# Transaction taxes in a price maker/taker market

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# TRANSACTION TAXES IN A PRICE MAKER/TAKER MARKET

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Abstract. We develop a price maker/taker model to study how a financial transaction tax affects markets where potential traders either take a price or quote prices for the next potential trader. We find taxes widen quoted and effective spreads by many times the tax; reduce volume, gains from quoting, and gains from trade; may decrease volatility slightly without market makers; increase volatility significantly with market makers; and, do not eliminate destabilizing speculators. These effects are amplified in markets with market makers. We find revenue-optimal rates of 55–70 basis points but that the socially-optimal tax would be zero. *JEL: C72, D44, G19*

Keywords: transaction tax, Tobin tax, market microstructure, limit order model, market makers, high-frequency trading, search costs

The taxation of financial transactions is a topic of perennial interest to regulators. Supporters of such a tax claim it would deter harmful speculation and raise significant tax revenues. Opponents of a transaction tax argue it would lead to reduced liquidity and make trading too costly for some investors. We model a price maker/taker market and then study how a transaction tax affects that market. This model gives economists a new

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tool to assess these claims and consider the effects of enacting, changing, or repealing a transaction tax. The model also yields insights into the effects of transaction taxes on markets with market makers (intermediaries). Since investors and intermediaries compete for liquidity, this work is especially relevant for markets with high-frequency traders or thin liquidity.[1]

Many supporters of a tax cite Tobin's (1974) proposal of a 1% tax on foreign exchange transactions to reduce short-term speculation post-Bretton Woods.[2] He hoped to "throw sand in the wheels of the market" by discouraging noise-creating speculation. This strategy was designed to achieve a policy objective: allowing nations greater leeway in exchange rate policies and in staving off monetary crises. While supporters of a broad-based securities transaction tax include politicians such as US Representative Peter DeFazio, German Chancellor Angela Merkel, and French President François Hollande, they also include other economists such as Larry Summers and Keynes (1936).[3] A common view among supporters is that most short-term financial transactions are made by speculators causing excessive volatility in the financial markets.[4]

Opponents argue that a Tobin-like transaction tax would increase price volatility due to reduced trading volume and increased bid-ask spreads. They also claim that it would increase price volatility and the cost of capital while decreasing security values.

[1] "Thin" liquidity means that there is little depth and the size of the inside bid-ask quotes are small.
[2] This idea was reiterated and expounded upon in Tobin (1978) and Tobin (1984).
[3] A broad-based tax is one that would be levied on all financial securities such as stocks, bonds, options and futures with only government securities exempt. Keynes was an academic, but he also advised on policy.
[4] The October 1987 market crash, the 6 May 2010 "flash crash," and the recent financial crisis are often cited as examples cum outcomes of harmful speculation.

We believe that our model is an innovative tool that can inform policy makers about the ramifications of transaction taxes. The model helps explain previously-observed effects of taxes, some of which have been puzzling to economists. In particular, our approach allows us to evaluate the effects of a tax on quoted and effective spreads, volume, volatility, gains from quoting, gains from trade, and deadweight loss.

The model features a sequence of traders who strategically choose price taking versus price making, an approach that mirrors market behavior observed by Anand et al. (2005) and Hasbrouck and Saar (2009). Our model is similar to Foucault (1999) but allows for a range of private reserve valuations as well as varying proportions of pure market makers. These sources of variation enable us to study the impact of a transaction tax on market makers and investors. Since maker/taker market models pose many challenges, this is the first such model that addresses the effects of transaction taxes.

We first analyze the model theoretically which lets us prove certain properties and bounds for the model behavior. We then extend the analysis to a few specific cases. For these cases, we first consider a distribution of reservation values which, while simple, allows us to find a closed-form solution and derive comparative statics. We then repeat the analysis with the addition of market makers. Finally, we consider a more complicated distribution of reservation values which must be analyzed numerically. We believe this model more accurately mirrors current capital markets and thus is more useful for quantifying potential effects of a transaction tax on markets.

