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16 April 2013

# Trejos-Wright with a 2-unit bound: existence and stability of monetary steady states

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April 27, 2013

**Abstract**

We investigate in details a Trejos-Wright random matching model of money with a consumer take-it-or-leave-it offer and the individual money holding set $\{0, 1, 2\}$. First we show generic existence of three kinds of steady states: (1) pure-strategy full-support steady states, (2) mixed-strategy full-support steady states, and (3) non-full-support steady states, and then we show relations between them. Finally we provide stability analyses. It is shown that (1) and (2) are locally stable, (1) being also determinate. (3) is shown to be unstable. (JEL classification: C62, C78, E40)

Keywords: random matching model; monetary steady state; local stability; determinacy; instability; Zhu (2003).

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‡The authors especially thank Neil Wallace for his guidance and encouragement. We are also grateful to Daniella Puzzello, Ed Green, Ricardo Cavalcanti, Rulin Zhou, and seminar participants at Cornell University and Penn State University for helpful comments and discussions.

# 1 Introduction

Trejos and Wright (1995) shows the existence of a monetary steady state in a random matching model under the assumption that an agent's money holding is in $\{0, 1\}$. In the same model, for consumer take-it-or-leave-it offers and for money holdings in $\{0, 1, \cdots, B\}$, Zhu (2003) provides sufficient conditions for existence of a full-support monetary steady state with a strictly increasing and strictly concave value function. By way of a variant of a neutrality argument, his result also implies the existence of non-full-support steady states in which all agents treat bundles of money, each bundle being $B/l \in \mathbb{N}$ units, as the smallest unit held and traded.

Among the questions that Zhu's existence result leaves open are the following. First, are his full-support steady states unique? Second, do both pure-strategy and mixed-strategy steady states exist generically? Third, are full-support steady states stable? Fourth, are the above non-full-support steady states stable? The smallest set of money holdings for which these questions arise is $\{0, 1, 2\}$, the smallest set for which the distribution of money holdings over people depends on the trades that are made. For this set, we answer all but the first question.

Under a condition that is weaker than Zhu's sufficient conditions, a full-support steady state exists. Both pure-strategy and mixed-strategy full-support steady states exist generically and any full-support steady state is stable. The non-full-support steady state, which necessarily has support $\{0, 2\}$, is unstable. Although the two-unit bound is restrictive, it, at least, provides conjectures for the general case.

# 2 The Zhu (2003) model

Time is discrete, dated as $t \geq 0$. There is a unit measure of non-atomic agents who are infinitely-lived. Also, there are divisible and non-storable consumption goods at each date. Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. At each date, if an agent produces an amount $q \geq 0$ of the good, the utility cost is $q$. If an agent consumes an amount $q \geq 0$ of the good, the period utility he gets is $u(q)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable on $\mathbb{R}_+$. Also, $u(0) = 0$, $u'(\infty) = 0$ and $u'(0)$ is sufficiently large but finite.[1] These

[1] The assumption $u'(0) < \infty$ is used only in the proof of proposition 2.

assumptions imply that there is a unique $\bar{x} > 0$ such that $u(\bar{x}) = \bar{x}$.

There exists a fixed stock of indivisible money that is perfectly durable. There is a bound on individual money holdings, denoted $B \in \mathbb{N}$, so the individual money-holding set is $\mathbb{B} \equiv \{0, 1, \cdots, B\}$. Let $m \in (0, 1)$ denote the per capita stock of money divided by the bound on individual money holdings so that the per capita stock is $Bm$.

In each period, agents are randomly matched in pairs. With probability $1/n$, where $n \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meetings. With probability $1 - 2/n$, the match is a no-coincidence meeting.[2] In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

Consider a date-$t$ single-coincidence meeting between a consumer (potential buyer) with $i$ units of money (pre-trade) and a producer (potential seller) with $j$ units of money (pre-trade), an $(i, j)$-*meeting*. If $i > 0$ and $j < B$, the meeting is called a *trade meeting*. In trade meetings, the consumer makes a take-it-or-leave-it offer. (There are no lotteries.) The producer accepts or rejects the offer. If the producer rejects it, both sides leave the meeting and go on to the next date.

For each $k \in \mathbb{B}$, let $w_k^t$ be the expected discounted value of holding $k$ units of money prior to date-$t$ matching. Using $w_k^t$'s, the consumer's problem in an $(i, j)$-meeting is

$$\max_{p \in \Gamma(i,j), q \in \mathbb{R}_+} \{u(q) + \beta w_{i-p}^{t+1}\} \tag{1}$$

$$\text{s.t.} \; -q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}, \tag{2}$$

where $\Gamma(i, j) \equiv \{p \in \mathbb{B} | p \leq \min\{i, B - j\}\}$ is the set of feasible payments. As (2) holds with equality in the solution, the consumer's problem reduces to

$$f^t(i, j) \equiv \max_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}$$

$$P^t(i, j) \equiv \operatorname*{argmax}_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{3}$$

Because the solution $P^t(i, j)$ may be multi-valued, Zhu introduces randomization. Let $\Lambda^t(i, j)$ denote the set of probability distributions on $P^t(i, j)$. A

