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Regulatory Capital Determination and Its Implications for Internal Ratings-Based Credit Risk Model Development and Validation

Honggao Cao

One key element of the internal ratings-based (IRB) Basel rule is a formula (or "supervisory mapping function") for determining minimum capital that a bank must hold to protect its solvency in an extremely adverse situation (Basel Committee on Banking Supervision, 2005; US Rules and Regulations 2007). Reflecting a negotiated settlement between regulators and the banking industry, this formula has a natural connection to several well-specified statistical models that can be leveraged by Basel model developers and validators to better understand the capital rule. For example, where does this formula come from? What risk does it try to capture? Why does the Basel II Accord stipulate that the formula be implemented on a basis of homogeneous segments for retail exposures or similar risk ratings of wholesale obligors? Is there any desirable property on the number of loans for a segment (or obligor group)? Why is LGD in the formula treated as a constant as opposed to a random variable? When covering expected loss – and determined independently – is the loss reserve related to the minimum regulatory capital in any particular manner?

In this paper, we try to expound these issues. At the core of our exercise is a derivation of the capital formula, which can be seen as an application of the single-risk-factor portfolio loss distribution model developed by Oldrich A.Vasicek (1987, 1991, and 2002), which is in turn an extension of the single-asset credit risk model developed by Robert C. Merton (1974).[1] We demonstrate that the regulatory capital as represented by the formula essentially tries to manage unexpected risks from asset correlation: In an event of no asset correlation, the regulatory capital would be reduced to zero. We show that homogeneity of the portfolio, along with an assumption of normality on asset value dynamics, is a necessary condition for the Vasicek model to be valid. In addition, the Vasicek formula is accurate only asymptotically. To use the formula, one has to presume that the target portfolio has a large number of identical loans. This not only explains why the Basel Accord asks for grouping retail loans or wholesale obligors of similar credit quality before applying the capital formula, but also puts constraints on the number of loans of any resulting portfolio segment or group: The formula is relevant only asymptotically; the portfolio size cannot be too small.

Contingent Claims and Probability of Default: The Single Asset Model

We start with the single-asset credit risk model developed by Merton, which regards an obligor's behavior in a contingent claims framework. According to this model, whether an obligor defaults on a debt over a fixed assessment horizon depends on whether the value of the underlying asset becomes lower than the debt obligation at the end of horizon. Specifically, assume that the value of the asset at time t, V_t , follows a geometric Brownian motion (GBM) as in equation (1),

[1] Similar discussions on the portfolio loss distributions that are related to regulatory capital determination can also be found in Céspedes (2002), Kochendörfer (2011); Schönbucher (2002); and Thomas & Wang (2005).

$$dV_t = V_t(\mu\, dt + \sigma\, dW_t) \qquad (1)$$

where W_t is a Wiener process (or standard Brownian motion) with constant drift μ and constant volatility σ. In particular,

$$W_t \sim N(0, dt) \qquad (2)$$

where $N(.)$ represents normal distribution. It follows, based on Ito's lemma, that the value of the asset at the end of horizon T, V_T, is

$$V_T = V_0\, e^{\mu T - 0.5\sigma^2 T + \sigma\sqrt{T} z} \qquad (3)$$

where V_0 is the value of the asset at the beginning of the horizon, and z is a standard normal random variable.

The obligor defaults (or $D = 1$) if V_T is lower than the expected debt obligation at T, denoted as $B \equiv e^b$. That is, substituting from (3), the probability of default is

$$\begin{aligned}
\Pr(D{=}1) &= \Pr(V_T < e^b) \\
&= \Pr(V_0\, e^{\mu T - 0.5\sigma^2 T + \sigma\sqrt{T} z} < e^b) \\
&= \Pr\left(z < \frac{b - \ln V_0 - \mu T + 0.5\sigma^2 T}{\sigma\sqrt{T}}\right) \\
&\equiv \Pr(z < \theta) = \Phi(\theta) \qquad (4)
\end{aligned}$$

where $\Phi(.)$ is cumulative density of the standard normal distribution.

The importance of equation (4) – as we will see below – is that an obligor's probability of default can be derived in closed form from the standard normal distribution.

