Optimal Unemployment Insurance With Different Types of Job

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Abstract

This paper considers the design of optimal unemployment insurance with multiple types of job. The environment is an extension of Hopenhayn and Nicolini [4]. We study the model by using the first-order approach. The optimal contract still displays benefits decreasing over the length of the unemployment spell. The wage tax after reemployment depends on the length of unemployment spell.

1 Introduction

A one-spell unemployment insurance (hereafter UI) model is a type of UI model in which employment is an absorbing state. This paper extends the one-spell UI model of Hopenhayn and Nicolini [4] by incorporating multiple types of job into the model. The significant extension is when search effort affects not only the probability of receiving an offer but also the kind of offer—–a low wage offer or a high wage offer. It is shown that wage tax dependent on history, and benefits that decrease over the duration of unemployment spell are robust in this extension. With multiple types of job, the extra contingency on offer type is not redundant. It is shown that whether the wage taxes are different across different types of job depends on the effect of search effort on offer type. For the sake of simplicity, this paper uses the first-order approach (hereafter FOA). The analysis concerning validity of FOA in Hopenhayn and Nicolini [4] applies only to their model. We give a different analysis for the significant extension based on Rogerson (1985). However, there are two gaps in my analysis. These gaps are also present in Hopenhayn and Nicolini [4].

The first gap concerns FOA. Our argument about validity of FOA uses the first-order derivatives of the planner's cost function. Showing differentiability of this cost function is not yet accomplished, because the objective function in the recursive problem is not convex so that the standard Benveniste-Scheinkman technique is not applied.

The second gap involves the positive aspect of the contract. Convexity of the cost function is used as a sufficient condition for decreasing benefits and history dependence of the wage tax. However, showing the convexity of cost function is missing because the objective function and the constraint correspondence are not convex in the recursive problem. Hopenhayn and Nicolini [4] suggested but did not try introducing lotteries to generate convexity.

The next section introduces the extended model. Subsection 3.1 presents analysis about the FOA. Subsection 3.2 is about decreasing benefits and history dependence. The appendix gives a detailed discussion of the propositions in Section 3.

2 Model

2.1 Environment

Time is discrete and is indexed by $t \in \{0, 1, 2...\}$. There are two player: a worker and a planner. The worker can be employed or unemployed. An employed worker has either a good job or a bad job. At the beginning of period 0, the worker is unemployed. The unemployed worker has 0 income and chooses search effort. The search outcome i can be good-job employment G, bad-job employment B or unemployment U. We denote by $H \equiv \{G, B, U\}$ the set of all search outcomes. The distribution of outcomes depends on the worker's unobservable search effort. The more effort a worker exerts, the more likely is the better outcome. Let $m(i; a)$ be the probability of outcome i when search effort a is taken, where $a \in E \equiv [0, \overline{a}]$. $m(i, a)$ satisfies $m_a(U; a) < 0$, $m_a(U; a) + m_a(B; a) < 0$ and $m_a(U; 0) = -\infty$. It displays diminishing returns to scale, i.e., $m_{aa}(U; a) > 0$ and $m_{aa}(U; a) + m_{aa}(B; a) >$

0[1].

If the worker is employed, i.e., $i \in \{G, B\}$, she earns wage w_i. The good job pays higher wage, that is, $w_G > w_B$. She can pursue on-the-job search. To simplify the situation, we assume $w_B \geq \beta w_G$ so that the planner wants the employed workers to stop searching for a better job[2], where β is the worker's discount factor.

The worker is infinitely lived. She draws utility from consumption and receives disutility from searching. The utility is additively separable between consumption and effort:

$$U(c, a) = u(c) - a$$

with $c \in Z \equiv [0, w_G]$. Function u satisfies $u(0) = 0$, $u'(c) > 0$ and $u''(c) < 0$. Inada conditions hold: $u'(0) = +\infty$, $u'(w_G) = 0$.

The risk neutral planner observes the worker's status i, but she cannot observe the worker's effort a. She faces a constant interest rate $r = \frac{1}{\beta} - 1 > 0$. The worker does not have direct access to asset markets. All borrowing and saving must be done through the planner. The planner offers the worker an incentive compatible insurance contract, which specifies net transfer to the worker each period. Bilateral commitment is assumed.

2.2 Planner's problem

For the sake of parsimonious notations, we will take it for granted that the employments are absorbing states[3]. We will focus on the history where the worker is unemployed for simplicity. We call the worker unemployed in a period if she is unemployed at the beginning of that period. Let $n \in N = \{0, 1, 2...\}$ be the public history of staying unemployed for n periods and remaining unemployed in period n[4]. All contracts can only be written based on publicly observed history. The worker's deviation, on the other hand, can depend on her private history[5]. The contract (a, c_G, c_B, c_U) specifies four functions: recommended effort $a : N \to E$, end-of-period unemployed

[1] Examples for two information structures include $m(U; a) = h(a), m(U; a) + m(B; a) = g(a)$ and $m(U; a) = h(a), m(B; a) = h(a)g$ respectively. Suppose that $h(a)$ and $g(a)$ are convex decreasing function with image in $[0, 1]$.

[2] See appendix.

[3] These are shown in **Proposition 1** of the appendix.

[4] n is used both for history and a natural number to save notations.

[5] The public equilibrium solution concept prevails in this literature.

consumption $c_U : N \to [0, w_G]$, constant future period consumption of end-of-period state-i $c_i : N \to [0, w_G]$, $i \in \{G, B\}$.

A contract is incentive compatible if the worker is willing to choose the recommended effort in each period. The planner's objective is to choose an incentive compatible contract which gives the worker at least his promised utility and minimizes the cost.

2.3 A Recursive formulation

The planner's problem is a repeated moral hazard problem. It is hard because it has infinite amount of constraints and they are history dependence. Following Spear and Srivastava [10], we introduce promised utility v as another state variable[6]. The variation of the continuation value of the promised utility overtime encodes the history dependence. Hence the problem can be formulated recursively. Let v_U be the continuation value of promised utility when tomorrow's state is U, where $v_U \in \overline{V} = [0, V^*]$[7], with $V^* = \frac{u(w_G)}{1-\beta}$. A recursive contract (a, c_G, c_B, c_U, v_U) specifies current recommended effort a, constant consumption in each future period in absorbing state-i c_i, with $i \in \{G, B\}$, current consumption c_U and continuation value v_U for remaining unemployed.

Let $U(a, c, v_U)$ denote the promised utility delivered by a recursive contract (a, c, v_U), where c is the shorthand for (c_G, c_B, c_U).

$$\begin{aligned} U(a, c, v_U) \equiv \; & m(G; a)(\frac{u(c_G)}{1-\beta}) + m(B; a)(\frac{u(c_B)}{1-\beta}) \\ & + m(U; a)(u(c_U) + \beta v_U) - a \end{aligned} \tag{2.1}$$

A recursive contract (a, c, v_U) is *incentive compatible*, if search effort a satisfies the following constraint "IC",

$$a \in \arg\max_{a \in E} U(a, c, v) \tag{2.2}$$

[6] This is the only state variable we need to keep tract of, since employments are two absorbing states. But we will keep tract of the labor market status in the appendix to show that employments are absorbing states.

[7] Among incentive-compatible continuation contracts, that with 0 consumption and 0 effort level every period gives 0 lifetime utility to the worker, while the one with w_G consumption and 0 effort level every period gives V^*.

Let $\Phi(v)$ be the set of all incentive compatible recursive contracts delivering promised utility v:

$$\begin{aligned}\Phi(v) &\equiv \{(a, c_G, c_B, c_U, v_U) \in E \times Z^3 \times \overline{V} | a \in \arg\max_{a \in E} U(a, c, v_U) \\ \text{and } v &\leq U(a, c, v_U), \text{with } c = (c_G, c_B, c_U)\}.\end{aligned}$$

Let $C^*(V)$ be the planner's expected cost associated with the optimal contract in $\Phi(V)$. The recursive problem is:

$$C^*(V) = \min_{(a, c_G, c_B, c_U, v_U)} m(G; a)(\frac{c_G - w_G}{1-\beta}) \tag{2.3}$$
$$+m(B; a)(\frac{c_B - w_B}{1-\beta}) + m(U; a)(c_U + \beta C^*(v_U))$$
$$\text{s.t. } (a, c_G, c_B, c_U, v_U) \in \Phi(V)$$

The recursive problem is still difficult because the constraint IC has infinite amount of inequality constraints. If $U(a, c, v)$ is concave in a, we can replace constraint IC by $U_a(a, c, v) = 0$ to obtain a problem with only two inequality constraints, i.e., FOA is valid. But it is not trivial to find out conditions under which $U(a, c, v)$ is concave in a.

For $\sum_{i \in \{G,B,U\}} m(i; a) = 1$, at least one of $m(i; a)$s is not strictly concave in a[8]. Therefore the worker's utility function $U(a, c, v)$ is not always strictly concave in a[9]. There exist assumptions under which $U(a, c, v)$ is concave in a under all (c, v). But these assumptions rule out the environment of economic interest[10].

[8] Suppose all $m(i, a)$s are strictly concave in a. Therefore $1 - m(G, a) - m(B, a) = m(U, a)$ is strictly convex in a. However $m(U, a)$ is concave in a, generating a contradiction.

[9] Suppose, without loss of generality, that $m(G; a)$ is not strictly concave in a. It is not hard to see under scheme $(w_G, 0, 0, 0)$, $U(a, c, v) = m(G, a)u(w_G)$ is not strictly concave in a.

[10] To make sure concave $U(a, c, v)$ in a for all (c, v), we need to assume that all $m(i, a)$s are concave in a. With $\sum_{i \in H} m(i, a) = 1$, we have $\sum_{i \in H} m_a(i, a) = 0$ and $\sum_{i \in H} m_{aa}(i, a) = 0$. Furthermore $m_{aa}(i, a) \leq 0$ for $i \in H$ implies $m_{aa}(i, a) = 0$. This leads to $m_a(i, a) = k_i$ where k_i is a constant. Therefore concave $U(a, c, v)$ in a under any (c, v) is equivalent to $m(i; a) = k_i a + l_i$ for all $i \in \{G, B, U\}$, where k_i and l_i are constants. That is, the marginal effects of effort a on both probability function and disutility function are fixed. $m(i; a)$ doesn't display diminishing returns to scale. Or disutility from searching cannot be expressed as a strictly convex function $v(a)$ through changing units of measurement. These environments are not standard.

To find out better assumptions, we consider the following transformation from (2.1).

$$\begin{aligned} U(a,c,v) &= m(G;a)(\frac{u(c_G)-u(c_B)}{1-\beta})+ \\ & [m(G;a)+m(B;a)](\frac{u(c_B)}{1-\beta}-(u(c_U)+\beta v_U)) \\ & +(u(c_U)+\beta v_U)-a \end{aligned} \tag{2.4}$$

If a scheme (c, v_U) has the ordering $\frac{u(c_G)}{1-\beta} \geq \frac{u(c_B)}{1-\beta} \geq (u(c_U)+\beta v_U)$, $U(a,c,v)$ is also concave in a by (2.4) because $m(G;a)$ and $[m(G;a)+m(B;a)]$ are concave in a. The main goal in section 3.1 is to find out conditions under which the optimal recursive scheme has such ordering.

3 Analysis

The significant case is when search effort affects not only the probability of receiving an offer but also the kind of offer. Accordingly the situation is divided into two structures: structure 1 where effort doesn't affect offer distribution and structure 2 where effort affects offer distribution. This paper uses FOA which replaces constraint IC by $U_a(a,c,v_U)=0$ and $U_a(a,c,v_U) \geq 0$ in the structure 1 and the structure 2, respectively.