[2] If $n \geq 3$, one foundation is that there are $n$ types of agents and $n$ types of consumption goods, that type-$k$ agents can produce type-$k$ goods only and consume type-$(k+1)$ goods only, and that the money is symmetrically distributed across the types.

mapping $\lambda^t$ is called (consumer's) optimal *strategy* if it maps each $(i, j) \in \mathbb{B} \times \mathbb{B}$ to an element of $\Lambda^t(i, j)$, so that

$$\sum_{p \in P^t(i,j)} \lambda^t(p; i, j) = 1. \tag{4}$$

For each $z \in \mathbb{B}$, let $\pi_z^t$ denote the fraction of agents holding $z$ units of money at the start of period $t$, so that $\pi^t$ is a probability distribution on $\mathbb{B}$ with mean $Bm$. Given a strategy, the law of motion for $\pi^{t+1}$ can be expressed as

$$\pi_z^{t+1} = \frac{n-2}{n}\pi_z^t + \frac{2}{n}\sum_{i=0}^{B}\sum_{j=0}^{B}\pi_i^t\pi_j^t\frac{\lambda^t(i-z; i, j) + \lambda^t(z-j; i, j)}{2}. \tag{5}$$

The second term of (5) tells who in single-coincidence meetings will end up with $z$ units: consumers who originally had $i$ units and spent $i - z$ units and producers who originally had $j$ units and acquired $z - j$ units.

The value function $w^t$ satisfies the Bellman equation

$$w_i^t = \frac{n-1}{n}\beta w_i^{t+1} + \frac{1}{n}\sum_{j=0}^{B}\pi_j^t f^t(i, j). \tag{6}$$

The first term of the r.h.s corresponds to either entering a no-coincidence meeting or becoming a producer, who is indifferent between trading and not trading. When $i = 0$, equation (6) reduces to $w_0^t = \beta w_0^{t+1}$, so the only nonexplosive case is $w_0^t = 0, \forall t$. For this reason, we focus on equilibria in which the value from owning no money is always zero and let $w^t \equiv (w_1^t, \cdots, w_B^t)$. Finally, we allow free disposal of money and consider equilibria in which agents are not willing to throw away money. That is, the value function must be nondecreasing in every period:

$$w_B^t \geq \cdots \geq w_1^t \geq w_0^t = 0. \tag{7}$$

**Definition 1** *Given $\pi^0$, an* equilibrium *is a sequence $\{(\pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies the consumer's optimality condition (4), the law of motion (5), the Bellman equation (6), and non-disposal of money (7). A tuple $(\pi, w)$ is a* monetary steady state *if $(\pi^t, w^t) = (\pi, w)$ for $t \geq 0$ is an equilibrium and $w \neq 0$. Pure-strategy steady states are those for which (3) has a unique solution for all meetings. Other steady states are called mixed-strategy steady states.*

# 3 Monetary steady states when $B = 2$

In Trejos and Wright (1995), the case $B = 1$, a necessary and sufficient condition for existence of a monetary steady state is

$$u'(0) > \frac{n(1-\beta)}{\beta(1-m)} + 1. \tag{8}$$

Our proposition says that (8) is also necessary and sufficient for existence of a full-support steady state in economy $B = 2$. To state it, it is helpful to express $\pi_0$ and $\pi_2$ in terms of $\pi_1$ using $\sum \pi_i = 1$ and $\sum i\pi_i = Bm$. We have

$$\begin{aligned} (\pi_0, \pi_2) &= (1 - m - \frac{\pi_1}{2}, m - \frac{\pi_1}{2}) && (9) \\ \text{where } \pi_1 &\in \Pi \equiv [0, 2\min\{m, 1-m\}]. && (10) \end{aligned}$$

Throughout this paper, the dependence of $\pi$ on $\pi_1$ is kept implicit to simplify the notations.

First we state two key equations regarding $\pi_1$, $w_1$ and $w_2$:

$$\begin{aligned} &[n(1-\beta) + (1-\pi_2)\beta]w_1 = \\ &\qquad \pi_0 u(\beta w_1) + \pi_1 u\left(\frac{(1-\pi_2)\beta}{n(1-\beta) + (1-\pi_2)\beta}\beta w_1\right) \end{aligned} \tag{11}$$

and

$$w_2 = \left(\frac{(1-\pi_2)\beta}{n(1-\beta) + (1-\pi_2)\beta} + 1\right) w_1. \tag{12}$$

For a given $\pi_1$, equation (11) has at most one positive solution for $w_1$. If it has a positive solution for $w_1$, then equation (12) defines positive $w_2$. Let

$$\pi_1^* \equiv (\sqrt{1 + 12m(1-m)} - 1)/3. \tag{13}$$

and let $(w_1^*, w_2^*)$ denote the positive solution to (11)-(12) for $\pi_1 = \pi_1^*$.