The Single-Risk Factor in the Asset Valuation Function

Consider a portfolio of *n* obligors. The probability of default for obligor *i*, p_i, is then

$$p_i = \Pr(D_i = 1) \qquad (5)$$

For reasons to be seen shortly, this p_i represents the obligor's <u>*unconditional*</u> default risk, or the expected default probability over time horizon and across different environments.

The asset value as determined in equation (3) for obligor i can be characterized as a random variable,

$$z_i = \theta_i \tag{6}$$

with the following properties:

$$E(z_i) = 0 \tag{6a}$$

$$E(z_i^2) = 1 \tag{6b}$$

$$E(z_i, z_j) = R \tag{6c}$$

$$\theta_i = \Phi^{-1}(p_i) \tag{6d}$$

The asset values of any two obligors in the portfolio are assumed to be correlated with a constant correlation R.

The relationship in (6)-(6c) can be represented as

$$z_i = \sqrt{R}\,Y + \sqrt{1-R}\,\varepsilon_i \tag{7}$$

where Y is a portfolio common factor that can be interpreted as an economic or scenario index, ε_i is a *normalized* obligor-specific idiosyncratic factor, and Y, ε_1, ε_2, …, ε_n are mutually independent standard normal variables.

As noted in Vasicek (2002), equation (7) is not a new assumption, but a property of the equicorrelated normal distribution as specified in (5)-(6c). Furthermore, $\sqrt{R}\,Y$ can be seen to be the obligor's exposure to the common (or systemic) risk, and $\sqrt{1-R}\,\varepsilon_i$ the obligor-specific risk.

Equations (5)-(7) constitute the well-known single-factor risk model. It has dominated industry efforts to model credit risk for determining both economic and regulatory capital.[2]

Conditional Probability of Default for a Single Obligor

Given the asset value correlation represented in (7), the probability that obligor i defaults on her loan conditional on a particular scenario $Y = y$ is often called conditional probability of default, which can be determined as follows:

[2] According to Gordy (2003), for example, this general framework has been used in the (J. P. Morgan) RiskMetrics Group's CreditMetrics, Credit Suisse Financial Product's CreditRisk$^+$, McKinsey's CreditPortfolio View, and KMV's Portfolio Manager. In Basel Committee on Banking Supervision (2005), this is referred to as the asymptotic single risk factor (or ASRF) framework.

$$p_i(y) \equiv p_i(Y = y) = \Pr(D_i = 1 \mid Y = y) \quad \text{(8a)}$$

$$= \Pr(z_i < \theta_i \mid Y = y) \quad \text{(8b)}$$

$$= \Pr(z_i < \theta_i \mid Y = y) = \Pr(\sqrt{R}\,Y + \sqrt{1-R}\,\varepsilon_i < \theta_i \mid Y = y) \quad \text{(8c)}$$

$$= \Pr(\varepsilon_i < \frac{\theta_i - \sqrt{R}\,y}{\sqrt{1-R}}) = \Phi(\frac{\theta_i - \sqrt{R}\,y}{\sqrt{1-R}}) \quad \text{(8d)}$$

$$= \Phi(\frac{\Phi^{-1}(p_i) - \sqrt{R}\,y}{\sqrt{1-R}}) \quad \text{(8e)}$$

where the equality in (8e) is due to equation (6d).

The unconditional probability of default, p_i as defined in (5), can be seen as the expectation of the conditional probabilities over all possible scenarios. That is,

$$p_i = \int p_i(Y = y) * f(Y)\, dY \quad \text{(9)}$$

where *f* is a density function of scenario.

Conditional PD for a Homogeneous Portfolio

The default rate of the portfolio, which consists of *n* obligors, is a random variable and may be expressed as

$$d(n) = \frac{1}{n}\sum_{i=1}^{n} D_i \quad \text{(10)}$$

If the events of default of all obligors in the portfolio were independent, the portfolio default rate would converge, by the central limit theorem, to a normal distribution as the portfolio size *n* increases. Because the defaults are not independent, however, the conditions of the central limit theorem are not satisfied, and – thus – $d(n)$ is not asymptotically normal. Given the single-risk factor assumptions as represented in equation (7), however, the default rate given a scenario – or conditional PD – for a homogeneous portfolio will converge to a limiting form. That is,

$$d(n;\ y) = \frac{1}{n}\sum_{i=1}^{n} D_i(Y = y) \quad \text{(11a)}$$

$$\xrightarrow{n \to \infty} p(y) \equiv d\{y\} \quad \text{(11b)}$$

$$= \Phi(\frac{\Phi^{-1}(p)-\sqrt{R}y}{\sqrt{1-R}}) \qquad (11c)$$

where $p(y)$ is the expected conditional portfolio PD, and, as in (9), p is the unconditional portfolio PD.

