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ON PURE STRATEGY EQUILIBRIA IN LARGE GENERALIZED GAMES

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Abstract. We consider a game with a continuum of players where at most a finite number of them are atomic. Objective functions are continuous and admissible strategies may depend on the actions chosen by atomic players and on aggregate information about the actions chosen by non-atomic players. When atomic players have convex sets of admissible strategies and quasi-concave objective functions, a pure strategy Nash equilibria always exists.

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1. Introduction

Schmeidler (1973) proved that in non-convex games with a continuum of players the set of pure strategy equilibria is non-empty provided that (i) all agents are non-atomic, and (ii) objective functions depend only on their own strategy and on the average of the actions chosen by the other players. These assumptions convexifies the game, as the integral of any correspondence is a convex set (Aumann (1965)).

In this paper, we extend Schmeidler's result to large generalized games with a finite number of atomic players. In our framework, both objective functions and admissible strategies may depend on the strategies of atomic players and on messages which aggregate information about strategies chosen by non-atomic players (i.e., not necessarily on the average of these actions). By extending the proof given by Rath (1992, Theorem 2) of Schmeidler (1973) classical result, we provide a short and direct proof of the existence of pure Nash equilibria in large generalized games. Our theorem generalizes substantially Schmeidler (1973, Theorem 2) and Rath (1992, Theorem 2) to generalized games with

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compact metric action spaces and doesn't rely on purification of mixed strategy equilibria. Still, our theorem is a special case of Balder (1999,2002) but our proof is much simpler.[1]

2. Pure strategy equilibria in large generalized games

Let $\mathcal{G}(T,(K_t,\Gamma_t,u_t)_{t\in T},H)$ be a generalized game with an infinite set of players $T=T_1\cup T_2$.

The set of players T_1 is a non-empty and compact metric space endowed with a σ-algebra Σ and a finite non-atomic measure λ such that (T_1,Σ,λ) is a complete measure space. Each player $t\in T_1$ has a closed and non-empty action space $K_t\subseteq\widehat{K}$, where $\widehat{K}$ is a compact metric space.[2] A profile of actions for players in T_1 is any function $f:T_1\to\widehat{K}$ such that $f(t)\in K_t$, for any $t\in T_1$.

There is a finite set T_2 of atomic players. Each $t\in T_2$ has a non-empty, closed and convex action space $K_t\subseteq\widehat{K}_t$, where $\widehat{K}_t$ is a compact Frechet space.[3] Since T_2 is finite, a profile of actions for the players in T_2 is a vector $a:=(a_t;t\in T_2)\in\prod\limits_{t\in T_2}K_t$.

Let $\mathcal{F}(T_i)$ be the space of all profiles of actions for players in T_i, with $i\in\{0,1\}$. Also, given $t\in T_2$, let $\mathcal{F}_{-t}(T_2)$ be the set of profiles of actions $a_{-t}:=(a_j;j\in T_2\setminus\{t\})$ for players in $T_2\setminus\{t\}$.

Actions chosen by non-atomic players are not necessarily advanced by the participants in the game. However, when making a decision, players will consider aggregate information of some characteristics of these actions. Thus, given an action profile of non-atomic players $f\in\mathcal{F}(T_1)$, each player in T will only take into account, for strategic purposes, aggregate information coded through the message $m(f):=\int\limits_{T_1}H(t,f(t))d\lambda$, where $H:T_1\times\widehat{K}\to\mathbb{R}^l$ is a continuous function.

Since we want to concentrate on action profiles for which messages are well defined, we say that f is a *strategic profile* of players in T_1 if both $f\in\mathcal{F}(T_1)$ and $H(\cdot,f(\cdot))$ is a Σ-measurable function.[4] Measurability restrictions are not necessary over the behavior of atomic players. For this reason, the set of strategic profiles of players in T_2 coincides with $\mathcal{F}(T_2)$.

The set of messages associated with strategic profiles of non-atomic players is given by

$$M=\left\{\int\limits_{T_1}H(t,f(t))d\lambda:\quad f\in\mathcal{F}(T_1)\ \ \wedge\ \ H(\cdot,f(\cdot))\text{ is }\Sigma\text{-measurable}\right\}\subseteq\mathbb{R}^l.^5$$

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[1]Below we point out the main difference.