**Proposition 1** *Inequality (8) is necessary and sufficient for (i) existence and uniqueness of a monetary steady state with support $\{0, 2\}$ and (ii) existence of a full-support monetary steady state. Under (8), a pure-strategy full-support steady state exists if $w_1^*$ exists and satisfies*

$$u(\beta w_2^*) - u(\beta w_1^*) < \beta w_1^*. \tag{14}$$

*It is a unique pure-strategy full-support steady state and is given by $(\pi^*, w^*)$. Otherwise there is a mixed-strategy full-support steady state.*

When $\beta$ is sufficiently close to one, the pure-strategy full-support steady state exists. To see this, fix all parameters except $\beta$ and let $\pi = \pi^*$. As $\beta \to 1$, equation (11) approaches $w_1^* = u(w_1^*)$, (12) approaches $w_2^* = 2w_1^*$, and (14) approaches $u(2w_1^*) < u(w_1^*) + w_1^*$. By strict concavity of $u$, this last inequality holds and hence the pure-strategy full-support steady state exists.

Although those inequality conditions for existence are stated in terms of primitives, it is helpful to have an example to show that (14) may or may not hold. Let $n = 2$, and $u(y) = y^{1/2}$. For such utility function, (11) and (12) can be explicitly solved, and the condition (14) for $(m, \beta)$ can be explicitly derived. Figure **??** shows that there are open regions of $(m, \beta)$ in which (14) holds and regions in which it does not hold. Moreover, this ought to be true for $u$ functions "close to" $u(y) = y^{1/2}$. This implies genericity of both kinds of full-support steady states.

Although the full-support steady states computed in figure **??** seem to be unique, we have been unable to establish such uniqueness in general. Nor do we have an example of multiplicity.

# 4 Stability

Our stability criterion is as follows.

**Definition 2** *A steady state $(\pi, w)$ is locally stable if there is a neighborhood of $\pi$ such that for any initial distribution in the neighborhood, there is an equilibrium path such that $(\pi^t, w^t) \to (\pi, w)$. A locally stable steady state is determinate, if for each initial distribution in this neighborhood, there is only one equilibrium that converges to it.*

This definition of stability only requires convergence of some equilibria, not all equilibria. This is because there are always equilibria that do not converge to a given monetary steady state. In particular, a non-monetary equilibrium always exists from any initial condition.

Notice that the above definition of local stability implies that the valued-money steady state in the Trejos-Wright $\{0, 1\}$ model is stable, because there is no 'neighborhood' of the steady state. Also, for that model, the only non-explosive path converging to that steady state is the one in which the value of money remains constant, which implies determinacy of that steady state.[3]

[3]For the Trejos-Wright $\{0, 1\}$ model, Lomeli and Temzelides (2002) show that the non-monetary steady state is indeterminate.

The following is our stability results for the $\{0, 1, 2\}$ economy.

**Proposition 2** *Full-support steady states are locally stable. The non-full-support steady state is unstable. Moreover, the pure-strategy full-support steady state is determinate.*

The standard approach to stability analysis of difference equation systems (see, for example, [5]) is to compare the number of eigenvalues of the dynamical system that are strictly smaller than one in absolute value, say $a$, and the number of initial conditions, say $b$. If $a = b$ ($a > b$), then there is a unique (an infinity of) convergent path(s). If $a < b$, then there is no convergent solution. This standard approach is applied to establish local stability of the pure-strategy full-support steady state.

The stability of mixed-strategy steady state is proved by showing that the mixed-strategy steady state can be attained in one step. The statement about non-full-support steady state shows that if the economy starts with a positive measure of people holding one unit of money, then the economy does not converge to the steady state in which a bundle of two units of money is treated as one in $\{0, 1\}$ model. The proof is by way of contradiction and relies on two features. First, the dynamical system necessarily involves *unit-root convergence* because the outflow from holdings of 1 unit, which comes from $(1, 1)$-meetings, approaches zero as the frequency of such meetings goes to zero. Second, the non-full-support steady state is on the *boundary* of the state space in two senses: the distribution does not have full support and the value of money is not strictly increasing. Hence, a convergent sequence must at all dates satisfy $\pi_1^t \geq 0$ and (7).

# 5 Proofs

Before turning to the proofs, we set out some steady state consequences that we use in the proofs. The steady-state law of motion reduces to

$$(\pi_1)^2 \lambda(1; 1, 1) = \left(1 - m - \frac{\pi_1}{2}\right) \left(m - \frac{\pi_1}{2}\right) \lambda(1; 2, 0), \tag{15}$$

which equates outflows from holdings of 1 (the lefthand side) to inflows into holdings of 1 (the righthand side). The Bellman equations are

$$w_1 = \frac{n-1+\pi_2}{n}\beta w_1 + \frac{\pi_0}{n}\max[u(\beta w_1), \beta w_1] \tag{16}$$
$$+\frac{\pi_1}{n}\max[u(\beta w_2 - \beta w_1), \beta w_1], \quad \text{and}$$

$$w_2 = \frac{n-1+\pi_2}{n}\beta w_2 + \frac{\pi_1}{n}\max[\{u(\beta w_2 - \beta w_1) + \beta w_1\}, \beta w_2]$$
$$+\frac{\pi_0}{n}\max[u(\beta w_2), \{u(\beta w_1) + \beta w_1\}, \beta w_2]. \tag{17}$$