It is important to note that results in (11a)-(11c) are valid only asymptotically. To use them to predict for an actual portfolio, portfolio size should not be too small.

The asymptotic portfolio-default-rate distribution conditional on scenario $Y = y$ can be derived in the following manner. Let $Q(x)=\Pr(d\{y\}\leq x)$ be the cumulative density of the distribution. Then

$$Q(x)=\Pr(d\{y\}\leq x)=\Pr(y\geq d^{-1}\{x\})=\Phi(\frac{\sqrt{1-R}\Phi^{-1}(x)-\Phi^{-1}(p)}{\sqrt{R}}) \qquad (12)$$

The probability density function is then

$$q(x)=\frac{\partial Q(x)}{\partial x}=\sqrt{\frac{1-R}{R}}\exp\{-\frac{1}{2R}(\sqrt{1-R}\Phi^{-1}(x)-\Phi^{-1}(p))^2+\frac{1}{2}(\Phi^{-1}(x))^2\}$$

(13) [3]

For a graphical representation of this portfolio-default-rate distribution, see the appendix for a set of { p, R } combinations.

Portfolio PD under Stress Condition

One conditional portfolio PD of significant interest is the PD under a particular stress condition, popularly defined as $Y=-\Phi^{-1}(0.999)$:

$$p(Y=-\Phi^{-1}(0.999)) = \Phi(\frac{\Phi^{-1}(p)+\sqrt{R}\,\Phi^{-1}(0.999)}{\sqrt{1-R}}) \qquad (14)$$

Intuitively, this stress condition corresponds to a very negative economic environment, or a negative economic environment that has a very small probability (< 0.001) to occur. As we shall see shortly, equation (14) is a key construct ubiquitous in efforts to derive credit risk regulatory capital. Because of this, we shall call the equation "*the stress portfolio PD*", and scenario $Y=-\Phi^{-1}(0.999)$ "*the stress condition*".

[3] See Vasicek (1991) or Schönbucher (2002).

The Stress Portfolio Loss and Regulatory Capital[4]

The conditional portfolio loss rate for a homogeneous portfolio can be calculated in a way similar to conditional portfolio PD:

$$l(n;\ y) = \frac{1}{n}\sum_{i=1}^{n}[D_i(Y=y) * LGD_i(Y=y)] \tag{15}$$

where $LGD_i(Y=y)$ is loss severity _given default and given the scenario_ for obligor *i*. In general, this loss rate has no closed-form result. If the loss severity given default and given the scenario is constant across all the obligors in this homogeneous portfolio, however, the portfolio loss rate can be derived asymptotically. More formally,

$$l(n;\ y) = \frac{1}{n}\sum_{i=1}^{n}[D_i(Y=y) * LGD_i(Y=y)] \tag{15a}$$

$$= [\frac{1}{n}\sum_{i=1}^{n} D_i(Y=y)] * LGD(Y=y) \tag{15b}$$

$$\xrightarrow[n\to\infty]{} \Phi(\frac{\Phi^{-1}(p)-\sqrt{R}y}{\sqrt{1-R}}) * LGD(Y=y) \tag{15c}$$

Now evaluating the conditional portfolio loss under "the stress condition" $Y = -\Phi^{-1}(0.999)$, we have

$$l(n;\ Y = -\Phi^{-1}(0.999)) \xrightarrow[n\to\infty]{}$$

$$\Phi(\frac{\Phi^{-1}(p)+\sqrt{R}\,\Phi^{-1}(0.999)}{\sqrt{1-R}}) * LGD(Y = -\Phi^{-1}(0.999)) \tag{16a}$$

$$= [\Phi(\frac{\Phi^{-1}(p)+\sqrt{R}\,\Phi^{-1}(0.999)}{\sqrt{1-R}}) - p] * LGD^{\oplus} + p * LGD^{\oplus} \tag{16b}$$

$$\equiv k + p * LGD^{\oplus} \tag{16c}$$

[4] A discussion slightly different from – but analogous in spirit to – this section can be found in "Technical Appendix: Downturn LGDs and the Basel II Risk Weighted Functions" in Basel Committee on Banking Supervision (2004).

