# Robustness of Stability to cost of carrying money in a Matching Model of Money

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# Robustness of Stability in a Matching Model of Commodity Money

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**Abstract**

We study stability of monetary steady states in a random matching model of money where money is indivisible, the bound on individual money holding is finite, and the trading protocol is buyer take-it-or-leave-it offers. The class of steady states we study have a non-full-support money-holding distribution. It is shown that there is no equilibrium path that converges to such steady states if the initial distribution has a different support.
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## 1 Introduction

Trejos and Wright (1995) shows the existence of a monetary steady state in a model where an agent's money holding is in $\{0, 1\}$. For buyer take-it-or-leave-it offers in that model and money holding in $\{0, 1, \cdots, B\}$, Zhu (2003) provides sufficient conditions—one of which is that $B$ is sufficiently large—for the existence of a steady state with a full-support money-holding distribution and a strictly increasing and strictly concave value function.

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However, as Zhu shows, the existence of such a full-support steady state also implies the existence of non-full-support steady states by way of the following neutrality argument. Take the full-support steady state of an economy. Then consider a different economy by multiplying by some integer $l$ both the bound on individual money holdings and the total stock of money. Besides its own full-support steady state, the new economy has a non-full-support steady state that is identical to the original full-support steady state except that all owned/traded quantities of money are multiplied by $l$. Call this equilibrium an $l$-neutral replica. This paper provides a stability analysis of the steady states of the Zhu economy for the smallest bound with such multiplicity; namely, $B = 2$. This is also the smallest bound for which the distribution of money is endogenous in the sense that it depends on the trades that are made.[1]

Although the two-unit bound is restrictive, it is enough to demonstrate a sharp contrast between the full-support and the non-full-support steady states. The full-support steady state is locally stable and also determinate in the sense that the equilibrium path converging to it is uniquely determined. In contrast, the non-full-support steady state (which has support $\{0, 2\}$) is unstable; if we start with a nearby distribution, which is necessarily one with a positive measure of people with one unit of money, then there is no equilibrium path that converges to the non-full-support steady state.[2]

# 2 The Zhu (2003) Model

The model is that in Zhu (2003), where a small carry cost is introduced. Time is discrete, dated as $t \geq 0$. There is a non-atomic unit measure of infinitely-lived agents. There is consumption good that is perfectly divisible and perishable. Each agent maximizes the discounted sum of expected utility with discount factor $\beta \in (0, 1)$. Utility in a period is $u(c) - q$, where $c \in \mathbb{R}_+$ is the amount of good consumed and $q \in \mathbb{R}_+$ is the amount of good produced. $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing and strictly concave. Also, $u(0) = 0$, $u'(\infty) = 0$ and $u'(0)$ is sufficiently large but finite.[3]

[1] The multiplicity of steady states bears some resemblance to that in Green-Zhou (2002). However, the models are very different, as are the stability results.

[2] The only existing stability analysis is for $\{0, 1\}$ money holdings (see Lomeli and Temzelides (2002)).

[3] The assumption $u'(0) < \infty$ is used only in the proof of proposition 1.

These assumptions imply that there is a unique $\bar{x} > 0$ such that $u(\bar{x}) = \bar{x}$.

There exists a fixed stock of indivisible money that is perfectly durable and that can potentially serve as a medium of exchange. Storing each unit of money in each period incurs disutility $\gamma > 0$, that we assume to be sufficiently small in the model. There is a bound on individual money holdings, denoted $B \in \mathbb{N}$, so the individual money-holding set is $\mathbb{B} \equiv \{0, 1, \cdots, B\}$. Let $m \in (0, 1)$ denote the per capita stock of money divided by the bound on individual money holdings so that the per capita stock is $Bm$.

In each period, agents are randomly matched in pairs. With probability $1/N$, where $N \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meeting. With probability $1 - 2/N$, the match is a no-coincidence meeting.[4] In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

Consider a date-$t$ single-coincidence meeting between a consumer (potential buyer) with $i$ units of money (pre-trade) and a producer (potential seller) with $j$ units of money (pre-trade), an $(i, j)$-*meeting*. If $i > 0$ and $j < B$, the meeting is called a *trade meeting*. In trade meetings, the consumer makes a take-it-or-leave-it offer. (There are no lotteries.) The producer accepts or rejects the offer. If the producer rejects it, both sides leave the meeting and go on to the next date.

For each $k \in \mathbb{B}$, let $w_k^t$ be the expected discounted value of holding $k$ units of money prior to date-$t$ matching. Using $w_k^t$'s, the consumer's problem in an $(i, j)$-meeting is

$$\max_{p \in \Gamma(i,j), q \in \mathbb{R}_+} \{u(q) + \beta w_{i-p}^{t+1}\} \tag{1}$$

$$\text{s.t.} \; -q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}, \tag{2}$$

where $\Gamma(i, j) \equiv \{p \in \mathbb{B} | p \leq \min\{i, B - j\}\}$ is the set of feasible payments. As (2) holds with equality in the solution, the consumer's problem reduces to

$$f^t(i, j) \equiv \max_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}$$

$$p^t(i, j) = \operatorname*{argmax}_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{3}$$

[4] One foundation is that there are $N$ types of agents and $N$ types of consumption goods, that type-$n$ agents can produce type-$n$ goods only and consume type-$(n + 1)$ goods only, and that the money is symmetrically distributed across the types.

