Why Ten $1'sAreNotTreatedasa$**10**

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Why Ten \$1's Are Not Treated as a \$10.

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Abstract

We study the stability of monetary steady states in a random matching model of money where money is indivisible, the bound on individual money holding is finite, and the trading protocol is buyer take-it-or-leave-it offers. The class of steady states we study have a non-full-support money-holding distribution and are constructed from the steady states of Zhu (2003). We show that no equilibrium path converges to such steady states if the initial distribution has a different support.
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1 Introduction

Trejos and Wright (1995) show the existence of a monetary steady state in a model where an agent's money holding is in $\{0, 1\}$. For buyer take-it-or-leave-it offers in that model but money holdings in $\{0, \Delta, 2\Delta, ..., B\Delta\}$, Zhu (2003) provides sufficient conditions for the existence of a steady state with a full-support money-holding distribution and a strictly increasing and strictly

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concave value function. This paper is concerned with the stability of a class non-full support steady states.

In Zhu's model, there are three exogenous nominal quantities: $(\Delta, B\Delta, m)$, where m is the per capita stock of money. If, for some positive integer l, we compare that economy to an otherwise identical economy with nominal quantities $(l\Delta, lB\Delta, lm)$, then we have neutrality. But what if we compare $(\Delta, B\Delta, m)$ to $(\Delta, lB\Delta, lm)$? As Zhu shows, any steady state for $(\Delta, B\Delta, m)$ is also a steady state for $(\Delta, lB\Delta, lm)$, one in which all owned/traded quantities of money are multiplied by l and the value function is a step function with jumps at and only at integer multiples of an l-bundle. We call this class of non-full-support steady states l-replica. Here we show that such a steady state is unstable if the support of the initial distribution differs from $\{0, l\Delta, 2l\Delta, ..., Bl\Delta\}$. In particular, if the economy starts with a positive measure of people holding what we call *change*, then there is no equilibrium that converges to a monetary steady state that is identical to that for economy $(\Delta, B\Delta, m)$.

Our result reinforces that in Wallace and Zhu (2004), who show for the same model that a commodity-money refinement rules out l-replica steady states. Both their result and ours are consistent with the observation that ten one-dollar bills have more roles than just being a perfect substitute for a ten-dollar bill.[1]

Depending on whether agents are indifferent between different trades, there are two kinds of Zhu steady states: those with a pure strategy and those with a mixed strategy. [2] gives an example where both kinds are generic. One can construct l-replicas from both kinds. Our instability result holds for l-replicas of pure-strategy steady state. The extension to cover the two kinds of steady states is not trivial. We have no conclusion about stability properties of l-replicas of mixed-strategy steady state.

2 Model

The model is that in Zhu (2003). Time is discrete, dated as $t \geq 0$. There is a non-atomic unit measure of infinitely-lived agents. There is a consumption good that is perfectly divisible and perishable. Each agent maximizes the discounted sum of expected utility with discount factor $\beta \in (0, 1)$. Utility

[1] The model in Green and Zhou (2002) also has a multiplicity of steady states. However, their model is very different, as is their stability result.

in a period is $u(c) - q$, where $c \in \mathbb{R}_+$ is the amount of good consumed and $q \in \mathbb{R}_+$ is the amount of good produced. $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing and strictly concave, and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, $u'(0)$ is sufficiently large but finite.

There is a fixed stock m of intrinsically useless money that is perfectly storable. Money is indivisible and the size of the smallest unit is normalized to one. Each agent can hold no more than $B \in \mathbb{N}$ units of money and it is assumed that $m \in (0, B)$. Let $\mathbb{B} = \{0, 1, \cdots, B\}$ denote the set of possible individual money holdings.

In each period, agents are randomly matched in pairs. With probability $1/N$, where $N \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meeting. With probability $1 - 2/N$, the match is a no-coincidence meeting.[2] In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

In a single-coincidence meeting between a consumer with i units of money and a producer with j units of money, an (i, j)-meeting, the consumer makes a take-it-or-leave-it offer consisting of the amount to be produced, q, and the amount of money to be paid, p. The offer must be feasible, $0 \leq p \leq \min\{i, B - j\}\}$, and must satisfy the producer's participation constraint, $-q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}$, where w_k^t is the expected discounted value of holding $k \in \mathbb{B}$ units of money, prior to date-t matching. Because the optimal offer leaves no positive gain to the producer, the consumer's problem reduces to choosing p in the set of optimal offers of money

$$p^t(i, j, w^{t+1}) = \underset{0 \leq p \leq \min\{i, B-j\}\}}{\operatorname{argmax}} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{1}$$

Because $p^t(i, j, w^{t+1})$ is discrete and may be multi-valued, randomizations over the elements of $p^t(i, j, w^{t+1})$ are allowed. Let $\lambda^t(p; i, j)$ be the probability that consumers with i (pre-trade) in meetings with producers with j offer p at date t. It has support in $p^t(i, j, w^{t+1})$ at the equilibrium,

$$\sum_{p \in p^t(i, j, w^{t+1})} \lambda^t(p; i, j) = 1. \tag{2}$$

[2] One foundation is that there are N types of agents and N types of consumption goods, that type-n agents can produce type-n goods only and consume type-$(n + 1)$ goods only, and that the money is symmetrically distributed across the types.

Let π_k^t denote the fraction of agents holding k units of money prior to date-t matching. The law of motion is

$$\begin{aligned}\pi_k^{t+1} = \pi_k^t \quad &+ \quad \frac{1}{N} \sum_{\{i,j|i>k\}} \pi_i^t \pi_j^t \lambda^t(i-k; i,j) \\ &+ \quad \frac{1}{N} \sum_{\{i,j|j<k\}} \pi_i^t \pi_j^t \lambda^t(k-j; i,j) \\ &- \quad \frac{1}{N} \sum_j \pi_k^t \pi_j^t \sum_{p>0} \lambda^t(p;k,j) \\ &- \quad \frac{1}{N} \sum_i \pi_i^t \pi_k^t \sum_{p>0} \lambda^t(p;i,k).\end{aligned} \tag{3}$$

The Bellman equation is

$$w_i^t = \frac{N-1}{N} \beta w_i^{t+1} + \frac{1}{N} \sum_{j=0}^{B} \pi_j^t \sum_p \lambda^t(p;i,j) \left\{ u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1} \right\}. \tag{4}$$

The first term on the r.h.s. corresponds to entering a no-coincidence meeting or becoming a producer who is indifferent between accepting and rejecting. Free disposal of money is allowed, so the value function must satisfy

$$w_i^t \geq w_{i-1}^t, \text{ for } i = 1, \cdots B, \text{ and } w_0^t = 0. \tag{5}$$

Definition 1 *Given π^0, an* equilibrium *is a sequence $\{(\lambda^t, \pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies (1)-(5). A* monetary steady state *is (λ, π, w) with $w \neq 0$ such that $(\lambda^t, \pi^t, w^t) = (\lambda, \pi, w)$ for all t is an equilibrium. Pure-strategy steady states are those for which (1) has a unique solution for all meetings. Other steady states are called mixed-strategy steady states. A Zhu steady state is a steady state for which π has a full support and w is strictly increasing and strictly concave.*[3]

This differs from Zhu's definition of equilibrium in that trades are an explicit part of the definition. We include the trades because we want stability to include the convergence of trades. Later, we will be doing a proof. In such a proof, the greater the number of objects that have to converge, the

[3]See the existence result of such steady state in [9].

easier it is to produce such a proof. And we do not view such a requirement as strong because the trades are what people do and are what an outside observer would see. Given that trades is required to be convergent along the equilbrium path, the only candidate for the limit is one of the strategies that support l-replica as a steady state. Thus, the description of optimal trades at the steady state, even for those meetings that occur with zero probability, will be provided in Lemma 1.

The above definition divides full-support steady states into two types. [2] investigates the $B = 2$ economy. Along with other discussion about these two types, it finds that the existence of a Zhu pure-strategy steady state is generic. Our main proposition will apply to non-full-support steady states that are associated with some pure-strategy steady state.

Call a bundle of l units of money a "*bundle*" and any amount less than l "*change.*" In what follows, non-prime letters (i, j, etc.) indicate numbers of "*bundles,*" and letters with primes (i', $j' \in \mathbb{L}$, etc.) indicate units of "*change.*" Replacing each unit of money in a monetary steady state by a bundle while keeping the smallest unit of money unchanged gives the following.