3.0.1 Some Notations

Notations are defined in the structure 2 first. Call the following constraint "IC of FOA":

$$\begin{aligned} 1 &\leq m_a(G;a)(\frac{u(c_G)}{1-\beta})+m_a(B;a)(\frac{u(c_B)}{1-\beta}) \\ & +m_a(U;a)(u(c_U)+\beta v_U) \end{aligned} \tag{3.1}$$

Let $\Gamma_{FOA}(V)$ be the set of all recursive contracts satisfying (3.1) and delivering promised utility V :

$$\begin{aligned} \Gamma_{FOA}(V) &\equiv \{(a,c_G,c_B,c_U,v_U) \in E \times Z^3 \times \overline{V} | U_a(a,c,v_U) \geq 0 \\ \text{and } V &\leq U(a,c,v_U), \text{ with } c=(c_G,c_B,c_U)\}. \end{aligned}$$

Note that $\Gamma_{FOA}(V^*) = \emptyset$. Also, $(0,0,0,0,0)$ and $(0, w_G, w_G, w_G, V^*)$ are the only incentive compatible contracts with promised utilities 0 and V^*, respectively. But they are not in $\Gamma_{FOA}(0)$ and $\Gamma_{FOA}(V^*)$. We redefine Γ_{FOA}:

$$\Phi_{FOA}(V) \equiv \begin{cases} \Gamma_{FOA}(0) \cup \{(0,0,0,0,0)\} & \text{if } V = 0 \\ \Gamma_{FOA}(V) & \text{if } V \in (0, V^*). \\ \{(0, w_G, w_G, w_G, V^*)\} & \text{if } V = V^* \end{cases} \tag{3.2}$$

Define mappings T and T_{FOA} in the space of bounded continuous functions, $C([0, V^*])$:

$$\begin{aligned} TC(V) = \min_{(a, c_G, c_B, c_U, v_U)} & m(G;a)(\frac{c_G - w_G}{1-\beta}) + m(B;a)(\frac{c_B - w_B}{1-\beta}) \\ & + m(U;a)(c_U + \beta C(v_U)) \\ & \text{s.t. } (a, c_G, c_B, c_U, v_U) \in \Phi(V) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} T_{FOA}C(V) = \min_{(a, c_G, c_B, c_U, v_U)} & m(G;a)(\frac{c_G - w_G}{1-\beta}) + m(B;a)(\frac{c_B - w_B}{1-\beta}) \\ & + m(U;a)(c_U + \beta C(v_U)) \\ & \text{s.t. } (a, c_G, c_B, c_U, v_U) \in \Phi_{FOA}(V). \end{aligned} \tag{3.4}$$

In the structure 1, we use the equality constraint IC of FOA, $U_a(a, c, v_U) = 0$. Φ_{FOA}, Γ_{FOA} and T_{FOA} are redefined accordingly. The standard existence result of Chapter 9.2 in Stokey, Lucas and Prescott applies to mappings T and T_{FOA}[11] in both structures.

3.0.2 First-order conditions

Let $C^{FOA}(.)$ be the fixed point of T_{FOA}. The following are first-order conditions:

[11] For more discussion of the cost function, see appendix 6.3.

F.O.C. for c_i and v_U

$$\begin{aligned}
\frac{1}{u'(c_G)} &= -\lambda - \mu \frac{m_a(G;a)}{m(G;a)} && (3.5)\\
\frac{1}{u'(c_B)} &= -\lambda - \mu \frac{m_a(B;a)}{m(B;a)} && (3.6)\\
\frac{1}{u'(c_U)} &= -\lambda - \mu \frac{m_a(U;a)}{m(U;a)} && (3.7)\\
&= C_v^{FOA}(v_U) && (3.8)
\end{aligned}$$

F.O.C. for a

$$\begin{aligned}
0 &= m_a(G;a)(\frac{c_G - w_G}{1-\beta}) + m_a(B;a)(\frac{c_B - w_B}{1-\beta}) \qquad (3.9)\\
&\quad + m_a(U;a)(c_U + \beta C^{FOA}(v_U)) + \mu U_{aa}(a,c,v_U)\\
&\quad + \lambda U_a(a,c,v_U)
\end{aligned}$$

Envelope condition

$$(T_{FOA}C^{FOA})'(V) = -\lambda \qquad (3.10)$$

Martingale property

$$\begin{aligned}
(T_{FOA}C^{FOA})'(V) &= +m(G;a)\frac{1}{u'(c_{Gt})} \qquad (3.11)\\
&\quad + m(B;a)\frac{1}{u'(c_{Bt})}\\
&\quad + m(U;a)\frac{1}{u'(c_{Ut})}
\end{aligned}$$

c_{ij} is state-i worker's consumption at period j.

The solution to the problem $(T_{FOA}C^{FOA})(V)$ satisfies the above first-order conditions[12]. It also satisfies $U_a(a,c,v_U) \geq 0$ or $U_a(a,c,v_U) = 0$. In the next subsection, we want to make sure that constraint IC is also satisfied.

3.1 First-Order Approach

The constraint IC is equivalent to

$$U(a,c,v) \geq U(a',c,v) \text{ for all } a' \in E.$$

[12]Interior solution is assumed here. Appendix shows that problem $T_{FOA}C^{FOA}(V)$ has interior solutions for $V \in (0, V^*)$ under certain conditions.

which involves a continuum number of inequality constraints. The FOA reduces the number of inequalities constraints by replacing constraint IC with worker's first order condition.

Another way to reduce the number of constraints is to reduce the number of possible effort levels in E. However, such simplification is not proper for this model. A 2-effort-level space E could be interpreted as either searching or not searching, or searching for a good job or a bad job. It doesn't include both interpretations. Adding one more effort level leads to a better interpretation, which however complicates the problem significantly.

Because of this, we choose to use the FOA. Propositions 3 concerns the validity of FOA in our model. We assume without proof that the differentiability is preserved by T_{FOA}. The proof is sketched. For a detailed discussion, please refer to the appendix.

Assumption 1 $\frac{m_a(U,a)}{m(U,a)} < \frac{m_a(B,a)}{m(B,a)} \leq \frac{m_a(G,a)}{m(G,a)}$(MLRP)

Assumption 2 $\frac{1}{u'}$ is concave.

Assumption 3 $u^{-1\prime}$ is convex[13].

Assumption 4 $\frac{w_B}{\beta} \geq w_B + \Pr(G)(w_G - w_B)$, where $\Pr(G) = \max_{a \in E} m(G, a)$

Proposition 3 In both structures, suppose that assumptions 1-4[14] hold and that mapping T_{FOA} preserves differentiability at interior promised values. Then the FOA is valid. The optimal contract implies $c_B^* = c_G^*$ in the structure 1 and $c_G^* > c_B^*$ in the structure 2.

The argument includes two steps. In the step 1, the solution to problem $(TC^{FOA})(V)$ has to be in $\Phi_{FOA}(V)$. In the step 2, the solution to problem $(T_{FOA}C^{FOA})(V)$ has to be in $\Phi(V)$.

Steps 1-2 imply that T_{FOA} and T have a common fixed point. To see this, note that problems $(TC^{FOA})(V)$ and $(T_{FOA}C^{FOA})(V)$ have a common objective function. Therefore the step 1 implies $(T_{FOA}C^{FOA})(V) \geq (TC^{FOA})(V)$,

[13] Functions satisfying assumption 2 and assumption 3 exist for example $u(c) = c^{1-p}$ where $p > 1/2$.

[14] The same set of assumptions are used for both structures in Proposition 3. But their roles in the structure 1 and in the structure 2 are different. Indeed, conditions 1-2, generated by assumptions 2-4, are used differently. As shown in the appendix, both conditions are used just to rule out $a^* = 0$ in the structure 1, while they also generate concave $U(a, c^*, v^*)$ in a under the optimal scheme (c^*, v^*) in the structure 2.

and the step 2 gives $(T_{FOA}C^{FOA})(V) \leq (TC^{FOA})(V)$. These two inequalities imply

$$C^{FOA}(V) = (T_{FOA}C^{FOA})(V) = (TC^{FOA})(V). \tag{3.12}$$

Hence $C^{FOA}(.)$ is also a fixed point of T. Since T and T_{FOA} are contraction mappings, as shown in appendix, $C^{FOA}(.)$ is the unique fixed point of T and T_{FOA}. Also, the equality (3.12) and steps 1-2 imply common policy functions for $(TC^{FOA})(V)$ and $(T_{FOA}C^{FOA})(V)$. It is valid to study the problem $(T_{FOA}C^{FOA})(V)$.

Appropriate Inada conditions rule out solutions with boundary effort level in the problem $(TC^{FOA})(V)$[15]. The first-order condition in the problem defined in constraint IC is satisfied. Therefore, we have the step 1.

To show the step 2, we want to make sure that $U(a, c^*, v_U^*)$ is concave in a[16], if (c^*, v_U^*) is the scheme solution to the problem $(T_{FOA}C^{FOA})(V)$. The ordering

$$\frac{u(c_G^*)}{1-\beta} \geq \frac{u(c_B^*)}{1-\beta} \geq (u(c_U^*) + \beta v_U^*) \tag{3.13}$$

is a sufficient condition for this by formula (2.4). Such ordering will be generated in the following.

3.1.1 Ordering $\frac{u(c_G^*)}{1-\beta} \geq \frac{u(c_B^*)}{1-\beta} \geq (u(c_U^*) + \beta v_U^*)$

In the structure 1, the effort doesn't affect the conditional probability of offer type. Since the worker is risk averse, the efficient risk sharing implies the same continuation value of promised utility for both types of employed workers. Therefore we have $c_G^* = c_B^*$[17]. The equality constraint IC of FOA becomes $-m_a(U, a^*)[\frac{u(c_B^*)}{1-\beta} - (u(c_U^*) + \beta v_U^*)] = 1$, leading to $\frac{u(c_B^*)}{1-\beta} \geq (u(c_U^*) + \beta v_U^*)$. Therefore any optimal scheme to the problem $(T_{FOA}C^{FOA})(V)$ satisfies $\frac{u(c_G^*)}{1-\beta} \geq \frac{u(c_B^*)}{1-\beta} \geq (u(c_U^*) + \beta v_U^*)$. $U(a, c^*, v_U^*)$ is concave in a. The first order condition, $U_a(a^*, c^*, v_U^*) = 0$, is a sufficient condition for the optimal effort solution to the problem defined by constraint IC.

The partial justification in Hopenhayn and Nicolini [4] applies to the structure 1, because constraint IC of FOA could be reduced into $-m_a(U, a^*)[\frac{u(c_B^*)}{1-\beta} -$

[15] For more discussions, see appendix subsection 6.4.

[16] Then $U_a(a^*, c^*, v_U^*) = 0$ is a sufficient and necessary condition for the optimal in problem $\max_{a \in E} U(a, c^*, v^*)$.

[17] Combining (3.5) and (3.6) leads to $c_G^* = c_B^*$.

$(u(c_U^*) + \beta v_U^*)] = 1$. This is not the case in the structure 2. A different argument is developed.

Call $C(.)$ satisfies condition 1 if

$$C_v(V) \geq u^{-1\prime}(V(1-\beta)) \text{ for } V \in \overline{V}.$$

Claim 1 *If $C^{FOA}(.)$ satisfies condition 1, the optimal plan in $(T_{FOA}C^{FOA})(V)$ has the desired ordering (3.13).*

With the inequality constraint IC of FOA, $U_a(a, c^*, v_U^*) \geq 0$, the multiplier of the constraint IC of FOA is non-positive, $\mu \leq 0$. Through comparing the first-order conditions (3.5)-(3.7), we can derive $u(c_G^*) \geq u(c_B^*) \geq u(c_U^*)$ from $\mu \leq 0$ and MLRP. Furthermore, since $C^{FOA}(.)$ satisfies condition 1 and (3.8) holds, we have $u^{-1\prime}(u(c_U^*)) = \frac{1}{u'(c_U^*)} = C_v^{FOA}(v_U^*) \geq u^{-1\prime}(v_U^*(1-\beta))$. Thus the unemployed worker's current utility is always higher than the average future utility: $u(c_U^*) \geq v_U^*(1-\beta)$. The optimal plan has ordering (3.13). Therefore $U(a, c^*, v_U^*)$ is concave in a. The sufficient and necessary condition for the optimal private effort a^* is $U_a(a^*, c^*, v_U^*) = 0$. But, this might not hold in structure 2 because inequality constraint is used. Claim 2 deals with this issue. Call $C(.)$ satisfies condition 2 if

$$C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)} \text{ for } V \in \overline{V}, \text{with } \nu(c) = \frac{1}{u'(c)}.$$

Claim 2 *If $C^{FOA}(.)$ satisfies condition 2, constraint IC of FOA is binding for the optimal solution in $(T_{FOA}C^{FOA})(V)$.*

By way of contradiction, suppose that the multiplier for constraint IC of FOA is 0. Then F.O.C.s (3.5)-(3.7) imply $c_U^* = c_B^* = c_G^*$. The F.O.C. (3.8) implies $\frac{\nu^{-1}(C_v^{FOA}(V))}{1-\beta} = \frac{c_U^*}{1-\beta}$. Thus condition 2 on $C^{FOA}(.)$ becomes $c_U^* + \beta C^{FOA}(v_U^*) \geq \frac{c_U^* - w_B}{1-\beta}$. Then we have the corresponding ordering for the planner: $c_U^* + \beta C^{FOA}(v_U^*) \geq \frac{c_B^* - w_B}{1-\beta} > \frac{c_G^* - w_G}{1-\beta}$. Applying formula (2.3) to the planner's cost, we have

$$\begin{aligned} C(a, c, v_U) &= m(G; a)(\frac{c_G - c_B}{1-\beta}) + \\ &\quad [m(G; a) + m(B; a)][\frac{c_B}{1-\beta} - (c_U + \beta C^{FOA}(v_U))] \\ &\quad + (c_U + \beta C^{FOA}(v_U)). \end{aligned} \tag{3.14}$$

$C(a,c,v)$ is defined as a function of contract (a,c,v). Its value is the associated cost. With $m_a(U;a) < 0$, $m_a(U;a) + m_a(B;a) < 0$ and the ordering for the planner, it is not hard to see $C_a(a^*,c^*,v_U^*) < 0$ from (3.14). The constraint IC of FOA in the structure 2 implies $U_a(a^*,c^*,v_U^*) \geq 0$. Therefore both parties prefer a higher effort under (c^*,v_U^*), leading to $0 > C_a(a^*,c^*,v_U^*) + \lambda U_a(a^*,c^*,v_U^*)$, a contradiction to F.O.C. (3.9).