[2]This is the most relavant distinction with Balder (1999). While we assume all non-atomic action spaces to be uniformly contained in a compact metric space $\widehat{K}$, he gets rid of this important restriccion. It follows that in our model, non-atomic agents strategic profiles are integrable, a strong assumption in standard applications.

[3]That is, it is a non-empty and compact metrizable locally convex topological vector space

[4]In other words, for every Borelian set $E\subseteq\mathbb{R}^l$ we have that $\{t\in T_1:H(t,f(t))\in E\}\in\Sigma$.

[5]Notice that, since $\widehat{K}$ and T_1 are compact metric spaces and H is continuous, for any profile of actions $f:T_1\to\widehat{K}$ the function $H(\cdot,f(\cdot)):T_1\to\mathbb{R}^l$ is bounded. As T_1 has finite measure, if $H(\cdot,f(\cdot))$ is measurable, then it is integrable. For this reason, in the definition of M we only require measurability of $H(\cdot,f(\cdot))$.

The messages about the strategic profiles of players in T_1 jointly with the strategic profiles of players in T_2 may restrict the set of admissible strategies available for a player $t \in T$. That is, given a vector $(m, a) \in M \times \mathcal{F}(T_2)$ the strategies available for a player $t \in T_1$ are given by a set $\Gamma_t(m, a) \subseteq \widehat{K}$, where $\Gamma_t : M \times \mathcal{F}(T_2) \twoheadrightarrow K_t$ is a continuous correspondence with non-empty and compact values. Analogously, given $(m, a_{-t}) \in M \times \mathcal{F}_{-t}(T_2)$, the set of strategies available for a player $t \in T_2$ is $\Gamma_t(m, a_{-t}) \subseteq K_t$, where $\Gamma_t : M \times \mathcal{F}_{-t}(T_2) \twoheadrightarrow K_t$ is a continuous correspondence with non-empty, compact and convex values. We refer to correspondences $(\Gamma_t; t \in T)$ as correspondences of admissible strategies. We assume that, for any $(m, a) \in M \times \mathcal{F}(T_2)$, the correspondence that associates to any non-atomic player $t \in T_1$ the set of admissible strategies $\Gamma_t(m, a)$ is measurable.

Given a topological space A, let $\mathcal{U}(A)$ be the collection of continuous functions $u : A \to \mathbb{R}$. Assume that $\mathcal{U}(A)$ is endowed with the sup norm topology. We suppose that each player $t \in T_1$ has an objective function $u_t \in \mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$, and each player $t \in T_2$ has an objective function $u_t \in \mathcal{U}(M \times \mathcal{F}(T_2))$ which is quasi-concave in its own strategy. Finally, we assume that the mapping $U : T_1 \to \mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$ defined by $U(t) = u_t$ is measurable. [6]

Definition. *A pure strategy Nash equilibrium for the large generalized game $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, H)$ is given by feasible strategic profiles $(f^*, a^*) \in \mathcal{F}(T_1) \times \mathcal{F}(T_2)$ such that,*

$$\begin{aligned} u_t(f^*(t), m(f^*), a^*) &\geq u_t(f, m(f^*), a^*), \quad \forall f \in \Gamma_t(m(f^*), a^*), \quad \forall t \in T_1; \\ u_t(m(f^*), a_t^*, a_{-t}^*) &\geq u_t(m(f^*), a, a_{-t}^*), \quad \forall a \in \Gamma_t(m(f^*), a_{-t}^*), \ \forall t \in T_2. \end{aligned}$$

Theorem 1. *Any generalized game $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, H)$ has a pure strategy Nash equilibrium.*

Proof. We divide the proof into four steps.

(1) *The space of messages $M \subset \mathbb{R}^l$ is non-empty, compact and convex*. Since for any $(m, a) \in M \times \mathcal{F}(T_2)$ the correspondence that associates to any $t \in T_1$ the set $\Gamma_t(m, a)$ is measurable, it follows from Aliprantis and Border (1994, Theorem 14.85, page 504) that this correspondence has a $\Sigma \times \mathcal{B}(\widehat{K})$-measurable graph. Thus, it follows from Aumann's Selection Theorem (see Aliprantis and Border (2006, Theorem 18.26, page 608)) that there exists a Σ-measurable function $g : T_1 \to \widehat{K}$ such that, $g(t) \in \Gamma_t(m, a) \subseteq K_t$, $\forall t \in T_1$. Hence, $\int_{T_1} H(t, g(t)) d\lambda$ is well defined and, therefore, M is non-empty. Since the integral of a correspondence in a non-atomic measurable space is a convex