Definition 2 *Let $s = (\hat{\lambda}, \hat{\pi}, \hat{w})$ be a Zhu pure-strategy steady state of (B, m) economy. For integer $l \geq 2$, an l-replica of s, denoted $s(l) = (\lambda^*, \pi^*, w^*)$, is a steady state of (Bl, ml) economy that satisfies (6) and (7):*

$$\pi^*_{il} = \hat{\pi}_i, \quad \textit{and} \quad \pi^*_{il+i'} = 0, \ \ \forall i' \in \mathbb{L}, \tag{6}$$

$$w^*_{il} = \hat{w}_i, \quad \textit{and} \quad w^*_{il+i'} = w^*_{il}, \ \ \forall i' \in \mathbb{L}, \tag{7}$$

with $\mathbb{L} \equiv \{1, \cdots, l-1\}$.

Equation (6) implies that an l-neutral replica has a non-full support, and equation (7) implies that the value of money has a step function form, so only a *bundle* has value. Equation (6) and (7) together imply that trading bundles replicates trading money in the associated monetary steady state. Because s is pure-strategy, randomization doe not occur in trading bundles.

The definition allows randomization in trading change both at the steady state and in its vicinity. Because the space of such randomization is continuous, there are a continuum of l-replicas, that have the same π and w but different λ, for each full-support steady state. Our main proposition are going to rule out any equilibrium path convergent to any of the associated l-replicas.

Definition 2 defines l-replica only for a pure-strategy steady state with full-support and a strictly increasing and strictly concave value function. And [2] provides an example where pure-strategy steady states are generic. Our argument only apply to l-replicas associated with pure-strategy steady state. We also assume strict concavity and it can be dropped in the main result. But this will only complicate the description about λ^* (See Lemma 1) and generality is not our sole purpose. The strict increasing value function does not appear restrictive and it guarantees that holding an extra unit of money (bundle) is strictly preferred at the steady state (l-replica). (See the strict positive matrix K in Lemma 3)

Lemma 1 shows that there can be a steady state corresponding to an l-replica, a steady state with the features that define an l-replica. For such the l-replica, the following has some instability result when s has pure strategy.

Proposition 1 *If $s(l)$ is an l-replica, and if π^0 has a support different from that of the l-replica, then there is no equilibrium that converges to $s(l)$.*

In other words, if the initial distribution π^0 has $\pi^0_{il+i'} > 0$ for some $i' \in \mathbb{L}$ and some $i \in \{0, 1, \cdots, B-1\}$, then the economy cannot reach the l-neutral replica steady state.

The standard approach to stability analysis of difference equation systems (see, for example, [5]) is to compare the number of eigenvalues of the dynamic system that are strictly smaller than one in absolute value, say a, and the number of initial conditions, say b. If $a = b$ ($a > b$), then there is a unique (an infinity of) convergent path(s). If $a < b$, then there is no convergent solution. Our analysis must go beyond those results for two reasons. First, our dynamic system necessarily involves *unit roots convergence*. Second, the l-replica steady state is on the *boundary* of the state space in two senses: π^* does not have full support and w^* is not strictly increasing. It is necessary to ensure that sign restriction (5) holds all along the path.

3 A simple example

We start with an example in which $B = 1$ (i.e., Trejos-Wright) and $l = 2$. This example is simple for several reasons. The Trejos-Wright steady state is pure-strategy and the trade is easy to describe. Therefore, it is easy to describe the trades in the $l = 2$ replica. In addition, the aggregate state of the $l = 2$ replica is one-dimensional and can be described by the fraction of people with one unit of money. Despite its simplicity, it captures some of the main ingredients of our stability analysis. The proof is by way of contradiction and we rule out any potential convergent path by showing that either the consumer's optimality conditions or no-disposal of money is violated in the limit.

Following Trejos-Wright, we assume that (8) has a positive solution. Because (λ^*, π^*, w^*) is identical to the Trejos-Wright steady state, we have $\lambda^*(2; 2, 0) = 1$, $(\pi_0^*, \pi_2^*) = (1 - m, m)$ and $w_0^* = 0$, and w_2^* is the unique positive solution to

$$\left(\frac{N/\beta - N}{\pi_0^*} + 1\right) \beta w_2^* = u(\beta w_2^*). \tag{8}$$

Also, (6) and (7) imply $\pi_1^* = 0$ and $w_1^* = 0$. Given such w_1^* and w_2^*, it is shown that $\lambda^*(1; 1, 1) = 1$ and $\lambda^*(1; 2, 1) = 1$. Randomization could occur in meeting$(1, 0)$.[4]

Assume by way of contradiction that there exists an equilibrium (λ^t, π^t, w^t) that converges to a 2-replica (λ^*, π^*, w^*) starting with $\pi_1^0 > 0$. The following argument excludes zero-unit payment in meeting $(1, 0)$ and identifies the one described in Table 1 as the equilibrium payment strategy. If the economy is close to (π^*, w^*), then (4) and $\pi_1^0 > 0$ imply $w_1^0 > 0$ because there is a positive probability that a consumer with one unit meets a producer with one unit and the consumer can get a positive amount of utility from such a meeting. Equation (8) implies $u(x) > x$ for all $x < \beta w_2^*$. Therefore $u(\beta w_1^t) > \beta w_1^t$ holds all along the path, so in meeting $(1, 0)$, paying one unit is strictly preferred to paying nothing. That is, $\lambda^t(1; 1, 0) = 1$ is the only possibility for such a convergent equilibrium path. This conclusion and convergence implies $\lambda^*(1; 1, 0) = 1$, a conclusion we could not get simply from the definition of a steady state. We call such payment strategy described in Table 1 Payment Strategy 1.

Using $\pi_0 + \pi_1 + \pi_2 = 1$ and $0\pi_0 + 1\pi_1 + 2\pi_2 = 2m$, the money-holding

[4]One can show that the step function form of the value function in (7) is in fact necessary in this example. The proof involves a guess-and-verify process: that is, fixing λ^* that is consistent with $\pi_1^* = 0$, then solving (4) for w_1^*, and finally checking that λ^* satisfies (2). Tedious as it is, such a process proves the uniqueness of the forementioned non-full-support steady state in this {0,1,2} case. However the non-full-support steady state is not unique in general. Throughout this paper, we only take such an l replica as an example and focus on stability.

Table 1: The equilibrium payment rule (Payment Rule 1)

		Seller's money holding		
		0	1	2
Buyer's money holdings	0	-	-	-
	1	1 unit	1 unit	-
	2	2 units	1 unit	-

distribution can be characterized by π_1^t only:

$$(\pi_0^t, \pi_1^t, \pi_2^t) = \left(1 - m - \frac{\pi_1^t}{2},\ \ \pi_1^t,\ \ m - \frac{\pi_1^t}{2}\right). \tag{9}$$

Let $\pi_1^t \in [0, 2\min(m, 1-m)]$ to ensure $\pi_0^t, \pi_2^t \in [0,1]$. Under Payment Strategy 1, the law of motion is

$$\pi_1^{t+1} = \Psi(\pi_1^t) \equiv \pi_1^t - \frac{2(\pi_1^t)^2}{N}. \tag{10}$$

Figure 1 shows the convergence behavior of the law of motion. The law of motion is locally stable at $\pi_1^* = 0$, but it has unit root convergence; the slope at the fixed point is one. This happens because there are no net inflows into holdings of one unit and there are net outflows, but only from meeting $(1,1)$. The unit root arises because the frequency of meeting $(1,1)$ converges to zero as π_1^t goes to zero.