The last step is to show that mapping T_{FOA} preserves conditions 1-2 under assumptions 2-4. Lemma 4.3 and Lemma 4.4 have the technical discussion. We follow Hopenhayn and Nicolini [4] and implicitly assume that T_{FOA} preserves differentiability.

3.1.2 Some Remarks

The structure 1 seems to be a special case of the structure 2. In fact, the argument for structure 2 also works for structure 1. We use the argument in Hopenhayn and Nicolini (1997) to the structure 1, because it generates concavity in a simpler way, and also it doesn't require showing binding constraint IC of FOA.

The literature studying justification of the FOA includes two mainstreams: Rogerson (1985) and Jewitt (1988). We apply the Rogerson method to the structure 2 for two reasons. First, Rogerson imposes less conditions on utility function than Jewitt does. Applying the Rogerson method requires less conditions on the cost function[18], an exogenous object in our setup. Second, Jewitt assumes the equality constraint IC of FOA, but showing that the multiplier is negative[19] becomes a problem[20]. Jewitt's method doesn't apply because the value functions are different across states in our model.

3.2 History dependence

We use FOA to consider two questions in the extended model. First, how the duration of unemployment affects the net transfers to the worker is con-

[18] Rogerson and Jewitt considered a static moral hazard problem. Our set-up and the static model turn out to be very similar. The cost function is corresponding to the worker's utility function.

[19] A negative multiplier helps to generate the desired ordering and other properties about the solution plan.

[20] We don't have this problem for the structure 1. Because a negative multiplier doesn't help to justify FOA, we can follow Hopenhayn and Nicolini [4] and use validity of the FOA to show that the multiplier is negative.

sidered. Proposition 4 shows that the optimal contract still exhibits benefits that decrease as unemployment spell lasts longer. Then, we turn to characterizing the wage tax rule. Proposition 5 shows that the optimal contract has wage tax of history dependence, i.e., (c^*_{Bt}, c^*_{Gt}) depends on t[21].

Let $C^*(V)$ be the common fixed point of mappings T_{FOA} and T. The results in this section depend on convexity and differentiability[22] of $C^*(.)$, which is also a problem in Hopenhayn and Nicolini [4]. Finding conditions that guarantee convexity in this model might not be easy due to the following. The graph of Φ_{FOA} is not generally convex. Moreover, its objective function in the mapping T_{FOA} might not be convex even if $C^*(.)$ is convex[23]. Therefore standard SLP technique doesn't apply. Hopenhayn and Nicolini [4] suggested but didn't try using lotteries to convexify the objective function and constraint correspondence[24]. We follow them and consider a model without lotteries.

Proposition 4 In both structures, suppose that assumptions 1-4 hold and that the mapping T_{FOA} preserves differentiability at the interior promised utility. Then if promised utility at period 0 V_0 is in $(0, V^*)$, the optimal consumption is decreasing over the unemployment spell. If additionally $C^*(V)$ is convex, promised continuation value V is decreasing over the unemployment spell.

Proof. With Proposition 3, we are justified to use FOA. Consider period-$t^* + 1$ problem $C^*(v_{Ut^*+1}) = T_{FOA}C^*(v_{Ut^*+1})$. By Lemma 4.2, $t^* + 1$-period multipler is strictly negative: $\mu_{t^*+1} < 0$.

$$\frac{1}{u'(c^*_{Ut^*+1})} = -\lambda_{t^*+1} - \mu_{t^*+1}\frac{m_a(U, a_{t^*+1})}{m(U, a_{t^*+1})}{}^{25}. \tag{3.15}$$

[21] We use c^*_{it} for $c^*_i(t)$, where $c^*_i(.)$ is specified by the contract.

[22] Milgrom and Segal [7] shows that differentiability can be supported by assumptions on the optimal contract. But no assumptions on exogenous objects is provided.

[23] The constraint set and the objective function are of the form $x * y$ with choice variables x and y. Use function $f(x, y) = xy$ as an example. It is not convex or concave, because its Hessain matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[24] They said in all their numerical exercises, cost functions turn out to be convex.

[25] The consumption is dependent on the whole history up to current period. Notations here ignore previous history to avoid complication.

Then $\mu_{t^*+1} < 0$ and $\frac{m_a(U,a)}{m(U,a)} < 0$ lead to $\frac{1}{u'(c^*_{Ut^*+1})} < -\lambda_{t^*+1}$ in (3.14). Combine this with the envelope condition, $-\lambda_{t^*+1} = C^*_v(v_{Ut^*+1})$, we have $\frac{1}{u'(c^*_{Ut^*+1})} < C^*_v(v^*_{Ut^*+1})$. Note also that the F.O.C.(3.8) for period-t^* problem still holds: $\frac{1}{u'(c^*_{Ut^*})} = C^*_v(v^*_{Ut^*+1})$. We have $\frac{1}{u'(c^*_{Ut^*+1})} < \frac{1}{u'(c^*_{Ut^*})}$. Then the concavity of u implies $c^*_{Ut^*+1} < c^*_{Ut^*}$.

With the F.O.C. (3.8) for period-t^* and period-t^*+1, $\frac{1}{u'(c^*_{Ut^*+1})} < \frac{1}{u'(c^*_{Ut^*})}$ imply $C^*_v(v^*_{Ut^*+1}) < C^*_v(v^*_{Ut^*})$. With convexity of $C^*(V)$ we have $v^*_{Ut^*+1} < v^*_{Ut^*}$. By mathematical induction, the above argument applies for all t. Therefore we have $c^*_{Ut+1} < c^*_{Ut}$ and $v^*_{Ut+1} < v^*_{Ut}$ for all t. ■

Proposition 5 In both structures, suppose that assumptions 1-4 hold and that the mapping T_{FOA} preserves differentiability at the interior promised utility. $C^*(V)$ is convex, and $m(B,a)$ is increasing in a. Then if the promised utility V is in $(0, V^*)$, then we can't have $(c^*_{Gt}, c^*_{Bt}) = (c^*_G, c^*_B)$[26] for all t, where c^*_i is a constant for $i \in \{G, B\}$. The tax on wage is not independent of the unemployment history[27].

Proof. By way of contradiction, suppose $c^*_{Bt} = c^*_B$ and $c^*_{Gt} = c^*_G$ for all t. Lemma 4.2 shows that period-t multiplier associated with constraint IC of FOA is strictly negative: $\mu_t < 0$.

By comparing the RHS of the F.O.C.s (3.5)-(3.8), we can derive $\frac{u(c^*_G)}{1-\beta} > \frac{u(c^*_B)}{1-\beta} > u(c^*_{Ut}) + \beta v^*_{Ut}$ from MLRP and $\mu_t < 0$. By proposition 4, $u(c^*_{Ut})$ and v^*_{Ut} and hence $u(c^*_{Ut}) + \beta v^*_{Ut}$ are decreasing over unemployment spell. Therefore we have $\frac{u(c^*_G)}{1-\beta} > \frac{u(c^*_B)}{1-\beta} > u(c^*_{Ut}) + \beta v^*_{Ut} > u(c^*_{Ut+1}) + \beta v^*_{Ut+1}$. Also $(a^*_t, c^*_t, v^*_{U(t+1)})$ satisfies constraint IC of FOA for all t. So a^*_t is increasing over unemployment spell.

MLRP implies

$$\frac{m_a(G,a)}{m(G,a)} > \frac{(m_a(G,a) + m_a(B,a))}{(m(G,a) + m(B,a))} \tag{3.16}$$

[26] c^*_{it} is the future consumption level if worker find a type-i job in period t, $i \in \{G, B\}$.
[27] The proof is an extension of the argument in [4].

Therefore we have

$$
\begin{aligned}
& [\frac{m(G,a)}{m(G,a)+m(B,a)}]' \\
= \;& \frac{m_a(G,a)(m(G,a)+m(B,a)) - (m_a(G,a)+m_a(B,a))m(G,a)}{(m(G,a)+m(B,a))^2} && (3.17)\\
> \;& 0 && (3.18)
\end{aligned}
$$

Let

$$E(\frac{1}{u'(c^*_{jk})}|j \in \{G,B\}) = \frac{1}{u'(c^*_{Gk})}\frac{m(G,a)}{m(G,a)+m(B,a)} + \frac{1}{u'(c^*_{Bk})}\frac{m(B,a)}{m(G,a)+m(B,a)}$$

With $c^*_G > c^*_B$ and increasing a^*_t in t, it is not hard to see that $E(\frac{1}{u'(c^*_{jk})}|j \in \{G,B\})$ is increasing over unemployment spell.

If we keep substituting the martingale property into previous period martingale property, we have

$$
\begin{aligned}
\frac{1}{u'(c^*_{Ut-1})} &= \sum_{k=0}^{T}[\Pi_{i=-1}^{i=k-2} m(U,a^*_{t+i})](1-m(U,a^*_{t+k-1}))* && (3.19)\\
& E(\frac{1}{u'(c^*_{j(t+k)})}|j \in \{G,B\}) + [\Pi_{i=-1}^{i=T-1} m(U,a^*_{t+i})]\frac{1}{u'(c^*_{U(t+T)})}.
\end{aligned}
$$

with $[\Pi_{i=0}^{i=-1} m(U,a^*_{t+i})] = 1$. The unemployed worker will eventually get an offer: $\sum_{k=0}^{T}[\Pi_{i=0}^{i=k-1} m(U,a^*_{t+i})](1-m(U,a^*_{t+k})) = 1$. Let T go to $+\infty$ in (3.18). And since $E(\frac{1}{u'(c^*_{jk})}|j \in \{G,B\})$ is increasing on k, we have

$$\frac{1}{u'(c^*_{Ut-1})} > E(\frac{1}{u'(c^*_{jt})}|j \in \{G,B\}) \qquad (3.20)$$

$c^*_G > c^*_B$ implies $\frac{1}{u'(c^*_{U(t-1)})} > E(\frac{1}{u'(c^*_{jt})}|j \in \{G,B\}) > \frac{1}{u'(c^*_{Bt})}$.

However, the F.O.C. for c_{Bt}

$$\frac{1}{u'(c^*_{Bt})} = -\lambda_t - \mu_t\frac{m_a(B,a_t)}{m(B,a_t)}. \qquad (3.21)$$

$m_a(B,a) > 0$ and the envelope condition $\frac{1}{u'(c^*_{U(t-1)})} = -\lambda_t$ imply $\frac{1}{u'(c^*_{Ut-1})} < \frac{1}{u'(c^*_{Bt})}$, generating a contradiction. ■

4 Conclusion

4.1 Conclusion

Our paper introduces two types of observable job offers into the model of Hopenhayn and Nicolini [4] and allows net transfers to be contingent on the type of offer. To simplify the analysis, we use the first-order approach. A set of conditions is proposed to support this approach in this dynamic model with two absorbing states. However, showing differentiability is missing in our paper, as well as in Hopenhayn and Nicolini [4]. When both types of job offers are always accepted, we illustrate that dependence on offer type might not be redundant. History-dependences of unemployment benefits and net transfers on wage are robust in the extended model. Further extension to a model with multiple absorbing states is immediate.