The sequential market structure is most applicable to markets which are thin, *i.e.* they have very small quantities of securities at the bid and ask, or markets where there are many substitutes (such as corporate bonds).[5] Black (1971) noted that markets in which traders compete on speed to get the best price are effectively thin. Therefore, this model also has policy implications for markets with high-frequency traders. A nice feature of our model is that we can examine search costs with no assumption about the arrival rates of a match; rather, matching happens endogenously by traders setting prices to achieve their equilibrium maximum benefit.[6]

We expected *ex-ante* that a transaction tax of $\tau$/share would increase the spread by $2\tau$ as price makers recover the tax through their quotes. We also expected this widened spread to yield (i) lower fill rates/volume, (ii) more limit orders (vs market orders), (iii) greater execution costs, and (iv) longer times for buyers and sellers to find each other (aka search costs). We were uncertain of what the effects would be on volatility and destabilizing speculation.

We find that participants whose trades are taxed widen their quotes by more than four times the nominal tax, a considerably larger effect than we had anticipated. Those traders also derive less value from providing liquidity (quoting) and are less likely to trade. Our extended model that is closest to real capital markets suggests that for small values of the tax, a 1 basis point (bp; 0.01%) tax increase decreases the volume traded by 0.5%–1%. Furthermore, that model suggests that the maximal revenue raised is about half of the naïve assumption of tax $\times$ pre-tax volume. For a 50 bp tax, the model suggests volume that falls by half; quoted spreads are about $3\times$ untaxed spreads; effective (trade-realized) spreads increase by about $3\times$ the nominal tax; the expected benefit of providing liquidity falls by more than

[5]Sequential trader models are applicable to markets with many substitutes because a potential trader does not come repeatedly to the market but only buys the cheapest of many substitutable assets.

[6]Search costs are a measure of liquidity defined by Lippman and McCall (1986) as "the time until an asset is exchanged for money."

half; gains from trade fall by half; and, search costs almost double. If half of the arriving traders are market makers, the realized volatility increases monotonically versus the tax to more than triple for a 50 bp tax. However, if none of the participants are market makers, the realized volatility may decrease slightly as taxes increase until about 15 bp — and then increase for taxes above 15 bp. For a tax of 50 bp, the realized volatility is about 1.2× the volatility without a tax. The revenue-optimal tax of 57–69 bp increases spreads by about half; reduces volume by about half; and, reduces the benefits of providing liquidity by about two-thirds.

Taxes also do not reduce the effects (externalities) of destabilizing speculators; rather, they accentuate those effects. As to the effects of intermediation in high-frequency or "thin" markets, we find that replacing half of potential investors with market makers increases the optimal spread by up to 25%. This effect occurs because investors and market makers compete for liquidity. However, market makers reduce the volatility of trade prices. In general, we find that market quality is increasingly sensitive to taxes with market makers.

## 1. Literature Review

Historically, securities transactions taxes have been proposed, enacted, modified, and even repealed in various countries.[7] An overview of the issues is given in Eichengreen et al. (1995).

Summers and Summers (1989), Stiglitz (1989), Kupiec (1995), Frankel (1996), Felix and Sau (1996), Palley (1999), and Baker (2000) have all proposed 'Tobin-like' taxes for various financial markets. ul Haq et al. (1996) and Spahn (2002) suggest a 0.1% to 0.2% tax would balance the opposing

[7]Countries which have considered or enacted financial transaction taxes include Australia, Brazil, China, France, Germany, India, Japan, Singapore, Sweden, Taiwan, and the United Kingdom.

objectives of lowering price volatility due to speculation *and* maintaining market liquidity. Pollin et al. (2003) consider a securities transaction tax for US financial markets as "one feature of a new financial architecture aimed at contributing to financial stabilization."

Studies opposed to transaction taxes include Friedman (1953), Grundfest and Shoven (1991), Campbell and Froot (1994), Kupiec (1995, 1996), Habermeier and Kirilenko (2001), and Forbes (2001). Both ul Haq et al. (1996) and Spahn (2002) point out that their proposed taxes would likely have little impact on speculative activity. (We agree based on our results for destabilizing speculators as in De Long et al. (1990).) Kupiec (1996) indicates that a tax might lower price volatility but that it would also decrease asset prices such that return volatility increases with taxes. Schwert and Seguin (1993) give a comprehensive overview of arguments both for and against a securities transaction tax.