Under Payment Strategy 1, the Bellman equation is

$$\left\{\begin{array}{lcl} w_1^t & = & \frac{N-1+\pi_2^t}{N}\beta w_1^{t+1} + \frac{\pi_0^t}{N}u(\beta w_1^{t+1}) + \frac{\pi_1^t}{N}u(\beta w_2^{t+1} - \beta w_1^{t+1}) \\ w_2^t & = & \frac{N-1+\pi_2^t}{N}\beta w_2^{t+1} + \frac{\pi_0^t}{N}u(\beta w_2^{t+1}) + \frac{\pi_1^t}{N}[u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1}] \end{array}\right. . \tag{11}$$

The Jacobian of the r.h.s. of (11) with respect to $w^{t+1} = (w_1^{t+1}, w_2^{t+1})$ evaluated at the $l = 2$ replica is

$$\begin{bmatrix} \bar{\lambda} & 0 \\ 0 & \underline{\lambda} \end{bmatrix}, \tag{12}$$

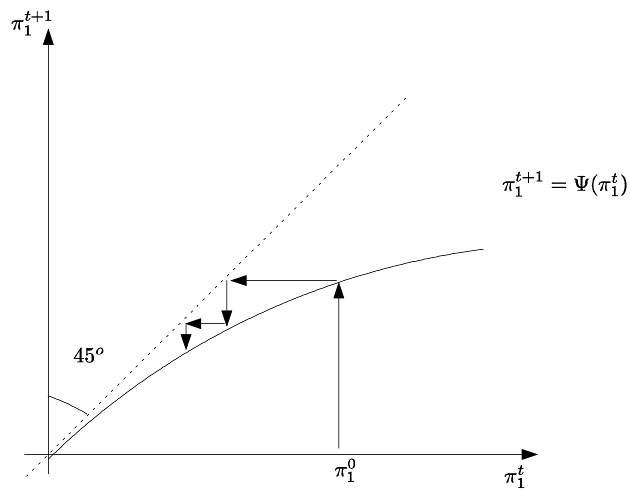

Figure 1: Law of Motion

where

$$\bar{\lambda} \equiv \frac{\pi_0^*}{N}u'(0)\beta + \frac{N-1+\pi_2^*}{N}\beta > 1 \tag{13}$$

$$\underline{\lambda} \equiv \frac{\pi_0^*}{N}u'(\beta w_2^*)\beta + \frac{N-1+\pi_2^*}{N}\beta \in (0,1). \tag{14}$$

If (8) has a positive solution, then (13) and (14) hold. Because the matrix (12) has an inverse and because we can extend the domain of u to include an open neighborhood around 0, the implicit function theorem can be applied to (11) to get

$$w^{t+1} = \Phi(\pi^t, w^t), \tag{15}$$

which we refer to as the forward-looking Bellman equation and which is valid in the neighborhood of an $l = 2$ replica.

(10) and (15) form our dynamic system. The Jacobian matrix of the joint system evaluated at the $l = 2$ replica is

$$\begin{bmatrix} \Psi_\pi & O \\ \Phi_\pi & \Phi_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma/\bar{\lambda} & 1/\bar{\lambda} & 0 \\ -\delta/(2\underline{\lambda}) & 0 & 1/\underline{\lambda} \end{bmatrix}, \tag{16}$$

where

$$\gamma \equiv \frac{1}{N}u(\beta w_2^*) > 0 \tag{17}$$

$$\delta \equiv \frac{1}{N}[u(\beta w_2^*) - \beta w_2^*] > 0, \tag{18}$$

and where the last inequality follows from (8).

Since, initially, $\pi_1^0 \neq 0$ and the law of motion has unit root convergence, the convergence trajectory will eventually be parallel to the eigenspace of (16) associated with the unit eigenvalue[5]. One associated eigenvector, which constitutes a base of the space, has the form

$$\begin{bmatrix} 1 \\ \frac{-\gamma}{\bar{\lambda}-1} \\ * \end{bmatrix},$$

where $*$ is a number irrelevant to our argument. Note that $-\gamma/(\bar{\lambda}-1) < 0$. Since the first variable of the linearized system, $\pi_1^t - \pi_1^*$, is always positive in the process of convergence, the second variable, $w_1^t - w_1^*$ (and therefore w_1^t) will eventually become negative, which violates (5). We conclude that there is no convergent equilibrium path.

4 Proof for the general case

The argument for the $\{0, 1, 2\}$ case uses knowledge of the trades in the pure-strategy monetary steady state. In the general case, we know very little about those trades. Nevertheless, as we now show, a proof can be constructed using similar ideas. The proof is by way of contradiction and relies on four lemmas. Lemma 1 describes what we know about the payment strategy in the l-replica steady state. Lemma 2 studies the law of motion and shows that unit root convergence holds. Lemma 3 describes the features of the Jacobian for the Bellman equation. Lemma 4 combines the results from the previous lemmas and considers the stacked system of the law of motion and the Bellman equation. The proofs of the lemmas are in the Appendix.

Even though no one holds change at $s(l)$, the equilibrium has implications for how change is traded. This is especially relevant for us because change

[5] See Subsection "dominant eigenvector" on page 165 of [4].

is actually traded in the vicinity of $s(l)$. The following gives equilibrium properties of change-trading.

Lemma 1 *Suppose that $s = (\hat{\lambda}, \hat{\pi}, \hat{w})$ is a pure-strategy steady state with strictly concave value function. Let $\hat{p}(i,j)$ be the unique solution of (1). The trade of $s(l)$ satisfies the following:*
(i) If $i' + j' < l$, then after the meeting, consumer has $i - \hat{p}(i,j)$ bundles and producer has $j + \hat{p}(i,j)$ bundles.
(ii) If $i' + j' \geq l$, then after the meeting, consumer has $i + 1 - \max(1, \hat{p}(i+1, j))$ bundles and producer has $j + \max(1, \hat{p}(i+1, j))$ bundles.

Although change has zero value at $s(l)$, when the sum of change in a meeting exceeds l, the producer can form an extra bundle by receiving enough change from the consumer. Therefore, when the total amount of change in a single-coincidence meeting exceeds l (case (ii)), the consumer behaves as if he had an extra bundle. The strictly concave value function makes sure that (ii) has a simple bundle-trading strategy. Any extension will lead to a more complicate *bundle*-trading strategy but it will not change the main message in the Lemma 1. That is, only *change* trading is arbitrary. Also, under the conditions of lemma 1, the strategy λ implies a degenerate bundle-trading strategy. We will keep this implicit in equations thoughout the paper to simplify the notations.

Before we proceed, it is helpful to rearrange components in the π^t and w^t vectors. First, we eliminate π_0^t and π_{Bl}^t in π^t using the following adding-up conditions:

$$\sum_{i=0}^{Bl} \pi_i^t = 1, \quad \text{and} \quad \sum_{i=0}^{Bl} i\pi_i^t = lm. \tag{19}$$

Then the remaining components of π^t are divided into l groups, where each group consists of π_i's that have the same amount of change. That is, we now let the state be described by (π_N^t, π_F^t), where $\pi_N^t \equiv (\pi^t(i'))_{i' \in \mathbb{L}}$ with $\pi^t(i') \equiv (\pi_{il+i'}^t)_{i \in \mathbb{B} \setminus B}$, and $\pi_F^t \equiv (\pi_{il}^t)_{i \in \mathbb{B} \setminus \{0, B\}}$. As regards w^t, we use the *incremental values of change*, $w_{il+i'}^t - w_{il}^t$ and group them into $l - 1$ groups according to the amount of change. Let $\Delta w^t \equiv (\Delta w_{i'}^t)_{i' \in \mathbb{L}}$, where $\Delta w_{i'}^t \equiv (w_{il+i'}^t - w_{il}^t)_{i \in \mathbb{B} \setminus \{B\}}$ and let $w_F^t = (w_{il}^t)_{i \in \mathbb{B}}$. Note that w^t can be recovered from $(\Delta w^t, w_F^t)$. The one-to-one transformation of variables from (π^t, w^t) to $(\pi_N^t, \Delta w^t, \pi_F^t, w_F^t)$ turns out to be useful because, as will be shown in Lemmas 2 and 3, the Jacobians of law of motion (20) and Bellman equation (21) have tractable triangular forms. In addition, the linearized system for $(\pi_N^t, \Delta w^t)$ does not depend upon the other two variables.

The main proof assumes by contradiction that there exists a sequence $\{\lambda^t, \pi^t, w^t\}$ convergent to a Lemma-1 steady state. Such sequence must satisfy six conditions: the convergences of value function, money holding distribution and payment strategy, and Bellman equation (4), law of motion (3), optimality conditions (1), which can involves some indifference conditions over solutions in (1) if randomization occurs. This section as well as the previous one show that the three convergence conditions together with Bellman equation and law of motion will lead to a contradiction to the optimality conditions of not throwing away money in the limit.