Because the solution $P^t(i,j)$ may be multi-valued, Zhu introduces randomization. Let $\Lambda^t(i,j)$ denote the set of probability distributions on $P^t(i,j)$. A mapping $\lambda^t$ is called (consumer's) optimal *strategy* if it maps each $(i,j) \in \mathbb{B} \times \mathbb{B}$ to an element of $\Lambda^t(i,j)$, so that

$$\sum_{p \in p^t(i,j)} \lambda^t(p; i, j) = 1. \tag{4}$$

For each $z \in \mathbb{B}$, let $\pi_z^t$ denote the fraction of agents holding $z$ units of money at the start of period $t$, so that $\pi^t$ is a probability distribution on $\mathbb{B}$ with mean $Bm$. Given a strategy, the law of motion for $\pi^{t+1}$ can be expressed as

$$\pi_z^{t+1} = \frac{N-2}{N}\pi_z^t + \frac{2}{N}\sum_{i=0}^{B}\sum_{j=0}^{B}\pi_i^t\pi_j^t\frac{\lambda^t(i-z;i,j)+\lambda^t(z-j;i,j)}{2}. \tag{5}$$

The second term of (5) tells who in single-coincidence meetings will end up with $z$ units: consumers who originally had $i$ units and spent $i-z$ units and producers who originally had $j$ units and acquired $z-j$ units.

The value function $w^t$ satisfies the Bellman equation

$$w_i^t = \frac{N-1}{N}\beta w_i^{t+1} + \frac{1}{N}\sum_{j=0}^{B}\pi_j^t f^t(i,j) - \gamma i, \tag{6}$$

The first term of the r.h.s corresponds to either entering a no-coincidence meeting or becoming a producer, who is indifferent between trading and not trading. When $i=0$, equation (6) reduces to $w_0^t = \beta w_0^{t+1}$, so the only nonexplosive case is $w_0^t = 0, \forall t$. For this reason, we focus on equilibria in which the value from owning no money is always zero and let $w^t \equiv (w_1^t, \cdots, w_B^t)$. Finally, we allow free disposal of money and consider equilibria in which agents are not willing to throw away money. That is, the value function must be nondecreasing in every period:

$$w_i^t \geq w_{i-1}^t, \text{ for } i = 1, \cdots B, \text{ and } w_0^t = 0. \tag{7}$$

**Definition 1** *Given $\pi^0$, an* equilibrium *is a sequence $\{(\pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies (3)-(7). A tuple $(\pi, w)$ is a* monetary steady state *if $(\pi^t, w^t) = (\pi, w)$ for $t \geq 0$ is an equilibrium and $w \neq 0$. Pure-strategy steady states are those for which (3) has a unique solution. Other steady states are called mixed-strategy steady states.*

## 3 Steady States for $\mathbb{B} = \{0, 1, 2\}$

In Trejos and Wright (1995), the case $B = 1$, a necessary and sufficient condition for existence of a monetary steady state is

$$u'(0) > \frac{n(1-\beta)}{\beta(1-m)} + 1. \tag{8}$$

Proposition 1 says that (8) is also a necessary and sufficient condition, under which a full-support steady state exists in economy $B = 2$ for a sufficiently small $\gamma > 0$, a model of commodity money. To state it, it is helpful to express $\pi_0$ and $\pi_2$ in terms of $\pi_1$ using $\sum \pi_i = 1$ and $\sum i\pi_i = Bm$. We have

$$\begin{aligned} (\pi_0, \pi_2) &= (1 - m - \frac{\pi_1}{2}, m - \frac{\pi_1}{2}) \\ \text{where } \pi_1 &\in \Pi \equiv [0, 2\min\{m, 1-m\}]. \end{aligned} \tag{9}$$

Throughout this paper, the dependence of $\pi$ on $\pi_1$ is kept implicit to simplify the notations.

Proposition 1 in [3] shows the existence of three steady states if $\gamma = 0$: pure-strategy, mixed-strategy, and non-full-support steady states. The following implies that these results are robust to the introduction of a small cost of carrying money.

**Proposition 1** *Generically, both pure and mixed strategy steady states exist.*

In particular, under (8), a pure-strategy full-support steady state exists for some $\gamma > 0$ if $w_1^*$ exists and satisfies

$$u(\beta w_2^*) - u(\beta w_1^*) < \beta w_1^*. \tag{10}$$

It is a unique pure-strategy full-support steady state and is given by $(\pi^*, w^*)$. Otherwise there is a mixed-strategy full-support steady state.

Small carrying cost is exogenous variable or endogenous variable?
Discussion about change in the nonfull-support steady state.
Some existence result in OLG model should be very helpful.

# 4 Stability

Our stability criterion is standard.