The k in equation (16c) would be the Basel II regulatory mapping function for determining the minimum capital requirement for credit risk *if and only if*[5]

- The unconditional portfolio default rate p is estimated as a long-run PD to approximate the portfolio default rate over all possible scenarios; and

- The LGD under *the stress scenario*, $LGD^{\oplus}$, is estimated as a constant derived from a very negative economic condition (or "economic downturn conditions").

It is worth noting that

- Equations (16a)-(16c) represent the ***asymptotic***, conditional (i.e., stress) portfolio loss for a homogenous portfolio. They may not hold if the portfolio is heterogeneous or portfolio size is too small;[6]

- The results reflect a decomposition of the conditional (stress) portfolio loss into the expected loss, $p * LGD^{\oplus}$, and the unexpected loss, k; and

- Assuming that the expected loss is covered by loan loss reserve, which is to be further discussed below, the regulatory capital k is set up so that the unexpected loss is covered should the stress condition emerge.

Equations (16a)-(16c) have another important implication as well: Regulatory capital is essentially meant to capture the systemic risk as represented in asset value correlation – if the assets in a portfolio are completely independent (or the portfolio is completely diversified), the conditional portfolio credit loss would be equal to expected loss, rendering the regulatory capital to disappear.

Implications for IRB Credit Risk Model Development and Validation

Internal ratings-based credit risk models consist of several distinct components. These include (i) portfolio segmentation to group individual loans into mutually exclusive homogeneous risk segments; (ii) portfolio PD quantification that captures long-term default rate for loans of each segment; and (iii) portfolio LGD that is estimated as a constant – and independently of PD – for each segment under economic downturn conditions. Based on our discussion in the previous sections, each of these components has a well-founded basis, serving to ensure that the regulatory capital combines with appropriately determined loss reserve to cover adequately total *conditional* credit loss that is expected from the stress condition.

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[5] For simplicity, we define k here by normalizing the maturity adjustment in the regulatory mapping function.
[6] The restriction that the size of a homogenized portfolio segment should not be too small can also be derived from the fact that the capital formula is concave in p. Capital charge for a small segment may lead to a granularity bias detrimental to the bank. See Kiefer and Larson (2003) for discussions along this line.

In particular, portfolio segmentation is motivated to create homogeneous sub-portfolios where obligors are similar in credit quality so that the foregone portfolio loss analysis can survive. As we have pointed out on several occasions, for each segment or sub-portfolio, the results on conditional portfolio PD, conditional portfolio loss rate, loss reserve and regulatory capital hold only asymptotically. This puts a constraint on the size of any potential segment: it should not be too small. It also explains why regulatory capital is not assessed at individual obligor or exposure level.[7]

Creating homogeneous segments also helps transitioning from conditional portfolio PD to conditional portfolio loss, which, as seen in equations (15a)-(15b), requires an assumption of constant conditional LGD. This assumption would generally be less credible if a portfolio is very heterogeneous.

The long-run PD in the capital formula has a connotation as the unconditional default rate for a given homogeneous segment, or default rate expected over all possible scenarios. According to the Basel accord, this long-run PD should be estimated over a minimum of five-year period. Although the choice of the five-year period may be a practical issue, it is important that the five-year period chosen cover as diverse scenarios as possible.

The regulatory decision that the conditional LGD is treated as a constant independent of conditional PD but estimated under economic downturn conditions reflects a compromise to an otherwise very difficult statistical problem. If the LGD were treated as a random variable correlated with PD, as one would normally assume, the entire analysis on the stress portfolio loss, including equations (16a)-(16c), would fall apart. The capital formula would be very difficult to specify. Estimating the LGD under economic downturn conditions reflects a regulatory intention to approximate $LGD(Y = -\Phi^{-1}(0.999))$. By not defining specifically what the economic downturn conditions are, it also leaves doors open for compensating the potential problem with the PD-LGD independence assumption: For instance, if the conditional LGD is negatively correlated with the conditional PD, one could define the economic downturn conditions as a scenario less severe than implied by $-\Phi^{-1}(0.999)$. Conversely, if the conditional LGD is positively correlated with the conditional PD, one could define the economic downturn conditions as a scenario more severe than implied by $-\Phi^{-1}(0.999)$.