The model we have studied contains a number of restrictions. We assume conditions under which the two types of employment are absorbing states. It would be interesting to introduce transitional states[28]. However, if we do so, it is hard to find a way to restrict the first-order derivatives of cost functions across different states, leading to new problems with justification of FOA and characterizing the optimal contract. Also our analysis, as well as that in Hopenhayn and Nicolini [4], relies on convexity and differentiability assumptions. Showing these properties of the cost function in our model requires different proof techniques from standard methods. One possible way, suggested by Hopenhayn and Nicolini [4], is to allow lotteries. This approach has not yet been explored.

5 References

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6 Appendix

6.1 Proposition 1

The section allows planner to tell worker to reject offer and presents conditions under which She doesn't want rejecting offer. Contract is redefined. Specifically, compared with previous definition, acceptance rule is added into the profile. Functions are contingent on offer history instead of worker's state history. Then transition matrix and continuation contract with effort observed are prepared for proposition 1.

Let S denote an acceptance rule: S:$O \times H \to H$, with search-outcome space $O = \{g, b, u\}$ and worker's state space $H = \{G, B, U\}$. G, B, U stand for becoming good-job employed, bad-job employed and unemployed. g, b, u

denote arrival of a good-job offer, a bad-job offer and no offer respectively. An acceptance rule specifies acceptance decision over each type of offer realization for each type of worker.

An acceptance rule is restrictive by worker's state. We denote the space of feasible acceptance rules: $D = \{S : O \times H \to H | S(j, i) \in \{U\} \cup \{i\} \cup \{I(j)\}\}$, where $I(.) : O \to H$ maps an element, denoted by a small letter j, in O to the element, denoted by the j's capital letter J, in H.

Denote worker's offer realized at date t j_t. The public history up to the beginning of period t is worker's offer history: $j^{t-1} = j_0, ... j_{t-1}$. Let the set of history$j^{t-1}$ be $\Phi^t = \{j^{t-1} | \forall k \in N,\ j_k \in O\}$, with $\Phi^0 = \emptyset$.

The planner recommends effort and acceptance rule to the worker at the beginning of each period, based on available information: j^{t-1}. Then search outcome is realized. With this additional information, consumption level is specified.In particular, A contract (c, a, d) specifies three sequences of functions: $\{c_t\}_{t=1}^{\infty}$, $\{a_t\}_{t=1}^{\infty}$ and $\{d_t\}_{t=1}^{\infty}$ where $c_t : \Phi^{t+1} \to Z$, $a_t : \Phi^t \to E$, $d_t : \Phi^t \to D$. c_t , a_t and d_t are current consumption, current recommended action and current acceptance rule for period t.

Offer realization and acceptance rule determine worker's state, which in turn determines wage payment. Therefore we want to keep track of it, although the state is not in the public history. Let i_t be worker's state right after current acceptance rule is carried out. Define the transition probability that offer j_t is realized and worker's state become i_t, conditional on state i_{t-1}, as function

$$q(i_t, j_t | i_{t-1}; a_t, d_t) = m(j_t; a_t) 1(d_t(j_t, i_{t-1}) = i_t)^{29}. \tag{6.1}$$

Since offer realization is correlated over time through effort strategy, we can calculate the probability of realized offer sequence j^t and state sequence i^t up to the beginning of period t+1 recursively: $\mu(i^t, j^t; a, d) = q(i_t, j_t | i_t; a_t, d_t) \mu(i^{t-1}, j^{t-1}; a, d)$, $\mu(U, \emptyset; a, d) = 1$, for any $\{a_t\}$ and $\{d_t\}$. The conditional probability $\mu(i^t, j^t | i^k, j^k; a, d)$ can be obtained similarly.

Consider the node right after a worker with continuation lifetime value V receives bad-job offer w_B. We change the environment. Suppose from next period onward effort is observed. If the worker is told to reject current job offer, the minimal cost of continuation contract in this environment is the

[29] j_{t-1} is omitted in q, because the transition matrix doesn't depend on j_{t-1}.

minimal value of the following problem:

$$\min_{(c,a,d)} c_k + \sum_{t=k}^{\infty} \beta^{t+1-k} \sum_{i^{t+1},j^{t+1}} \mu(i^{t+1}, j^{t+1}|i^k, j^k; a, d)(c(j^{t+1}) - w_{i_{t+1}})$$

$$\begin{aligned} \text{s.t } V &\leq u(c_k) + \sum_{t=k}^{\infty} \beta^{t+1-k} \sum_{i^{t+1},j^{t+1}} \mu(i^{t+1}, j^{t+1}|i^k, j^k; a, d) u(c(j^{t+1})) \\ & - \sum_{t=k}^{\infty} \beta^{t+1-k} \sum_{i^t,j^t} \mu(i^t, j^t|i^k, j^k; a, d) a_t(j^t) \end{aligned}$$

Without further incentive problem, the solution implies full insurance against offer risk. The cost formula therefore can be further simplified. Suppose a^*, d^* are recommended effort and recommended acceptance rule in the solution scheme. The minimal cost to the above problem is

$$\begin{aligned} & \frac{u^{-1}(V(1-\beta) + (1-\beta)\sum_{t=k}^{\infty} \beta^{t+1-k} \sum_{i^t,j^t} \mu(i^t, j^t|i^k, j^k; a, d) a_t(j^t))}{1-\beta} \\ - & \sum_{t=k}^{\infty} \beta^{t+1-k} \sum_{i^{t+1},j^{t+1}} \mu(i^{t+1}, j^{t+1}|i^k, j^k; a^*, d^*) w_{i_{t+1}} \qquad (6.2) \end{aligned}$$

The following proposition allows us to focus on the case where both types of job are accepted every period. The proof is by comparing the option of accepting current bad-job offer, that of rejecting current offer rejected with effort observed in the future and that of rejecting current offer with effort unobserved in the future.

Proposition 1 With $w_G \geq w_B \geq \beta w_G$, the optimal contract tells worker to accept bad job offer or to stay bad-job employed.

Proof. Suppose that the worker has a bad job at hand and is promised continuation lifetime value V. If the job is accepted and worker won't quit this job, the continuation cost is $\frac{u^{-1}(V(1-\beta))-w_B}{1-\beta}$.

Consider

$$\frac{u^{-1}(V(1-\beta)+(1-\beta)\sum_{t=k}^{\infty}\beta^{t+1-k}\sum_{i^t,j^t}\mu(i^t,j^t|i^k,j^k;a,d)a_t(j^t))}{1-\beta}$$
$$-\sum_{t=k}^{\infty}\beta^{t+1-k}\sum_{i^{t+1},j^{t+1}}\mu(i^{t+1},j^{t+1}|i^k,j^k;a^*,d^*)w_{i_{t+1}}$$
$$\geq \frac{u^{-1}(V(1-\beta))}{1-\beta}-\frac{\beta w_G}{1-\beta} \quad (6.3)$$
$$\geq \frac{u^{-1}(V(1-\beta))-w_B}{1-\beta} \quad (6.4)$$

The second inequality is from $w_G > w_B \geq \beta w_G$. The first one is by $u' > 0$ and the following:

$$\sum_{t=k}^{\infty}\beta^{t+1-k}\sum_{i^{t+1},j^{t+1}}\mu(i^{t+1},j^{t+1}|i^k,j^k;a^*,d^*)w_{i_{t+1}}$$
$$= E_a\{\sum_{t=1}^{\infty}\beta^t w_i\} \quad (6.5)$$
$$\leq E_a\{\sum_{t=1}^{\infty}\beta^t w_G\} \quad (6.6)$$
$$= \frac{\beta w_G}{1-\beta} \quad (6.7)$$

If the bad-job offer is rejected, the minimal cost of continuation contract with future effort unobserved is higher than that with future effort observed, which is higher than accepting current bad-job offer by (6.4). Therefore accepting and staying employed whenever it is possible is optimal. ■

6.2 Employed Worker Problem

Consider an incentive-compatible continuation contract with continuation lifetime value V_G for a good-job employed worker. With no further incentive problem for employed worker, it is optimal to give worker a constant level of consumption $c_G = u^{-1}(V_G(1-\beta))$. The optimal period net transfer will be $c_G - w_G$. With $c_G - w_G > 0$, the planner subsidizes good-job employed worker; With $c_G - w_G < 0$, the planner taxes good-job employed worker.

Expected cost function of the optimal contract providing V_G to good-job employed worker is $\frac{u^{-1}(V_G(1-\beta))-w_G}{1-\beta}$, which is strictly increasing and strictly convex in V_G. The derivative with respect to V is $\frac{1}{u'(c_G)}$.

The similar statement holds for a bad-job employed worker. The expected cost and derivative are $\frac{u^{-1}(V_B(1-\beta))-w_B}{1-\beta}$ and $\frac{1}{u'(c_B)}$, respectively, with $V_B = \frac{u(c_B)}{1-\beta}$.

6.3 Existence

In this section, we consider the fixed point issues of the mapping T and a mapping redefined from T_{FOA} . First we show that the redefined mapping has a continuous correspondence. Then we show that it satisfies blackwell sufficient conditions, and hence it is a contraction mapping on bounded continuous function space. Similar results hold for T by a similar discussion.

6.3.1 Continuity

We want to show that $\Phi_{FOA}(V)$ is continuous at $V = V^*$.

First, we show that it is upper hemi-continuous at $V = V^*$. Suppose sequences $\{v_i\}$ and $\{(a^i, c^i_G, c^i_B, c^i_U, v^i_U)\}$ with $v_i \to V^*$ and $(a^i, c^i_G, c^i_B, c^i_U, v^i_U) \in \Phi_{FOA}(v_i)$. Since $E\times Z^3\times\overline{V}$ is compact, there exists a subsequence $\{(a^{i_j}, c^{i_j}_G, c^{i_j}_B, c^{i_j}_U, v^{i_j}_U)\}$ and $(a', c'_G, c'_B, c'_U, v'_U) \in E\times Z^3\times\overline{V}$; $(a^{i_j}, c^{i_j}_G, c^{i_j}_B, c^{i_j}_U, v^{i_j}_U)$ converges to $(a', c'_G, c'_B, c'_U, v'_U)$. Continuous $U(a, c, v)$ implies $U(a^{i_j}, (c^{i_j}_G, c^{i_j}_B, c^{i_j}_U), v^{i_j}_U) \to U(a', (c'_G, c'_B, c'_U), v'_U)$. Because $(a^{i_j}, c^{i_j}_G, c^{i_j}_B, c^{i_j}_U, v^{i_j}_U)$ satisfies promise-keeping constraint, we have $V^* \geq U(a^{i_j}, (c^{i_j}_G, c^{i_j}_B, c^{i_j}_U), v^{i_j}_U) \geq v_{i_j}$[30]. $v_{i_j} \to V^*$ and $U(a^{i_j}, (c^{i_j}_G, c^{i_j}_B, c^{i_j}_U), v^{i_j}_U) \to U(a', (c'_G, c'_B, c'_U), v'_U)$ lead to $U(a', (c'_G, c'_B, c'_U), v'_U) = V^*$. Note that the only plan in $E \times Z^3 \times \overline{V}$ with lifetime value V^* is $(0, w_G, w_G, w_G, V^*)$. This implies $(a', c'_G, c'_B, c'_U, v'_U) = (0, w_G, w_G, w_G, V^*) \in \Phi_{FOA}(V^*)$. $\Phi_{FOA}(V)$ is upper hemi-continuous at $V = V^*$.

Now consider lower hemi-continuity at $V = V^*$. We want to show that for any $\{v_i\}$ with $v_i \to V^*$, there exists a sequence $\{(a^i, c^i_G, c^i_B, c^i_U, v^i_U)\}$ satisfying $(a^i, c^i_G, c^i_B, c^i_U, v^i_U) \in \Phi_{FOA}(v_i)$ and $(a^i, c^i_G, c^i_B, c^i_U, v^i_U) \to (0, w_G, w_G, w_G, V^*)$. Actually, for any sequence $\{(a^i, c^i_G, c^i_B, c^i_U, v^i_U)\}$ with $(a^i, c^i_G, c^i_B, c^i_U, v^i_U) \in \Phi_{FOA}(v_i)$, any subsequence of $\{(a^i, c^i_G, c^i_B, c^i_U, v^i_U)\}$ has a sub-subsequence

[30] Any incentive-compatible contract in $E \times Z^3 \times \overline{V}$ is associated with lifetime value no higher than V^*.

convergent to $(0, w_G, w_G, w_G, V^*)$ by the argument for u.h.c. at $V = V^*$. This implies $(a^i, c_G^i, c_B^i, c_U^i, v_U^i) \to (0, w_G, w_G, w_G, V^*) \in \Phi_{FOA}(V^*)$. $\Phi_{FOA}(V)$ is lower-hemi continuous at V^*.