**Definition 2** *A steady state $(\pi, w)$ is locally stable if there is a neighborhood of $\pi$ such that for any initial distribution in the neighborhood, there is an equilibrium path such that $(\pi^t, w^t) \to (\pi, w)$. A locally stable steady state is determinate, if for each initial distribution in this neighborhood, there is only one equilibrium that converges to it.*

This definition of stability only requires convergence of some equilibria, not all equilibria. This is because there are always equilibria that do not converge to a given monetary steady state. In particular, a non-monetary equilibrium always exists from any initial condition.

The entire sequence is within such neighborhood.

Notice that the above definition of local stability implies that the valued-money steady state in the Trejos-Wright $\{0, 1\}$ model is stable, because there is no 'neighborhood' of the steady state. Also, for that model, the only non-explosive path converging to that steady state is the one in which the value of money remains constant, which implies determinacy of that steady state.[5] The following is our stability results for the $\{0, 1, 2\}$ economy.

**Proposition 2** *Generically, both full-support steady states locally stable and determinate, while the non-full-support steady state is locally stable and indeterminate.*

Both proofs start from a first-order difference equation in $(\pi_1^t, w_1^t, w_2^t)$ that is derived from the $B = 2$ versions of (5) and (6) and (7). In this system, only $\pi_1^t$ has an exogenous initial value and is a 'predetermined' variable. The proof of the first part is standard (See [6]). We show that the stable manifold is one-dimensional.

The meaning of generic.

Some stability result in OLG model should be very helpful.

Relationship with fiat money model.

[5] For the Trejos-Wright $\{0, 1\}$ model, Lomeli and Temzelides (2002) show that the non-monetary steady state is indeterminate.

(Move the following into proof section.)

The proof of the second part is not standard. The idea is borrowed from [2]. There are two necessary features of any convergent path. First, the convergence of $\pi_1^t$ is slow, because no inflow into holdings of 1 unit and because the outflow, which comes from (1,1)-meetings, approaches zero as the frequency of such meetings goes to zero. This implies that the dominant root that determines the speed of convergence is equal to one. Second, the steady state is on the boundary of the state space. Hence, even if the system appears to be convergent, we have to check that the convergence is such that $\pi_1^t,\ w_1^t \geq 0$. using the eigenvector that corresponds to the dominant unit root, that this condition fails.

Technique for show determinacy.

# 5 Proofs

Before turning to the proofs, we set out some steady state consequences that we use in the proofs. The steady-state law of motion reduces to

$$(\pi_1)^2\lambda(1;1,1) = \left(1 - m - \frac{\pi_1}{2}\right)\left(m - \frac{\pi_1}{2}\right)\lambda(1;2,0), \tag{11}$$

which equates outflows from holdings of 1 (the lefthand side) to inflows into holdings of 1 (the righthand side). The Bellman equations are

$$\begin{aligned} w_1 = &\frac{n-1+\pi_2}{n}\beta w_1 + \frac{\pi_0}{n}\max[u(\beta w_1), \beta w_1] \\ &+\frac{\pi_1}{n}\max[u(\beta w_2 - \beta w_1), \beta w_1] - \gamma, \quad \text{and} \end{aligned} \tag{12}$$

$$\begin{aligned} w_2 = &\frac{n-1+\pi_2}{n}\beta w_2 + \frac{\pi_1}{n}\max[\{u(\beta w_2 - \beta w_1) + \beta w_1\}, \beta w_2] \\ &+\frac{\pi_0}{n}\max[u(\beta w_2), \{u(\beta w_1) + \beta w_1\}, \beta w_2] - 2\gamma. \end{aligned} \tag{13}$$

As to full-support steady states, Proposition 1 of [3] establishes that in the model of fiat money with $\gamma = 0$, In what follows, we will show that the existence of such monetary steady states is robust to the introduction of carrying cost.

The proof is by continuity.

**Proof of Proposition 1.** We are going to construct a pure-strategy steady state with one-unit payment being the optimal in all trade meetings, and a mixed-strategy steady state with two-unit payment in $(2,0)$-meetings being also optimal . Corresponding inequalities are

$$\begin{array}{llcll}
(1,1)\text{-meeting} & u(\beta w_2-\beta w_1) & > & \beta w_1 & (14)\\
(1,0)\text{-meeting} & u(\beta w_1) & > & \beta w_1 & (15)\\
(2,1)\text{-meeting} & u(\beta w_2-\beta w_1) & > & \beta w_2-\beta w_1 & (16)\\
(2,0)\text{-meeting} & u(\beta w_1)+\beta w_1 & \geq & u(\beta w_2) & (17)\\
 & \& \; u(\beta w_1)+\beta w_1 & > & \beta w_2. & (18)
\end{array}$$