Empirical studies disagree as well. Umlauf's (1993) study of Sweden imposing a 1% transaction tax in 1984 (and doubling it in 1986) found that 30% of equity trading volume moved to London, the market for interest rate options dried up, market volatility did not decline, and volume did not return to pre-tax levels when the tax was repealed in 1987.[8] Liu and Zhu (2009) found the October 1999 deregulation of full commissions in Japan significantly increased price volatility in the equities market. Jones and Seguin (1997), however, found that reducing commissions on the New York Stock Exchange (NYSE) and American Stock Exchange (AMEX) in 1975 was followed by reduced market volatility in the following year.

Habermeier and Kirilenko (2001) posit three reasons for this disagreement. First, securities transaction taxes are often enacted with other policy

[8]For the years 1988–1990 between 48% and 52% of trading volume in Swedish equities occurred in London. This may explain Swedish opposition to a transaction tax.

shifts confounding causal inferences about changes in market measures. Second, measuring the tax reduction of noise trading is difficult since there is no way to determine if decreased market volume is due to informed traders or noise traders. Third, if asset prices change because of the tax, there is no way to determine if this was due to anticipation of the tax, trading moving to other venues, or untaxed securities. We would add two more reasons. First, comparing empirical studies is harder because securities transaction taxes vary in size and scope — from 0.13 bp to 528 bp. Second, reducing commissions is not the same as reducing a tax since lower commissions reduce rent-extraction and expose financial firms to more competition. Thus initial conditions of profitability and competition might explain the differences between Liu and Zhu (2009) and Jones and Seguin (1997). All of these issues make excellent arguments in favor of theoretical studies.

Very few microstructure-based studies have been done. Dupont and Lee (2007) investigated a transaction tax using a Glosten and Milgrom (1985) model incorporating spread and depth. They found that higher information asymmetry made the tax more likely to decrease market liquidity. Mannaro et al. (2008) used heterogeneous agent types to study transaction taxes in simulated markets. For a single market, they found volatility increased as the number of orders decreased; for two competing markets, traders tended to avoid the taxed market — which exhibited higher volatility than the untaxed market. Cipriani and Guarino (2008) found that a tax caused a laboratory financial market (sequential trading, one market maker) to cease trading during large disparities between an asset's price and true value. However, reduced noise trading caused by the tax offset some of the induced market inefficiency. Pellizzari and Westerhoff (2009) showed that if adequate liquidity were maintained a tax could help stabilize both double-auction and dealer markets.[9] Otherwise, a tax reduced trading volume which increased volatility as each trade had greater price impact. Demary (2010) used an agent-based framework to show that tax rates above 0.1% destabilized the market and that taxes had stronger effects on more risk-averse traders.

Our model also allows inference about search costs. Typically, search models assume sequential search and bargaining and explicitly detail the search process (for example, assuming Poisson arrival rates to the market). Buyers and sellers seek to trade one unit of an asset with pairings assigned by a matching process. Paired traders bargain in an attempt to agree on a price of the asset and re-enter the market until an agreement is reached. Examples of such models include Diamond (1982), Rubenstein and Wolinsky (1985), Gale (1987a,b), Binmore and Herrero (1988), Lu and McAfee (1996), Mortensen and Wright (2002), Duffie et al. (2005), and Duffie et al. (2010). Our model is more similar to how Lo and MacKinlay (1990) looked at nonsynchronous trading.

## 2. Model of a Price Maker/Taker Market

Our limit order book model is similar to the simple price maker/taker model of Foucault (1999).[10] The economy has one risky asset with fundamental value $v$. Traders arrive sequentially, one per unit of time, and have a spectrum of idiosyncratic reserve values $v + d_t$ where $d_t \overset{iid}{\sim} F$. We assume that $F$ is a symmetric distribution with a mean of zero and finite variance: a trader with $d_t < 0$ would prefer to sell, a trader with $d_t > 0$ would prefer to buy.