[6] Suppose that there is a finite number of types on the set of non-atomic agents, T_1. That is, there is a finite partition of T_1 into measurable sets $\{I_1, \ldots, I_r\}$ such that, two players belonging into the same element of the partition are identical. In this case, the restriction about measurability of U is trivially satisfied.

set (see Aumann (1965, Theorem 1)), we conclude that M is convex. Indeed, let $Q : T_1 \twoheadrightarrow \mathbb{R}^l$ defined by $Q(t) = H(t, K_t)$, for any $t \in T_1$. Then $M = \int_{T_1} Q(t) d\lambda$ is convex.[7]

Let $\widetilde{Q} : T_1 \twoheadrightarrow \mathbb{R}^l$ be the correspondence defined by $\widetilde{Q}(t) = H(T_1, \widehat{K})$, for any $t \in T_1$. Then $M = \int_{T_1} Q(t) d\lambda \subseteq \int_{T_1} \widetilde{Q}(t) d\lambda = \text{convexhull}(H(T_1, \widehat{K}))$. Therefore, since H is continuous, M is a subset of a compact set. Thus, to guarantee that M is compact it remains to prove that M is closed. Let $\{m_k\}_{k\in\mathbb{N}} \subseteq M$ be a sequence that converges to a vector $m \in \mathbb{R}^l$. Since $m_k \in M$, $m_k = \int_{T_1} h_k(t) d\lambda$, where $h_k : T_1 \to \mathbb{R}^l$ is a mesurable function and $h_k = H(\cdot, f_k(\cdot))$ for some $f_k \in \mathcal{F}(T_1)$. For each t, $\{H(t, f_k(t))\}_{k\in\mathbb{N}} \subseteq Q(t)$, which is a compact set. Thus, every limit point of $\{h_k(t)\}_{k\in\mathbb{N}}$ is contained in $Q(t)$. Also, since H is continuous, T_1 is compact, and $\bigcup_{t\in T_1} K_t \subseteq \widehat{K}$, it is easy to see that $\{h_k\}_{k\in\mathbb{N}}$ is uniformly bounded by an integrable function. By Aumann (1976), the limit point of $\int_{T_1} h_k(t) d\lambda$ belongs to $\int_{T_1} Q(t) d\lambda$. Therefore, the space of messages M is compact.

(2) *Best-reply correspondences are closed with non-empty and compact values. Furthermore, atomic players' best-reply correspondences have convex values.*

Given $t \in T_1$, let $B_t : M \times \mathcal{F}(T_2) \twoheadrightarrow K_t$ with $B_t(m, a) = \underset{f(t)\in\Gamma_t(m,a)}{\text{argmax}}\ u_t(f(t), m, a)$ be the best-reply correspondence of non-atomic layer t. Analogously, for any atomic player $t \in T_2$, his best-reply correspondence $B_t : M \times \mathcal{F}_{-t}(T_2) \twoheadrightarrow K_t$ is defined by $B_t(m, a_{-t}) = \underset{a_t\in\Gamma_t(m,a_{-t})}{\text{argmax}}\ u_t(m, a_t, a_{-t})$. As a consequence of Berge's Maximum Theorem, best-reply correspondences have closed graph and non-empty compact values. Moreover, the convexity of admissible strategies correspondences, jointly with the quasi-concavity of objective functions, guarantee that atomic players' best-reply correspondences have convex values.

(3) *The correspondence* $\Omega(m, a) := \int_{T_1} H(t, B_t(m, a)) d\lambda$ *is closed, non-empty and convex valued.*

Given $(m, a) \in M \times \mathcal{F}(T_2)$, by assumption the correspondence $\Phi_{(m,a)} : T_1 \twoheadrightarrow \widehat{K}$ defined by $\Phi_{(m,a)}(t) = \Gamma_t(m, a)$ is measurable and has non-empty and compact values. By the Measurable Maximum Theorem (see Aliprantis and Border (2006, Theorem 18.19, page 605)) the mapping $t \twoheadrightarrow H(t, B_t(m, a))$ has a measurable selector. Therefore, Ω has non-empty values.

