing in n when it is nonnegative. (37) implies $\underline{\theta} < \overline{\theta}$. Hence $(\underline{\theta}, \overline{\theta})$ is nonempty. For any $\theta \in (\underline{\theta}, \overline{\theta})$,

$$v(\theta, 1, x, \delta) < v(\overline{\theta}, 1, x, \delta) = 0 = v(\underline{\theta}, \lambda, x, \delta) < v(\theta, \lambda, x, \delta), \tag{38}$$

where the inequalities hold because $v(\theta, n, x, \delta)$ is strictly increasing in θ.

For signal $\theta \in (\underline{\theta}, \overline{\theta})$, if other patient agents play wait, it is the best response for each patient agent to play wait. There exists an equilibrium y

such that $y(\theta) = 0$ for any $\theta \in (\underline{\theta}, \overline{\theta})$. Similarly, there exists an equilibrium y' such that $y'(\theta) = 1$ for any $\theta \in (\underline{\theta}, \overline{\theta})$. ■

$\delta > \lambda$ allows the payoff to depend on others' actions. Hence a run can occur as a result of coordination on the perfect signal. One way to eliminate such a channel is to set $\delta = \lambda$. The following constructs an example where setting $\delta = \lambda$ is too restrictive. A run is desirable only if the long-run return turns out to be low. Such an event occurs with high probability in the following example. Hence $\delta = \lambda$ will eliminate a desirable run with high probability.

Proof of Prop 5. It does no harm to assume that R is sufficiently big that $[Ru(1)]/u(R) > 1$. Construct a sequence $\{p_l(.)\}$ such that $E_\theta(p_l(\theta))u(R) > u(x)$, $\lim_{l\to\infty} p_l(\theta) = 1$ if $\theta \geq 1 - \frac{u(1)}{u(R)}$, and $\lim_{l\to\infty} p_l(\theta) = 0$ otherwise[6].

If the fundamental is $\theta < p_l^{-1}(\frac{u(1)}{u(R)}) - 2\varepsilon$, then all agents observe signals $\theta' < p_l^{-1}(\frac{u(1)}{u(R)}) - \varepsilon$ and know $\theta < p_l^{-1}(\frac{u(1)}{u(R)})$. They choose withdraw because

$$u(\frac{1 - x\lambda}{1 - \lambda}R)p_l(\theta) < \frac{u(1)}{u(R)}u(\frac{1 - x\lambda}{1 - \lambda}R) \leq u(x), \tag{39}$$

where the second inequality holds because $x \geq 1$ and hence $\frac{1-x\lambda}{1-\lambda}R < R$. Therefore, withdraw is the best response for all agents under any $(x, \delta) \in A$.

If $\theta \geq 1 - \frac{u(1)}{u(R)} + 2\varepsilon$, all agents observe signals $\theta' \geq 1 - \frac{u(1)}{u(R)} + \varepsilon$ and know that the fundamental $\theta \geq 1 - \frac{u(1)}{u(R)}$. Let $k \in (0, R - 1)$. There exists l^* such that if $l > l^*$, (40) holds for any $(x, \delta) \in A_k$. In other words, under any $(x, \delta) \in A_k$, they choose wait when the fundamental is $\theta \geq 1 - \frac{u(1)}{u(R)} + 2\varepsilon$.

$$u(\frac{1 - x\lambda}{1 - \lambda}R)p_l(\theta) > u(x) \tag{40}$$

[6]Such a sequence exists if p_l converges to 1 when $\theta \geq 1 - \frac{u(1)}{u(R)}$ faster than it does to 0 when $\theta < 1 - \frac{u(1)}{u(R)}$ so that p_l is productive in the long-run for all l.

