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Abstract

While the *Generalized Error Distribution (GED)* has been used quite extensively in time series applications and has demonstrated a sound flexibility in the estimation process, there is so far no attempt to use this function in the construction of Copulas. Copulas are probability functions that link one multivariate distribution function to univariate distribution functions called marginals. These marginal functions are assumed to be continuous and to follow a uniform behaviour within [0,1]. In this paper we propose a new Copula function that, to our knowledge, has not been used in the literature of Copulas, until now: the *bivariate GED-Copula*. This function embeds other well-known distributions including the gaussian distribution. In order to assess the relative performance of this new Copula we investigate financial contagion in foreign exchange, stocks, bonds and sovereign debt markets in Latin America. Standard decision criteria provides strong evidence in favour of the *GED-Copula* against other Elliptical and Arquimidean alternatives.

JEL Classification: C22, C46, C52, C65

Key Words: GED-Distribution, Copula Function, Multivariate Distribution, Contagion, Financial Markets.

I. Introduction

Empirical applications of Copula functions in economics and finance have grown impressively during the last decade—for a recent review on these see the work of Patton (2009) and an earlier sound critique in Mikosch (2006) Practitioners and academics have been attracted to them looking for alternative measures of risk and association where two o more variables are involved. Copulas have proved useful in the analysis of risk and in the valuation of derivative products[1] such as options, swaps and structured notes, it is also a fresh parsimonious approach to investigate phenomena like financial contagion[2] and risk management[3]. Financial time series applications have also integrated the concept of Conditional Copulas developed by Patton (2006).

A *Copula* is a probability function that links one multivariate distribution function to univariate distribution functions called marginals. These marginal functions are assumed to be continuous and to follow a uniform behaviour within [0,1] (Nelsen 1999:1).

The multivariate distribution that is most often employed to link or copulate the two marginals is the normal distribution—see the work by Chen and Fan (2006) for instance where they fit a constant normal Copula. Some other authors have used alternative distributions such as the *t*-copula or the *Mixed Normal* to capture the association between two marginal functions more naturally—see Rodriguez (2007) and Patton (2006) for instance.

A *t-copula* captures upper tail dependence and the thickness of tails commonly found in financial applications more easily. The proposed copula functions are in the end assumptions about the dependence behaviour of the marginals and researchers usually employ either copula function based on the empirical features and the specific application at hand (asymmetries, persistence, fat tails, etc.). If a researcher wants to discriminate between different copulas she should fit some of the existing copula functions first and then, using standard criteria, try to assess which fits better a given application. Since these are in general non-nested functions, such selection process will not guaranty that the final copula chosen is the best alternative. Then it would be ideal to count with a distribution that

[1] See Cherubini, et. al. (2004) for applications of copulas in mathematical finance and derivatives.

[2] For instances of copula methods in the study of contagion see Rodríguez (2007), Chollete, *et. al.* (2005) and Arakeliand and Dellaportas (2005).

[3] Starting with the work of Hull and White (1998), in this area the applications focus mostly on VaR analysis. Recent contributions on this field and on risk management in general can be found in Embrechts and Höing (2006) and Alexander (2008).

allows the researcher to capture the thickness of tails more naturally, to model other features and also to allow the possibility of testing other nested Copula alternatives.

In this paper we propose a new distribution function that, to our knowledge, has not been used in the literature of copulas so far: the bivariate *Generalized Error Distribution (GED) Copula*. The *GED* distribution has been used quite extensively in univariate time series with financial applications and has demonstrated an excellent flexibility in the estimation process. In this paper we move away from the assumption of normality (a distribution that is never truly corroborated with financial data) and allow the researcher to employ direct testing on the distribution followed by the data. Another advantage of our new *GED Copula* is that it can behave just as any symmetrical distribution (including the normal) and can also be easily adjusted to replicate other well-known distributions by constraining some specific parameters.

Hence, the main contribution of this paper is the proposal of the new bivariate *GED Copula*. In order to assess its goodness of fit and shape we examine financial contagion in Latin America during the current financial crisis in different markets: sovereign debt markets, stocks, bonds and exchange rates. In general we find that the new GED copula provides a much better fit than the Copulas widely used in the empirical literature.

The following section presents the basic theory of Copulas and the elements that give support to the introduction of the GED Copula. The third section of the paper introduces the new Copula function, which is estimated in section four assessing its goodness of fit and applications to examine financial contagion in four Latin American markets. Section five concludes the paper.

II. Copula Functions

Standard methods to examine the association between two random variables make strong assumptions about the variables themselves and about the nature of the association. Among these assumptions linearity and normality in the individual and joint distribution are the most prominent. It has been documented quite extensively that many variables, including those in financial applications, are not distributed as normal and they in fact show fat tails and much greater dependence (persistence)—see for instance Hogg & Klugman (1984) y Longin & Solnik (2001).

Other problems related to standard measures of linear association are also the invariance to non-linear transformations, which implies that under some transformations correlation measures will change. Violations of the normality assumption can in turn overestimate the outcomes, depending on whether the actual distribution possesses fatter tails for instance.

This set of problems is avoided efficiently by Copula functions as they are invariant to increasing transformations, linear or non-linear, and because the dependence structure is separated from the marginal distributions—see Nelsen (1999). These functions have their origins in the 50's but have gained popularity in economics and finance just recently.[4]

A Copula is basically a joint probability function that links a multivariate distribution function to one-dimensional marginal distribution functions which are assumed to follow uniform distributions within (0,1)—Nelsen (1999:1). Copula functions can be multivariate but in this paper we focus on the bivariate case to keep the analysis basic.

Hence a Copula function is C: $\mathbf{I}^2 \rightarrow \mathbf{I}$, where $\mathbf{I} = [0,1]$,[5] with the following properties:

$$C(u,0) = C(0,u) = 0\,, \quad [1]$$

$$C(u,1) = C(1,u) = u\,, \quad [2]$$

and for each u_1, u_2, v_1, v_2 in $\mathbf{I}$ with $u_1 \leq u_2$ and $v_1 \leq v_2$,

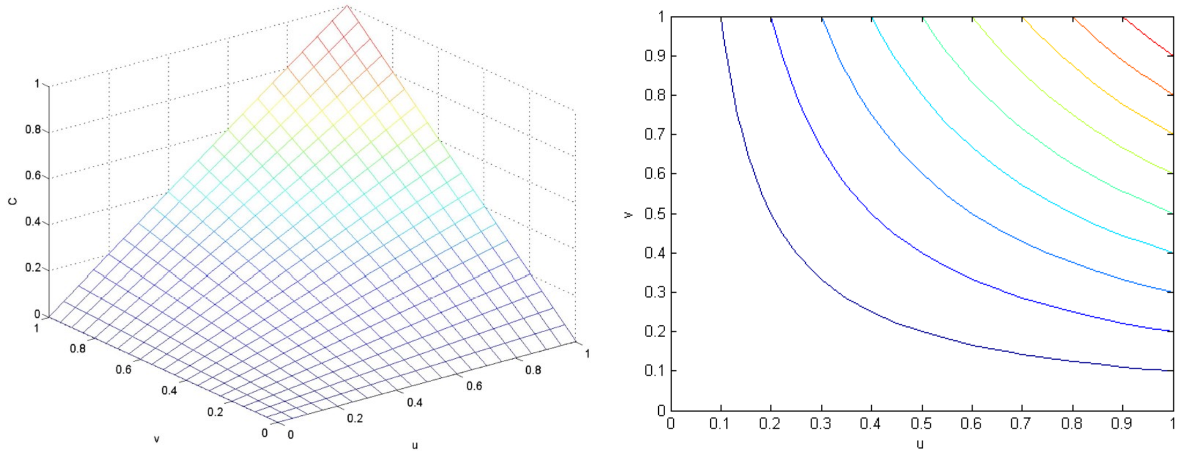
$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0\,. \quad [3]$$

One of the most basic copula functions found in the literature is the *Product Copula* defined as $C=uv$, with density function and contour plots shown in figure 1.