The correspondence Ω has convex values, since for any $(m, a) \in M \times \mathcal{F}(T_2)$, the set $\Omega(m, a)$ is the integral of the correspondence $t \twoheadrightarrow H(t, B_t(m, a))$.

Fix $t \in T_1$. Since B_t has closed graph, the correspondence that associate to each $(m, a) \in M \times \mathcal{F}(T_2)$ the set $H(t, B_t(m, a))$ has closed graph too.[8] On the other hand, since $T_1 \times \widehat{K}$ is compact and H is continuous, there is a bounded function $v : T_1 \to \mathbb{R}^l$ such that $-v(t) \leq H(t, f(t)) \leq v(t)$, for any $t \in T_1$, $f \in \mathcal{F}(T_1)$ and $\int_{T_1} v(t) d\lambda$ is finite. Therefore, the correspondence that associates to

[7] This follow immediately from the definition of integral of a correspondence (Aumann (1965)) and the fact that we do not require action profiles to be measurable.

[8] It is a direct consequence of the continuity of H and the fact that $B_t(m, a) \subseteq \widehat{K}$ for any $(t, m, a) \in T_1 \times M \times \mathcal{F}(T_2)$.

each $(m,a) \in M \times \mathcal{F}(T_2)$ the integral on T_1 of the correspondence $t \twoheadrightarrow H(t, B_t(m,a))$ has closed graph (a consequence of the main result in Aumann (1976)). In other words, Ω is closed.

(4) *The generalized game* $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t\in T}, H)$ *has a pure strategy Nash equilibrium.*

Define $\Gamma : M \times \mathcal{F}(T_2) \twoheadrightarrow M \times \mathcal{F}(T_2)$ by $\Gamma(m,a) = (\Omega(m,a), (B_t(m, a_{-t}))_{t\in T_2})$. Then Γ is closed and has nonempty, convex and compact values. Applying Kakutani's Fixed Point Theorem, we conclude that Γ has a fixed point, i.e. there exists $(m^*, a^*) \in M \times \mathcal{F}(T_2)$ such that $(m^*, a^*) \in \Gamma(m^*, a^*)$. That is, for some $f^* \in \mathcal{F}(T_1)$, $m^* = \int_{T_1} H(t, f^*(t))d\lambda$ and $f^*(t) \in B_t(m^*, a^*)$, for any $t \in T_1$. Also, for any $t \in T_2$, $a^*_t \in B_t(m^*, a^*_{-t})$. These properties ensure that (f^*, a^*) is a pure strategy Nash equilibrium of $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t\in T}, H)$. □

There are generalizations of our results that are quite straightforward. For example, similar arguments to those made in Remark 6 in Rath's article would allow us to avoid fixing a topology over the space of objective functions. On the other hand, we could also relax the continuity hypothesis of our coding function H. In particular, as in Balder (1999, 2002), we could assume that H is a vector valued function of Carathéodory functions.

References

[1] Aliprantis, C. and K. Border (2006): *Infinite Dimensional Analysis,* Third Edition, Springer-Verlag, Berlin, Heidelberg.

[2] Aliprantis, C. and K. Border (1994): *Infinite Dimensional Analysis,* Second Edition, Springer-Verlag, Berlin, Heidelberg.

[3] Aumann, R.J. (1965): "Integrals of set-valued functions," *Journal of Mathematical Analysis and Applications* volume 12, pages 1-12.

[4] Aumann, R.J. (1976): "An elementary proof that integration preserves uppersemicontinuity," *Journal of Mathematical Economics*, volume 3, pages 15-18.

[5] Balder, E.J. (1999): "On the existence of Cournot-Nash equilibria in continuum games," *Journal of Mathematical Economics*, volume 32, pages 207-223.

[6] Balder, E.J. (2002): "A unifying pair of Cournot-Nash equilibrium existence results," *Journal of Economic Theory*, volume 102, pages 437-470.

[7] Rath, K.P. (1992): "A direct proof of the existence of pure strategy equilibria in games with a continuum of players," *Economic Theory*, volume 2, pages 427-433.

[8] Schmeidler, D.(1973): "Equilibrium point of non-atomic games," *Journal of Statistical Physics*, volume 17, pages 295-300.