As to full-support steady states, Lemma 1 will establish that zero-unit payment is suboptimal and one-unit payment is optimal in all trade meetings in any full-support steady state, two-unit payment in $(2,0)$-meetings being also optimal for a mixed-strategy full-support steady state. Corresponding inequalities are

$$\begin{array}{lrcl}
(1,1)\text{-meeting} & u(\beta w_2 - \beta w_1) & > & \beta w_1 \quad (18) \\
(1,0)\text{-meeting} & u(\beta w_1) & > & \beta w_1 \quad (19) \\
(2,1)\text{-meeting} & u(\beta w_2 - \beta w_1) & > & \beta w_2 - \beta w_1 \quad (20) \\
(2,0)\text{-meeting} & u(\beta w_1) + \beta w_1 & \geq & u(\beta w_2) \quad (21) \\
 & \& \; u(\beta w_1) + \beta w_1 & > & \beta w_2. \quad (22)
\end{array}$$

If these inqualities hold, the Bellman equation (16)-(17) becomes (11)-(12). Claims 1 and 2 are used in lemma 1.

**Claim 1** *If equations (11) and (12) are satisfied for some $\pi$ such that $\pi_1 > 0$, then (18) and (22) hold.*

**Proof.** Suppose by way of contradiction that (18) does not hold:

$$u(\beta w_2 - \beta w_1) = u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right) \leq \beta w_1.$$

Then, we have

$$\begin{aligned}
\beta w_1 &< \frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}u(\beta w_1) \\
&< u\left(\frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\beta w_1\right) \\
&< u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right) = u(\beta w_2 - \beta w_1),
\end{aligned}$$

where the first inequality is by substituting the supposition into (11) and the second is by $u(0) = 0$ and strict concavity of $u$. This is contradiction and thus (18) should hold.

Inequality (22) follows from

$$\begin{aligned} u(\beta w_1) &> u(\beta w_2 - \beta w_1) \\ &> \beta w_1 \\ &> \beta w_2 - \beta w_1, \end{aligned}$$

where the first and the third inequalities are by (12) and the second is (18). ■

**Claim 2** *Inequalities (18) and*

$$u(\beta w_1) + \beta w_1 \geq \beta w_2 \tag{23}$$

*imply (19) and (20).*

**Proof.** Suppose by way of contradiction that (19) does not hold: $u(\beta w_1) \leq \beta w_1$. Then (18) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with (23) gives $u(\beta w_1) > \beta w_1$, which is a contradiction.

Suppose by contradiction that (20) does not hold: $u(\beta w_2 - \beta w_1) \leq \beta w_2 - \beta w_1$. Then (23) implies $\beta w_2 - \beta w_1 \leq \beta w_1$. But (18) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction. ■

**Lemma 1** *For any monetary steady state with a full-support distribution, the solution set to (3) for (2,0)-meetings is either $\{1\}$ or $\{1, 2\}$. The solution set to (3) for other trade meetings is $\{1\}$. Any monetary steady state with a non-full-support distribution has $w_1 = 0$ and $w_2$ that is the unique positive solution to*

$$w = \frac{n - 1 + m}{n} \beta w + \frac{1 - m}{n} u(\beta w). \tag{24}$$

**Proof.** We need to show that any full-support monetary steady state satisfies (18)-(22). We start from (18) and assume by way of contradiction that it does not hold. Then (19) must hold, because substituting (19) with a reversed weak inequality and the supposition into (16) gives $0 = w_1 = w_2$, a contradiction to being a monetary steady state. Then the supposition and (19) gives

$$\beta w_2 - \beta w_1 < \beta w_1. \tag{25}$$

Note that (19) implies $0 < \beta w_1 < \bar{x}$, with $\bar{x} = u(\bar{x})$. Thus we have $0 \leq \beta w_2 - \beta w_1 < \bar{x}$, which in turn implies (20) with weak inequality. This weak inequality and (25) gives (22). Because $u$ is strictly concave, that (18) does not hold implies $u(\beta w_2) - u(\beta w_1) < \beta w_1$. This together with (22) implies $\lambda(1; 2, 0) = 1$. For $\pi_1$ to be strictly positive in (15), we must have $\lambda(1; 1, 1) > 0$ and hence our supposition implies (18) must hold with equality. So far, trading one unit of money is optimal in all trade meetings. Bellman equation (16)-(17) implies equations (11) and (12) for some full-support $\pi > 0$. Claim 1 implies (18), a contradiction.

From (18), we can show the remaining inequalities. Because (18) implies $\lambda(1; 1, 1) = 1$, for $\pi_1$ to be strictly positive in (15), we must have (21) and (23). Claim 2 gives (19) and (20). Therefore, trading one unit is optimal in all trade meetings and Bellman equation becomes equations (11)-(12) with full-support $\pi > 0$, so claim 1 implies (22). Overall (18)-(22) are necessary conditions for a full-support monetary steady state.

We turn to non-full-support steady states. Support $\{0, 2\}$ implies $(\pi_0, \pi_1, \pi_2) = (1 - m, 0, m)$ and hence $\lambda(1; 2, 0) = 0$ follows from (15). Equations (16)-(17) imply that both $w_1$ and $w_2$ must satisfy $w = \frac{n-1+m}{n}\beta w + \frac{1-m}{n}\max[u(\beta w), \beta w]$.