If the optimality conditions of not throwing way money in a particular finite date are violated, then we have a contradiction. If not, this section will show that such conditions are violated as $t \to \infty$. Different from the simple example, our analysis here might involve randomization. The argument in the simple example uses the fact that $\pi_1^t > 0$ for some t implies $w_1^t > 0$. Then $w_1^t \to 0$ and the shape of u rule out no trade and hence leave trading one unit of money as the only option. One can extend the argument to general l-replica to show that $\pi_{jl+j'}^t > 0$ implies $w_{l-j'}^t > 0$ for $l - j' > 1$, and hence can rule out no trade in meeting $(l - j', 0)$. This is not sufficient to exclude randomization over trading different amounts of change. In fact, we cannot rule out randomization in general. Consider 6-replica of a Trejos-Wright steady state for instance. Start with $\pi_3^0 > 0$ and $\pi_{j'}^0 = 0$ for $j' = 1, 2, 4$ and 5. Such deviation is equivalent to that in the simple example. Along the convergent path, we can has step value function and hence randomization could occur in meetings with overall change less than three units of money.

When randomization occurs, we can view $\{\pi^t, w^t\}$ as a nonautonomous system. Particularly, we start with (4) and (3) assuming that such payment strategy sequence $\{\lambda^t\}$ is chosen optimally by agents. In general, randomization might be involved and the payment strategy could change across dates. The law of motion and Bellman equation become time dependence or nonautonomous due to the variation in change-trading strategy. But no matter how the strategy evolves over time, it is assumed by contradiction to be convergent. By continuity, the implied system must have a convergent sequence of difference equations defined by law of motion and Bellman equation, and a convergent sequence of the associated phase diagrams. With all these convergence, we can apply a trick often used in engineering[6]. That is to introduce an additional variable that itself forms an autonomous system. Any variable that defines a convergent autonomous sequence will be appropriate. Without loss of generality, we use $y^t = \frac{1}{t+1}$, that satisfies the difference equation

$$y^{t+1} = \Upsilon(y^t) \equiv \frac{y^t}{1+y^t}.$$

The sequence $\{y^t\}$ is autonomous and converges to the fixed point $y^* = 0$ corresponding to the fact that $\{\lambda^t\}$ converges to λ^*.

We express the law of motion and the Bellman equation after introducing y^t and Δw^t as

$$(\pi_N^{t+1}, \pi_F^{t+1}) = \Psi(y^t, \pi_N^t, \pi_F^t) \tag{20}$$

and

$$(\Delta w^t, w_F^t) = \phi(y^t, \pi_N^t, \Delta w^{t+1}, \pi_F^t, w_F^{t+1}). \tag{21}$$

We let Ψ and ϕ to refer to each component of the mappings. For example $\phi^{\Delta w}$ indicates the subvector that defines Δw^t.

Following the literature, we will approximate the dynamic system by the linearized system. Conclusions will be reached by studying the associated Jacobians. We first examine the properties of the Jacobian of Ψ, the most important of which concerns the existence of unit eigenvalues, and then turn to $\phi^{\Delta w}$.

Lemma 2 *(i) The Jacobian of Ψ evaluated at (y^*, π^*) has the following triangular form:*

$$\Psi_\pi = \begin{bmatrix} \Psi_{\pi_N}^{\pi_N} & O \\ \Psi_{\pi_N}^{\pi_F} & \Psi_{\pi_F}^{\pi_F} \end{bmatrix} \text{ with } \Psi_{\pi_N}^{\pi_N} = \begin{bmatrix} \Psi_{\pi(1)}^{\pi(1)} & \cdots & \Psi_{\pi(l-1)}^{\pi(1)} \\ O & \ddots & \vdots \\ O & O & \Psi_{\pi(l-1)}^{\pi(l-1)} \end{bmatrix}, \tag{22}$$

where

$$\Psi_{\pi(v')}^{\pi(k')} = \left(\frac{\partial \pi_{kl+k'}^{t+1}}{\partial \pi_{vl+v'}^t} \right)_{k,v=0,1,\cdots,B-1}.$$

(ii) $\Psi_{\pi_N}^{\pi_N}$ has at least one unit eigenvalue.
(iii) If $\Psi_{\pi_N}^{\pi_N}$ has a positive eigenvalue that is strictly less than one, then the associated eigenvector has both strictly positive and strictly negative elements.

[6] See section 1.2 in [6].

The proof of the above lemma involves studying the derivative of Ψ. Parts (ii) and (iii) are similar to their analogues in the $\{0,1,2\}$ case. In order for the measure of people holding positive change to go to zero, it is necessary for them to be matched with others with change. However, the frequency of such meetings goes to zero as the measure of people with change goes to zero. Thus, π_N^t involves unit roots convergence.

We now investigate the Jacobians of ϕ, equation (21). For that purpose, we do not need the explicit form of equation (21); because the change of variables from w^t to $(\Delta w^t, w_F^t)$ is a linear transformation, we can first linearize the original Bellman equation and then do the variable transformation. Let $w_{x+p} - w_x \equiv \Delta w(x,p)$. The original Bellman equation can be written as

$$\begin{aligned} & w^t_{il+i'} = \\ & \sum_{j\in\mathbb{B}\setminus\{B\},j'\in\mathbb{L}\cup\{0\}} \frac{\pi^t_{jl+j'}}{N}\sum_p \lambda^t(p;il+i',jl+j')u(\beta\Delta w^{t+1}(jl+j',p)) \\ & + \sum_{j\in\mathbb{B}\setminus\{B\},j'\in\mathbb{L}\cup\{0\}} \frac{\pi^t_{jl+j'}}{N}\beta\sum_p \lambda^t(p;il+i',jl+j')w^{t+1}_{il+i'-p} + \frac{N-1+\pi^t_{Bl}}{N}\beta w^{t+1}_{il+i'}. \end{aligned} \tag{23}$$

Taking the linear expansion of equation (23) at the steady state and then subtracting w_{il}^t from $w_{il+i'}^t$ for $i' \in \mathbb{L}$, we get the linear expansion of (21) or, equivalently, the Jacobian of (21). A careful study of the linear expansion proves several important properties of the Jacobian, as summarized in the following lemma.

Lemma 3 *(i) The linear approximation of $\phi^{\Delta w}$ around the l-replica has the form*

$$\Delta w^t = \phi^{\Delta w}_{\Delta w}\Delta w^{t+1} + \phi^{\Delta w}_{\pi_N}\pi^t_N. \tag{24}$$

That is, it does not depend on π_F^t or w_F^{t+1}.
(ii) The Jacobian $\phi^{\Delta w}_{\pi_N}$ consists of $(l-1)\times(l-1)$ blocks and has a lower-right triangular form:

$$\phi^{\Delta w}_{\pi_N} = \begin{bmatrix} O & \cdots & O & K \\ \vdots & / & / & \vdots \\ O & / & & \vdots \\ K & \cdots & \cdots & K \end{bmatrix}. \tag{25}$$

Furthermore, if s has a strictly increasing value function, then K is a B by B matrix that only has strictly positive elements.
(iii) The Jacobian $\phi_{\Delta w}^{\Delta w}$ has a lower-left triangular form:

$$\phi_{\Delta w}^{\Delta w} = \begin{bmatrix} \phi_{\Delta w_1}^{\Delta w_1} & O & \cdots & O \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & O \\ \phi_{\Delta w_1}^{\Delta w_{l-1}} & \cdots & \cdots & \phi_{\Delta w_{l-1}}^{\Delta w_{l-1}} \end{bmatrix}, \tag{26}$$

where each diagonal block $\phi_{\Delta w_{i'}}^{\Delta w_{i'}}$ is a B by B matrix that only has nonnegative elements.

An strictly increasing value function, one of the two major conditions in proposition 1, makes sure that consumers are strictly better off with one extra bundle. This in turn implies that K only has strictly positive elements. The proof uses Lemma 1 to get the triangular forms.

All of the above lemmas are about non-full-support steady states that is associated with a pure-strategy steady state. Then bundle-trading is unaffected if the consumer with some change meets a producer without change. When we drop such restrictions, s could have multiple solutions in the indifferent set p. And bundle-trading could vary if the consumer has different amount of change. Then our trick of introducing incremental value of holding change is not enough to simplify the analysis on these steady states. The incremental value of holding change could also depend on the incremental value of holding bundles in the linearized Bellman equation. Specifically, equation (24) would depend on w_F. The properties in lemma 1 is not enough to guarantee the triangular form of matrix (27), whose last block of the fourth row might not be a zero matrix. It might no longer be valid to only focus on matrix A, a submatrix of the joint system.