[9]Maintenance of adequate liquidity for such findings to be relevant is more likely in a dealer market where a market maker is expected to maintain a two-sided market.

[10]Traders in the Foucault (1999) framework have only two possible reservation values, $v \pm L$ which occur with equal probability. We allow a range of private valuations and quotes based on these private valuations because for only two possible valuations, as in Foucault (1999), a tax would either have no effect or eliminate all trading.

We assume that traders have heterogeneous reasons for trading: alpha (real or perceived), business risks to hedge, and inventory risk to eliminate being a few examples of such reasons. We also assume traders have access to inventory or borrowed stock which allows them to sell without constraints.[11] For simplicity and tractability, quotes live for only one period; and, to prevent gaming, the market continues at each period with non-unit probability $\rho$. Traders realize the utility of their trade (whether through expected return or by benefiting from a risk-reducing hedge) immediately following trading.

2.1. **Strategic Quoting: Make versus Take.** Each trader seeks to trade one unit of the risky asset. Trading may be done by taking the prevailing bid-ask quote (*i.e.* sending a market order) or by making a new bid-ask quote (sending limit orders which replace the prevailing quote).[12] Thus each trader plays a game against the next trader. To clarify the exposition, we will call these two traders Ilsa (the time $t$ trader) and Rick (the time $t+1$ trader).[13]

Traders pay tax at both position entry and exit: if Ilsa trades with Rick, each is debited a transaction tax of $\tau$ on both entry and exit of their positions. Consequently, one could conjecture that if Ilsa quotes a bid and ask, she will shade her quotes, *i.e.* pass on some amount of the tax to Rick by quoting a bid of $v - \delta_t$ and an ask of $v + \beta_t$, where $\delta_t$ and $\beta_t$ are functions of the tax $\tau$. Ilsa solves for these equilibrium offsets to decide the optimal amount of the tax to (potentially) pass on to Rick. This strategic price shading causes $\delta_t$ and $\beta_t$ to be functions of Ilsa's reservation value

[11]Institutional traders generally have access to borrowable stock via brokerage customer's holdings as well as market making inventory.
[12]This yields phenomena seen in markets such as failure to trade when no one of the opposite preference arrives in the market. Since trading is not guaranteed, the model helps explain how equilibrium fill rates/volumes are affected by changes in market setup.
[13]While we use this language to discuss the game played by each trader, this should not be construed as implying a repeated game. The setup is a sequential trader model: as time goes by, a sequence of unique traders arrive. Thus if a trader Sam first plays the game at any time $s$, we know that Sam will never play it again.

$v+d_t$, the distribution of Rick's unknown reservation value $v+d_{t+1}$, and the transaction tax $\tau$. A diagram of Ilsa's possible gains is shown in Figure 1.

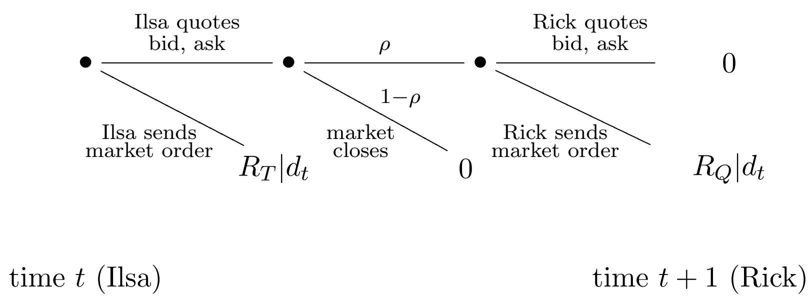



FIGURE 1. Ilsa's gains in different scenarios. Her gain for price taking is $R_T|d_t$; her expected gain for quoting prices is $R_Q|d_t$. With probability $\rho \leq 1$, the market continues.