[4] For the first contributions see the works by Wassily Hoeffding (1940, 1941); and also Abe Sklar (1959) who was the first author using the Copula term. Among early applications of Copulas in finance and economics we found Tibiletti (1995) and Joe et al. (1996), more recent contributions can be found in Patton (2006), Patton (2009) and Embrechts (2008).

[5] That is, the C function has two variables that can take values and also results within the unit inteval [0,1]. For more details on the notation the reader is referred to Simon & Blume (1994).

Figure 1: The Product Copula.



(a) Density Function (b) Contour Plot

One important principle in Copula theory is the Sklar Theorem. This theorem is the foundation of dependence measures and association of variables as it describes how multivariate distribution functions are related with univariate marginal functions. The Sklar Theorem states that if H is a joint distribution function and F and G are its marginal distributions then there is a copula function defined as:

$$H(x, y) = C(F(x), G(y)) \text{ ,} \quad [4]$$

if F and G are continuous then C is unique, otherwise it is unique in *RanF x RanG*.

The univariate marginal distributions of a joint distribution has the following form:

$$\text{if} \quad DomF = S_1, \quad F(x) = H(x, b_2) \text{ ,} \quad [5]$$

$$\text{if} \quad DomG = S_2, \quad G(y) = H(b_1, y) \text{ ,} \quad [6]$$

where b_1 is the greater element of S_1 and b_2 is the greater element of S_2, generally $b = \infty$. H is a joint distribution function from $\boldsymbol{R}^2$ to $\boldsymbol{I}$, hence F and G are also distribution functions going from $\boldsymbol{R}$ to $\boldsymbol{I}$.

If F and G are continuous functions then from the Sklar Theorem the following formula is derived, useful to construct copulas from bivariate distribution functions:

$$C(u,v) = H(F^{-1}(u), G^{-1}(v)) \; ,$$

[7]

where F^{-1} and G^{-1} are the inverse functions of *F* and *G* respectively.

Copulas are invariant functions to strictly increasing transformations of their marginal functions. Also, if α is a strictly increasing transformation and β is a strictly decreasing transformation then:

$$C_{\alpha(x)\beta(y)}(u,v) = u - C_{xy}(u,1-v) \, ,$$

[8]

similarly if α is a strictly decreasing transformation and β is a strictly increasing transformation then

$$C_{\alpha(x)\beta(y)}(u,v) = v - C_{xy}(1-u,v) \, ,$$

[9]

if both α and β are strictly decreasing transformations then

$$C_{\alpha(x)\beta(y)}(u,v) = u + v - 1 + C_{xy}(1-u,1-v) \, ;$$

[10]

the proofs to (8), (9) and (10) can be found in Nelsen (1999:22).

There are several classes of copulas. Among the most popular in the literature are in the Arquimidean and Elliptical classes. We now briefly describe these copulas.

2.1 Arquimidean Copulas

Arquimidean Copulas have grown in popularity as they are easy to generate and manipulate. This class of copulas are characterized by being created from a *generator function*. The generator φ is a strictly decreasing function of $\boldsymbol{I}$ in $[0, \infty]$ and $\varphi(1) = 0$, with Arquimidean copula defined as:

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)),$$

[11]

where φ^{-1} is the inverse of φ.

The two most widely used Arquimidean Copulas are *Gumbel* and *Clayton Copulas*. The *Gumble Copula* and its generator function are defined as follows:

$$C(u,v) = \exp(-((-\ln u)^{\theta} + (-\ln v)^{\theta})^{1/\theta}), \qquad \varphi(t) = (-\ln t)^{\theta} \quad [12].$$

The *Clayton Copula* and its generator function are defined as:

$$C(u,v) = \max((u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, 0), \qquad \varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1) \qquad [13].$$

These copulas have been applied in the financial literature by Fantazzini (2004) to investigate the dependence structure of stocks using the Kendall Tau and by Shemyakin & Young (2006) to model joint survival distributions.

2.2 Elliptic Copulas

The *Gaussian* (or normal) *Copula* and the *Student-t Copula* are the most widely known elliptic copulas. An elliptic copula is a function with variables with an elliptical distribution. The normal Copula is given by:

$$C(u,v|r) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-(t^2 - 2\rho ts + s^2)}{2(1-\rho^2)}\right) dsdt \qquad -1 < r < 1,$$

[14]

where Φ^{-1} is the inverse of the normal distribution and ρ is the correlation of the two variables *t* and *s*.

The *Student-t* Copula is defined by:

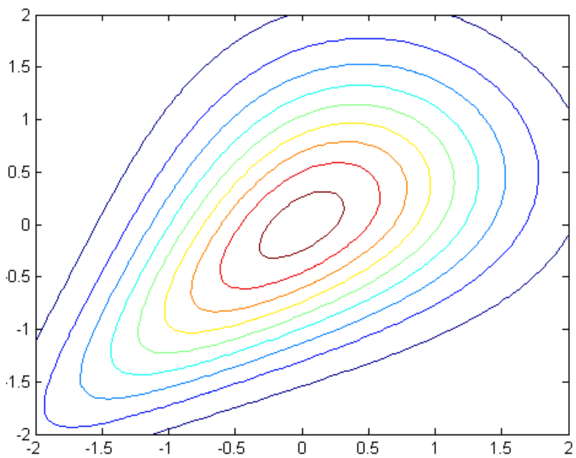
$$C_{v,\rho}(u,v) = \int_{-\infty}^{t_v^{-1}(u)} \int_{-\infty}^{t_v^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left[1 + \frac{x^2 - 2\rho xy + y^2}{v(1-\rho^2)}\right]^{-\frac{v+2}{2}} dydx,$$

[15]

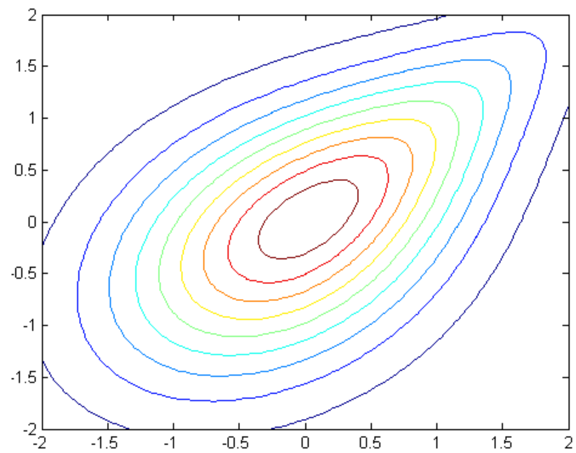
where t^{-1} is the inverse of the univariate distribution *Student-t* and *v* is a parameter that describes the shape of the distribution or degrees of freedom. The *Student-t* Copula is useful because it captures through v a regular phenomenon of financial time series: the thickness of tails. Heavy tails have been documented in several contributions, among them Bradley & Taqqu (2002) y Hyung & de Vries (2007).