Suppose by way of contradiction that $w_1 > 0$. It must be $u(\beta w_1) > \beta w_1$, because otherwise (24) implies $w_1 = 0$. Similarly, we have $u(\beta w_2) > \beta w_2$. Then both $w_1$ and $w_2$ are the unique positive solution to (24), which leads to $w_1 = w_2$. This contradicts to $\lambda(2; 2, 0) = 1$, because one-unit payment would be strictly better in $(2, 0)$-meetings in such a case. Thus $w_1 = 0$. Then setting $w_1 = \pi_1 = 0$ in Bellman equation (17) implies that $w_2$ must be the unique positive solution to (24). ■

The proof of proposition 1 uses the intermediate function theorem to construct full-support steady states.

**Proof of proposition 1.** It is not hard to see that $w_1 = 0$ and $w_2$ as a positive solution to (24) satisfy (18), (20), $u(\beta w_2) > \beta w_2$, and $u(\beta w_1) \geq \beta w_1$. Therefore, by lemma 1, a non-full-support monetary steady state exists if and only if (24) has a strictly postive solution. Differentiating (24) at $w = 0$ gives an equivalent condition (8) for the existence of such a solution.

Now we turn to full-support steady states. First we show necessity of (8). By lemma 1, full-support steady states satisfy (18)-(22). If all these optimality conditions are substituted into (16) and (17), then one can get (11), which must have a (unique) positive solution for some $\pi_1 > 0$. Differentiating (11) at $w_1 = 0$ gives the necessary and sufficient condition for such existence:

$$u'(0) > \frac{[n(1-\beta)+(1-\pi_2)\beta]^2}{\{\pi_0[n(1-\beta)+(1-\pi_2)\beta]+\pi_1(1-\pi_2)\beta\}\beta}. \tag{26}$$

Subtracting the r.h.s. of (8) from the r.h.s. of (26) gives

$$\frac{n(1-\beta)}{\beta}\cdot\frac{\pi_1 n(1-\beta)+\beta\pi_1\pi_0}{[\pi_0 n(1-\beta)+\beta(1-\pi_2)^2](2-2m)} > 0.$$

Therefore (8) is implied by (26) and is necessary.

Now we consider sufficiency. Our argument uses the intermediate value theorem to show the existence of full-support steady state. Under (8), when $\pi_1 = 0$, (11) and (12) have a (unique) positive solution $(\hat{w}_1, \hat{w}_2)$. Differentiation at such a solution gives

$$u'(\beta\hat{w}_1) < \frac{n(1-\beta)}{\beta(1-m)} + 1. \tag{27}$$

Then by the mean value theorem, we have

$$\begin{aligned} u(\beta\hat{w}_1)+\beta\hat{w}_1-u(\beta\hat{w}_2) &= \beta\hat{w}_1 - u'(\xi)(\beta\hat{w}_2-\beta\hat{w}_1), \quad \xi\in(\beta\hat{w}_1,\beta\hat{w}_2)\\ &> \beta\hat{w}_1 - u'(\beta\hat{w}_1)(\beta\hat{w}_2-\beta\hat{w}_1)\\ &> 0, \end{aligned}$$

where the second inequality follows from (12) and (27). Therefore $(\hat{w}_1, \hat{w}_2)$ satisfies $u(\beta\hat{w}_1)+\beta\hat{w}_1 > u(\beta\hat{w}_2)$.

As $\pi_1$ increases, the solution $(\hat{w}_1, \hat{w}_2)$ as a function of $\pi_1$ changes continuously. Suppose that there exists $\bar{\pi}_1 \in (0, \pi_1^*)$ such that the l.h.s. and r.h.s. of (26) are equal and such that $(\hat{w}_1, \hat{w}_2)$ exists for all $\pi_1 \in (0, \bar{\pi}_1)$.[4] For such $\bar{\pi}_1$, the r.h.s. of (11) as a function of $w_1$ should be tangent to the l.h.s., and therefore, as $\pi_1$ approaches $\bar{\pi}_1$, $(\hat{w}_1, \hat{w}_2)$ approaches a zero vector. Differentiating (11) for $\pi = \bar{\pi}_1$ at $w_1 = 0$ gives

$$\begin{aligned} &\frac{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}{\beta(1-m+\frac{\bar{\pi}_1}{2})}\\ &= \left[\frac{1-m-\frac{\bar{\pi}_1}{2}}{1-m+\frac{\bar{\pi}_1}{2}} + \frac{\beta\bar{\pi}_1}{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}\right]u'(0), \end{aligned} \tag{28}$$

[4]If $w_1^*$ does not exist, which means that (26) does not hold for $\pi_1 = \pi_1^*$, then such $\bar{\pi}_1$ exists.

where the coefficient of $u'(0)$ is proven to be smaller than one for any $n > 0$. This implies

$$u'(0) > \frac{n(1-\beta) + \beta(1-m+\frac{\bar{\pi}_1}{2})}{\beta(1-m+\frac{\bar{\pi}_1}{2})}. \tag{29}$$