We are now ready to obtain a stacked system and study its Jacobian. We apply the implicit function theorem to solve for $(\Delta w^{t+1}, w_F^{t+1})$ as a function of $(y^t, \pi_N^t, \Delta w^t)$ from Bellman equation (21) around the l-replica and refer to it as the forward-looking Bellman equation, denoted by Φ. Then we stack the law of motion and the forward-looking Bellman equation:

$$\begin{aligned} y^{t+1} &= \Upsilon(y^t) \\ (\pi_N^{t+1}, \pi_F^{t+1}) &= \Psi(y^t, \pi_N^t, \pi_F^t) \\ (\Delta w^{t+1}, w_F^{t+1}) &= \Phi(y^t, \pi_N^t, \Delta w^t, \pi_F^t, w_F^t). \end{aligned}$$

By Lemmas 2(i) and 3(i), the Jacobian of the stacked system has the following form:

$$\begin{bmatrix} \Upsilon' & O & O & O & O \\ * & \Psi^{\pi_N}_{\pi_N} & O & O & O \\ * & -[\phi^{\Delta w}_{\Delta w}]^{-1}\phi^{\Delta w}_{\pi_N} & [\phi^{\Delta w}_{\Delta w}]^{-1} & O & O \\ * & \Psi^{\pi_F}_{\pi_N} & O & \Psi^{\pi_F}_{\pi_F} & O \\ * & * & * & * & * \end{bmatrix}, \tag{27}$$

where the $*$s are blocks irrelevant to our analysis.

There might be a potential problem associated with linearization approach. That is, if $\{\lambda^t\}$ can be specified such that π^t reaches and stays at π^* after finite dates, then the original system is stable, even though the linearized system might have a different dynamic properties. However, reaching steady state π^* in finite dates is impossible, because always a proportion of π^t_N miss the trading opportunity, and, to make sure the money stock unchanged along the path, they cannot throwing away change.

The equilibrium path starts with deviation $\pi^0_N \neq 0$ and hence $\Delta w^0 \neq 0$ by Bellman equation. Therefore, the crucial submatrix is

$$A \triangleq \begin{bmatrix} \Psi^{\pi_N}_{\pi_N} & O \\ -[\phi^{\Delta w}_{\Delta w}]^{-1}\phi^{\Delta w}_{\pi_N} & [\phi^{\Delta w}_{\Delta w}]^{-1} \end{bmatrix}.$$

Since the upper-right block of A is a zero matrix, the eigenvalues of A are those of $\Psi^{\pi_N}_{\pi_N}$ and $[\phi^{\Delta w}_{\Delta w}]^{-1}$. By Lemma 2(ii)-(iii), $\Psi^{\pi_N}_{\pi_N}$ has unit eigenvalues and they are essential to the law of motion. Then, relevant to our analysis are the eigenvectors of A associated with those unit eigenvalues. The following lemma characterizes the subvector of any such eigenvector that corresponds to the evolution of $\Delta w^t_{i'}$. It provides a sufficient condition under which the relevant subvector (i.e., the trajectory of the values of change) has strictly negative elements. This will lead to a contradiction in the proof of Proposition 1.

Lemma 4 *Denote an eigenvector of $\Psi^{\pi_N}_{\pi_N}$ associated with a unit eigenvalue by $z = (z_1^T, \cdots, z_{l-1}^T)^T \geq 0$, where each $z_{i'}$ is a B-dimensional column vector. Let $\varphi_{\Delta w_{i'}} \triangleq (I - \phi^{\Delta w_{i'}}_{\Delta w_{i'}})^{-1} K z_{l-i'}$, with $i' \in \mathbb{L}$.*
(i) The matrix A has a unit eigenvalue, and its associated eigenvector has

the form $(z, \varpi_1, \cdots, \varpi_{l-1})$*, with*

$$\begin{bmatrix} \varpi_1 \\ \vdots \\ \varpi_{l-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi_{\Delta w_{l-1}} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \varphi_{\Delta w_{l-2}} \\ * \end{bmatrix} + \cdots + \begin{bmatrix} \varphi_{\Delta w_1} \\ * \\ \vdots \\ * \end{bmatrix}. \tag{28}$$

(ii) If $z_{l-i'} \geq 0$*,* $z_{l-i'} \neq 0$ *and* $\varphi_{\Delta w_{i'}} \geq 0$*, then* $\varphi_{\Delta w_{i'}} > 0$*.*
(iii) If $z_{l-i'} \geq 0$*,* $z_{l-i'} \neq 0$ *and* $\lambda^*(i'; i', 0) = 1$*, then* $\varphi_{\Delta w_{i'}}$ *has strictly negative elements.*

Proof of Proposition 1. Suppose by way of contradiction that there exists an equilibrium path (λ^t, π^t, w^t) convergent to a Lemma 1 steady state (λ^*, π^*, w^*). Such sequence must satisfy (4), (3) and indifference condition across solutions in (1) if randomization occurs. The following shows that just (4) and (3) are enough to give rise to a contradiction. With these features, the sequence $\{\pi^t, w^t\}$ can be viewed as a nonautonomous system and can be transformed into an autonomous system. Then we consider the linearized system (27). The crucial submatrix is A, corresponding to $(\pi_N^t, \Delta w^t)$, assumed by contradiction to converge to zero vector.

The dominant eigenvalue is assumed by contradiction to be positive and no greater than one. Suppose the dominant eigenvalue is strictly less than one. It is enough to consider the evaluation of π_N^t in this case. By Lemma 2 (iii), the associated eigenvector of $\Psi_{\pi_N}^{\pi_N}$ has strictly negative elements. The dominant mode argument (see [4]) implies that π_N^t eventually becomes parallel to the associated eigenspace. It follows that some elements in π_N^t will become strictly negative, contradicting $\pi_N^t \geq 0$.

Therefore the law of motion involves unit root convergence. The dominant mode argument implies that $(\pi_N^t, \Delta w^t)$ eventually becomes parallel to an eigenspace, vectors of which have the form of $(z, \varpi_1, \cdots, \varpi_{l-1})$ in Lemma 4. And $(z, \varpi_1, \cdots, \varpi_{l-1})$ must be nonnegative because $(\pi_N^t - \pi_N^*, \Delta w^t - \Delta w^*)$ is nonnegative along the convergent path. Let $h' = \max\{i' | z_{i'} \neq 0\}$. Then by Lemma 4(i), we have

$$\begin{bmatrix} \varpi_1 \\ \vdots \\ \varpi_{l-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi_{\Delta w_{l-1}} \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ \varphi_{\Delta w_{l-h'}} \\ * \end{bmatrix}. \tag{29}$$

Case 1: Suppose $h' < l-1$. Equation (29) implies $0 \leq \varpi_{l-h'} = \varphi_{\Delta w_{l-h'}}$ and $\varpi_{l-i'} = 0$ for $i' > h'$. By Lemma 4(ii), $\varphi_{\Delta w_{l-h'}} \geq 0$ implies $\varphi_{\Delta w_{l-h'}} > 0$ and hence $\varpi_{l-h'} > 0$. With the dominant mode argument, Δw^t becomes parallel to $(0, \cdots, 0, \varpi_{l-h'}, \cdots, \varpi_{l-1})$. In other words, the value of less than $l-h'$ units of change will become negligible compared with the values of $l-h'$ units of change. Furthermore, that $\Delta w^t(0, l-h')$ is strictly positive and converges to zero implies $u'(0)\beta\Delta w^t(0, l-h') > \beta\Delta w^t(0, l-h')$. Hence, the value of trading all the change dominates the value of no trade in meeting $(l-h', 0)$. Therefore we have $\lambda^*(l-h'; l-h', 0) = 1$. Then lemma 4(iii) implies that $\varphi_{\Delta w_{l-h'}}$ has negative elements, a contradiction to $\varphi_{\Delta w_{l-h'}} \geq 0$.
Case 2: Suppose $h' = l-1$. Argument similar to those of case 1 leads to $\lambda^*(1; 1, 0) = 1$ for all $j \in \mathbb{B}\backslash\{B\}$. Lemma 4(iii) implies that $\varphi_{\Delta w_1}$ has negative elements, which contradicts $\varphi_{\Delta w_1} \geq 0$. ■

5 Concluding remarks

This paper shows the instability of the non-full-support steady states that are l-replicas of some pure-strategy monetary steady state with full-support money-holding distribution. A possible extension is to consider the case when the full-support steady state is a mixed-strategy steady state. Our argument can rule out the convergent path where the same strategy in trading bundles is played across meetings $(vl+i', jl)$ for different $i' \in \mathbb{L}$. But if the strategy of trading bundles in such meeting depends on change-holding positions i', the same argument does not go through. In particular, The linearized Bellman equation Δw^t in Lemma 3 will depend on w_F and (27) fails to have a triangular form. Our matrix computation does not fit in this case and further study is needed.