$\Phi_{FOA}(V)$ is l.h.c. at $V \in [0, V^*)$.

For $V = 0$, consider any sequence $\{v_j\}$ with $v_j \to 0$. If we have $(a, c_G, c_B, c_U, v_U) = (0, 0, 0, 0, 0)$, we can construct a sequence $\{(0, c_j, c_j, 0, 0)\}$ with c_j satisfying $\frac{u(c_j)}{1-\beta}(1 - m(U, 0)) = v_j$[31], which implies $c_j \to 0$, hence $(0, c_j, c_j, 0, 0) \to (0, 0, 0, 0, 0)$. Moreover with $U_a(0, (c_j, c_j, 0), 0) = +\infty > 0$, we have $(0, c_j, c_j, 0, 0) \in \Phi_{FOA}(v_j)$; If we have $(a, c_G, c_B, c_U, v_U) \in \Gamma_{FOA}(0)$, it satisfies $U(a, c, v_U) \geq 0$ and $U_a(a, c, v_U) \geq 0$. Because $U(a, c, v_U)$ and $U_a(a, c, v_U)$ are continuous, we can construct a sequence convergent to (a, c_G, c_B, c_U, v_U) by similar perturbation. Therefore $\Phi_{FOA}(V)$ is l.h.c. at $V = 0$.

For $V \in (0, V^*)$, $\Phi_{FOA}(V)$ is defined by two continuous functions. Similar perturbation can be constructed so that $\Phi_{FOA}(.)$ is l.h.c. at $V \in (0, V^*)$. In summary, $\Phi_{FOA}(V)$ is l.h.c. at $V \in [0, V^*]$. Therefore its closure $\overline{\Phi}_{FOA}(V)$ is l.h.c. at $V \in [0, V^*]$[32].

Then we want to show that $\overline{\Phi}_{FOA}(V)$ is u.h.c.. $\overline{\Phi}_{FOA}(V)$ is immediately u.h.c. at V^*[33], because $\Phi_{FOA}(V)$ is upper hemi-continuous at $V = V^*$. We need to show that $\overline{\Phi}_{FOA}(V)$ is u.h.c. at $V \in [0, V^*)$. Define $\Psi(v) = \{(a, c_G, c_B, c_U, v_U) \in E \times Z^3 \times \overline{V} | v \leq U(a, c, v_U)\}$ and $\Gamma_{FOA} = \{(a, c_G, c_B, c_U, v_U) \in E \times Z^3 \times \overline{V} | 0 \leq U_a(a, c, v_U)\} \cup \{(0, 0, 0, 0, 0)\}$. With $V \in [0, V^*)$, it is not hard to see $\Phi_{FOA}(V) = \Psi(V) \cap \Gamma_{FOA}$, which implies $\overline{\Phi}_{FOA}(V) = \overline{\Psi}(V) \cap \overline{\Gamma}_{FOA}$. $\Psi(V)$ is defined by $V \leq U(a, c, v_U)$, and $U(a, c, v_U)$ is continuous. So $\Psi(V)$ and hence $\overline{\Psi}(V)$ have a closed graph on $[0, V']$ with $V < V' < V^*$. Also since $\overline{\Gamma}_{FOA}$ is closed, $\overline{\Phi}_{FOA}(V)$ also has a closed graph on $[0, V']$. Furthermore, because $\overline{\Phi}_{FOA}(V)$ is in a compact space $E \times Z^3 \times \overline{V}$, $\overline{\Phi}_{FOA}(V)$ is u.h.c. at $V \in [0, V^*)$[34].

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[31] This is feasible for sufficiently large j.

[32] See exercise 25 page 300 in "Real Analysis with economics application".

[33] See exercise 13 page 293 in "Real Analysis with economics application".

[34] See proposition 3 (a) about closed graph, page 295 in "Real Analysis with economics application".

6.3.2 Contraction

Define $\widehat{T}_{FOA}$ with constraint set $\Phi_{FOA}(V)$ replaced by $\overline{\Phi}_{FOA}(V)$ in T_{FOA}[35]. The following considers the fixed points of the redefined mapping $\widehat{T}_{FOA}$ and mapping T.

Proposition 2 Consider the mapping $\widehat{T}_{FOA}$ with constraint set $\Phi_{FOA}(V)$ replaced by $\overline{\Phi}_{FOA}(V)$ in T_{FOA}. This mapping is a contraction mapping on a space of bounded continuous functions, hence its unique fixed point is a bounded continuous function; T is a contraction mapping on a space of bounded functions, hence its unique fixed point is a bounded function.

Proof. Suppose $B(\overline{V})$ is the space of continuous functions on $\overline{V}$. Notice that $\overline{\Phi}_{FOA}(.)$ is continuous, compact and non-empty. The theorem of maximization implies $\widehat{T}_{FOA} : B(\overline{V}) \to B(\overline{V})$. For $\forall\ C(V) \in B(\overline{V})$, we want to show that $\widehat{T}_{FOA}$ satisfies blackwell sufficient conditions:

Discounting: We have

$$\begin{aligned}(\widehat{T}_{FOA}C)(V) + \beta k \ \geq\ & m(g;a^*)(\frac{c_G^* - w_G}{1-\beta}) && (6.8)\\ & +m(b;a^*)(\frac{c_B^* - w_B}{1-\beta}) \\ & +m(u;a^*)\{c_U^* - w_U + \beta C(v_U^*)\} \\ & +m(u;a^*)\beta k \\ \geq\ & \widehat{T}_{FOA}(C+k)(V) && (6.9)\end{aligned}$$

with $k > 0$ and (a^*, c^*, v^*)[36], the solution to the above problem $(\widehat{T}_{FOA}C)(V)$.

[35] But the solution plan is in $\Phi_{FOA}(V)$ if $C(V)$ satisfies two inequality. At $V = V^*$, the solution is $(0, w_G, w_G, w_G, V^*) \in \Phi_{FOA}(V^*)$. At $V = 0$, the solution is $(0,0,0,0,0) \in \Phi_{FOA}(0)$. In later section, we shows that the solution is an interior solution at $V \in (0, V^*)$, therefore it is also in $\Phi_{FOA}(V)$.

[36] This solution to $\widehat{T}_{FOA}C(V)$ exists because we have continuous objective function and compact constraint set.

Monotonicity: Suppose $C^1, C^2 \in B(\overline{V})$ with $C^1 \leq C^2$. We have

$$\begin{aligned}
(\widehat{T}_{FOA}C^2)(V) &= m(G;a')(\frac{c'_G - w_G}{1-\beta}) &(6.10)\\
&+m(B;a')(\frac{c'_B - w_B}{1-\beta})\\
&+m(U;a')(c'_U - w_U + \beta C^2(v'_U))\\
&\geq m(G;a')(\frac{c'_G - w_G}{1-\beta}) &(6.11)\\
&+m(B;a')(\frac{c'_B - w_B}{1-\beta})\\
&+m(U;a')(c'_U - w_U + \beta C^1(v'_U))\\
&\geq (\widehat{T}_{FOA}C^1)(V) &(6.12)
\end{aligned}$$

with (a', c', v') the optimal plan to the problem $(\widehat{T}_{FOA}C^2)(V)$.

The mapping of $\widehat{T}_{FOA}$ is a contraction mapping on $B(\overline{V})$. Its fixed point is a bounded continuous function. T is also a contraction mapping in bounded function space by a similar argument. ■

6.4 Interior Solution

6.4.1 Interior effort

We want to have interior effort in $TC(V)$ and $\widehat{T}_{FOA}C(V)$ with $V \in (0, V^*)$.

For $TC(V)$, if the optimal implementable effort is an interior effort level, it satisfies first-order condition in worker's private problem. Therefore the policy function of problem $TC(V)$ is in $\Phi_{FOA}(V)$.

For $\widehat{T}_{FOA}C(V)$, if the optimal effort is an interior effort level, the policy function of $\widehat{T}_{FOA}C(V)$ is in $\Phi_{FOA}(V)$, the constraint set of $\widehat{T}_{FOA}C(V)$, implying $\widehat{T}_{FOA}C(V) = T_{FOA}C(V)$. Thus the existence results for $\widehat{T}_{FOA}$are carried over to T_{FOA}.

Define a subspace of continuous functions space

$$\begin{aligned}
M(\overline{V}) &= \{C(V) \in B(\overline{V}) | C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}\\
, C_v(V) &\geq u^{-1\prime}(V(1-\beta)), \text{with } \nu(c) = \frac{1}{u'(c)}, \text{ and } C_v(0) = 0\}.
\end{aligned}$$

Lemma 1 With $C(V) \in M(\overline{V})$, we have $a^* \in (0, \overline{a})$ in solution to problems $TC(V)$, $\widehat{T}_{FOA}C(V)$[37] and $T_{FOA}C(V)$ for all $V \in (0, V^*)$, in both structures.

Proof. With $m_a(i, \overline{a}) = 0$, $i \in \{G, B, U\}$, we have $U_a(\overline{a}, c, v) < 0$ for any scheme of (c, v). The worker tend to reduce effort level. Thus $a^* = \overline{a}$ is not implementable and hence it is not optimal in problem $TC(V)$ for any function $C(V)$ and $V \in [0, V^*]$. Similar statement applies to $T_{FOA}C(V)$.

By way of contradiction, suppose $a^* = 0$ in problem $TC(V)$ for $V \in (0, V^*)$. Consider the problem $Q(V)$

$$\begin{aligned} \min_{(c, v_U)} m(G;0)(\frac{c_G - w_G}{1-\beta}) + m(B;0)(\frac{c_B - w_B}{1-\beta}) \\ + m(U;0)(c_U + \beta C(v_U)) \\ \text{s.t. } V \leq U(0, c, v_U). \end{aligned} \tag{6.13}$$

Denote the minimal cost $q(V)$. It is obvious that $q(V) \leq TC(V)$. In problem $Q(V)$, the first-order conditions are

$$1 + \lambda u'(c_i^*) \leq 0, \text{ if } c_i^* = w_G \tag{6.14}$$

$$1 + \lambda u'(c_i^*) \geq 0, \text{ if } c_i^* = 0 \tag{6.15}$$

$$1 + \lambda u'(c_i^*) = 0, \text{ otherwise} \tag{6.16}$$

$$C_v(v_u^*) + \lambda \leq 0, \text{ if } v_U^* = V^* \tag{6.17}$$

$$C_v(v_u^*) + \lambda \geq 0, \text{ if } v_U^* = 0 \tag{6.18}$$

$$C_v(v_u^*) + \lambda = 0, \text{ otherwise} \tag{6.19}$$

for $i \in H$.

$c_i^* = w_G$ implies $u'(c_i^*) = 0$ violating first-order condition (6.14). Therefore we have $c_i^* < w_G$. Similarly we have $v_U^* < V^*$.

Now suppose $c_i^* = 0$. The F.O.C. (6.15) implies $\lambda = 0$. Then the F.O.C. (6.15) becomes $1 \geq 0$ for all $i \in H$. This implies $c_i^* = 0$ for all $i \in H$. For v_U, with the F.O.C.s (6.18)-(6.19), we have either $C_v(v_u^*) = -\lambda = 0$ or $v_u^* = 0$. Both imply $v_u^* = 0$, because $C_v(V) \geq u^{-1\prime}(V(1-\beta))$ implies $C_v(V) > 0$ for all $V \in (0, V^*]$. Then the associated lifetime value is equal to 0, violating the promise-keeping constraint with $V > 0$.

[37] $\widehat{T}$ is defined in existence section.

Similarly suppose $v_u^* = 0$. $\lambda \leq 0$ and F.O.C. (6.18) imply $C_v(0) = 0 \geq -\lambda \geq 0$, hence $\lambda = 0$. By (6.15), we have $c_i^* = 0$, violating the above $c_i^* > 0$. Hence problem $Q(V)$ has $c_i^* = c^* \in (0, w_G)$ for $i \in H$ and $v_U^* \in (0, V^*)$.

Now consider the planner's cost $q(V)$. $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$ implies $C(v_u^*) \geq \frac{c^*}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$[38]. Therefore we have $\frac{c^*-w_G}{1-\beta} < \frac{c^*-w_B}{1-\beta} \leq c^* + \beta C(v_U^*)$. This implies $q_a(a, c^*, v_U^*) < 0$ for $a \in [0, \overline{a})$ and $q_a(0, c^*, v_U^*) = -\infty$[39].