Because the game played every time period is between the current and next-period trader, the solution takes a form which does not depend on past states. Therefore, we can assume that prices are static (*i.e.* $\Delta v = 0$) without loss of generality and solve for equilibrium $\delta$ and $\beta$ (without time subscripts). Were $\Delta v \neq 0$, we could merely shift the $d_t$ distribution (as well as $\delta$ and $\beta$) by $E(\Delta v)$. Were $\text{Var}(\Delta v) > 0$, we could scale the $d_t$ distribution to have a variance of $\text{Var}(\Delta v) + \text{Var}(d_t)$. Therefore, while we work with static prices, the results here are applicable to stochastically-evolving prices.

We determine optimal trade strategies by working forward from time $t$. The time $t$ trader, Ilsa has imperfect knowledge: She sees her reservation value $v+d_t$ but not Rick's $(v+d_{t+1})$; however, the distribution of reservation values is common knowledge. Taking the quoted bid or ask price has a known benefit of price taking $R_T|d_t$,

$$R_T|d_t = \max(\underbrace{d_t - \beta_{t-1} - 2\tau}_{\text{buy at prior ask}}, \underbrace{-d_t - \delta_{t-1} - 2\tau}_{\text{sell at prior bid}}). \tag{1}$$

If Ilsa decides to quote, she chooses not to take the current bid or ask price for benefit $R_T|d_t$; she prefers to quote optimal bid and ask prices for the time $t+1$ trader, Rick, for an expected benefit of $R_Q|d_t$. This optimal expected value defines the boundary between sending a market order and quoting and varies with the reservation value $v + d_t$. Her utility is then the greater of the known and expected benefit:

$$U_t = \max(R_T|d_t, R_Q|d_t). \tag{2}$$

Since the iid distribution of all trader's reservation values is common knowledge, Ilsa solves for *Rick's* optimal fixed-point quote revenue $R_Q^{0*}$ given the unconditional distribution of $d_{t+1}$ and maximizes $R_Q$ in equation (30).

When the next trader, Rick, enters the market at time $t+1$, he decides whether to trade against Ilsa's quote or quote a bid and ask for the following trader. Ilsa does not know Rick's reservation value $v + d_{t+1}$. She can find the unconditional expectation of his optimal quote revenue, $R_Q^{0*}$; however, defining this and proceeding further requires that we now intoduce some formalism.

2.2. **Formal Definition of the Maker/Taker Game.** To clarify our thinking and the exposition, we may define the game formally. We define the reservation price offsets, the iid $d_t$'s, on a probability space $(\Omega, \mathcal{F}, F)$. For convenience, we assume that the state space is $\Omega = \mathbb{R}$ or some finite subset of $\mathbb{R}$. The sigma field $\mathcal{F}$ may be the Borel sigma field or any $\pi - \lambda$ system defined on $\Omega$. $F$ and $f$ are the unconditional cumulative distribution function (cdf) and probability density function (pdf) for $\mathcal{F}$.

Player 1 (Ilsa) observes a signal $T_1 = d_t \in \mathcal{F}$; she does not observe the signal $T_2 = d_{t+1} \in \mathcal{F}$ received by player 2 (Rick). The common prior on $(\Omega, T_1 \times T_2)$ is the bivariate distribution $F \times F$.

The action space $A_i$ for each player is that for setting bid and ask prices $(\delta, \beta) \in A_{i\in\{1,2\}} = \mathbb{R}^2 \cup \{\varnothing, \varnothing\}$; an action of $\{\varnothing, \varnothing\}$ implies taking the existing price and not making a price for the following trader. Each player's utility is defined as $U_{i\in\{1,2\}} : A_1 \times A_2 \times \mathcal{F} \to \mathbb{R}$. Since quoted prices and taxes are bounded, we know that $U$ is bounded.

Ilsa's chooses the greater of $R_T|d_t$ and her maximal $R_Q|d_t$. (Her actions only affect $R_Q|d_t$.) Ilsa finds Rick's expected benefit of quoting $R_Q^{0*}$ as:

$$R_Q^{0*} = E(R_Q|d_t) = \int_\Omega \max_{\delta,\beta} R_Q(\delta, \beta|d_t)dF. \tag{3}$$

2.3. **Characterizing Propositions.** With the game well-defined, there are a number of properties of the game we can investigate. We characterize the game by proving a number of propositions. (Proofs are in Appendix A.)