6 Appendix

Proof of Lemma 1. (Necessity) The optimality of the bundle trading specified in (i) is obvious and proof is omitted. For the optimality of (ii), note first that under $i'+j' \geq l$, the consumer pays at least $l-j'(\leq i')$ units in order for the producer to form one bundle and produce a positive amount of goods. Suppose that the consumer pays another s bundles and s' units of change to the producer such that (s, s') solves the following.

$$
\begin{aligned}
& \max_{\substack{s\in\{0,1,\cdots,\min(i,B-j-1)\} \\ s'\in\{0,\cdots,i'+j'-l\} \\ (j+1)l+sl+s'\le Bl}} u[\beta(w^*_{(j+1)l+sl+s'} - w^*_{jl+j'})] + \beta w^*_{il+i'-(l-j')-(sl+s')} \\
= & \max_{s+1\in\{1,\cdots,\min(i+1,B-j)\}} u[\beta(w^*_{jl+(s+1)l} - w^*_{jl})] + \beta w^*_{((i+1)-(s+1))l}, \qquad (30)
\end{aligned}
$$

where the equality follows from the step-function form (7). If $\hat{p}(i+1, j)$ is not equal to zero, then $s+1$ is equal to $\hat{p}(i+1, j)$.

If $\hat{p}(i+1, j) = 0$, then $s+1 = 0$ is not possible. In this case, we will consider any second best solution. Given that the objective function in (30) is concave in s, $s = 0$ will achieve the optimum in (30). Overall, the consumer's post-trade bundle holding is $i + 1 - \max(1, \hat{p}(i+1, j))$ and the producer's post-trade bundle holding is $j + \max(1, \hat{p}(i+1, j))$.

(Sufficiency) Equation (7) implies that change is worthless and hence the consumer cares only about his post-trade bundle holding and is indifferent about his post-trade change holding. Hence as long as λ^* satisfies (i) and (ii), the optimality of λ^* holds. ■

Proof of Lemma 2. First, we rewrite the law of motion (3) in terms of "change" and "bundles" as follows:

$$
\begin{aligned}
& \pi^{t+1}_{kl+k'} = \pi^t_{kl+k'} \\
+ & \frac{1}{N} \sum_{i',i,j',j,il+i'>kl+k'} \pi^t_{il+i'} \pi^t_{jl+j'} \boldsymbol{\lambda}^t(il+i'-kl-k'; il+i', jl+j') \\
+ & \frac{1}{N} \sum_{i',i,j',j,jl+j'<kl+k'} \pi^t_{il+i'} \pi^t_{jl+j'} \boldsymbol{\lambda}^t(kl+k'-jl-j'; il+i', jl+j') \\
- & \frac{1}{N} \sum_{j',j} \pi^t_{kl+k'} \pi^t_{jl+j'} \sum_{p>0} \boldsymbol{\lambda}^t(p; vl+v', jl) \\
- & \frac{1}{N} \sum_{i',i} \pi^t_{il+i'} \pi^t_{kl+k'} \sum_{p>0} \boldsymbol{\lambda}^t(p; il, vl+v'), \qquad (31)
\end{aligned}
$$

where π^t_0 and π^t_{Bl} are given by two adding-up conditions (19). The conclusion is reached by differentiating this law of motion and evaluating the result at π^*. Because most of the terms in the law of motion are quadratic and $\pi^*_{il+i'} = 0$ for $i' > 0$, after differentiating and evaluating them at π^*, many of the terms will disappear. Throughout this paper, the derivatives are evaluated at the steady state. We suppress such dependence to simplify the notations. One can show that for $(v', v) \neq (k', k)$ and $v', k' \in \mathbb{L}$,

$$\begin{aligned}
& \frac{\partial \pi_{kl+k'}^{t+1}}{\partial \pi_{vl+v'}^{t}} \\
= & \frac{1}{N}\sum_{j=0}^{B} \pi_{jl}^{*} \boldsymbol{\lambda}^{*}(vl+v'-kl-k'; vl+v', jl) \\
+ & \frac{1}{N}\sum_{i=0}^{B} \pi_{il}^{*} \boldsymbol{\lambda}^{*}(il-kl-k'; il, vl+v') \\
+ & \frac{1}{N}\sum_{j=0}^{B} \pi_{jl}^{*} \boldsymbol{\lambda}^{*}(kl+k'-jl; vl+v', jl) \\
+ & \frac{1}{N}\sum_{i=0}^{B} \pi_{il}^{*} \boldsymbol{\lambda}^{*}(kl+k'-vl-v'; il, vl+v') \\
\geq & \ 0,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
& \frac{\partial \pi_{kl+k'}^{t+1}}{\partial \pi_{kl+k'}^{t}} \\
= & \ 1 - \frac{1}{N}\sum_{i=0}^{B} \pi_{il}^{*} \sum_{p>0, p\neq il-kl-k'} \boldsymbol{\lambda}^{*}(p; il, kl+k') \\
& -\frac{1}{N}\sum_{j=0}^{B} \pi_{jl}^{*} \sum_{p>0, p\neq kl+k'-jl} \boldsymbol{\lambda}^{*}(p; kl+k', jl) \quad \geq 0.
\end{aligned} \tag{33}$$

Consider the case where $k' > v'$. Note that all the terms in (32) are associated with meetings where the total amount of change is v'. By Lemma 1, nobody ends up with k' units of change after such meetings. Thus, all the terms are zero, and hence Ψ_π in (22) has a block-triangular form.

To prove (ii) and (iii), we first establish several claims:

Claim 1 *If $k' < v'$, then $\Psi_{\pi(v')}^{\pi(k')} \geq 0$. Moreover, the equality for the (k, v) element holds if and only if for all $i \in \mathbb{B}$, meetings between il and $vl + v'$ leave no one with $kl + k'$ units of money with probability one.*

Claim 2 *Each diagonal block is positive. Moreover, in the* v*th column of* $\Psi_{\pi(v')}^{\pi(v')}$*, the sum of elements is no greater than one, and it is equal to one if and only if for all* $i \in \mathbb{B}$*, meetings between* il *and* $vl + v'$ *leave one agent with* v' *units of change with probability one.*

Claim 3 *The* v*th column of* $\Psi_{\pi(v')}^{\pi(k')}$ *for all* $k' = 1, \cdots, v' - 1$ *is equal to zero if and only if in the* v*th column of* $\Psi_{\pi(v')}^{\pi(v')}$*, the sum of elements is equal to one.*

Claim 1 is exactly what equation (32) states. The first part of Claim 2 follows from the inequalities in (32) and (33). The second part is shown as follows:

$$
\begin{aligned}
& [\text{The sum of the } v\text{th column of } \Psi_{\pi(v')}^{\pi(v')}] \\
= \; & \frac{\partial \pi_{vl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} + \sum_{k=0, k\neq v}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} \\
= \; & 1 - \frac{1}{N} \sum_{i=0}^{B} \pi_{il}^{*} \sum_{p>0, p\neq il-vl-v'} \boldsymbol{\lambda}^{*}(p; il, vl+v') \\
& \qquad + \frac{1}{N} \sum_{i=0}^{B} \pi_{il}^{*} \sum_{k=0, k\neq v}^{B-1} \boldsymbol{\lambda}^{*}(il - kl - v'; il, vl+v') \\
& \qquad + \frac{1}{N} \sum_{i=0}^{B} \pi_{il}^{*} \sum_{k=0, k\neq v}^{B-1} \boldsymbol{\lambda}^{*}(kl - vl; il, vl+v') \\
& - \frac{1}{N} \sum_{j=0}^{B} \pi_{jl}^{*} \sum_{p>0, p\neq vl+v'-jl} \boldsymbol{\lambda}^{*}(p; vl+v', jl) \\
& \qquad + \frac{1}{N} \sum_{j=0}^{B} \pi_{jl}^{*} \sum_{k=0, k\neq v}^{B-1} \boldsymbol{\lambda}^{*}(vl - kl; vl+v', jl) \\
& \qquad + \frac{1}{N} \sum_{j=0}^{B} \pi_{jl}^{*} \sum_{k=0, k\neq v}^{B-1} \boldsymbol{\lambda}^{*}(kl + v' - jl; vl+v', jl)
\end{aligned}
$$

$$= 1 - \frac{1}{N}\sum_{i=0}^{B}\pi^*_{il}\sum_{p>0, p\neq il-kl-v' \text{ or } (k-v)l \text{ for } k\in\mathbb{B}}\boldsymbol{\lambda}^*(p; il, vl+v')$$
$$- \frac{1}{N}\sum_{j=0}^{B}\pi^*_{jl}\sum_{p>0, p\neq kl+v'-jl \text{ or } (v-k)l \text{ for } k\in\mathbb{B}}\boldsymbol{\lambda}^*(p; vl+v', jl) \quad .$$

Note that $\sum_{p>0, p\neq il-kl-v' \text{ or } (k-v)l \text{ for } k\in\mathbb{B}}\boldsymbol{\lambda}^*(p; il, vl+v') = 0$ is equivalent to the fact that the meeting $(il, vl+v')$ leaves no one with v' units of change with probability one. A similar statement holds for the second term in the above. Therefore we have Claim 2. Combining Claims 1 and 2 leads to Claim 3.