$\frac{1}{u'(c^*)} = C_v(v_u^*)$, by (6.16) and (6.19), and $C_v(V) \geq u^{-1\prime}(V(1-\beta))$ imply

$$\frac{u(c^*)}{1-\beta} \geq u(c_U^*) + \beta v_U^*. \tag{6.20}$$

If (6.20) is a strict inequality, we have $U_a(0, c^*, v_U^*) = +\infty$. The optimal effort solution to worker's private problem is $a^* > 0$. The incentive-compatible plan (a^*, c^*, v_U^*) generates a higher lifetime value to worker but a strictly lower cost than $q(V)$ with $q_a(a, c^*, v_U^*) < 0$ for $a \in [0, \overline{a})$, generating a contradiction.

If (6.20) is an equality, we have $U_a(0, c^*, v_U^*) = -1$. The planner can increase c_G^* and reduce c_B^* and c_U^* by a small amount[40] while maintaining the same ex-anted promise value to the worker and the same lifetime cost. Again we have $U_a(0, c^*, v_U^*) = +\infty$. Also this change preserves ordering $\frac{c_G^*-w_G}{1-\beta} < \frac{c_B^*-w_B}{1-\beta} \leq c_U^* + \beta C(v_U^*)$, hence $q_a(a, c^*, v_U^*) < 0$ for $a \in [0, \overline{a})$ and $q_a(0, c^*, v_U^*) = -\infty$. Similar argument for strict inequality (6.20) generates a contradiction. Therefore we have $a^* > 0$.

Similar arguments can be used to rule out effort solution $a^* = 0$ to $\widehat{T}_{FOA}C(V)$ and that to $T_{FOA}C(V)$ for $V \in (0, V^*)$. ■

6.4.2 Interior plan

Now we want to show that the solution to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, for $V \in (0, V^*)$.

Lemma 2 In both structures, suppose $C(V) \in M(\overline{V})$. The solution to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, for all $V \in (0, V^*)$;

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[38] $C(v_u^*) \geq \frac{\nu^{-1}(C_v(v_u^*))}{1-\beta} - \frac{w_B}{\beta(1-\beta)} = \frac{\nu^{-1}(\frac{1}{u'(c^*)})}{1-\beta} - \frac{w_B}{\beta(1-\beta)} = \frac{c^*}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$

[39] $q(a, c, v_U)$ is the cost of plan (a, c, v_U): $q(a, c, v_U) = m(G, a)\frac{c_G - w_G}{1-\beta} + m(B, a)\frac{c_B - w_B}{1-\beta} + m(U, a)(c_U + \beta C^{FOA}(v_U))$

[40] This is feasible because c_G is an interior point in Z and $\begin{cases} U(a^*, c^*, v_U^*) = U(a^*, c, v_U^*) \\ q(a^*, c^*, v_U^*) = q(a^*, c, v_U^*) \end{cases}$ is a group of equation with 3 variable.

If the plan solution to problem $T_{FOA}C(0)$ have effort level $a^* > 0$, then the optimal plan is also an interior point.

Proof. With $V \in (0, V^*)$, Lemma 1 implies that problem $T_{FOA}C(V)$ has effort solution $a^* \in (0, \overline{a})$; Problem $T_{FOA}C(0)$ has effort solution $a^* \in (0, \overline{a})$. Hence $\frac{m_a(i,a)}{m(i,a)}$s are finite nonzero numbers.

When boundary constraints are included in problem $T_{FOA}C(V)$, the first-order conditions for consumption are

$$1 + \lambda u'(c_i) + \mu \frac{m_a(i,a)}{m(i,a)} u'(c_i) \leq 0, \text{ if } c_i^* = w_G \tag{6.21}$$

$$1 + \lambda u'(c_i) + \mu \frac{m_a(i,a)}{m(i,a)} u'(c_i) \geq 0, \text{ if } c_i^* = 0 \tag{6.22}$$

$$1 + \lambda u'(c_i) + \mu \frac{m_a(i,a)}{m(i,a)} u'(c_i) = 0, \text{ otherwise} \tag{6.23}$$

for all $i \in H$. It is obvious that the first-order conditions (6.21) imply $c_i^* \neq w_G$ for all i. Similarly $C_v(V^*) = +\infty$ implies $v_U^* \neq V^*$.

First we want to show $c_i^* \neq 0$ for $i \in \{G, B\}$ in structure 2. The inequality constraint IC of FOA implies $\mu \leq 0$. By way of contradiction, suppose $c_G^* = 0$ for example. (6.22) implies $\lambda = \mu = 0$[41]. The F.O.C (6.22) for c_i becomes $1 > 0$, which implies $c_i^* = 0$ for all $i \in H$. Also the F.O.C.s for v_U^* become

$$C_v(v_U) \leq 0, \text{ if } v_U^* = V^* \tag{6.24}$$

$$C_v(v_U) \geq 0, \text{ if } v_U^* = 0 \tag{6.25}$$

$$C_v(v_U) = 0, \text{ otherwise.} \tag{6.26}$$

For $C_v(V) \geq u^{-1\prime}(V(1-\beta))$, $v_U^* > 0$ implies $C_v(v_U^*) > 0$, violating the F.O.C.s (6.24) and (6.26). Hence we have $v_U^* = 0$. $v_U^* = c_i^* = 0$ implies $U_a(a^*, c^*, v_U^*) = -1$, contradicting to constraint IC of FOA. Therefore, we have $c_i^* \neq 0$ for $i \in \{G, B\}$.

Then we want to show $c_i^* \neq 0$ for $i \in \{G, B\}$ in structure 1, where $\Phi_{FOA}(V)$ are defined by equality constraint IC of FOA. Therefore the multiplier μ can be positive. By way of contradiction, suppose $c_G^* = 0$ for example. $\mu < 0$ contradicts to F.O.C.(6.22). Then consider $\mu = 0$. This implies $\lambda = 0$,

[41] Otherwise, we have $\lambda + \mu \frac{m_a(G,a)}{m(G,a)} < 0$, hence $(\lambda + \mu \frac{m_a(G,a)}{m(G,a)}) u'(c_G) = -\infty$, violating the first-order condition (6.21) for $i = G$.

because $\lambda < 0$ implies $\lambda + \mu\frac{m_a(G,a)}{m(G,a)} < 0$, hence $(\lambda + \mu\frac{m_a(G,a)}{m(G,a)})u'(c_G) = -\infty$, violating the first-order condition (6.22) for $i = G$. With $\lambda = \mu = 0$, F.O.C. (6.22) becomes $1 > 0$. Again, we have $c_i = 0$ for all $i \in H$ and $v_U^* = 0$. Constraint IC of FOA is violated. Consider $\mu > 0$. Suppose $(\lambda + \mu\frac{m_a(G,a)}{m(G,a)}) \geq 0$. $\frac{m_a(G,a)}{m(G,a)} = \frac{m_a(B,a)}{m(B,a)}$ leads to $(\lambda + \mu\frac{m_a(B,a)}{m(B,a)}) \geq 0$, which implies $1 + \lambda u'(c_i) + \mu\frac{m_a(i,a)}{m(i,a)}u'(c_i) > 0$. Only (6.22) can be satisfied. Then we have $c_B^* = c_G^* = 0$, violating constraint IC of FOA. Suppose $(\lambda + \mu\frac{m_a(G,a)}{m(G,a)}) < 0$, which implies $(\lambda + \mu\frac{m_a(G,a)}{m(G,a)})u'(c_G) = -\infty$. The first-order condition (6.22) for c_G is violated. Given $c_G^* = 0$. A contradiction is generated for $\mu > 0$, $\mu = 0$ and $\mu < 0$ respectively. The procedure can be applied to $c_B^* = 0$. Hence we have $c_i^* \neq 0$ for $i = \{G, B\}$ for structure 1.

We want to show $c_U^* \neq 0$. By way of contradiction, suppose $c_U^* = 0$. For all $i \in H$, we perturb $u(c_i)$ around the solution plan by a sufficiently small amount of Δu_i satisfying $\Delta u_B = k\Delta u_G$ and Δu_G, $\Delta u_U > 0$[42]. The real number k will be determined properly later. This perturbation generates changes in expected cost C, worker's first-order derivative U_a, and the lifetime value U:

$$\Delta C = (m(G,a^*)\frac{1}{u'(c_G^*)} + m(B,a^*)\frac{k}{u'(c_B^*)})\Delta u_G + m(U,a^*)\frac{\Delta u_U}{u'(0)} \quad (6.27)$$

$$\Delta U_a = m_a(G,a^*)\Delta u_G + m_a(B,a^*)k\Delta u_G + m_a(U,a^*)\Delta u_U \quad (6.28)$$

$$\Delta U = m(G,a^*)\Delta u_G + m(B,a^*)k\Delta u_G + m(U,a^*)\Delta u_U. \quad (6.29)$$

We want to make sure that the perturbed plan is still in Φ_{FOA}: $\Delta U_a \geq 0$, $\Delta U \geq 0$. This is equivalent to the following.

$$\Delta u_G \geq -\frac{m_a(U,a^*)\Delta u_U}{m_a(G,a^*) + m_a(B,a^*)k} \quad (6.30)$$

$$\Delta u_G \geq -\frac{m(U,a^*)\Delta u_U}{m(G,a^*) + m(B,a^*)k} \quad (6.31)$$

$c_i^* \in (0, w_G)$ implies $\frac{1}{u'(c_i^*)} > 0$ for $i \in \{G, B\}$. Let k go to $-\infty$. We have $(m(G,a^*)\frac{1}{u'(c_G^*)} + m(B,a^*)\frac{k}{u'(c_B^*)}) \to -\infty$. (6.26) implies $\Delta C \to -\infty$, with $\frac{1}{u'(0)} = 0$. It is easy to check $-\frac{m_a(U,a^*)}{m_a(G,a^*)+m_a(B,a^*)k}$, $-\frac{m(U,a^*)}{m(G,a^*)+m(B,a^*)k} \to 0$.

[42] This can be done without violating boundary constraint, because c_G^* and c_B^* are interior points of Z.

Hence for each $a^* \in (0, \overline{a})$, there exists a $k < 0$ such that (6.29), (6.30) and $\Delta C < 0$ are satisfied. Thus, we have a perturbation that doesn't violate any of the constraints but generates strictly lower expected cost, leading to a contradiction. Because $C_v(0)- = 0$ plays the same role of $\frac{1}{u'(0)} = 0$, similar perturbation implies $v_U^* \neq 0$. Therefore, we have $v_U^* \in (0, V^*)$. ■

6.5 Differentiability at boundary

Differentiability If the optimal plan of $T_{FOA}C(0)$ is $(0, 0, 0, 0, 0)$, then we have $(T_{FOA}C)_v(0) = 0$; We have $(T_{FOA}C)_v(V^*) = +\infty$.

Proof. $T_{FOA}C(V)$ is differentiable at $V = 0$. Let $\Delta V = V_1 - V_0$. For $V_1 > V_0 = 0$, we have

$$\begin{aligned} 0 &\leq T_{FOA}C(V_1) - T_{FOA}C(V_0) && (6.32) \\ &\leq \sum_{i \in \{G,B\}} m(i,0) \frac{[u^{-1}(u(0) + \Delta V(1-\beta)) - u^{-1}(u(0))]}{1-\beta} && (6.33) \\ &\quad + m(U,0)[u^{-1}(u(0) + \Delta V) - u^{-1}(u(0))] \\ &= \sum_{i \in \{G,B,U\}} m(i,0) u^{-1\prime}(V_i^{\xi 0})(V_1 - V_0). && (6.34) \end{aligned}$$

(6.32) is by that $T_{FOA}C(V)$ is non-decreasing on V. Notice that $(0, u^{-1}(\Delta V(1-\beta)), u^{-1}(\Delta V(1-\beta)), u^{-1}(\Delta V), 0)$ is a closure point of $\Phi_{FOA}(V_1)$. It can be approximated by a sequence of plan in $\Phi_{FOA}(V_1)$. Also the objective function is continuous. Hence (6.33) holds. Let V_1 go to V_0. We have $V_i^{\xi 0} \to 0$.

Therefore the above inequality implies $\frac{T_{FOA}C(V_1) - T_{FOA}C(V_0)}{V_1 - V_0} \to 0$, and thus $(T_{FOA}C)_v(0) = 0$.