Contour plots are useful tools to obtain more information about the shape and other properties of elliptical and non-elliptical copulas. The contour plots of the Normal, Clayton and Gumbel Copulas with gaussian marginal distributions are presented in the graphs below:

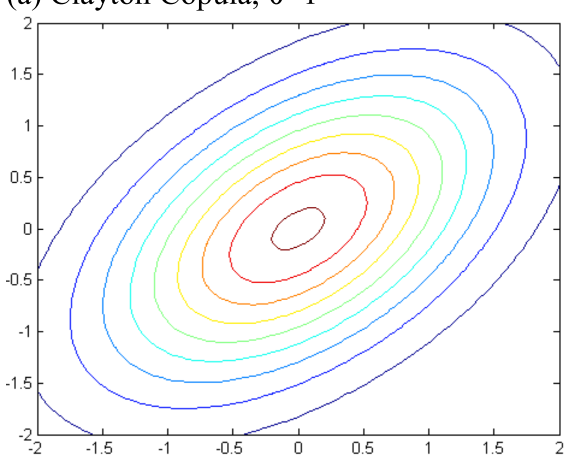
Figure 2: Contour Plots of Arquimidean and Elliptical Copulas.



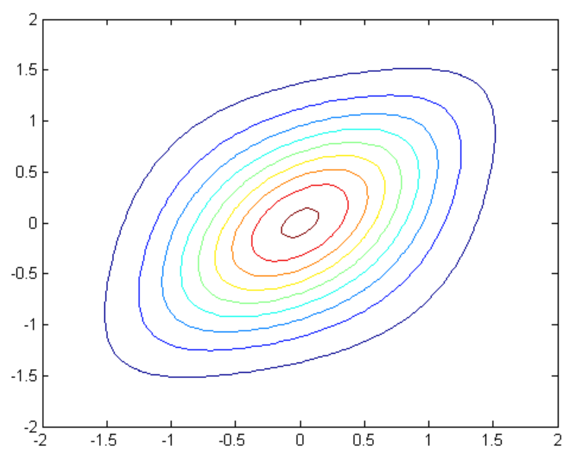
(a) Clayton Copula, θ=1



(b) Gumbel Copula, θ=1.5



(c) Normal Copula, ρ=0.5



(d) Student-t Copula

Some interesting features can be observed from these plots. The Clayton Copula for instance shows more steep contours on the negative quadrant indicating that negative events occur simultaneously. For instance, falls in the US stock market would be associated with falls of the Mexican stock market. The contour plot of the Gumbel Copula captures the reverse effect, that is, positive events have a high chance of arriving together: a recovery in the US stock market would be associated with a Mexican Stock Exchange recovery. The normal copula shows a very symmetric association of the variables, where the probability of occurrence is the same. Finally, for the Student Copula one observes the same probability for joint appreciations and depreciations and a lower probability for the appreciation of one currency and the depreciation of the other.

2.3 Dependence in the theory of Copulas

The measures of dependence and association most commonly found are non parametric such as *Spearman-rho* ρ, developed by Charles Spearman (1904), and *Kendall Tau (τ)*, developed by Maurice Kendall (1938). In Copula theory the *Spearman-rho* is defined as:

$$\rho = 12 \int\int_{I^2} C(u,v)\,dv\,du - 3 \quad [16]$$

while the *Kendall Tau* has the following form:

$$\tau = 4 \int\int_{I^2} C(u,v)\,dC(u,v) - 1 \quad [17]$$

$$= 4E(C(u,v)) - 1,$$

It can be observed that both correlations depend exclusively on the Copula and not the marginal distributions—greater detail on this can be found in Nelsen (1999, pp. 125-138).

<u>Tail Dependence</u>

This type of dependence appears when distributions possess heavy tails and it refers to the association between extreme observations of the variables. There are two types of tail-dependence: the upper tail dependence that measures the possibility of two extreme positive values occurring simultaneously and lower tail dependence that measures the possibility of two extreme negative events occurring simultaneously.

According to Melchiori (2003, p.3) the upper tail dependence coefficient, λ_U, is given by:

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u}.$$

[18]

If $\lambda_U > 0$ there is a probability that extreme values are presented simultaneously in the upper tail, that is, if λ_U is in between *(0,1]* then two random variables *X* and *Y* would be asymptotically dependent on the upper tail; if $\lambda_U = 0$, then *X* and *Y* would be asymptotically independent.

The lower tail dependence, λ_L, is obtained from the following formula.

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u}.$$

[19]

Similarly, there is a probability that extreme negative values occur simultaneously when λ_L is in the inverval *(0,1)*, that is when *X* and *Y* are asymptotically dependent on the lower tail. If $\lambda_L = 0$, then *X* and *Y* are asymptotically independent in the lower tail.

III. The GED Copula

Among some of the salient features of financial returns are volatility, clustering, persistence, asymmetries and heavy tails in the distribution. To replicate and account for heavy tails many applications have adopted conditional distributions different to the gaussian distribution in estimations of univariate and multivariate models. Various studies have found that from the several alternative distributions, the *Generalized Error Distribution (GED)* dominates other distributions (such as the normal, *Student-t* or *skewed* versions of these distributions) in the modelling of financial returns and heavy tails—see Bouaddi & Rombouts (2007) for a recent instance.

But even when the *GED-distribution* has been used extensively in finance literature to investigate the behaviour of risky assets, exchange rates and other assets—see works by Nelson (1991), Liesenfeld and Jung (2000) and Komunjer (2007)—there are in fact very few references describing the statistical behaviour of the GED distribution. Among the few authors that examine this function we found Chiodi (2000) who examines the statistical properties and multivariate tests of the *GED-distribution*.

We did not find a reference on *GED-Copulas* in the literature and, in this respect, we aim at contributing by providing a new Copula able to capture heavy tails and also able to replicate the features of other well-known distributions.

3.1 Derivation of the GED-Copula

The *GED-Copula* introduced in this section has the ability to replicate heavy tails, as well as thin tails and also, its shape can vary from a bell that is identical to the normal distribution to a bell that resembles a uniform distribution. Hence the new *GED-Copula* proposed in this paper is evidently highly flexible.

The GED-Copula hence developed is based on the General Error Distribution described by Lindsay (2001, p. 212) who presents the following density function:

$$f(x) = \frac{1}{a\Gamma(1+\frac{1}{2b})2^{1+\frac{1}{2b}}} \exp\left(-\frac{1}{2}\left|\frac{x}{a}\right|^{2b} \right),$$

[20]

where *a* is the scale parameter and *b* is the shape parameter. These two parameters capture and describe the form of the distribution. *Γ* is the gamma function and *x* is a random variable. Unlike the gaussian distribution, this is a generalized function for the distribution of errors that is not limited to just one single distribution but instead is able to replicate many distributions.