Under these inequalities, the Bellman equation (12)-(13) becomes

$$[N(1-\beta)+(1-\pi_2)\beta]w_1 = \pi_0 u(\beta w_1)+\pi_1 u\left(\frac{\beta((1-\pi_2)\beta w_1-N\gamma)}{N(1-\beta)+(1-\pi_2)\beta}\right)-N\gamma \tag{19}$$

and

$$[N(1-\beta)+(1-\pi_2)\beta](w_2-w_1)=(1-\pi_2)\beta w_1-N\gamma. \tag{20}$$

For a given $\pi_1$, there exist at most two positive $w_1$ satisfying equation (19). If such $w_1$ exists, then equation (20) defines positive $w_2$. Let

$$\pi_1^* \equiv (\sqrt{1+12m(1-m)}-1)/3. \tag{21}$$

and let $(w_1^*, w_2^*)$ denote the bigger of the two solutions, if they exist, to (19)-(20) for $\pi_1=\pi_1^*$. The following two claims imply that (14)-(18) are robust to a small $\gamma>0$, if we can find $\pi_1$ and $w_1$ solving (19). Then we will show that the existence of such $\pi_1$ and $w_1$. ■

**Claim 1** *Suppose $\gamma>0$ is sufficiently small. If equations (19) and (20) are satisfied for some $\pi$ such that $\pi_1>0$, then (14) and (18) hold.*
***Proof.*** *Let $\gamma=0$ and suppose by way of contradiction that (14) does not hold:*

$$u(\beta w_2-\beta w_1)=u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right)\leq \beta w_1.$$

*Then, we have*

$$
\begin{aligned}
\beta w_1 &< \frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}u(\beta w_1) \\
&< u\left(\frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\beta w_1\right) \\
&< u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right) = u(\beta w_2 - \beta w_1),
\end{aligned}
$$

*where the first inequality is by substituting the supposition into (19) and the second is by* $u(0) = 0$ *and strict concavity of* $u$*. This is contradiction and thus (14) should hold.*

*Inequality (18) follows from*

$$
\begin{aligned}
u(\beta w_1) &> u(\beta w_2 - \beta w_1) \\
&> \beta w_1 \\
&> \beta w_2 - \beta w_1,
\end{aligned}
$$

*where the first and the third inequalities are by (20) and the second is (14). Because these are strict inequalities, when* $\gamma > 0$ *is sufficiently small, they still hold.* ■

**Claim 2** *Inequalities (14) and*

$$u(\beta w_1) + \beta w_1 \geq \beta w_2 \tag{22}$$

*imply (15) and (16).*

**Proof.** Suppose by way of contradiction that (15) does not hold: $u(\beta w_1) \leq \beta w_1$. Then (14) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with (22) gives $u(\beta w_1) > \beta w_1$, which is a contradiction.

Suppose by contradiction that (16) does not hold: $u(\beta w_2 - \beta w_1) \leq \beta w_2 - \beta w_1$. Then (22) implies $\beta w_2 - \beta w_1 \leq \beta w_1$. But (14) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction.

When $\beta = 1$ and $\gamma = 0$ in (19)-(20), it is not hard to see that solution $(w_1^*, w_2^*)$ exists and satisfies $u(w_1^*) = w_1^* = w_2^*/2$. Then (17) holds with strict inequality.

Now consider a $\beta$ sufficiently close to one and a sufficiently small $\gamma > 0$. Because such change shift the r.h.s. of (19) shifted downward and the l.h.s.

upward, the $(w_1^*, w_2^*)$ still exists and becomes slightly smaller, and hence (17) still holds with strict inequality. Claim 1 implies (14) and (18). Then claim 2 gives the remaining conditions (15)-(16). Therefore, we have a pure-strategy full-support steady state when $\beta$ is sufficiently close one and $\gamma > 0$ is sufficiently small.

Then we turn to mixed-strategy steady state and start with setting $\gamma = 0$. Again we will show some conditions involving strict inequalities, so that when we introduce a sufficiently small $\gamma$, these conditions still hold.

Fixing $\pi_1$ at any value in $\Pi$, we can find $\hat{w}_1 > 0$ that solves (19). Given the concavity of $u$, the existence of $\hat{w}_1 > 0$ is equivalent to

$$u'(0) > \frac{[n(1-\beta) + (1-\pi_2)\beta]^2}{\{\pi_0[n(1-\beta) + (1-\pi_2)\beta] + \pi_1(1-\pi_2)\beta\}\beta}. \tag{23}$$

The r.h.s. of (23) starts from positive infinity and approaches one as $\beta$ increases in $(0, 1)$. Therefore, we can find a $\beta^*$ of intermediate value so that (23) does not hold particularly when $\pi_1 = \pi_1^*$.

Subtracting the r.h.s. of (23) for $\pi_1 = 0$ from the r.h.s. for $\pi_1 = \pi_1^*$ gives

$$\frac{n(1-\beta)}{\beta} \cdot \frac{\pi_1^* n(1-\beta) + \beta\pi_1^*\pi_0^*}{[\pi_0^* n(1-\beta) + \beta(1-\pi_2^*)^2](2-2m)} > 0.$$

The r.h.s. of (23) reaches its strict minimum at $\pi_1 = 0$ when other parameters are fixed. Therefore, we can adjust $\beta^*$ so that (23) holds when $\pi_1 = 0$.