Nevertheless, it is still worth noting that treating the LGD under the stress condition as a constant (or non-stochastic) may not be consistent with reality. As Gordy (2003) puts it,

> "In practice, LGD not only may be highly uncertain, but may also be subject to systemic risk. For example, the recovery value pf defaulted commercial real estate loans depends on the value of the real estate collateral, which is likely to be lower (higher) when many (few) other real estate projects have failed."

[7] Another popular explanation of why portfolio segmentation is needed for Basel II credit risk modeling is related to mitigating idiosyncratic risk across obligors. As seen in equation (7), the idiosyncratic risk ε_i is typically a non-zero variable. When assessed in a large obligor pool, this risk tends to be averaged out. We do not believe the explanation to be convincing, for the idiosyncratic risks can be averaged out in a large portfolio even without segmentation.

Regulatory Capital and Loss Reserve

Regulatory capital and expected loss combine to form the total credit loss expected from the stress condition. As indicated in equation (16c), the two components have a clearly specified relationship: for any given segment, given the minimum regulatory capital – that is, given the long-run PD, p, and the stress LGD, $LGD^{\oplus}$ – the expected loss is given, making the expected loss, and the total credit loss expected from the stress condition completely determined by p and $LGD^{\oplus}$ alone.

In practice, the expected loss is often approximated by loss reserve, which is often determined independently of the regulatory capital determination.[8] Comparing the expected loss in equation (16c) and actual loss reserve can be an important way to assess the adequacy of the capital requirement and the consistency in risk measurements across businesses.[9]

Concluding Remarks

In this paper, we have shown that the Basel IRB credit risk capital rule for determining minimum regulatory capital has a clear connection to several well-specified statistical models. Recognizing this connection, we have come to understand why the Basel Accord asks the credit risk models to be built by segmenting a portfolio into homogeneous sub-portfolios; by estimating long-run PD for each sub-portfolio, and by estimating LGD from economic downturn conditions.

We have also shown that, in order to use the underlying capital formula in an appropriate manner, the size of a homogenized sub-portfolio cannot be too small, and the long-run PD for each sub-portfolio should be estimated over a long period that covers economic scenarios as diverse as possible. Finally, we have demonstrated that the loss reserve, while often determined independently of the capital determination process, has a well-specified relationship with the minimum regulatory capital, a relationship that can be leveraged in credit-risk model validation.

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[8] More specifically, in calculating loss reserve, LGD is not always derived from the economic downturns, nor is the PD always derived as a long run average.

[9] Potential inconsistencies are expected between the expected loss as seen in equation (16c) and loan loss reserve. The latter is often managed through estimating expected, point-in-time (PIT) credit loss, with the underlying PD estimates generally different from the unconditional – through-the-cycle (TTC) – PD, p.