Parameter *a* captures the thickness of tails and kurtosis, while *b* controls the shape/width of the bell. When *b* lies within *(0,1)* we obtain leptokurtic distributions and when *b* is greater than *1* we obtain platyckurtic distributions. Figure 3 shows density functions and contour plots of the GED distribution for different values of *a* and *b* respectively showing the flexibility of this distribution.

For instance, for the values of *a=1* and *b=1* the shape of the GED distribution is the same as the gaussian distribution. When *a=10* and *b= 3* the GED distribution approximates very well the uniform distribution and for a=0.5 and b= 0.1 the GED distribution approaches a Laplace distribution. As we see the GED distribution allows for a great range of distributions with different bell forms and different thickness of tails.

Lindsey (2001, p.212) in turn presents the bivariate GED distribution function:

$$H_{GED}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}\frac{1}{\pi a^2\sqrt{1-\rho^2}\Gamma\left(1+\frac{1}{b}\right)2^{\frac{1}{b}}}\exp\left(-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{a^2(1-\rho^2)}\right)^b\right)dsdt\ ,$$

[21]

where π is the pi number, Γ is the Gamma function, a is the scale parameter, b is the shape parameter, x and y are random variables and ρ is the correlation between x and y.

In order to derive the *GED-Copula* we simply apply the Sklar Theorem [7] to this bivariate GED Distribution in [21]. That is, the Sklar Theorem states that

$$C(u,v)=H(F^{-1}(u),G^{-1}(v))\ ,$$

H is the GED bivariate ditribution in [21] and F and G are the marginal distributions of H, that is

$$\text{if}\quad DomF=S_1,\quad F(x)=H(x,b_2)\ ,$$

$$\text{if}\quad DomG=S_2,\quad G(y)=H(b_1,y)\ ,$$

where b_1 is the greater element of S_1 and b_2 is the greater element of S_2, generally $b=\infty$. Hence:

$$F(x)=\int_{-\infty}^{x}\int_{-\infty}^{\infty}\frac{1}{\pi a^2\sqrt{1-\rho^2}\Gamma\left(1+\frac{1}{b}\right)2^{\frac{1}{b}}}\exp\left(-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{a^2(1-\rho^2)}\right)^b\right)dsdt$$

$$F(X)=\int_{-\infty}^{x}\frac{1}{a\Gamma(1+\frac{1}{2b})2^{1+\frac{1}{2b}}}\exp\left(-\frac{1}{2}\left|\frac{y}{a}\right|^{2b}\right)dt\ ,\qquad [22]$$

and given that H is symmetric then F=G

$$G(Y)=\int_{-\infty}^{x}\frac{1}{a\Gamma(1+\frac{1}{2b})2^{1+\frac{1}{2b}}}\exp\left(-\frac{1}{2}\left|\frac{x}{a}\right|^{2b}\right)dt\qquad [23]$$

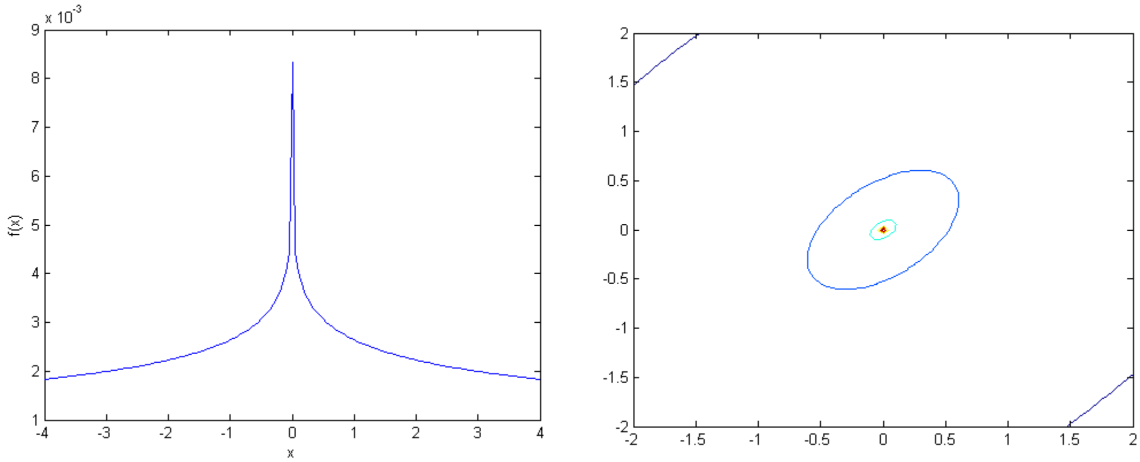
Using (23) and Substituting *G* and *F* in the Sklar Theorem [7], we get the *GED-Copula*:

$$C(u,v)=\int_{-\infty}^{F^{-1}(u)}\int_{-\infty}^{F^{-1}(v)}\frac{1}{\pi a^2\sqrt{1-\rho^2}\Gamma\left(1+\frac{1}{b}\right)2^{\frac{1}{b}}}\exp\left(-\frac{1}{2}\left(\frac{u^2-2\rho uv+v^2}{a^2(1-\rho^2)}\right)^b\right)dsdt\ ,\ [24]$$

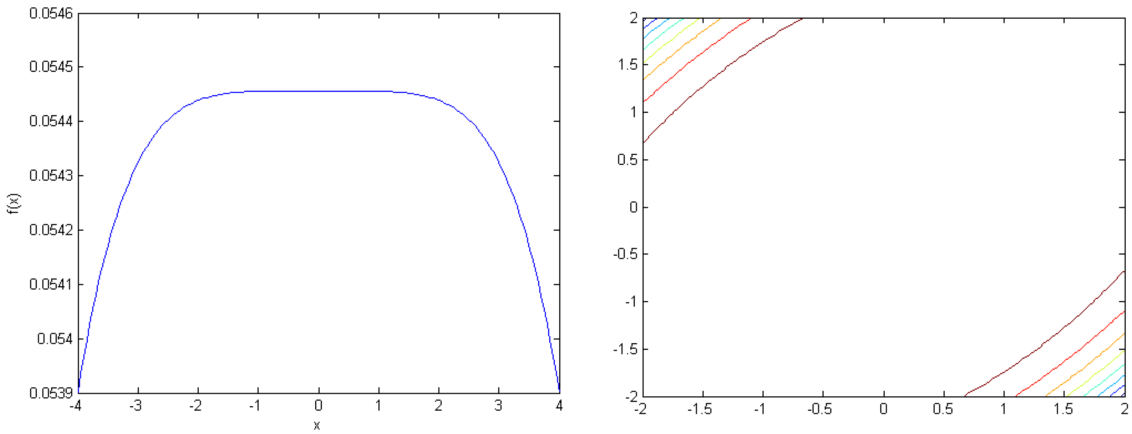
where $F^{-1}(u)$ and $F^{-1}(v)$ are inverse univariate marginal functions that can be derived from [22] and [23].

Figure 3 below present the contour densities of the GED distribution for *a=0.5* and *b=0.1*, *a=10* and *b =3*, and *a=b=1*. In all cases $\rho=0.5$. As it can be appreciated from the graph panel (c), these last values generate the normal distribution.