For such $\beta^*$, when $\pi_1 = 0$, the above implies the existence of $(\hat{w}_1, \hat{w}_2)$. A necessary condition for the existence of a positive solution is

$$u'(\beta\hat{w}_1) < \frac{n(1-\beta)}{\beta(1-m)} + 1. \tag{24}$$

Then by the mean value theorem, we have

$$\begin{aligned} u(\beta\hat{w}_1) + \beta\hat{w}_1 - u(\beta\hat{w}_2) &= \beta\hat{w}_1 - u'(\xi)(\beta\hat{w}_2 - \beta\hat{w}_1), \quad \xi \in (\beta\hat{w}_1, \beta\hat{w}_2) \\ &> \beta\hat{w}_1 - u'(\beta\hat{w}_1)(\beta\hat{w}_2 - \beta\hat{w}_1) \\ &> 0, \end{aligned}$$

where the second inequality follows from (20) and (24). Therefore (17) holds with strict inequality when $\pi_1 = 0$.

As $\pi_1$ increases, the solution $(\hat{w}_1, \hat{w}_2)$ as a function of $\pi_1$ changes continuously. Note that the existence condition will be violated at $\pi_1^*$. Therefore before $\pi_1$ reaches $\pi_1^*$, $\hat{w}_1$ and $\hat{w}_2$ will merge into zero. In other words, there exists $\bar{\pi}_1 \in (0, \pi_1^*)$ such that the l.h.s. and r.h.s. of (23) are equal and such that $(\hat{w}_1, \hat{w}_2)$ exists for all $\pi_1 \in (0, \bar{\pi}_1)$. The condition at such $\bar{\pi}_1$ is

$$\begin{aligned} & \frac{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}{\beta(1-m+\frac{\bar{\pi}_1}{2})} \\ = \quad & \left[\frac{1-m-\frac{\bar{\pi}_1}{2}}{1-m+\frac{\bar{\pi}_1}{2}} + \frac{\beta\bar{\pi}_1}{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}\right] u'(0), \end{aligned} \tag{25}$$

where the coefficient of $u'(0)$ is proven to be smaller than one for any $n > 0$. This implies

$$u'(0) > \frac{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}{\beta(1-m+\frac{\bar{\pi}_1}{2})}. \tag{26}$$

Concavity of $u$ and (20) give the following inequality:

$$\begin{aligned} & \frac{u(\beta\hat{w}_1)+\beta\hat{w}_1-u(\beta\hat{w}_2)}{\beta\hat{w}_1} < \frac{\beta\hat{w}_1 - u'(\beta\hat{w}_2)\beta(\hat{w}_2-\hat{w}_1)}{\beta\hat{w}_1} \\ \to \quad & 1-u'(0)\frac{\beta(1-m+\frac{\bar{\pi}_1}{2})}{n(1-\beta)+\beta(1-m+\frac{\bar{\pi}_1}{2})}, \text{ as } \pi_1 \to \bar{\pi}_1. \end{aligned} \tag{27}$$

Because by (26) the above limit is strictly smaller than zero, we have $u(\beta\hat{w}_1)+\beta\hat{w}_1 < u(\beta\hat{w}_2)$, for $\pi_1$ sufficiently close to $\bar{\pi}_1$. Note that the two inequalities corresponding to $\pi_1 = 0$ and $\pi_1$ sufficiently close to $\bar{\pi}_1$ are strict, and hence they are robust to the introduction of a small $\gamma > 0$. In this case, the intermediate value theorem can be applied, and we can find a $\pi_1 > 0$ such that the solution to (19)-(20) for a small $\gamma > 0$ satisfies

$$u(\beta\hat{w}_1)+\beta\hat{w}_1 = u(\beta\hat{w}_2). \tag{28}$$

Finally we show that such pair $(\pi, \hat{w})$ that satisfies (28) is a mixed-strategy full-support steady state. Claim 1 implies (14) and (18). Then claim 2 gives the remaining conditions (15)-(16). Therefore we have (14)-(18) with equality in (17). ■

increasing value function gives stability result.

**Proof of Proposition 2.** For the pure-strategy full-support steady state, trading one unit in all trade meetings is a strictly preferred strategy at the steady state (see Definition 1 and Lemma 1), so it is also optimal in its neighborhood. That is, $\lambda^t(1;1,0) = \lambda^t(1;1,1) = \lambda^t(1;2,1) = \lambda^t(1;2,0) = 1$ for all $t \geq 0$.

Similarly, we can also pin down the optimal trading strategy that is constantly played along a path that converges to the non-full-support steady state, if there is any such path. When $\gamma > 0$, trading two units in $(2,0)$-meeting and trading one unit in all other meetings is a strictly preferred strategy at the steady state and hence it is also the unique optimal in its neighborhood. When $\gamma = 0$, the same strategy becomes weakly preferred at the steady state but it will be the unique optimal in the neighborhood. To see this, suppose by way of contradiction that there exists an equilibrium path that converges to the non-full-support steady state (i.e., $\pi_1 = 0$ and $w_1 = 0$) from some initial distribution such that $\pi_1^0 \neq 0$. As is shown in the proof of proposition 1, trading one unit is strictly preferred in $(1,1)$- and $(2,1)$-meetings, and paying two units is strictly preferred in $(2,0)$-meetings at $(\pi, w)$. Therefore, they are also optimal in the neighborhood of $(\pi, w)$, so $\lambda^t(1;1,1) = \lambda^t(1;2,1) = \lambda^t(2;2,0) = 1$ for all $t \geq 0$. Moreover, the following argument shows $\lambda^t(1;1,0) = 1$ should be the case for all $t \geq 0$. When the economy is close to but not equal to $(\pi, w)$, we have $\pi_1^t > 0$ for all $t \geq 0$ so (6) implies $w_1^t > 0$ for all $t > 0$, because there is always a positive probability that consumer with one unit meets producer with one unit and the consumer can get positive amount of utility from such a meeting. Equation (8) implies $u(x) > x$ for all $x < \beta w_2$ and therefore $u(\beta w_1^t) > \beta w_1^t$ holds all along the path. So in $(1,0)$-meetings, paying one unit is strictly preferred to paying nothing along the path.