Concavity of $u$ and (12) give the following inequality:

$$\begin{aligned} \frac{u(\beta\hat{w}_1) + \beta\hat{w}_1 - u(\beta\hat{w}_2)}{\beta\hat{w}_1} &< \frac{\beta\hat{w}_1 - u'(\beta\hat{w}_2)\beta(\hat{w}_2 - \hat{w}_1)}{\beta\hat{w}_1} \\ \to \quad 1 - u'(0)\frac{\beta(1-m+\frac{\bar{\pi}_1}{2})}{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}&, \text{ as } \pi_1 \to \bar{\pi}_1. \end{aligned} \tag{30}$$

Because by (29) the above limit is strictly smaller than zero, we have $u(\beta\hat{w}_1) + \beta\hat{w}_1 < u(\beta\hat{w}_2)$, for $\pi_1$ sufficiently close to $\bar{\pi}_1$. In this case, the intermediate value theorem can be applied, and we can find a $\pi_1 > 0$ such that the solution satisfies

$$u\left(\beta\hat{w}_2\right) - u(\beta\hat{w}_1) = \beta\hat{w}_1. \tag{31}$$

Suppose now that such $\bar{\pi}_1$ does not exist, so that the positive solution to (11)-(12) $(\hat{w}_1, \hat{w}_2)$ exists for all $\pi_1 \in \Pi$ and in particular for $\pi_1 = \pi_1^*$. If (14) fails to hold so we have $u(\beta\hat{w}_1) + \beta\hat{w}_1 \leq u(\beta\hat{w}_2)$, for $\pi_1 = \pi_1^*$, then again the the intermediate value theorem implies (31) for some $\pi_1$.

Finally we show that such pair $(\pi, \hat{w})$ that satisfies (31) is a mixed-strategy full-support steady state. Claim 1 implies (18) and (22). Then claim 2 gives the remaining conditions (19)-(20). Therefore we have (18)-(22) with equality in (21). Corresponding $\lambda(1; 2, 0)$ is uniquely determined by (15).

We show that if we have (14), then $(\pi^*, w^*)$ is a pure-strategy full-support steady state. Claim 1 implies (18) and (22). Then claim 2 gives the remaining conditions (19)-(20). Therefore transferring one unit is strictly preferred in all trade meetings. By lemma 1, it is the unique pure-strategy full-support steady state.

Overall, a mixed-strategy steady state exist when pure-strategy steady state doesn't. ■

**Proof of Proposition 2.** First we show stability of the mixed-strategy full-support steady state. Suppose that the initial distribution $\pi_1^0$ is sufficiently close to the steady state distribution $\pi_1$. For the mixed-strategy steady state, agents can choose the initial randomization $\lambda^0(1; 2, 0)$ so that $\pi_1^0$ can jump to $\pi_1$ in one period, and $(\pi^t, w^t, \lambda^t) = (\pi, w, \lambda)$ for all $t \geq 1$. Such randomization is the optimal choice by the agents, because $w^1 = w$ satisfies the indifference condition. Then the initial value $w^0$ can be determined from the initial distribution $\pi_1^0$ and $w^1$ via the Bellman equation. (Note that $w^0$ does not affect agents' decisions.) Thus the mixed-strategy steady state is stable.

The proof of stability of the pure-strategy full-support steady state and the proof of instability of the non-full-support steady state share some common procedures. First, we will pin down the optimal trading strategy along a possible convergent equilibrium path. Based on that strategy, we construct a dynamical system and use linear approximation to study its dynamic properties.

For the pure-strategy full-support steady state, trading one unit in all trade meetings is a strictly preferred strategy at the steady state (see Definition 1 and Lemma 1), so it is also optimal in its neighborhood. That is, $\lambda^t(1;1,0) = \lambda^t(1;1,1) = \lambda^t(1;2,1) = \lambda^t(1;2,0) = 1$ for all $t \geq 0$.

Similarly, we can also pin down the optimal trading strategy that is constantly played along a path that converges to the non-full-support steady state, if there is any such path. To see this, suppose by way of contradiction that there exists an equilibrium path that converges to the non-full-support steady state (i.e., $\pi_1 = 0$ and $w_1 = 0$) from some initial distribution such that $\pi_1^0 \neq 0$. As is shown in the proof of proposition 1, trading one unit is strictly preferred in $(1,1)$- and $(2,1)$-meetings, and paying two units is strictly preferred in $(2,0)$-meetings at $(\pi, w)$. Therefore, they are also optimal in the neighborhood of $(\pi, w)$, so $\lambda^t(1;1,1) = \lambda^t(1;2,1) = \lambda^t(2;2,0) = 1$ for all $t \geq 0$. Moreover, the following argument shows $\lambda^t(1;1,0) = 1$ should be the case for all $t \geq 0$. When the economy is close to but not equal to $(\pi, w)$, we have $\pi_1^t > 0$ for all $t \geq 0$ so (6) implies $w_1^t > 0$ for all $t > 0$, because there is always a positive probability that consumer with one unit meets producer with one unit and the consumer can get positive amount of utility from such a meeting. Equation (8) implies $u(x) > x$ for all $x < \beta w_2$ and therefore $u(\beta w_1^t) > \beta w_1^t$ holds all along the path. So in $(1,0)$-meetings, paying one unit is strictly preferred to paying nothing along the path.