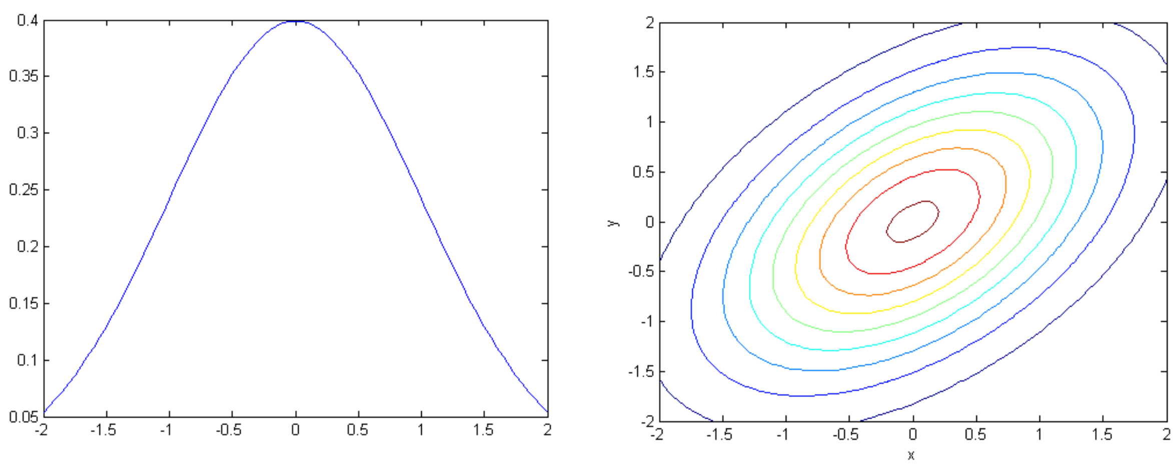
Figure 3. Contour plots of GED distribution for different values of *a* and *b*.



(a) GED density function with a = 0.5, b = 0.1 y $\rho = 0.5$



(b) GED density function with a = 10 y b = 3 and ρ= 0.5



(c) GED density function with a=, b=1 and ρ=0.5

3.1 Optimization

To estimate the parameters *a*, *b* and ρ of the *GED-Copula* we optimize the following function via Maximum Likelihood:

$$L = \ln \prod c(u_i, v_i) =$$

$$= \sum \ln \left(\frac{1}{\pi a^2 \sqrt{1-\rho^2}\,\Gamma\left(1+\frac{1}{b}\right) 2^{\frac{1}{b}}} \right) - \frac{1}{2} \left(\frac{F^{-1}(u_i)^2 - 2\rho F^{-1}(u_i) F^{-1}(v_i) + F^{-1}(v_i)^2}{a^2(1-\rho^2)} \right)^b$$

[25]

and we iterate

$$\begin{bmatrix} r_{n+1} & b_{n+1} \end{bmatrix} = \begin{bmatrix} r_n & b_n \end{bmatrix} - \begin{bmatrix} \partial L_{r,r} & \partial L_{r,b} \\ \partial L_{r,b} & \partial L_{b,b} \end{bmatrix}^{-1} \begin{bmatrix} \partial L_r & \partial L_b \end{bmatrix}$$

[26]

until $L_{n+1} < L_n$. L, here, is the natural logarithm of the Likelihood function, the function to be maximized in order to find the best fitting parameters. The Newton–Raphson method, [26] uses derivatives to find the maximum of L.

Once all the parameters have been estimated, we calculate the Akaike and Schwarz statistics given by:

$$AIC = 2k - 2L \,, \quad [27]$$

$$BIC = k \ln n - 2L \,. \quad [28]$$

Where *k* is the number of parameters estimated and *n* is the number of observations.

An additional useful test statistic is the likelihood ratio (LR) given by:

$$LR = 2(L_0 - L_1)\,,$$

[29]

where L_0 represents the optimized likelihood function (null hypothesis) of the unrestricted model, L_1 is the optimized likelihood of the restricted model (alternative hypothesis) and *LR* is the test statistic distributed as χ_n, with *n* degrees of freedom equal to the number of restrictions (Hamilton 1991, pp.144-145).

L_0 has been defined as the likelihood of the *GED-Copula. This new Copula* contains within it a family of copulas as we have seen. Some restrictions can be placed on the parameters such as making *b=0.5* and in this case the *GED-Copula* becomes the *Laplace-Copula* with thick tails. When *b=1* the *GED-Copula* becomes the *Normal-Copula*.

IV. Estimation Results

We use in this section daily data consisting of log-returns for different asset prices: exchange rates, interest rates, stocks and sovereign bond markets. The aim is to assess the relative performance of the *GED-Copula* proposed in this paper with other popular copulas frequently used in the literature such as Arquimidean Copulas (*Clayton* and *Gumbel*), as well as two Elliptical Copulas (*Normal* and *Student-t Copulas*).

4.1 Data

We start by investigating sovereign debt risk contagion in four Latin American countries: Argentina, Brazil, Mexico and Venezuela. For this purpose we obtained daily data of the Emerging Bond Market Index (EMBI+) by JP Morgan for the mentioned countries available for the period 17/04/2008 - 25/04/09. Then, in order to examine financial contagion in other financial markets data corresponding to period 03/01/2005-01/06/2009 is collected for different assets: exchange rates (US$ Dollar/Mexican Peso and US$ Dollar/EURO), interest rates (fund rates for Mexican and US markets), and stock markets (S&P500 and Mexican stock price index). Further, in order to investigate whether dependence association measures change before and after the financial crisis of 2008 we divide the overall sample in two sub-samples.

In Table 1 we present descriptive statistics of daily returns for the whole samples considered in this study. It is observed that EMBI+ returns are greater in all cases and show in general a greater risk than stock or exchange rates. Interestingly, US funding rates showed the lowest investment log returns and the greatest reinvestment risk for the period considered. At the same time, except for the EMBI+ return in Mexico, all log-returns exhibit excess kurtosis, some degree of skewness and in general departures from normality as indicated by normality tests in the last three columns of the table.

Departure from normality, fat tails and nonlinear behavior of asset returns are features that strongly suggest to take into account alternative conditional and non conditional distributions to the normal. The *Student-t* for instance has been used by

Bollerslev (1987), Hsieh (1989), while the GED distribution has been employed by Baillie and Bollerslev (1989) and Nelson (1991).

The literature examining sovereign bonds, exchange rates, stock and interest rates is mainly done in developed countries but some studies have been also carried out to investigate contagion in emerging markets using Copulas, mainly in stock markets—see Ozun & Ozbakis (2007), Mendes, et al (2007) and Rodriguez (2007) for some recent examples. In this sense our paper also contributes to the literature of financial crisis and contagion in emerging markets.

4.2 Estimation

The first step in the estimation of our GED copula is to obtain the marginal distributions *u* and *v*. These are found by a) calculating log returns of financial assets using

$$r_t = \ln(\frac{x_t}{x_{t-1}}) \,, \tag{30}$$

where r_t is the log-return, x_t is the asset price in period *t* and *ln* is the natural log. b) in line with recent applications using Copulas on financial markets, we employ residuals obtained from univariate GARCH models and use them as marginal distributions—see Lucchetti (2002), Patton (2006) y Harvey & Chakravarty (2008) for some examples. In this paper we fit the GARCH(1,1) model of Bollerslev (1990) which has become the empirical workhorse in the modeling of financial time series with time varying volatility. The marginal distributions hence estimated are distributed normal with mean μ and variance σ^2, that is $u = Z(\varepsilon_x)$ and $v = Z(\varepsilon_y)$, where ε_y and ε_y are residuals of GARCH(1,1) models with $Z_{\varepsilon x} \sim N(\mu, \sigma)$ y $Z_{\varepsilon y} \sim N(\mu, \sigma)$.