For both steady states, a unique strategy is constantly played along any potential convergent path, so we can construct dynamic system from the law of motion and Bellman equation under the given strategy:

$$\pi_1^{t+1} = \pi_1^t - \frac{2(\pi_1^t)^2}{n} + \frac{2}{n}\left(1 - m - \frac{\pi_1^t}{2}\right)\left(m - \frac{\pi_1^t}{2}\right)\lambda(1;2,0) \tag{29}$$

$$w_1^t = \frac{n-1+\pi_2^t}{n}\beta w_1^{t+1} + \frac{\pi_0^t}{n}u(\beta w_1^{t+1}) + \frac{\pi_1^t}{n}u(\beta w_2^{t+1} - \beta w_1^{t+1}) - \gamma \tag{30}$$

$$\begin{aligned} w_2^t = {} & \frac{n-1+\pi_2^t}{n}\beta w_2^{t+1} + \frac{\pi_0^t}{n}\max[u(\beta w_1^{t+1}) + \beta w_1^{t+1}, u(\beta w_2^{t+1})] \\ & + \frac{\pi_1^t}{n}[u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1}] - 2\gamma. \end{aligned} \tag{31}$$

We have $u(\beta w_1^{t+1}) + \beta w_1^{t+1} < u(\beta w_2^{t+1})$ and $\lambda(1;2,0) = 0$ for the non-full-support steady state, and $u(\beta w_1^{t+1}) + \beta w_1^{t+1} > u(\beta w_2^{t+1})$ and $\lambda(1;2,0) = 1$ for the pure-strategy full-support steady state. Denote (29) by $\pi_1^{t+1} = \Phi^\lambda(\pi_1^t) : \Pi \to \Pi$ and (30)-(31) by $w^t = \phi^\lambda(\pi_1^t, w^{t+1}) : \Pi \times W \to W$, where $w^t \equiv (w_1^t, w_2^t)$ and $W \equiv \{(w_1, w_2) | 0 \leq w_1 \leq w_2\}$. As is ensured below, the implicit function theorem can be applied to (30)-(31) generically for both steady states. In the vicinity of each of the steady states, we can solve $w^{t+1}$ as a function of $(\pi_1^t, w^t)$ to obtain $w^{t+1} = \Psi^\lambda(\pi_1^t, w^t) : \Pi \times W \to W$. The joint system is

$$\begin{pmatrix} \pi_1^{t+1} \\ w^{t+1} \end{pmatrix} = \begin{pmatrix} \Phi^\lambda(\pi_1^t) \\ \Psi^\lambda(\pi_1^t, w^t) \end{pmatrix}. \tag{32}$$

Its jacobian is

$$A^\lambda \equiv \begin{bmatrix} \Phi_\pi^\lambda & O \\ -(\phi_w^\lambda)^{-1}\phi_\pi^\lambda & (\phi_w^\lambda)^{-1} \end{bmatrix}. \tag{33}$$

Straightforward differentiation leads to

$$\begin{aligned} \Phi_\pi^\lambda(\pi_1) &= 1 - \frac{3\pi_1^* + 1}{n}\lambda(1;2,0) \\ &= 1 - \frac{\sqrt{1 + 12m(1-m)}}{n}\lambda(1;2,0). \end{aligned} \tag{34}$$

Thus the pure-strategy steady state has an eigenvalue strictly less than one and the non-full-support steady state has a unit eigenvalue.For the pure-strategy steady state, we have $\lambda(1;2,0) = 1$, and straightforward differentiation gives

$$\phi_w^\lambda = \begin{pmatrix} \frac{n-1+\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta \Delta w^*) & \frac{\pi_1^*}{n}\beta u'(\beta \Delta w^*) \\ \frac{1-\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta \Delta w^*) & \frac{n-1+\pi_2^*}{n}\beta + \frac{\pi_1^*}{n}\beta u'(\beta \Delta w^*) \end{pmatrix}, \tag{35}$$

where $\Delta w^* \equiv w_2^* - w_1^*$. This matrix is generically invertible, confirming that the application of the implicit function theorem and (33) are valid. Because the top-right submatrix of $A^\lambda$ is a zero matrix, one eigenvalue of $A$ is given by (34), which is smaller than one, and the other two eigenvalues are those of $(\phi_w^\lambda)^{-1}$, which are the reciprocals of eigenvalues of $\phi_w^\lambda$. In what follows, we are going to show that eigenvalues of $\phi_w^\lambda$ are smaller than one in absolute value.