Similar argument implies $(T_{FOA}C)_v(V^*) = +\infty$. The solution to $T_{FOA}C(V^*)$ is $(0, w_G, w_G, w_G, V^*)$. Also we have $(0, u^{-1}(u(w_G) - \Delta V(1-\beta)), u^{-1}(u(w_G) - \Delta V(1-\beta)), u^{-1}(u(w_G) - \Delta V), V^*) \in \overline{\Phi}_{FOA}(V_0)$, for $V_0 < V_1 = V^*$. The same logic as the above argument implies the following inequality.

$$\begin{aligned} & T_{FOA}C(V_1) - T_{FOA}C(V_0) \\ \geq & \sum_{i\in\{G,B\}} m(i,0)\frac{[u^{-1}(u(w_G)) - u^{-1}(u(w_G) - \Delta V(1-\beta))]}{1-\beta} \quad (6.35) \\ & +m(U,0)[u^{-1}(u(w_G)) - u^{-1}(u(w_G) - \Delta V)] \\ = & \sum_{i\in\{G,B,U\}} m(i,0)u^{-1\prime}(V_i^{\xi 1})(V_1 - V_0) \quad (6.36) \end{aligned}$$

As $V_0 \to V^*$, we have $V_i^{\xi 1} \to u(w_G)$, which implies $\sum_{i\in\{G,B,U\}} m(i,a^0)u^{-1\prime}(V_i^{\xi 1}) \to +\infty$, and hence $\frac{T_{FOA}C(V_1)-T_{FOA}C(V_0)}{V_1-V_0} \to +\infty$. ■

6.6 About first-order approach

6.6.1 lemmas

Lemma 3.1 and Lemma 4.1 say that worker's value function is concave under solution scheme. Lemma 4.1 says that the multiplier associated with constraint IC of FOA is negative. Lemma 4.3 and Lemma 4.4 imply that conditions 1-2 are preserved by mapping T_{FOA}.

Lemma 3.1 In structure 1, for $V \in [0, V^*)$, if the optimal plan to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, then it is also in $\Phi(V)$. It implies $c_B^* = c_G^*$.

Proof. Let (a^*, c^*, v_U^*) be the solution to $T_{FOA}C(V)$. Since this is an interior point, we can use the first-order conditions

$$\frac{1}{u'(c_G^*)} = -\lambda - \mu\frac{m_a(G;a^*)}{m(G;a^*)} \quad (6.37)$$

$$\frac{1}{u'(c_B^*)} = -\lambda - \mu\frac{m_a(B;a^*)}{m(B;a^*)}. \quad (6.38)$$

In structure 1, we have $\frac{m(G,a')}{m(G,a)} = \frac{m(B,a')}{m(B,a)}$ for all a, $a' \in E$. This implies $\frac{m(G,a')-m(G,a)}{m(G,a)\Delta a} = \frac{m(B,a')-m(B,a)}{m(B,a)\Delta a}$ with $\Delta a = a' - a$. As $\Delta a \to 0$, we have $\frac{m_a(G;a^*)}{m(G;a^*)} = \frac{m_a(B;a^*)}{m(B;a^*)}$.

Therefore (6.37) and (6.38) imply $\frac{1}{u'(c_G^*)} = \frac{1}{u'(c_B^*)}$, thus $c_G^* = c_B^* = c'$, where c' is some constant. The optimal plan (a^*, c^*, v_U^*) satisfies constraint IC of FOA

$$U_a(a^*, c^*, v_U^*) = -m_a(U, a^*)[\frac{u(c')}{1-\beta} - (u(c_U^*) + \beta v_U^*)] - 1 = 0. \tag{6.39}$$

Therefore $m_a(U, a^*) < 0$ implies the ordering $\frac{u(c')}{1-\beta} > u(c_U^*) + \beta v_U^*$. Worker's function $U(a, c^*, v_U^*)$ is concave in a. Hence $U_a(a^*, c^*, v_U^*) = 0$ is a necessary and sufficient condition for global optimal effort a^* in worker's private effort problem. The promise-keeping constraint is also satisfied. Therefore (a^*, c^*, v_U^*) is in $\Phi(V)$. ■

Lemma 4.1 shows concave worker's utility function in private effort level. Compared with the static moral hazard model, where MLRP is sufficient to validify FOA, we need conditions 1-2 from cost function: $C(V) \in M(\overline{V})$.

Lemma 4.1 In structure 2, suppose $C(V) \in M(\overline{V})$. For $V \in [0, V^*)$, if the optimal plan to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, and the MLRP (assumption 1) holds, then the optimal plan is in $\Phi(V)$.

Proof. Since we use inequality constraint IC of FOA, we have $\mu \leq 0$. Through the first-order conditions for c_i (3.5)-(3.7), MLRP implies $c_G^* \geq c_B^* \geq c_U^*$. With F.O.C. (3.8), $C_v(V) \geq u^{-1\prime}(V(1-\beta))$ implies $u(c_G^*) \geq u(c_B^*) \geq v_U^*(1-\beta)$. Therefore we have the ordering $\frac{u(c_G^*)}{1-\beta} \geq \frac{u(c_B^*)}{1-\beta} \geq u(c_U^*) + \beta v_U^*$. Hence $U(a, c^*, v_U^*)$ is also concave in a.

Assumptions in Lemma 4.2 are satisfied, which implies a binding constraint IC of FOA: $U_a(a^*, c^*, v_U^*) = 0$. Therefore the optimal effort level a^* maximizes $U(a, c^*, v_U^*)$ given (c^*, v_U^*). (a^*, c^*, v_U^*) is in $\Phi(V)$. ■

Lemma 4.2 In structure 2, suppose $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$ with $\nu(c) = \frac{1}{u'(c)}$. For $V \in [0, V^*)$, if the optimal plan to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, then we have $\mu < 0$.

Proof. Denote (a^*, c^*, v_U^*) the solution to $T_{FOA}C(V)$. By way of contradiction, suppose $\mu = 0$. Then F.O.C.s for consumption (3.5)-(3.7) implies $c_G^* = c_B^* = c_U^* = c^*$ and $\lambda < 0$.

With F.O.C. (3.8), $\frac{1}{u'(c^*)} = C_v(v_U^*)$, and $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$, we have $c^* + \beta C(v_U^*) \geq \frac{c^*-w_B}{1-\beta}$[43].

Consider the partial derivative of the planner's cost function with respect to a. We have

$$\begin{aligned} & m_a(G;a^*)(\frac{c^*-w_G}{1-\beta}) + m_a(B;a^*)(\frac{c^*-w_B}{1-\beta}) \\ & +m_a(U;a^*)[c^* + \beta C^*(v_U^*)] \leq \qquad (6.40) \\ & m_a(G;a^*)(\frac{c^*-w_G}{1-\beta}) - m_a(G;a^*)(\frac{c^*-w_B}{1-\beta}) \\ < \; & 0. \qquad (6.41) \end{aligned}$$

(6.39) is by $m_a(U;a^*) < 0$ and $c^* + \beta C(v_U^*) \geq \frac{c^*-w_B}{1-\beta}$. Also plan (a^*, c^*, v_U^*) satisfies constraint IC of FOA, $U_a(a^*, c^*, v^*) \geq 0$[44]. Therefore we have

$$\begin{aligned} 0 \; > \; & m_a(G;a^*)(\frac{c_G^*-w_G}{1-\beta}) + m_a(B;a^*)(\frac{c_B^*-w_B}{1-\beta}) \qquad (6.42) \\ & +m_a(U;a^*)(c_U^* + \beta C(v_U^*)) \\ & +\lambda U_a(a^*, c^*, v^*) \end{aligned}$$

generating a contradiction to F.O.C. for a (3.9). ■

Lemma 4.3 shows that assumption 3 is crucial for T_{FOA} to preserve inequality $C_v(V) \geq u^{-1\prime}(V(1-\beta))$.

Lemma 4.3 In structure 2, suppose

1) We have $(T_{FOA}C)_v(V) = \sum_{i\in H} m(i,a^*)\frac{1}{u'(c_i^*)}$, where (a^*, c^*, v_U^*) is the optimal plan in problem $T_{FOA}C(V)$;

2) The optimal plan to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, for $V \in (0, V^*)$;

3) The solution to $(T_{FOA}C)(0)$ is $(0,0,0,0,0)$;

[43] $C(v_U^*) \geq \frac{\nu^{-1}(C_v(v_U^*))}{1-\beta} - \frac{w_B}{\beta(1-\beta)} = \frac{\nu^{-1}(\frac{1}{u'(c^*)})}{1-\beta} - \frac{w_B}{\beta(1-\beta)} = \frac{c^*}{1-\beta} - \frac{w_B}{\beta(1-\beta)} \rightarrow c^* + \beta C(v_U^*) \geq \frac{c^*-w_B}{1-\beta}$

[44] Note that only plan $(0,0,0,0,0)$ in $\Phi_{FOA}(0)$ doesn't satisfies constraint IC of FOA. But it is not an interior point in $E \times Z^3 \times \overline{V}$. It doesn't satisfy the assumptions in Lemma 4.2.

4) Assumption 3, $C_v(V) \geq u^{-1\prime}(V(1-\beta))$, $C_v(0) = 0$ hold.

Then we have $T_{FOA}C_v(V) \geq u^{-1\prime}(V(1-\beta))$.

Proof. For $V \in (0, V^*)$, (a^*, c^*, v_U^*) is an interior solution. The first-order condition (3.8) implies $\frac{1}{u'(c_i^*)} = u^{-1\prime}(u(c_U^*)) = C_v(v_U^*)$. For $V = V^*$ the only plan is a boundary point. But we still have $\frac{1}{u'(w_G)} = +\infty = C_v(V^*)$ because of $C_v(V^*) \geq u^{-1\prime}(V^*(1-\beta)) = +\infty$. For $V = 0$, we also have$\sum_{i \in H} m(i,0)\frac{1}{u'(0)} = C_v(0) = 0 = \frac{1}{u'(0)}$, since the solution to $(T_{FOA}C)(0)$ is $(0,0,0,0,0)$.

Consider the martingale property:

$$
\begin{aligned}
(T_{FOA}C)'(V) &= \sum_{i \in H} m(i;a^*)\frac{1}{u'(c_i^*)} && (6.43)\\
&= \sum_{i \in \{G,B\}} m(i;a^*)u^{-1\prime}(u(c_i^*)) && (6.44)\\
&\quad + m(U;a^*)\max\{C_v(v_U^*), u^{-1\prime}(u(c_U^*))\} &&\\
&\geq \sum_{i \in \{G,B\}} m(i;a^*)u^{-1\prime}(u(c_i^*)) && (6.45)\\
&\quad + m(U;a^*)\max\{u^{-1\prime}(v_U^*(1-\beta)), u^{-1\prime}(u(c_U^*))\} &&\\
&\geq \sum_{i \in \{G,B\}} m(i;a^*)u^{-1\prime}(u(c_i^*)) && (6.46)\\
&\quad + m(U;a^*)u^{-1\prime}(\max\{v_U^*(1-\beta), u(c_U^*)\})\} &&\\
&\geq \sum_{i \in \{G,B\}} m(i;a^*)u^{-1\prime}(u(c_i^*)) && (6.47)\\
&\quad + m(U;a^*)u^{-1\prime}((u(c_U^*) + \beta v_U^*)(1-\beta)) &&\\
&\geq u^{-1\prime}(\sum_{i \in \{G,B\}} m(i;a^*)u(c_i^*) && (6.48)\\
&\quad + m(U;a^*)(u(c_U^*) + \beta v_U^*)(1-\beta)) &&\\
&\geq u^{-1\prime}(V(1-\beta)) && (6.49)
\end{aligned}
$$

where $(a^*, c_G^*, c_B^*, c_U^*, v_U^*)$ is the solution to problem $(T_{FOA}C)(V)$. (6.44) is by $\frac{1}{u'(c_i^*)} = C_v(v_U^*) = \max\{C_v(v_U^*), u^{-1\prime}(u(c_U^*))\}$. $C_v(V) \geq u^{-1\prime}(V(1-\beta))$ implies $\max\{C_v(v_U^*), u^{-1\prime}(u(c_U^*))\} \geq \max\{u^{-1\prime}(v_U^*(1-\beta)), u^{-1\prime}(u(c_U^*))\}$, hence (6.45).

Increasing $u^{-1\prime}$ implies $\max\{u^{-1\prime}(v_U^*(1-\beta)), u^{-1\prime}(u(c_U^*))\} = u^{-1\prime}(\max\{v_U^*(1-\beta), u(c_U^*)\}\})$, hence (6.46). (6.47) is by increasing $u^{-1\prime}$ and the following:

$$\max\{a,b\} \geq a(1-\beta) + \beta b, \text{ with } a, b \in R \text{ and } \beta \in (0,1). \tag{6.50}$$

(6.48) is by assumption 3: convexity of $u^{-1\prime}$. (6.49) is by promise-keeping constraint: $\frac{\sum_{i\in\{G,B\}} m(i;a^*)u(c_i^*)}{(1-\beta)} + m(U;a^*)(u(c_U^*) + \beta v_U^*) \geq V$. ■

Lemma 4.4 shows that assumptions 3-4 are crucial for mapping T_{FOA} to preserve inequality $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$.