4.3 Contagion in Sovereign Bond Markets

In Table 2 we present parameter estimates for all different copulas including the GED Copula and all pairs of countries considered. We use the same marginal functions[6] and we report robust standard errors in parenthesis, the Akaike and Schwartz decision criteria, as well as the optimized likelihood function.

[6] We do not report the GARCH(1,1) estimations of Bollerslev (1986) in order to save space but the results are readily available from the authors.

Table 1. Descriptive Statistics of Levels and Log Returns.

	Mean	Min.[b]	Max.[c]	Std. Dev.[d]	Skewness	Kurtosis	Obs.[e]	SW[f]	Z[g]	Prob > z
Levels										
Emerging Market Bond Index (EMBI+)										
Arg[a]	1220.6754	543.4783	1967.3913	557.2150	-0.0068	-1.8225	374	0.8077	9.2720	0.0000
Bra	348.6631	180.6452	680.6452	116.8379	0.1249	-1.1920	374	0.8948	7.8420	0.0000
Mex	287.7350	125.8065	600.0000	119.7935	0.0965	-1.3326	374	0.8916	7.9120	0.0000
Ven	1172.7796	543.4783	1891.3043	496.2237	-0.0002	-1.7860	374	0.8225	9.0830	0.0000
LA	526.3780	257.6923	900.0000	202.9354	-0.0682	-1.7738	374	0.8243	9.0590	0.0000
Exchange Rates										
USD/Eur	0.7543	0.6254	0.8571	0.0588	-0.4686	-0.5956	1151	0.9556	8.6160	0.0000
USD/MXN Peso	11.2503	9.8745	15.4900	1.0673	2.0238	3.1523	1151	0.6790	13.5450	0.0000
Interest Rates										
US	3.4438	0.0800	5.4100	1.7325	-0.6376	-0.8431	1151	0.8742	11.2120	0.0000
Mex	7.7714	5.1500	10.2000	0.9713	0.5895	-0.0973	1151	0.9048	10.5190	0.0000
Stock Markets										
Standard&Poors	1259.9739	676.5300	1565.1500	197.3830	-0.9706	0.3901	1151	0.9016	10.6000	0.0000
IPC	22690.9784	11739.9900	32836.1200	6209.9904	-0.0635	-1.2964	1151	0.9350	9.5670	0.0000
Log Returns										
Emerging Market Bond Index (EMBI+)										
Arg	0.0031	-0.0606	0.1138	0.0202	0.7854	3.5346	373	0.9561	5.7600	0.0000
Bra	0.0013	-0.1064	0.1178	0.0257	0.3284	3.4550	373	0.9508	6.0340	0.0000
Mex	0.0022	-0.1170	0.1054	0.0300	-0.1560	2.0132	373	0.9740	4.5170	0.0000
Ven	0.0022	-0.1029	0.1092	0.0219	-0.0130	5.4526	373	0.9174	7.2600	0.0000
LA	0.0020	-0.0855	0.0846	0.0204	0.3399	3.9173	373	0.9376	6.5960	0.0000
Exchange Rates										
USD/Eur	0.0000	-0.0403	0.0474	0.0065	0.0472	5.8299	1150	0.9438	9.2040	0.0000
USD/MXN Peso	0.0001	-0.0563	0.0664	0.0069	1.0929	18.0772	1150	0.8206	12.0930	0.0000
Interest Rates										
US	-0.0021	-0.6212	0.5776	0.0803	-1.3980	23.5291	1150	0.5809	14.2070	0.0000
Mex	-0.0005	-0.2776	0.2630	0.0257	-0.5930	33.4655	1150	0.6429	13.8080	0.0000
Stock Markets										
Standard&Poors	-0.0002	-0.0947	0.1096	0.0152	-0.1862	10.7570	1150	0.8418	11.7810	0.0000
IPC	0.0006	-0.0727	0.1044	0.0161	0.1933	4.8075	1150	0.9394	9.3920	0.0000

a Argentina, Brazil, Mexico, Venezuela and Latin America are abbreviated as Arg, Bra, Mex, Ven and LA respectively. b minimum. c Maximum. d Standard Deviation. e Observations. F Shapiro Wilks test of normality. g Normality test.

Table 2: Estimation Results and Likelihood Ratio Tests.

	Arg-Bra[a]	Arg-Mex	Arg-Ven	Arg-LA	Bra-Mex	Bra-Ven	Bra-LA	Mex-Ven	Mex-LA	Ven-LA
Clayton										
θ	1.4991	1.0704	1.1834	1.8481	3.5582	1.553	5.5019	1.0345	3.2454	2.1872
$L(\theta)$[b]	-1705.5	-1679.5	-1789.4	-1792.1	-1461	-1650.1	-1427.9	-1635.7	-1536.2	-1689.3
AIC[c]	3413	3361.01	3580.77	3586.18	2924.06	3302.15	2857.8	3273.47	3074.34	3380.59
BIC[d]	3416.92	3364.93	3584.69	3590.1	2927.98	3306.07	2861.72	3277.39	3078.26	3384.52
Gumbel										
θ	1.7495	1.5352	1.5917	1.9241	2.7791	1.7765	3.751	1.5173	2.6227	2.0936
$L(\theta)$	-1659.4	-1641.3	-1732.7	-1724.9	-1394.5	-1601.6	-1390	-1581	-1478.9	-1626.7
AIC	3320.72	3284.69	3467.46	3451.71	2790.94	3205.28	2782.09	3163.91	2959.84	3255.34
BIC	3324.64	3288.61	3471.38	3455.63	2794.86	3209.21	2786.01	3167.83	2963.76	3259.26
Normal										
ρ	0.6753	0.5549	0.645	0.7221	0.8519	0.7292	0.9388	0.6414	0.8591	0.8157
$L(\theta)$	-936.87	-982.99	-948.22	-914.18	-815.09	-911.75	-658.94	-955.62	-807.69	-850.33
AIC	1875.74	1967.97	1898.43	1830.37	1632.18	1825.5	1319.88	1913.24	1617.38	1702.66
BIC	1879.67	1971.9	1902.36	1834.29	1636.1	1829.42	1323.8	1917.16	1621.3	1706.58
Student t										
ρ	0.6808	0.5603	0.6522	0.7263	0.8527	0.7324	0.9391	0.6443	0.8594	0.8174
V	3.7E+07	4.1E+07	3.7E+07	3.5E+07	3.7E+07	3.5E+07	3.2E+07	3.7E+07	3.7E+07	3.4E+07
$L(\theta)$	-936.84	-982.97	-948.18	-914.16	-815.09	-911.74	-658.94	-955.61	-807.69	-850.32
AIC	1877.69	1969.94	1900.35	1832.33	1634.18	1827.48	1321.87	1915.22	1619.38	1704.64
BIC	1885.53	1977.79	1908.2	1840.17	1642.02	1835.32	1329.72	1923.07	1627.22	1712.49
GED										
ρ	0.5188	0.4039	0.5059	0.5677	0.6904	0.5592	0.8415	0.4958	0.7087	0.6504
	(-0.040)[e]	(-0.0473)	(-0.0449)	(-0.0578)	(-0.0585)	(-0.0790)	(-0.250)	(-0.0376)	(-0.026)	(-0.068)
b	2.9958	2.9576	2.8459	3.105	2.779	2.5511	2.8537	2.7697	3.3166	2.6545
	(-1.9807)	(-4.4710)	(-2.3084)	(-9.6055)	(-2.6241)	(-4.3471)	(-99.86)	(-1.6157)	(-2.044)	(-4.512)
$L(\theta)$	-650.52	-695.77	-670.63	-624.41	-563.66	-669.65	-426.8	-688.11	-527.98	-612.8
AIC	1307.05	1397.55	1347.25	1254.83	1133.31	1345.31	859.607	1382.21	1061.96	1231.59
BIC	1318.81	1409.31	1359.02	1266.59	1145.08	1357.07	871.372	1393.98	1073.72	1243.36
Likelihood Ratio Test (GED vs:)										
Laplace	2013.72	2018.12	1994.16	2023.61	1915.61	1891.97	1841.61	1957.51	1980.17	1865.44
Normal	572.643	574.395	555.099	579.496	502.864	484.17	464.265	535.011	559.423	475.051