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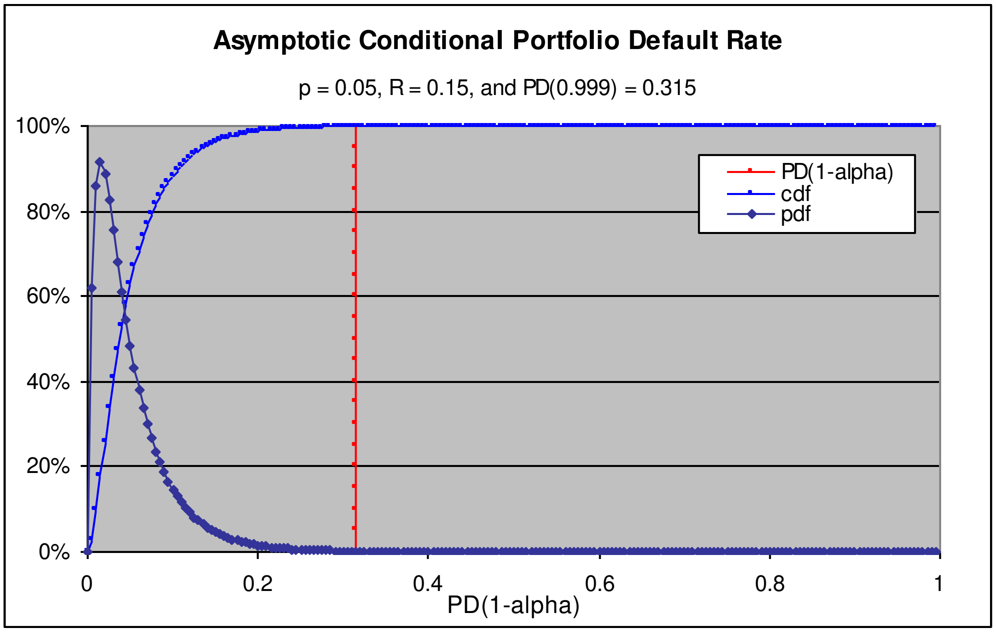
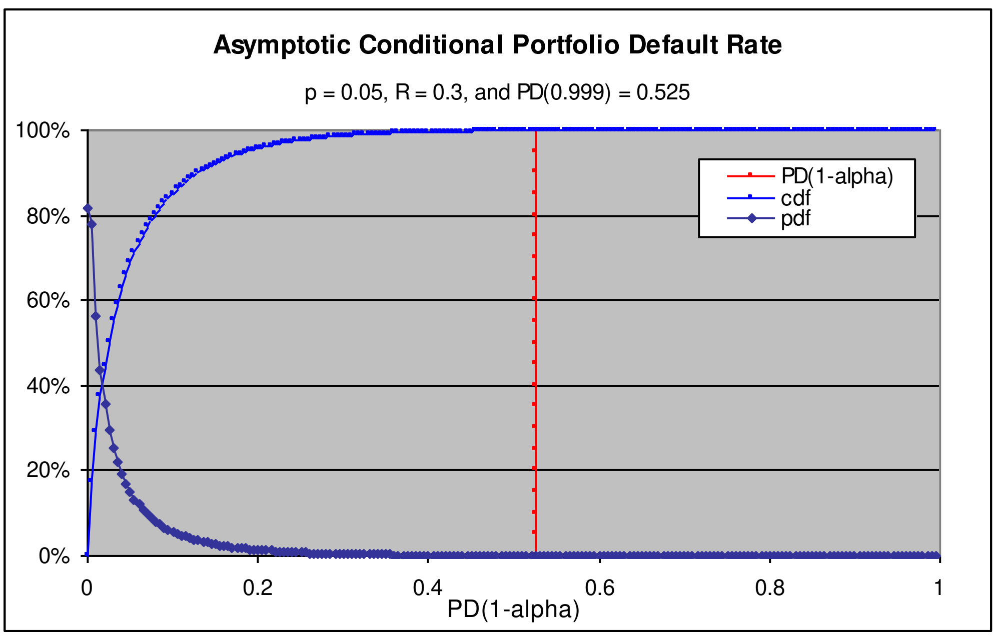
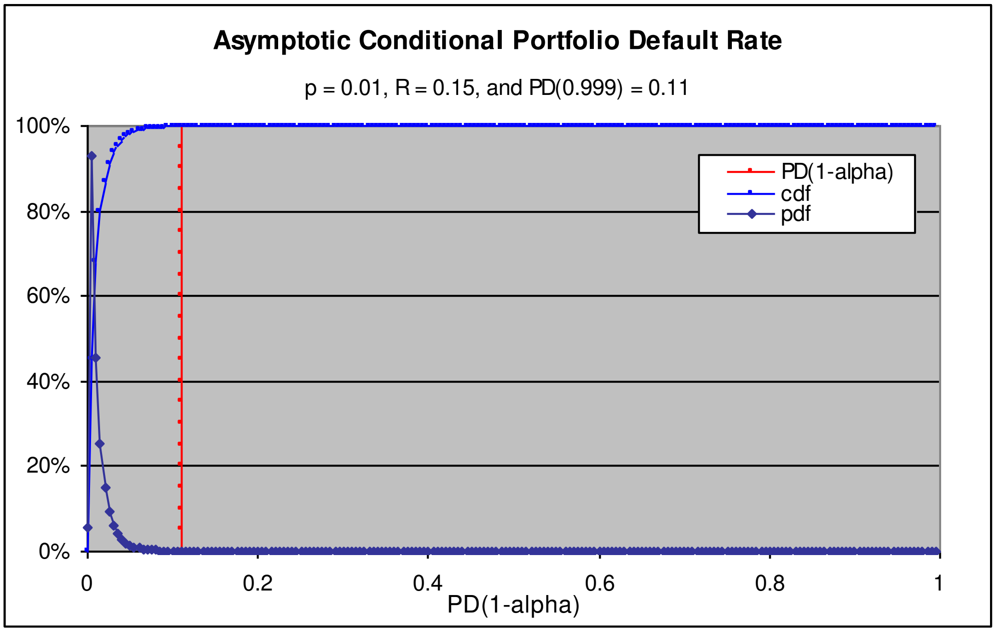
Appendix. The Asymptotic Distribution of Conditional Default Rate for a Homogeneous Portfolio Based on the Single-Risk Factor Model: Examples [10]