(ii) In the following discussion, we denote the transpose of a vector by superscript T. For any $x \in \mathbb{N}$, we let $1_x = (1, \cdots, 1)^T$ and $0_x = (0, \cdots, 0)^T$, both of which are x-dimensional. Because Ψ^π_π is an upper-triangular block matrix, the eigenvalues of Ψ^π_π are those of $\Psi^{\pi(v')}_{\pi(v')}$, with $v' \in \mathbb{L}$. Consider $\Psi^{\pi(1)}_{\pi(1)}$, the block corresponding to one-unit change. Claim 2 implies that in each column of $\Psi^{\pi(1)}_{\pi(1)}$, the sum of elements is equal to one: $1_B^T \Psi^{\pi(1)}_{\pi(1)} = 1_B^T = 1 \cdot 1_B^T$. Hence, $\Psi^{\pi(1)}_{\pi(1)}$ has a unit eigenvalue, and therefore, Ψ^π_π has a unit eigenvalue.

(iii) Suppose by way of contradiction that Ψ^π_π has an eigenvalue that is smaller than one, say $\tau \in (0,1)$, and that its associated eigenvector has only non-negative elements such that the law of motion could have exponential convergence to π^* along this eigenvector. Denote that eigenvector by $\eta = (\eta_1^T, \cdots, \eta_{l-1}^T)^T \geq 0$, where for each $v' \in \mathbb{L}$, $\eta_{v'}$ is a B-dimensional vector. We have $B(l-1)$ equations:

$$0_{B(l-1)} = \left(\Psi^\pi_\pi - \tau I\right)\eta.$$

Sum up the first B equations, the second B equations, etc.

$$0_{l-1} = \begin{bmatrix} 1_B^T\Psi^{\pi(1)}_{\pi(1)} - 1_B^T\tau & 1_B^T\Psi^{\Pi(1)}_{\Pi(2)} & \cdots & 1_B^T\Psi^{\pi(1)}_{\pi(l-1)} \\ 0_B^T & 1_B^T\Psi^{\pi(2)}_{\pi(2)} - 1_B^T\tau & \cdots & \\ 0_B^T & 0_B^T & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1_B^T\Psi^{\pi(l-2)}_{\pi(l-1)} \\ 0_B^T & 0_B^T & \cdots & 1_B^T\Psi^{\pi(l-1)}_{\pi(l-1)} - 1_B^T\tau \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \vdots \\ \eta_{l-1} \end{bmatrix} \tag{34}$$

By Claim 1, we have $1_B^T\Psi_{\Pi(v')}^{\Pi(k')} \geq 0$ for $k' < v'$ and thus the upper right blocks of the first matrix of the above equation are positive matrices. Moreover, the previous argument implies that $1_B^T\Psi_{\pi(1)}^{\pi(1)} - \tau 1_B^T = 1_B^T - \tau 1_B^T = (1-\tau)1_B^T > 0$. Therefore, we have $\eta_1 = 0$.

We use mathematical induction and assume that $\eta_1 = \cdots = \eta_{v'-1} = 0$. In what follows, we want to show $\eta_{v'} = 0$. The following discussion is divided into two cases.

Case 1: Consider any $v \in \mathbb{B}\backslash B$ such that $\sum_{k=0}^{B-1} \frac{\partial\pi_{kl+v'}^{t+1}}{\partial\pi_{vl+v'}^{t}} \neq 1$. Then by Claim 3, there exists k' smaller than $v^{'}$ such that the vth column of $\Psi_{\Pi(v')}^{\Pi(k')}$, which corresponds to the (k', v') block in the first matrix of (34), has strictly positive elements. The sum of elements in this column $\sum_{k=0}^{B-1} \frac{\partial\pi_{kl+k'}^{t+1}}{\partial\pi_{vl+v'}^{t}}$ is strictly greater than 0. If we look into the k'th row in the r.h.s. of (34), given that $\eta_1 = \cdots = \eta_{v'-1} = 0$, the resulting vector is equal to $\sum_{r'=v'}^{l-1} 1_B^T\Psi_{\Pi(r')}^{\Pi(k')}\eta_{r'}$. It satisfies the following condition:

$$0 = \sum_{r'=v'}^{l-1} 1_B^T\Psi_{\Pi(r')}^{\Pi(k')}\eta_{r'} \geq 1_B^T\Psi_{\Pi(v')}^{\Pi(k')}\eta_{v'} \geq \sum_{k=0}^{B-1} \frac{\partial\pi_{kl+k'}^{t+1}}{\partial\pi_{vl+v'}^{t}}\eta_{v'}(v) \geq 0, \tag{35}$$

where $\eta_{v'}(v)$ is the vth element of vector $\eta_{v'}$. The equality is by (34). The first inequality is because $1_B^T\Psi_{\Pi(r')}^{\Pi(k')}\eta_{r'} \geq 0$ for all $r' \geq v^{'}$. If we write out the matrix multiplication in $1_B^T\Psi_{\Pi(v')}^{\Pi(k')}\eta_{v'}$, (32), (33) and $\eta_{v'} \geq 0$ imply the second inequality. Since $\sum_{k=0}^{B-1} \frac{\partial\pi_{kl+k'}^{t+1}}{\partial\pi_{vl+v'}^{t}}$ is strictly greater than 0, we must have $\eta_{v'}(v) = 0$.

Case 2: Consider any $v \in \mathbb{B}\backslash B$ such that $\sum_{k=0}^{B-1} \frac{\partial\pi_{kl+v'}^{t+1}}{\partial\pi_{vl+v'}^{t}} = 1$. If we look into the v'th row in the r.h.s. of (34), the resulting vector is equal to $(1_B^T\Psi_{\Pi(v')}^{\Pi(v')} -$

$\tau 1_B^T)\eta_{v'} + \sum_{r'=v'+1}^{l} 1_B^T \Psi_{\Pi(r')}^{\Pi(v')} \eta_{r'}$. It satisfies

$$\begin{aligned} 0 &= (1_B^T \Psi_{\Pi(v')}^{\Pi(v')} - \tau 1_B^T)\eta_{v'} + \sum_{r'=v'+1}^{l-1} 1_B^T \Psi_{\Pi(r')}^{\Pi(v')} \eta_{r'} \\ &\geq (1_B^T \Psi_{\Pi(v')}^{\Pi(v')} - \tau 1_B^T)\eta_{v'} \\ &\geq (\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} - \tau)\eta_{v'}(v) \\ &\geq 0. \end{aligned} \tag{36}$$

The equality is by (34). The first inequality is because $1_B^T \Psi_{\Pi(r')}^{\Pi(v')} \eta_{r'} \geq 0$ for all $r' > v^{'}$. If we write out the matrix multiplication in $(1_B^T \Psi_{\Pi(v')}^{\Pi(v')} - \tau 1_B^T)\eta_{v'}$, we have the sum over $(\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} - \tau)\eta_{v'}(v)$ for all $v \in \mathbb{B}\backslash B$. For those $v \in \mathbb{B}\backslash B$ such that $\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} \neq 1$, we have $\eta_{v'}(v) = 0$ by Case 1, and therefore $(\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} - \tau)\eta_{v'}(v) = 0$ for such v. For those $v \in \mathbb{B}\backslash B$ such that $\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} = 1$, we have $(\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} - \tau)\eta_{v'}(v) \geq 0$. Combining these results for the two types of v gives the second inequality.

Since $\sum_{k=0}^{B-1} \frac{\partial \pi_{kl+v'}^{t+1}}{\partial \pi_{vl+v'}^{t}} - \tau$ is strictly greater than 0, (36) implies $\eta_{v'}(v) = 0$.