a Argentina, Brazil, Mexico, Venezuela and Latin America are abbreviated as Arg, Bra, Mex, Ven and LA respectively. b Optimized Likelihood Value. c Akaike Information Criterion. d Bayes Information Criterion. e t-test statistic.

Table 2 shows that, among all Arquimedian and Elliptical Copulas considered in this study, our *GED-Copula* provides the lowest values of Akaike and Schwartz Criteria. The closest competitor is the *Normal-Copula*, followed very closely by the *Student-t Copula*, but even in this last case that takes into account fat tails the *GED-Copula* describes the data better and provides a much satisfactory fit than these two popular elliptical alternatives.

The optimized likelihood function is the highest for the *GED-Copula* which reinforces our conclusion in favor of our own elliptical proposal. In order to assess more formally our results we restrict the *GED-Copula* parameters to obtain the Laplace and Normal Copula functions and show Likelihood Ratio tests at the bottom of Table 2. We find again strong evidence in favor of the *GED-Copula*.

In figure 4 we show the contour plots of *GED-Copula* in sovereign bond markets for the pairs Argentina-Brazil, Argentina-Mexico and Brazil-Mexico. The shape of the bell curve in the first two cases implies that deviations from the mean are not big, which is an indication of small returns tending to grow and decrease jointly. In the case of Brazil-Mexico, we find thicker tails suggesting that returns grow and decrease jointly in greater proportions. In other words, sovereign bond market risk contagion is greater between Mexico and Brazil, than for any of the two pair of countries considered.

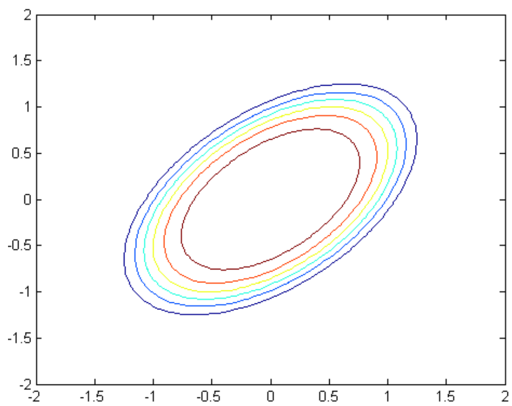
4.4 Contagion in the Foreign Exchange, Bonds and Capital Markets

We now report the results obtained for the foreign exchange, bonds and capital markets, for the samples sizes described above and also for the two-sample results designed to examine contagion before and during the crisis. Hence, we heuristically define the break points as follows: bond markets on 19/09/2007 (when US interest rates started to fall), foreign exchange rate on 27/08/2008 and stock market indexes on 06/06/2008.

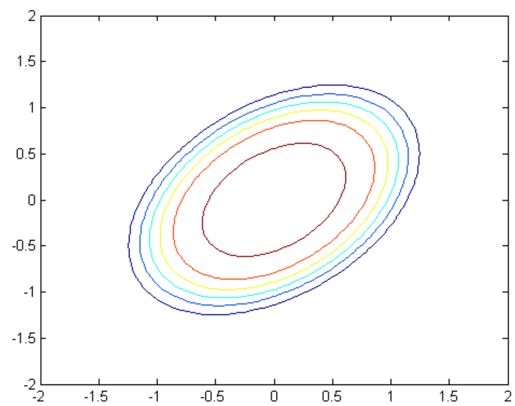
Table 3 presents the nine results obtained using the *GED-Copula* for these markets and comparing it with other elliptical and Arquimidean alternatives. As with sovereign risk contagion, for the cases of foreign exchange, bonds and stock markets, we found strong evidence indicating that *GED-Copula* provides better results as measured by optimized likelihood value, Akaike and Schwartz criteria.

To test whether contagion parameters in each copula change before and during the crisis we have formally tested the equality of these using a standard Wald test. The null that ρ and b are the same in both subperiods is rejected at 5% level. We observe the association of the two markets measured by ρ is more intense during the crisis for the three markets. In the foreign exchange and stock markets, joint depreciations (and appreciations) are more frequent during the crisis period than before and, in the bond market, joint lower interest rates in one market and higher interest rates in the other were more frequent during the crisis.

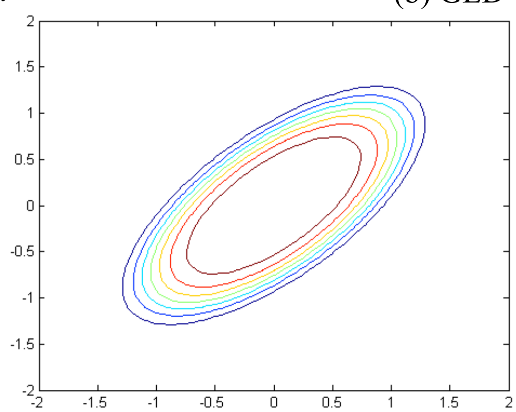
Figure 4. Contour Plots of GED Copulas in Sovereign Bond Markets.



(a) GED Copula: Arg-Bra



(b) GED Copula: Arg-Ven



(c) GED Copula: Bra-Mex

The coefficient b measuring the shape of the distribution is greater during the crisis period in the foreign exchange and stock markets. This implies that during crisis periods the tails of the joint distribution were thicker (a higher probability of extreme events) than in more stable periods. Interestingly, in the bond market we observe a lower *b* parameter during the crisis, which indicates a lower probability of extreme events during stable periods. This is a revealing finding that reflects some of the events observed during this crisis. That is, while stock and exchange rate markets have shown a high volatility during the crisis, bond markets have remained stable and in fact money has shift from risky markets to treasury markets with a lower risk (credit and reinvestment).

All in all, given the results presented above, it is evident that the *GED Copula* introduced in this paper provides a better fit than the most commonly used Arquimidean and Elliptical Copulas in empirical finance.

Table 3. GED-Copula Estimation Results before and during the crisis.