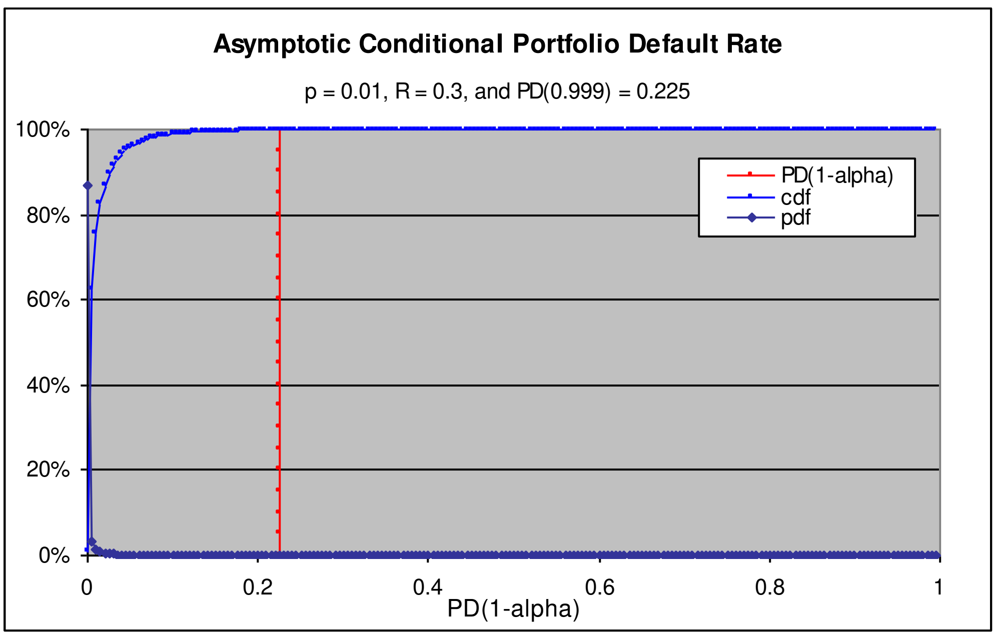




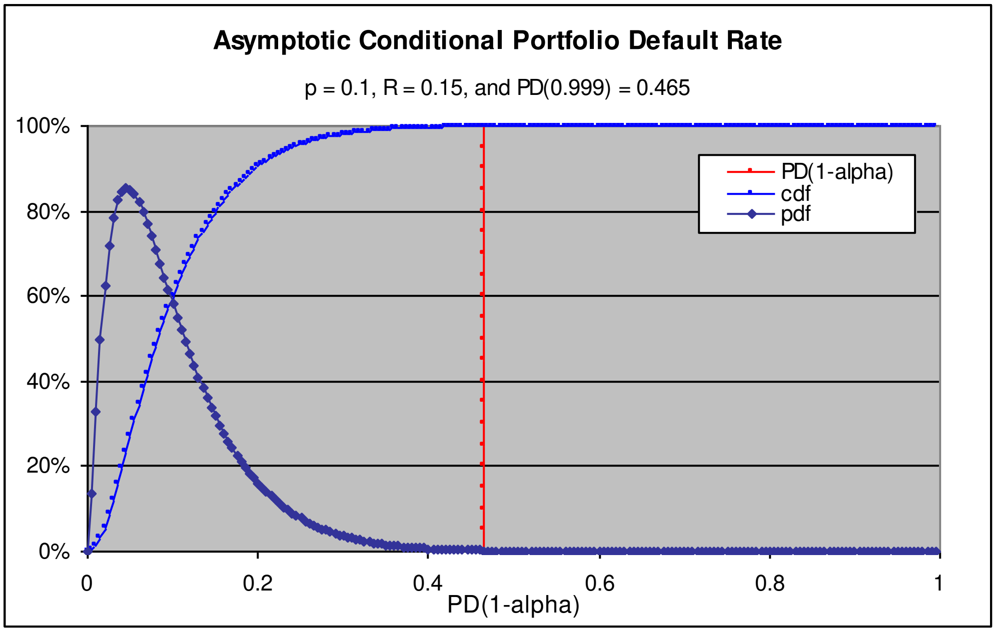
[10] The probability density function is plotted on the secondary y-axis, whose values are suppressed on purpose. In general, when the loss distribution concentrates over a range of low loss rates, the probability density can not only be greater than one (1), but also be huge. The *PD(1-alpha)* line is approximated.