In both cases, a unique strategy is constantly played along any potential convergent path, so we can construct dynamic system from the law of motion and Bellman equation under the given strategy:

$$\begin{aligned}
\pi_1^{t+1} &= \pi_1^t - \frac{2(\pi_1^t)^2}{n} + \frac{2}{n}\left(1-m-\frac{\pi_1^t}{2}\right)\left(m-\frac{\pi_1^t}{2}\right)\lambda(1;2,0) &(32)\\
w_1^t &= \frac{n-1+\pi_2^t}{n}\beta w_1^{t+1} + \frac{\pi_0^t}{n}u(\beta w_1^{t+1}) + \frac{\pi_1^t}{n}u(\beta w_2^{t+1}-\beta w_1^{t+1}) &(33)\\
w_2^t &= \frac{n-1+\pi_2^t}{n}\beta w_2^{t+1} + \frac{\pi_0^t}{n}\max[u(\beta w_1^{t+1})+\beta w_1^{t+1}, u(\beta w_2^{t+1})]\\
&\quad + \frac{\pi_1^t}{n}[u(\beta w_2^{t+1}-\beta w_1^{t+1})+\beta w_1^{t+1}]. &(34)
\end{aligned}$$

We have $u(\beta w_1^{t+1}) + \beta w_1^{t+1} < u(\beta w_2^{t+1})$ and $\lambda(1;2,0) = 0$ for the non-full-support steady state, and $u(\beta w_1^{t+1}) + \beta w_1^{t+1} > u(\beta w_2^{t+1})$ and $\lambda(1;2,0) = 1$ for the pure-strategy full-support steady state. Denote (32) by $\pi_1^{t+1} = \Phi^\lambda(\pi_1^t) : \Pi \to \Pi$ and (33)-(34) by $w^t = \phi^\lambda(\pi_1^t, w^{t+1}) : \Pi \times W \to W$, where $w^t \equiv (w_1^t, w_2^t)$ and $W \equiv \{(w_1, w_2)|0 \le w_1 \le w_2\}$. As is ensured below, the implicit function theorem can be applied to (33)-(34) generically for both steady states. In the vicinity of each of the steady states, we can solve $w^{t+1}$ as a function of $(\pi_1^t, w^t)$ to obtain $w^{t+1} = \Psi^\lambda(\pi_1^t, w^t) : \Pi \times W \to W$. The joint system is

$$\begin{pmatrix}\pi_1^{t+1}\\ w^{t+1}\end{pmatrix} = \begin{pmatrix}\Phi^\lambda(\pi_1^t)\\ \Psi^\lambda(\pi_1^t, w^t)\end{pmatrix}.$$

Its jacobian is

$$A^\lambda \equiv \begin{bmatrix}\Phi_\pi^\lambda & O\\ -(\phi_w^\lambda)^{-1}\phi_\pi^\lambda & (\phi_w^\lambda)^{-1}\end{bmatrix}. \tag{35}$$

Straightforward differentiation leads to

$$\begin{aligned}
\Phi_\pi^\lambda(\pi_1) &= 1 - \frac{3\pi_1^*+1}{n}\lambda(1;2,0)\\
&= 1 - \frac{\sqrt{1+12m(1-m)}}{n}\lambda(1;2,0). &(36)
\end{aligned}$$

Thus the pure-strategy steady state has an eigenvalue strictly less than one and the non-full-support steady state has a unit eigenvalue. In what follows, we will compute the other two eigenvalues for the pure-strategy steady state and then turn to the non-full-support steady state.

For the pure-strategy steady state, we have $\lambda(1;2,0)=1$, and straightforward differentiation gives

$$\phi_w^\lambda = \begin{pmatrix} \frac{n-1+\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) & \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) \\ \frac{1-\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) & \frac{n-1+\pi_2^*}{n}\beta + \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) \end{pmatrix}, \tag{37}$$

where $\Delta w^* \equiv w_2^* - w_1^*$. This matrix is generically invertible, confirming that the application of the implicit function theorem and (35) are valid. Because the top-right submatrix of $A^\lambda$ is a zero matrix, one eigenvalue of $A$ is given by (36), which is smaller than one, and the other two eigenvalues are those of $(\phi_w^\lambda)^{-1}$, which are the reciprocals of eigenvalues of $\phi_w^\lambda$. In what follows, we are going to show that eigenvalues of $\phi_w^\lambda$ are smaller than one in absolute value.