That the slope of the r.h.s. of (19) at the positive fixed point $\beta w_1^*$ should be smaller than the slope of the l.h.s. gives

$$\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta} > \pi_0^* u'(\beta w_1^*) + \pi_1^* \frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta} u'(\beta \Delta w^*). \tag{36}$$

The eigenvalues of a general $2\times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by

$$\eta_+, \eta_- = \frac{a+d \pm \sqrt{(a-d)^2+4bc}}{2}.$$

Because

$$\begin{aligned}
& (a-d)^2+4bc \\
= \; & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - 2\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]^2 \\
& +4\left[\frac{1-\pi_2^*}{n}\beta + \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) \\
= \; & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)\right]^2 + 4\frac{1-\pi_2^*}{n}\beta\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) > 0,
\end{aligned}$$

both eigenvalues are real. They are smaller than one in absolute value if and only if $a+d<2$ and $(1-a)(1-d)-bc>0$. Checking these conditions for (35) gives

$$\begin{aligned}
& 1-a+1-d \\
= \; & 2\left(1-\frac{n-1+\pi_2^*}{n}\beta\right) - \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) \\
> \; & 2\frac{n(1-\beta)+(1-\pi_2^*)\beta}{n} - \frac{\pi_0^*}{n}\beta u'(\beta w_1^*) - \frac{\pi_1^*}{n}\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}\beta u'\left(\beta\Delta w^*\right) \\
> \; & \frac{n(1-\beta)+(1-\pi_2^*)\beta}{n} > 0,
\end{aligned}$$

$$
\begin{aligned}
& (1-a)(1-d)-bc \\
= & \left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)+\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right)\left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right) \\
& -\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\left[\frac{1-\pi_2^*}{n}\beta+\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right] \\
= & \frac{(n(1-\beta)+(1-\pi_2^*)\beta)\beta}{n^2}\times \\
& \quad \left(\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta}-\pi_0^* u'(\beta w_1^*)-\pi_1^*\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}u'(\beta\Delta w^*)\right) \\
> & \ 0,
\end{aligned}
$$

where the last inequalities of the above two conditions follow from (36). Therefore, the eigenvalues of $(\phi_w^\lambda)^{-1}$ are greater than one in absolute value. This full-support steady state has a one-dimensional stable manifold. Because we have one initial condition, this full-support steady state is locally stable and determinate.

Next we consider the non-full-support steady state. Equation (34) computes the unit eigenvalue for the law of motion. Furthermore, as figure 5 illustrates, the law of motion (29) features unit-root convergence; the slope at the fixed point is unity. Note also that when $\gamma = 0$, this steady state is on the boundary of the state space $\Pi \times W$, which makes it necessary to explictly study the limiting behavior by seeing the eigenspace of the linearized system (33).[6]

First we compute

$$
\phi_\pi^\lambda = \begin{bmatrix} r \\ s \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{n}u(\beta w_2-\beta w_1)-\frac{1}{2n}[u(\beta w_1)+\beta w_1] \\ \frac{1}{n}[u(\beta w_2-\beta w_1)+\beta w_1]-\frac{1}{2n}[u(\beta w_2)+\beta w_2] \end{bmatrix} > 0
$$

and

$$
\phi_w^\lambda = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} \equiv \tag{37}
$$

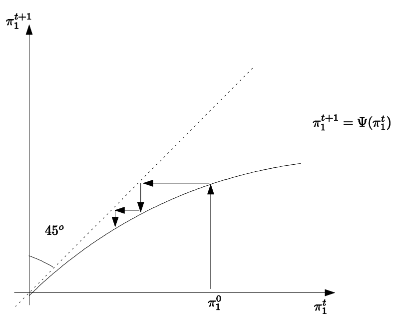
$$
\begin{bmatrix} \frac{(n-1+m)\beta}{n}+\frac{1-m}{n}\beta u'(\beta w_1) & 0 \\ 0 & \frac{(n-1+m)\beta}{n}+\frac{1-m}{n}u'(\beta w_2)\beta \end{bmatrix}.
$$

[6] Note that this analysis is not needed for the pure-strategy full-support steady state because that steady state is in the interior of $\Pi \times W$.

Because $w_2$ is a positive solution to (13), $a' > 1$ and $d' \in (0,1)$ hold. We have

$$A^{\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ -r/a' & 1/a' & 0 \\ -s/d' & 0 & 1/d' \end{bmatrix}. \tag{38}$$

In the context of the discrete-time dynamic system theory, the unit root is a "border" case in which the higher-order terms should be examined. In our case, the higher-order term seems to imply unit-root convergence (i.e., Figure 5) and the stable manifold of (32) is two dimensional. When $\gamma > 0$




is sufficiently small, the initial condition on $\pi_1^0$ imposes one restriction on the convergent paths, and reduces the degree of freedom by one. The steady state is locally stable and indeterminate.