For any $(x, \delta) \in A$, if the above strategy is assumed, then

$$
\begin{aligned}
& \lim_{l\to\infty,\varepsilon\to 0} | \int_{1-\frac{u(1)}{u(R)}+2\varepsilon}^{1} \{\lambda u(x) + (1-\lambda)u(\frac{1-x\lambda}{1-\lambda}R)p_l(\theta)\}d\theta \\
& + \int_{p_l^{-1}(\frac{u(1)}{u(R)})-2\varepsilon}^{1-\frac{u(1)}{u(R)}+2\varepsilon} \{\min(n,\delta)u(x) \\
& + [1-\min(n,\delta) - \lambda \max(0, 1-\frac{\delta}{n})]u(\frac{1-\overline{x}\lambda}{1-\lambda}R)p_l(\theta)\}d\theta \\
& + \int_{0}^{p_l^{-1}(\frac{u(1)}{u(R)})-2\varepsilon} \{\delta u(x) + (1-\delta)(1-\lambda)u(\frac{1-x\delta}{1-\delta}R)p_l(\theta)\}d\theta \\
& - [\lambda u(x) + (1-\lambda)u(\frac{1-x\lambda}{1-\lambda}R)]\frac{u(1)}{u(R)} \\
& - \delta u(x)(1-\frac{u(1)}{u(R)})| \\
= \; & \lim_{l\to\infty,\varepsilon\to 0} | -2\varepsilon\lambda u(x) + \delta u(x)[p_l^{-1}(\frac{u(1)}{u(R)}) - 2\varepsilon - (1-\frac{u(1)}{u(R)})] \\
& (1-\lambda)u(\frac{1-x\lambda}{1-\lambda}R)[\int_{1-\frac{u(1)}{u(R)}+2\varepsilon}^{1} p_l(\theta)d\theta - \frac{u(1)}{u(R)}] \\
& (1-\delta)(1-\lambda)u(\frac{1-x\delta}{1-\delta}R) \int_{0}^{p_l^{-1}(\frac{u(1)}{u(R)})-2\varepsilon} p_l(\theta)d\theta \\
& + \int_{p_l^{-1}(\frac{u(1)}{u(R)})-2\varepsilon}^{1-\frac{u(1)}{u(R)}+2\varepsilon} \{\min(n,\delta)u(x) \\
& + [1-\min(n,\delta) - \lambda \max(0, 1-\frac{\delta}{n})]u(\frac{1-\overline{x}\lambda}{1-\lambda}R)p_l(\theta)\}d\theta| \\
= \; & 0, \qquad (41)
\end{aligned}
$$

where the convergence is uniform[7].

Consider any contract $(x, \lambda) \in A \backslash A_k$ When the fundamental is $\theta \geq 1 - \frac{u(1)}{u(R)} + 2\varepsilon$, wait may or may not be the best response for all agents. However the above welfare of (x, λ), assuming that all agents choose wait, is an upper

[7] For $\forall \epsilon > 0$, there exist $l^* > 0 \& \; \varepsilon^* > 0$ that are independent of (x, δ) such that if $l > l^*$ and $\varepsilon \in (0, \varepsilon^*)$, then the difference between the welfare of (x, δ) and its limit is less than ϵ for all $(x, \delta) \in A$.

bound on the equilibrium welfare. The partial derivative of the limiting welfare with respect to x at $(R/(1-\lambda+\lambda R),\lambda)$ is

$$\lambda u'(R/(1-\lambda+\lambda R)) - \lambda u'(R/(1-\lambda+\lambda R))\frac{Ru(1)}{u(R)} < 0. \tag{42}$$

For a sufficiently small k, (x,λ) is close to $(R/(1-\lambda+\lambda R),\lambda)$, and hence (42) approximates the partial derivative at (x,λ). Then, there exist $(\overline{x},\lambda) \in A_k$ and $\rho > 0$ so that the difference between the limiting welfares of $(\overline{x},\lambda)$ and (x',λ) is greater than ρ for all $(x',\lambda) \in A\backslash A_k$. Given the convergence in (41), there exist $l' > 0$ and $\varepsilon' > 0$ so that if $l > l'$ and $\varepsilon' > \varepsilon > 0$, the welfare of $(\overline{x},\lambda)$ is strictly higher than that of any $(x',\lambda) \in A\backslash A_k$. Note that $(\overline{x},\lambda) \in A_k$ has the best response consistant with the first equality of (41), and the welfare is achieved in equilibrium.

Consider contracts in A_k. There exist $\delta' > \lambda$ and $k' < k$ so that $(x,\delta') \in A_{k'}$ for any x such that $(x,\lambda) \in A_k$. The difference between the limiting welfares of (x,δ') and (x,λ) is greater than $(\delta'-\lambda)u(1)[1-u(1)/u(R)] > 0$ by the first equality of (41). Then, there exist $l'' > 0$ and $\varepsilon'' > 0$ so that if $l > l''$ and $\varepsilon'' > \varepsilon > 0$, the welfare of (x,δ') is strictly higher than that of (x,λ) for any x such that $(x,\lambda) \in A_k$.

Note that k is chosen arbitrarily. Then for such a k', there exist l'' so that when $l > l''$, (x,δ') has best response consistant with the first equality of (41).

Overall, when l is sufficiently large and $\varepsilon > 0$ is sufficiently small, for each $(x,\lambda) \in A$, there is a contract in A with suspension parameter δ' that has a higher welfare than (x,λ) in equilibrium. ■

The above proof constructs an example by letting the p-curve approach the horizontal axis in Figure 1. The following lets the curve approach the horizontal line $p \equiv 1$. The production technology is a version of DD technology with small uncertainty and noisy signals. The equilibrium approaches the DD suspension equilibrium in the sense that the probability of a run approaches zero.