That the slope of the r.h.s. of (11) at the positive fixed point $\beta w_1^*$ should be smaller than the slope of the l.h.s. gives

$$\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta} > \pi_0^* u'(\beta w_1^*) + \pi_1^* \frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta} u'(\beta\Delta w^*). \tag{38}$$

The eigenvalues of a general $2\times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by

$$\eta_+, \eta_- = \frac{a+d \pm \sqrt{(a-d)^2+4bc}}{2}.$$

Because

$$\begin{aligned} & (a-d)^2+4bc \\ = & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - 2\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]^2 \\ & +4\left[\frac{1-\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) \\ = & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)\right]^2 + 4\frac{1-\pi_2^*}{n}\beta\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) > 0, \end{aligned}$$

both eigenvalues are real. They are smaller than one in absolute value if and only if $a+d<2$ and $(1-a)(1-d)-bc>0$. Checking these conditions for

(37) gives

$$
\begin{aligned}
& 1-a+1-d \\
= \; & 2\left(1-\frac{n-1+\pi_2^*}{n}\beta\right)-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*) \\
> \; & 2\frac{n(1-\beta)+(1-\pi_2^*)\beta}{n}-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}\beta u'\left(\beta\Delta w^*\right) \\
> \; & \frac{n(1-\beta)+(1-\pi_2^*)\beta}{n}>0,
\end{aligned}
$$

$$
\begin{aligned}
& (1-a)(1-d)-bc \\
= \; & \left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)+\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right)\left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right) \\
& -\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\left[\frac{1-\pi_2^*}{n}\beta+\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right] \\
= \; & \frac{(n(1-\beta)+(1-\pi_2^*)\beta)\beta}{n^2}\times \\
& \quad \left(\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta}-\pi_0^* u'(\beta w_1^*)-\pi_1^*\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}u'(\beta\Delta w^*)\right) \\
> \; & 0,
\end{aligned}
$$

where the last inequalities of the above two conditions follow from (38). Therefore, the eigenvalues of $(\phi_w^\lambda)^{-1}$ are greater than one in absolute value. This full-support steady state has a one-dimensional stable manifold. Because we have one initial condition, this full-support steady state is locally stable and determinate.

Next we consider the non-full-support steady state. Equation (36) computes the unit eigenvalue for the law of motion. Furthermore, as figure **??** illustrates, the law of motion (32) features unit-root convergence; the slope at the fixed point is unity. Note also that this steady state is on the boundary of the state space $\Pi\times W$, which makes it necessary to explictly study the limiting behavior by seeing the eigenspace of the linearized system (35).[5]

First we compute

$$
\phi_\pi^\lambda=\begin{bmatrix}\frac{1}{n}u(\beta w_2)\\ \frac{1}{2n}[u(\beta w_2)-\beta w_2]\end{bmatrix}\equiv\begin{bmatrix}r\\ s\end{bmatrix}>0
$$

[5]Note that this analysis is not needed for the pure-strategy full-support steady state because that steady state is in the interior of $\Pi\times W$.

and

$$\phi_w^\lambda = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} \equiv \tag{39}$$

$$\begin{bmatrix} \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}\beta u'(0) & 0 \\ 0 & \frac{(n-1+m)\beta}{n} + \frac{1-m}{n} u'(\beta w_2)\beta \end{bmatrix}.$$

Because $w_2$ is a positive solution to (24), $a' > 1$ and $d' \in (0,1)$ hold. We have

$$A^\lambda = \begin{bmatrix} 1 & 0 & 0 \\ -r/a' & 1/a' & 0 \\ -s/d' & 0 & 1/d' \end{bmatrix}. \tag{40}$$

Since $\pi_1^0 \neq 0$ and the law of motion has unit root convergence, the convergence trajectory will eventually be parallel to the eigenspace of (40) associated with the unit eigenvalue[6]. The associated eigenvector, which constitutes a base of the space, has the form

$$\begin{bmatrix} 1 \\ \frac{-r}{a'-1} \\ \frac{s}{1-d'} \end{bmatrix}.$$

In the context of the discrete-time dynamical system theory, the unit root is a "border" case in which the higher-order terms should be examined. In our case, the higher-order term seems to imply unit-root convergence (i.e., Figure **??**). However, the fact that convergent trajectory of $(\pi_1^t, w_1^t, w_2^t - w_2)$ will be parallel to the above eigenvector implies that $\pi_1^t$ and $w_1^t$ will eventually have different signs, contradicting to $\pi_1^t, w_1^t > 0$ for all $t$. ■

# 6 Concluding remarks

We show that the necessary and sufficient condition for the monetary steady state of the Trejos-Wright $\{0,1\}$ economy, namely (8), is also necessary and sufficient for the existence of a full-support steady state of the $\{0,1,2\}$ economy, showing that Zhu (2003)'s sufficient condition is not necessary for the

[6]See Subsection "Dominant Eigenvector" on page 165 of [4].

bound of two. Moreover, both the pure-strategy and mixed-strategy full-support steady states are generic. Given our result, a reasonable conjecture should be that even for a higher bound, the condition (8) is necessary and sufficient for the existence of full-support steady states. For values of parameters that lead to lower values of money (i.e., high $n$, low $\beta$ and high $m$), randomizations may simply occur.

Generalizing Proposition 2 to a higher bound case is not simple. When the bound is two, we can identify candidate strategies that support steady states and get explicit expressions for the relevant difference-equation system. For a general bound, we do not know the supporting strategies. Therefore, if analogue proofs are to be provided, they must be constructed differently.[7]

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[7] [2] is an attempt to generalise the instability of non-full-support steady states (Proposition 2) to a general bound case.