Lemma 4.4 In structure 2, suppose

1) we have $(T_{FOA}C)_v(V) = \sum_{i\in H} m(i,a^*)\frac{1}{u'(c_i^*)}$, where (a^*, c^*, v_U^*) is the optimal plan in problem $T_{FOA}C(V)$;

2) The optimal plan to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$, for $V \in (0, V^*)$;

3) The solution to $(T_{FOA}C)(0)$ is $(0,0,0,0,0)$;

4) Assumption 2 and assumption 4 hold;

5) We have $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$, $C_v(0) = 0$ and $C_v(V^*) = +\infty$.

Then we have $(T_{FOA}C)(V) \geq \frac{\nu^{-1}((T_{FOA}C)_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$.

Proof. We have the following

$$\begin{aligned}
& \frac{\nu^{-1}((T_{FOA}C)_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)} \\
= \; & \frac{\nu^{-1}(\sum_{i\in H} m(i;a^*)u^{-1\prime}(u(c_i^*)))}{1-\beta} - \frac{w_B}{\beta(1-\beta)} & (6.51) \\
\leq \; & \frac{\sum_{i\in H} m(i;a^*)\nu^{-1}(u^{-1\prime}(u(c_i^*)))}{1-\beta} - \frac{w_B}{\beta(1-\beta)} & (6.52) \\
= \; & \frac{\sum_{i\in H} m(i;a^*)c_i^*}{1-\beta} - \frac{w_B}{\beta(1-\beta)} & (6.53) \\
\leq \; & \frac{\sum_{i\in\{G,B\}} m(i;a^*)[c_i^* - w_i]}{1-\beta} + \frac{m(U;a^*)[c_U^* - w_B]}{(1-\beta)} & (6.54) \\
\leq \; & T_{FOA}C(V) & (6.55)
\end{aligned}$$

Assumption 2, concavity of $\frac{1}{u'}$, implies that its inverse ν^{-1} is convex, hence (6.52). Assumption 4 implies $-\frac{w_B}{\beta} \leq -w_B - m(G,a^*)(w_G - w_B)$, therefore (6.54). Property $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$ and $C_v(v_U^*) = \frac{1}{u'(c_U^*)}$ imply $c_U^* + \beta C(v_U^*) \geq \frac{c_U^* - w_B}{1-\beta}$, hence (6.55). ■

6.6.2 First-order approach

Proposition 3 In structure 1 (2), suppose that assumptions 1-4 hold and mapping T_{FOA} preserves differentiability at interior promised values. Then the FOA is valid. It implies $c_B^* = c_G^*$ ($c_G^* \geq c_B^*$).

In the following, I will use the previous lemmas to prove Proposition 3.

Step 1 implies that the solution plan to $\widehat{T}_{FOA}C(V)$ is in $\Phi_{FOA}(V)$. Therefore the existence result about $\widehat{T}_{FOA}$ can be applied to T_{FOA}.

Step 1 With $C(V) \in M(\overline{V})$, the solution to $\widehat{T}_{FOA}C(V)$ is in $\Phi_{FOA}(V)$ for $V \in [0, V^*]$.

For $V \in (0, V^*)$, the solution to $\widehat{T}_{FOA}C(V)$ has an interior effort by Lemma 1 and hence it is in $\Phi_{FOA}(V)$.

For $V = V^*$, $\Phi(V^*) = \Phi_{FOA}(V^*) = \{(0, w_G, w_G, w_G, V^*)\}$ implies step 1.

For $V = 0$, if the optimal effort to problem $\widehat{T}_{FOA}C(0)$ is 0, then the optimal plan is $(0,0,0,0,0) \in \Phi_{FOA}(0)$, because any plan with $a = 0$ generates higher cost than $(0,0,0,0,0)$; If the optimal effort is strictly positive, since $\overline{\Phi}_{FOA}(0) \backslash \Phi_{FOA}(0)$ only contains plans with effort level 0, the optimal plan is also in $\Phi_{FOA}(0)$.

Step 2 says that the first-order condition, $U_a(a^*, c^*, v_U^*) = 0$, is a necessary condition for the optimal contract.

Step 2 With $C(V) \in M(\overline{V})$, the solution to $(TC)(V)$ is in $\Phi_{FOA}(V)$ for $V \in [0, V^*]$.

For $V = V^*$, $\Phi_{FOA}(V^*) = \Phi(V^*) = \{(0, w_G, w_G, w_G, V^*)\}$ implies step 2.

For $V \in (0, V^*)$, Lemma 1 says the effort solution to $(TC)(V)$ is an interior point, hence the optimal plan satisfies the first-order condition in worker's private problem, and it is in $\Phi_{FOA}(V)$.

For $V = 0$, if the optimal plan is $(0,0,0,0,0)$, then it is also in $\Phi_{FOA}(0)$. If the optimal plan is not $(0,0,0,0,0)$, it must achieve value $V' > 0$[45]. This

[45] $(0,0,0,0,0)$ is the only incentive-compatible plan with promise value value 0.

plan is also the solution to problem $(TC)(V')$, because of $\Phi(V') \subseteq \Phi(0)$. We have a problem with $V' \in (0, V^*)$ again. The plan is in $\Phi_{FOA}(V')$. Therefore it is also in $\Phi_{FOA}(0)$ because of $\Phi_{FOA}(V') \subseteq \Phi_{FOA}(0)$.

Step 3 makes sure that the solution to problem $T_{FOA}C(V)$ is still in $\Phi(V)$.

Step 3 In structures 1-2, the solution to $T_{FOA}C(V)$ is in $\Phi(V)$ for $V \in [0, V^*]$ with $C(V) \in M(\overline{V})$ and MLRP.

For $V = V^*$, the solution to $T_{FOA}C(V^*)$ is in $\Phi(V^*)$ because of $\Phi(V^*) = \Phi_{FOA}(V^*) = \{(0, w_G, w_G, w_G, V^*)\}$.

For $V \in (0, V^*)$, the first part of Lemma 2 says that the solution to problem $T_{FOA}C(V)$ is an interior point of $E \times Z^3 \times \overline{V}$. Therefore assumptions in Lemma 3.1 (4.1) are satisfied. The solution to $T_{FOA}C(V)$ is in $\Phi(V)$ for $V \in (0, V^*)$.

For $V = 0$, if the optimal effort is $a^* = 0$, then the optimal plan is $(0, 0, 0, 0, 0) \in \Phi_{FOA}(0)$[46]; if we have $a^* > 0$, by the second part of Lemma 2, the optimal plan is an interior point. All assumptions in Lemma 3.1 (4.1) hold again. Thus the solution to $T_{FOA}C(0)$ is in $\Phi(0)$.

Step 4 In structures 1-2, the solution to $T_{FOA}C(0)$ is $(0, 0, 0, 0, 0)$ with $C(V) \in M(\overline{V})$ and MLRP.

By way of contradiction, suppose $(0, 0, 0, 0, 0)$ is not the solution to $T_{FOA}C(0)$. Because any other incentive-compatible plan with $a = 0$ produces strictly higher cost than $(0, 0, 0, 0, 0)$, the solution (a^*, c^*, v_U^*) to problem $T_{FOA}C(0)$ must have $a^* > 0$.Thus the assumptions in second part of Lemma 2 are satisfied. So plan (a^*, c^*, v_U^*) is an interior point of $E \times Z^3 \times \overline{V}$ and hence it is in $\Phi(0)$ by Lemma 3.1 (4.1).

The solution to $(TC)(0)$ is in $\Phi_{FOA}(0)$ by step 1. And $T_{FOA}C(0)$ and $TC(0)$ have a common objective function. So $(a^*, c^*, v_U^*) \in \Phi(0)$ generates no higher cost than $TC(0)$. (a^*, c^*, v_U^*) is also the solution to $TC(0)$.

Under the interior scheme (c^*, v_U^*), she can guarantee a strictly positive lifetime value by choosing $a = 0$ under scheme (c^*, v_U^*). Because the plan (a^*, c^*, v_U^*) is incentive-compatible, it must generate promised value $V' > 0$.

Because the plan is an interior point, we have $\min\{u(c_i^*), (u(c_U^*) + \beta v_U^*)\} > 0$. Then we can reduce the same sufficiently small amount of utility for each state in the solution plan without violating boundary constraint. Such plan

[46] Any plan with $a = 0$ generates higher cost than $(0, 0, 0, 0, 0)$.

also satisfies $U_a(a, c, v_U) \geq 0$ and $U_a(a, c, v_U) \geq 0$. But it generate lower cost, contradicting to the fact that (a^*, c^*, v_U^*) is the solution to $(TC)(0)$. Hence the optimal plan (a^*, c^*, v_U^*) should be $(0, 0, 0, 0, 0)$.

Step 5 $T_{FOA} : M(\overline{V}) \to M(\overline{V})$ under assumptions 1-4.

Consider the problem $T_{FOA}C(V)$ with $C(V) \in M(\overline{V})$. First, by Lemma 2, the solution to problem $T_{FOA}C(V)$ is an interior point in $E \times Z^3 \times \overline{V}$ for $V \in (0, V^*)$.

Second, step 4 implies that the solution to $T_{FOA}C(0)$ is $(0, 0, 0, 0, 0)$.

Third, we want to have $(T_{FOA}C)_v(V) = \sum_{i \in H} m(i, a^*) \frac{1}{u'(c_i^*)}$. By the Lemma in differentiability section, we have $(T_{FOA}C)_v(V) = \sum_{i \in H} m(i, a^*) \frac{1}{u'(c_i^*)}$ for $V \in \{0, V^*\}$; Assuming that $(T_{FOA}C)()$ is differentiable, we also have $(T_{FOA}C)_v(V) = \sum_{i \in H} m(i, a^*) \frac{1}{u'(c_i^*)}$ for $V \in (0, V^*)$ by theory 1 in Milgrom and Segal [7].

Fourth, we have $C_v(0) = 0$ and $C_v(V^*) = +\infty$, because $C(V) \in M(\overline{V})$ implies $C_v(0) = 0$ and $C_v(V) \geq u^{-1\prime}(V(1-\beta))$.

With the above four conditions and assumptions 2-4, assumptions in Lemma 4.3 and Lemma 4.4 are satisfied, which implies $(T_{FOA}C)(V) \geq \frac{\nu^{-1}((T_{FOA}C)_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$ and $(T_{FOA}C)(V) \geq u^{-1\prime}(V(1-\beta))$. Moveover, the third condition implies $(T_{FOA}C)_v(0) = 0$. Hence, we have $T_{FOA} : M(\overline{V}) \to M(\overline{V})$.

With $C(V) \in M(\overline{V})$, the solution to $(TC)(V)$ is in $\Phi_{FOA}(V)$, and the solution to $T_{FOA}C(V)$ is in $\Phi(V)$, for $V \in [0, V^*]$. Since $T_{FOA}C(V)$ and $(TC)(V)$ have the same objective function, they have the same policy function.

The following shows dependency on information structure.

Lemma 5 In structure 2, the solution to problem $T_{FOA}C(V)$ has $c_{Gt}^* > c_{Bt}^*$ for $V \in (0, V^*)$, with $C(V) \in M(\overline{V})$ and assumption 1 (MLRP),

Proof. With $C(V) \in M(\overline{V})$, Lemma 2 says the solution to problem $T_{FOA}C(V)$ is an interior point of $E \times Z^3 \times \overline{V}$ for $V \in (0, V^*)$. It is valid to use the first-order conditions (3.5)-(3.6), which imply $\frac{1}{u'(c_{Gt}^*)} - \frac{1}{u'(c_{Bt}^*)} = \mu(\frac{m_a(B;a)}{m(B;a)} - \frac{m_a(G;a)}{m(G;a)})$.

Since we have an interior optimal plan for $V \in (0, V^*)$, and $C(V) \geq \frac{\nu^{-1}(C_v(V))}{1-\beta} - \frac{w_B}{\beta(1-\beta)}$, the assumptions in lemma 4.2 are satisfied. Therefore we have $\mu < 0$, which leads to $\frac{1}{u'(c_{Gt}^*)} - \frac{1}{u'(c_{Bt}^*)} > 0$ in structure 2. Since u is a strictly concave function, $\frac{1}{u'(c_{Gt}^*)} - \frac{1}{u'(c_{Bt}^*)} > 0$ implies $c_{Bt} < c_{Gt}$. ■