	Exchange Rates			Funding Rates			Stock Markets		
	Overall Sample	Pre Crisis	Crisis	Overall Sample	Pre Crisis	Crisis	Overall Sample	Pre Crisis	Crisis
ρ^a	0.1205	0.0285	0.1302	-0.0579	-0.0345	-0.0843	0.5475	0.4918	0.6119
	(0.0441)	(0.0363)	(0.0733)	(0.0532)	(0.0709)	(0.0670)	(0.0774)	(0.0441)	(0.1049)
b^b	2.3407	2.6500	2.7755	2.3209	2.6783	2.2701	2.4405	2.9348	2.5310
	(0.4903)	(0.6676)	(1.5091)	(0.6104)	(1.2877)	(0.7526)	(2.3661)	(2.1260)	(3.0331)
$L(\theta)^c$	-2393.9	-1978.0	-398.6	-2330.2	-1359.9	-955.7	-2069.5	-1587.5	-433.5
AIC^d	4793.8	3962.0	803.2	4666.3	2725.7	1917.4	4145.0	3181.0	872.9
BIC^e	4809.0	3976.6	813.1	4681.5	2739.4	1929.7	4160.2	3195.4	883.6
$\rho_1=\rho_2^f$		51.7145*			1.8E10^9*			72.12*	
$b_1=b_2$		4.5001*			24.8697*			22.6464*	

* Significant at the 5% level. Notes: a Correlation coefficient. b thickness of tail parameter. c Optimized Likelihood Function. d Aikaike Information Criterion. e Bayes Information Criterion. f Wald Test.

5. Conclusions

Copulas are functions that link two marginal functions with the ability to describe non-linear behavior, asymmetry and non-normality. We introduce in this paper a new Copula based on a very commonly used distribution in empirical finance: the GED distribution. Despite the extended use of this distribution there has not bee a *GED-Copula* proposed in the literature, until now. By using four markets (sovereign bonds, exchange rates, stock and bond markets) we compare our new *GED-Copula* with the most commonly employed elliptical and Arquimidean Copulas. The main advantage of this new Copula resides in its capacity to replicate the behavior of several other distributions, from the exponential and normal, to the uniform distribution.

Standard decision criteria such as Schwartz, Akaike, the Likelihood Function and Likelihood Ratios, all show strong evidence in favor of the GED Copula against the normal, t-distribution and the main Arquimidean copulas employed in the financial literature.

We have used standard marginal distributions by fitting standard GARCH(1,1) processes with no asymmetries, persistence or other well known phenomena. Potential extensions can be done in these areas by allowing the tail parameters to exhibit inertia, leverage effects and even long memory. We are confident all these extensions should show the GED Copula a very competitive if not the best alternative.

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Appendix A. Estimation of Arquimidean and Elliptical Copulas

A.1. Arquimidean Copulas

Kendall Tau

The distribution function of *C(u,v)* is represented by *Kc(t)*

$$K_C(t) = t - \frac{\varphi(t)}{\varphi'(t)},$$

For a detailed proof see Nelsen (1999, p.102). Substituting this expression in [17] we get:

$$\begin{aligned}\tau &= 4E(C(u,v)) - 1 = 4\int_0^1 t dK_C(t) - 1 \\ &= 4tKc(t)_0^1 - 4\int_0^1 K_C(t)dt - 1 \\ &= 3 - 4\int_0^1 t - \frac{\varphi(t)}{\varphi'(t)}dt = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}d\end{aligned} \quad [A.1].$$

Where φ is the generator function of the Archimidean Copula. For the Clayton Copula the Kendall Tau is obtained as follows:

$$\begin{aligned}\tau &= 1 + 4\int_0^1 \frac{\frac{1}{\theta}(t^{-\theta} - 1)}{-t^{-\theta-1}}dt = 1 + 4\int_0^1 \frac{-t + t^{\theta+1}}{\theta}dt \\ &= 1 + \frac{4}{\theta}\left(\frac{1}{\theta+2} - \frac{1}{2}\right) = \frac{\theta}{\theta+2}\end{aligned} \quad [A.2];$$

and for the Gumbel copula we get:

$$\begin{aligned}\tau &= 1 + 4\int_0^1 \frac{(-\ln t)^{\theta}}{-\frac{1}{t}\theta(-\ln t)^{\theta-1}}dt = 1 + 4\int_0^1 \frac{t\ln t}{\theta}dt \\ &= 1 + \frac{4}{\theta}\left(\frac{t^2}{2}\ln t_0^1 - \int_0^1 \frac{t}{2}dt\right) = 1 + \frac{4}{\theta}\left(0 - \frac{1}{4}\right) = 1 - \frac{1}{\theta}\end{aligned} \quad [A.3].$$

Having defined the Gumbel and Clayton Copula the Arquimidean Copulas can be estimated. Solving we find Kendall Tau for the Clayton Copula:

$$\tau = \frac{\theta}{\theta + 2} \rightarrow \theta = \frac{2\tau}{1 - \tau}, \qquad \text{[A.4]}$$

and for the Gumbel Copula:

$$\tau = 1 - \frac{1}{\theta} \rightarrow \theta = \frac{1}{1 - \tau}; \qquad \text{[A.5]}$$

that is, Kendall Tau for two random variables depends on the number of concordant (*c*) and discordant (*d*) pairs as follows:

$$\tau = \frac{c - d}{c + d}. \qquad \text{[A.6]}$$

Hence, using A.4 we can easily calculate θ of Arquimidean Copulas.

A.2 Elliptical Copulas

Using the Normal Copula defined in [14] we observe that the only parameter to be estimated is ρ, the correlation coefficient of the two random variables, where $\Phi^{-1}(u)$ and $\Phi^{-1}(v)$, Φ represent the normal distribution with mean 0 and variance 1. We use a practical approach in the estimation of this copula by first obtaining an estimate of the correlation coefficient and use it in a second step to derive the Normal Copula.

For the *Student-t* Copula and *GED-Copula* we use Maximum Likelihood and the Newton-Raphson[7] method to estimate the Copula parameters. The Log Likelihood Function of the *Student-t Copula* is:

$$L = \ln \prod c(u_i, v_i) = \\ = \sum \ln\left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right) - \frac{\nu + 2}{2} \ln\left(1 + \frac{t_\nu^{-1}(u_i)^2 - 2\rho t_\nu^{-1}(u_i) t_\nu^{-1}(v_i) + t_\nu^{-1}(v_i)^2}{\nu(1-\rho^2)}\right)^b.$$

[A.7]

[7] The algorithm goes back to the first description made by Thomas Simpson in 1740.

It is important to note that this distribution function contains inverse functions in the marginals, hence parameter estimates are somewhat different to those in which standard functions would be used.

We aim at maximizing *L(v,ρ)* and this is achieved when $dL_r = dL_v = 0$. The optimisation method is Rapson-Newton:

$$[\rho_{n+1} \quad \nu_{n+1}] = [\rho_n \quad \nu_n] - \begin{bmatrix} \partial L_{\rho,\rho} & \partial L_{\rho,\nu} \\ \partial L_{\rho,\nu} & \partial L_{\nu,\nu} \end{bmatrix}^{-1} [\partial L_r \quad \partial L_\nu]. \qquad [A.8]$$

We iterate this equation and stop when L is maximum, that is, when $L_{n+1} - L_n < 0.00001$.