Proof of Prop 6. For each l, there exists a unique threshold equilibrium and denote its cut-off value by $\widehat{\theta}_l$. The proof of proposition 2 implies $\widehat{\theta}_l \in [\varepsilon_l^*, 1-\varepsilon_l^*]$. The fundamental θ is uniform distributed over $[\widehat{\theta}_l - \varepsilon_l^*, \widehat{\theta}_l + \varepsilon_l^*]$

given the signal $\widehat{\theta}_l$. Then

$$\begin{aligned}
0 &= \lim_{l\to\infty} U(\widehat{\theta}_l, \widehat{\theta}_l, x, \delta) \\
&= [\lim_{l\to\infty} p_l(\widehat{\theta}_l)](\int_{\lambda}^{\delta} u[\frac{1-xn}{1-n}R]dn + \int_{\delta}^{1} \{u[\frac{1-x\delta}{1-\delta}R]\frac{\delta}{n}dn) \\
&\quad - \int_{\lambda}^{\delta} u(x)dn - \int_{\delta}^{1} u(x)\frac{\delta}{n}dn, \qquad (43)
\end{aligned}$$

where integration by substitution implies the second equality of (43). Then $\lim_{l\to\infty} p_l(\widehat{\theta}_l)$ exists and belongs to $(0,1)$, which implies $\lim_{l\to\infty} \widehat{\theta}_l = 0$ because $\lim_{l\to\infty} p_l(\theta) = 1$ for all $\theta \in (0,1]$. Also note that $\lim_{l\to\infty} \varepsilon_l^* = 0$, which implies

$$\lim_{l\to+\infty} \int_{\widehat{\theta}_l+\varepsilon_l^*}^{1} (p_l(\theta)-1)d\theta = 0. \qquad (44)$$

Then the limiting welfare of (x^*, δ) in the perturbed model is equal to the first-best welfare in DD:

$$\begin{aligned}
&\lim_{l\to\infty}\{\int_{\widehat{\theta}_l+\varepsilon_l^*}^{1} [\lambda u(x^*) + (1-\lambda)u(\frac{1-x^*\lambda}{1-\lambda}R)p_l(\theta)]d\theta \\
&+ \int_{0}^{\widehat{\theta}_l-\varepsilon_l^*} [\delta u(x^*) + (1-\lambda)(1-\delta)u(\frac{1-x^*\delta}{1-\delta}R)p_l(\theta)]d\theta \\
&+ \int_{\widehat{\theta}_l-\varepsilon_l^*}^{\widehat{\theta}_l+\varepsilon_l^*} [\frac{\delta}{\max(n(\theta,\widehat{\theta}_l),\delta)}\lambda u(x^*) \\
&+u[\frac{1-x^*\min(\delta, n(\theta,\widehat{\theta}_l))}{1-\min(\delta, n(\theta,\widehat{\theta}_l))}R]p_l(\theta)[(1-\lambda) - \frac{(n(\theta,\widehat{\theta}_l)-\lambda)\delta}{\max(n(\theta,\widehat{\theta}_l),\delta)}] \\
&+u(x^*)\frac{(n(\theta,\widehat{\theta}_l)-\lambda)\delta}{\max(\delta, n(\theta,\widehat{\theta}_l))}]d\theta\} \\
&-\lambda u(x^*) - (1-\lambda)u(\frac{1-x^*\lambda}{1-\lambda}R) \\
=\ & \lim_{l\to\infty}\{\int_{\widehat{\theta}_l+\varepsilon_l^*}^{1} (1-\lambda)u(\frac{1-x^*\lambda}{1-\lambda}R)(p_l(\theta)-1)d\theta\} \\
=\ & 0. \qquad (45)
\end{aligned}$$

Because $\lim_{l\to\infty} \widehat{\theta}_l = 0$ and $\lim_{l\to\infty} \varepsilon_l^* = 0$, the first equality of (45) holds. The second equality of (45) follows from (44). ■

6 Concluding remarks

Suspension is widely used during financial crises and has attracted attention in the literature. This paper introduces another potential role of suspension with respect to Global Games literature. Suspension generates the upper interval as an endogenous result, while previous studies directly assume such property[8]. In this paper, by reserving enough resources for future withdrawals, suspension is sufficient for uniqueness without the new GP assumption that appears for the first time in their proof.

The optimality of suspension in DD is robust to the introduction of a small amount of uncertainty and a small amount of noise. However in general both the GP policy and my generalization of it seem not to be optimal. Such contracts do not attempt to elicit information received by agents and to condition payoffs and the suspension parameter on that information. The potential gain from such an attempt would be substantial, especially when the model has a lot of return uncertainty. Therefore, extending the contract to allow for eliciting information would be interesting.

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