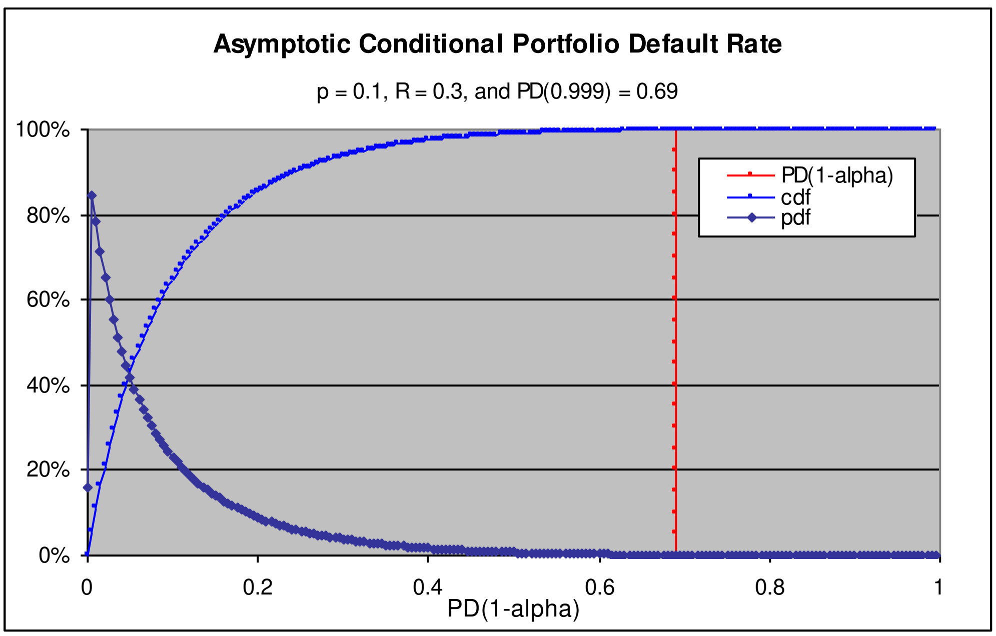


Asymptotic Conditional Portfolio Default Rate
p = 0.01, R = 0.15, and PD(0.999) = 0.11
100%
80%
60%
40%
20%
0%
0
0.2
0.4
0.6
0.8
1
PD(1-alpha)
PD(1-alpha)
cdf
pdf



Asymptotic Conditional Portfolio Default Rate
p = 0.01, R = 0.3, and PD(0.999) = 0.225
100%
80%
60%
40%
20%
0%
0
0.2
0.4
0.6
0.8
1
PD(1-alpha)
PD(1-alpha)
cdf
pdf


Asymptotic Conditional Portfolio Default Rate
p = 0.1, R = 0.15, and PD(0.999) = 0.465
100%
80%
60%
40%
20%
0%
0
0.2
0.4
0.6
0.8
1
PD(1-alpha)
PD(1-alpha)
cdf
pdf


Asymptotic Conditional Portfolio Default Rate
p = 0.1, R = 0.3, and PD(0.999) = 0.69
100%
80%
60%
40%
20%
0%
0
0.2
0.4
0.6
0.8
1
PD(1-alpha)
PD(1-alpha)
cdf
pdf