When $\gamma = 0$, the steady state is at the boundary of the state space $\Pi \times W$. Keeping the entire convergent path in the space becomes problematic. In particular, unit root convergence implies that the convergence trajectory will eventually be parallel to the eigenspace of (38) associated with the unit eigenvalue[7]. The associated eigenvector, which constitutes a base of the space, has the form

$$\begin{bmatrix} 1 \\ \frac{-r}{a'-1} \\ \frac{s}{1-d'} \end{bmatrix}.$$

The fact that convergent trajectory of $(\pi_1^t, w_1^t, w_2^t - w_2)$ will be parallel to the above eigenvector implies that $\pi_1^t$ and $w_1^t$ will eventually have different signs, contradicting to $\pi_1^t, w_1^t > 0$ for all $t$. Therefore, the steady state is not stable.

[7] See Subsection "Dominant Eigenvector" on page 165 of [5].

Finally, we consider the mixed-strategy steady state. Our existence proof shows that generically it involves complete randomization: $\lambda^*(1; 2, 0) \in (0, 1)$. Along any convergent paths starting from sufficiently nearby neighborhood, complete randomization, which could vary along the path, in $(2, 0)$-meeting and one-unit payment in all other meetings must be the optimal, because $\pi^t$ will jump out of the neighborhood otherwise. Therefore (30)-(31) are still valid along the path. It turns out that this is not consistent the indifference condition (28) for $t > 1$ generically.

In what follows, we are going to exclude three types of convergent paths in a sufficiently small neighborhood of the steady state: ones where $\pi^t$ reaches the steady state but $w^{t+1}$ does not at some $t > 1$ date, i.e $\pi^t = \pi^*$ and $w^{t+1} \neq w^*$, ones where $w^{t+1}$ reaches the steady state but $\pi^t$ does not for some $t > 1$, i.e $\pi^t \neq \pi^*$ and $w^{t+1} = w^*$, and those where both $\pi^t$ and $w^{t+1}$ never arrive at the steady state. And finally we will show that the unique convergent path will be that where $(\pi^t, w^t)$ jumps into the steady state at date 1.

The first two types can be ruled out by examing the linearized Bellman equation at the steady state,

$$\begin{pmatrix} dw_1^t \\ dw_2^t \end{pmatrix} = \phi_\pi^\lambda d\pi_1^t + \phi_w^\lambda \begin{pmatrix} dw_1^{t+1} \\ dw_2^{t+1} \end{pmatrix}, \tag{39}$$

and the linearied indifference conditions (28) for date $t$ and date $t + 1$,

$$[u'(\beta w_1^*) + 1] \begin{pmatrix} dw_1^t \\ dw_2^{t+1} \end{pmatrix} = u'(\beta w_2^*) \begin{pmatrix} dw_2^t \\ dw_2^{t+1} \end{pmatrix}. \tag{40}$$

The first two types have $dw^t \neq 0$ and suppose $dw_1^t \neq 0$ without loss of generality. (39) and (40) must be satisfied by four difference quotients: $\frac{d\pi_1^t}{dw_1^t}$, $\frac{dw_2^t}{dw_1^t}$, $\frac{dw_1^{t+1}}{dw_1^t}$ and $\frac{dw_2^{t+1}}{dw_1^t}$. At date $t > 1$ on a first-type path, we have $\pi^t = \pi^*$ which pins down $\frac{d\pi_1^t}{dw_1^t} = 0$, while on a second-type path, $dw_1^{t+1} = dw_2^{t+1} = 0$ fixes $\frac{dw_1^{t+1}}{dw_1^t}$ and $\frac{dw_2^{t+1}}{dw_1^t}$ at zero. Along any third-type path that never arrive at the steady state, none of the difference quotients are pinned down. In all three cases, at most four variables need to solve four equations simultaneously. This is not true generically, because the coefficients in (39) and (40) are the values and derivatives of function $u$ at specific points, and we can always perturb $u$ by altering these derivatives arbitrarily without the values. Therefore, the three types of path do not exist generically.

Finally, we will show that $(\pi^t, w^t)$ reaches the steady state at date 1. Note that the steady state involves complete randomization generically. Therefore starting with the initial distribution $\pi_1^0$ sufficiently close to the steady state distribution, agents can choose the initial randomization $\lambda^0(1;2,0) \in (0,1)$ so that $\pi_1^0$ can jump to $\pi_1^*$ in one period, and then $(\pi^t, w^t) = (\pi^*, w^*)$ for all $t \geq 1$. Such randomization is the optimal choice by the agents, because $w^1 = w^*$ satisfies the indifference condition. Then the initial value $w^0$ can be determined from the initial distribution $\pi_1^0$ and $w^1$ via the Bellman equation. (Note that $w^0$ does not affect agents' decisions.) Thus generically, the mixed-strategy steady state is stable and determinate$\pm\pm$. ■

# 6 Concluding remarks

This paper shows instability of the nonfull-support steady states that are $l$-replicas of some strict monetary steady state in Zhu (2003). A counterpart of our analysis would be a stability analysis of full-support steady states.[8] In a companion paper, we do that for the $\{0, 1, 2\}$ case. We provide a necessary and sufficient condition for the existence of full-support steady states and showing that they are locally stable. So far, there is no stability analysis for full-support steady states, whether strict or not, in the general case.

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