Combining these two cases, we have $\eta_{v'} = 0$. ■

Proof of Lemma 3. (i) Using Lemma 1, we can rewrite Bellman equation

(4) by linearizing the equations and then subtracting w^t_{il} from $w^t_{il+i'}$:

$$\begin{aligned}
& w^t_{il+i'} - w^t_{il} \\
= & \sum_{\substack{j\in\mathbb{B}\setminus B \\ p\in\hat{p}(i,j) \\ v'\in\{0,\cdots,i'\}}} \frac{\pi^*_{jl}}{N} u'(\beta\Delta w^*(jl,pl))\beta\lambda^*(pl+v';il+i',jl)\Delta w^{t+1}(jl+pl,v') \\
& + \sum_{\substack{j\in\mathbb{B}\setminus B \\ p\in\hat{p}(i,j) \\ v'\in\{0,\cdots,i'\}}} \frac{\pi^*_{jl}}{N}\beta\lambda^*(pl+v';il+i',jl)\Delta w^{t+1}(jl-pl,i'-v') \\
& + \frac{\pi^*_{Bl}+N-1}{N}\beta(w^{t+1}_{il+i'} - w^{t+1}_{il}) \\
& + \sum_{jl+j'\notin\{0,Bl\}} \kappa_{(il+i',jl+j')}(\pi^t_{jl+j'} - \pi^*_{jl+j'}), \qquad (37)
\end{aligned}$$

where

$$\begin{aligned}
\kappa_{(il+i',jl+j')} \equiv & \frac{1}{N}\{\sum_p \lambda^*(p;il+i',jl+j')\left[u(\beta\Delta w^*(jl+j',p)) + \beta w^*_{il+i'-p}\right] \\
& - \sum_p \lambda^*(p;il,jl+j')\left[u(\beta\Delta w^*(jl+j',p)) + \beta w^*_{il-p}\right]\}.
\end{aligned}$$

By Lemma 1, $\kappa_{(il+i',jl+j')} = 0$ if $j' = 0$. Thus, the r.h.s. of (37) does not depend on π^t_F. Lemma 1 implies the same pure strategy in trading bundles across meetings $(il+i', jl)$ for different change holding i'. Thus, the r.h.s. depends on w^{t+1} only throught incremental values Δw^{t+1}.

(ii) Looking into the coefficients with respect to π^t_N in (37) gives $\phi^{\Delta w}_{\pi_N}$. Because of (7) and Lemma 1, two facts follow: (I) $\kappa_{(il+i',jl+j')} = 0$ for all i', j' such that $i'+j' < l$, and (II) as long as both $i'+j'$ and $i''+j''$ are greater than l, $\kappa_{(il+i',jl+j')} = \kappa_{(i''+il,j''+jl)}$. Therefore, it is valid to let $K_{i,j}$ be equal to $\kappa_{(il+i',jl+j')}$ for $i'+j' \geq l$, and we have the statement.

The strict positiveness of $K_{i,j}$ is due to the following. The first term of $K_{i,j}$ is the consumer's payoff in the meeting $(il+i', jl+j')$, and the second term is that in the meeting $(il, jl+j')$. Because $i'+j' \geq l$, $l-j'$ units of money have the same value as a bundle to the producer. The consumer gets a higher payoff from giving that much change than she does in the meeting $(il, jl+j')$.

(iii) Looking into the coefficients with respect to the incremental values of change gives $\phi_{\Delta w}^{\Delta w}$. Because the total amount of change in a meeting cannot increase after trade, we have

$$\frac{d(w_{il+i'}^{t}-w_{il}^{t})}{d(w_{vl+v'}^{t+1}-w_{vl}^{t+1})}=0$$

if $i'<v'$. Hence, all the blocks to the right of the diagonal are zero. For the diagonal blocks, we have

$$\begin{aligned}\left(\phi_{\Delta w_{i'}}^{\Delta w_{i'}}\right)_{i+1,v+1} &= \frac{d(w_{il+i'}^{t}-w_{il}^{t})}{d(w_{vl+i'}^{t+1}-w_{vl}^{t+1})}\\ &= \sum_{j\in\mathbb{B}\backslash B}\frac{\pi_{jl}^{*}}{N}u'(\beta\Delta w^{*}(jl,vl-jl))\beta\lambda^{*}(vl+i'-jl;il+i',jl)\\ &+ \sum_{j\in\mathbb{B}\backslash B}\frac{\pi_{jl}^{*}}{N}\beta\lambda^{*}(il-vl;il+i',jl)\\ &+ \frac{\pi_{Bl}^{*}+N-1}{N}\beta\mathbf{1}\{i=v\} \qquad (38)\\ &\geq 0,\end{aligned}$$

where the terms in the summation correspond to inflows into $vl+i'$ generated by meetings $(il+i',jl)$ for $j\in\mathbb{B}\backslash B$. ■

Proof of Lemma 4. (i) Suppose $\eta=(z,\eta_2)$ is the eigenvector of A associated with a unit eigenvalue of $\Psi_{\pi_N}^{\pi_N}$:

$$\begin{bmatrix}\Psi_{\pi_N}^{\pi_N} & 0\\ -[\phi_{\Delta w}^{\Delta w}]^{-1}\phi_{\pi_N}^{\Delta w} & [\phi_{\Delta w}^{\Delta w}]^{-1}\end{bmatrix}\begin{bmatrix}z\\ \eta_2\end{bmatrix}=\begin{bmatrix}z\\ \eta_2\end{bmatrix}.$$

Hence, we have $-[\phi_{\Delta w}^{\Delta w}]^{-1}\phi_{\pi_N}^{\Delta w}z+[\phi_{\Delta w}^{\Delta w}]^{-1}\eta_2=\eta_2$. Note that the inverse of the lower-triangular matrix is a lower-triangular matrix. Therefore, we have

$$\eta_2=(I-\phi_{\Delta w}^{\Delta w})^{-1}\phi_{\pi_N}^{\Delta w}z=\begin{bmatrix}0\\ \vdots\\ 0\\ (I-\phi_{\Delta w_{l-1}}^{\Delta w_{l-1}})^{-1}Kz_1\end{bmatrix}+\cdots+\begin{bmatrix}(I-\phi_{\Delta w_1}^{\Delta w_1})^{-1}Kz_{l-1}\\ *\\ \vdots\\ *\end{bmatrix}.$$

(ii) Assume by way of contradiction that $(\varphi_{\Delta w_{i'}})_i=0$ for some i. By the definition of φ, We have

$$(I-\phi_{\Delta w_{i'}}^{\Delta w_{i'}})\varphi_{\Delta w_{i'}}=Kz_{l-i'}, \qquad (39)$$

where the r.h.s. is strictly positive because of $z_{l-i'} \geq 0$ and Lemma 3 (ii). The i-th element of the l.h.s. of (39) is $-\sum_{v\neq i}\left(\phi^{\Delta w_{i'}}_{\Delta w_{i'}}\right)_{i,v}(\varphi_{\Delta w_{i'}})_v$ which cannot be strictly positive because $\left(\phi^{\Delta w_{i'}}_{\Delta w_{i'}}\right)_{i,v}$ is non-negative for $v \neq i$ (Lemma 3 (iii)).

(iii) Suppose by contradiction $\varphi_{\Delta w_{i'}} \geq 0$. (ii) implies $\varphi_{\Delta w_{i'}} > 0$. By letting $i = 0$ and $\lambda^*(i'; i', 0) = 1$ in (38), it can be shown that the first element of $(I - \phi^{\Delta w_{i'}}_{\Delta w_{i'}})\varphi_{\Delta w_{i'}}$ is

$$\left[1 - \beta\left(\frac{\pi^*_{Bl} + N - 1 + \pi^*_0 u'(0)}{N}\right)\right](\varphi_{\Delta w_{i'}})_1 - \sum_{v=1}^{B-1}\left(\phi^{\Delta w_{i'}}_{\Delta w_{i'}}\right)_{i+1,v+1}(\varphi_{\Delta w_{i'}})_{v+1} < 0, \tag{40}$$

since $u'(0)$ is assumed to be any large number.($u'(0) > [N/\beta - N + (1 - \pi^*_{Bl})]/\pi^*_0$ is sufficient here. This condition resembles (13) in the example section.) However, the strict inequality (40) contradicts the fact that the r.h.s. of (39) is strictly positive. ■

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