We return to Ilsa's decision: She must take the existing quote or make a new quote for Rick. Rick is in the same position as Ilsa: he does not know the type of the following trader (Sam) and so makes his decision according to the logic in Table 1. We can also show that one and only one of the clauses in Table 1 will always be satisfied

| Condition | Action |
|---|---|
| $v - \delta - (v + d_{t+1}) - 2\tau > R_Q^{0*}$ $\implies d_{t+1} < -R_Q^{0*} - \delta - 2\tau$ | Sell at Ilsa's bid |
| $v + d_{t+1} - (v + \beta) - 2\tau > R_Q^{0*}$ $\implies d_{t+1} > R_Q^{0*} + \beta + 2\tau$ | Buy at Ilsa's ask |
| Otherwise | Quote new bid and ask for Sam |

Table 1. Rick's decision rules for taking Ilsa's quoted prices. If none of the conditions are satisfied, Rick will quote his own bid and ask instead.

**Proposition 1.** *If taxes, bid-ask spreads, and the optimal quote revenue $R_Q^{0*}$ are non-negative, then Rick (and all traders) have a clear course of action: only one of the conditions in Table 1 is satisfied.*

Ilsa's conditionally expected quote revenue, $R_Q|d_t$, is then:

$$\begin{aligned}R_Q|d_t &= \rho P(\text{Rick sells at bid})\big(\overbrace{v+d_t}^{\text{utility or sale price}} - \overbrace{(v-\delta)}^{\text{bought from Rick}} - 2\tau\big) \\ &\quad + \rho P(\text{Rick buys at ask})\big(\underbrace{v+\beta}_{\text{sold to Rick}} - \underbrace{(v+d_t)}_{\text{utility or buy price}} - 2\tau\big).\end{aligned} \tag{4}$$

$$\begin{aligned}&= \rho F(-R_Q^{0*} - \delta - 2\tau)(\delta + d_t - 2\tau) \\ &\quad + \rho\left(1 - F(R_Q^{0*} + \beta + 2\tau)\right)(\beta - d_t - 2\tau).\end{aligned} \tag{5}$$

Ilsa maximizes her quote revenue by setting the partial derivatives, $\frac{\partial R_Q|d_t}{\partial \delta}$ and $\frac{\partial R_Q|d_t}{\partial \beta}$, to 0 and solving. This implies that her optimal strategy is to bid at $v - \delta$ and ask for $v + \beta$, where the bid and ask offsets are respectively,[14]

$$\delta = \frac{F(G(\delta))}{f(G(\delta))} - d_t + 2\tau, \qquad \text{and } \beta = \frac{F(G(\beta))}{f(G(\beta))} + d_t + 2\tau, \text{ where} \tag{6}$$

$$G(x) = -R_Q^{0*} - x - 2\tau \qquad \text{(expected gain boundary)} \tag{7}$$

defines the boundary separating where Ilsa expects Rick would gain by trading with her. Specifically, if Rick's $d_{t+1} < G(\delta)$, Ilsa expects that Rick would gain by selling to her at her bid; and, if Rick's $d_{t+1} > -G(\beta)$, Ilsa expects Rick would gain by buying from her at her asking price.

From this, we can see that the bid and ask are skewed in line with Ilsa's reservation value: For a positive $d_t$, she will quote a bid and ask higher than for $d_t = 0$; and, for a negative $d_t$ she will quote a bid and ask lower than for $d_t = 0$.

**Proposition 2** (Spread Skewed with $d_t$)**.** *If $d_t > 0$, then the optimal bid and ask parameters are such that $\beta > \delta$. Similarly, if $d_t < 0$, then the optimal*

[14]Note that we have used the symmetry of $F$ to express the probability of Rick trading at Ilsa's asking price in equation (5) via $F$ instead of $1 - F$.

*bid and ask parameters are such that* $\beta < \delta$*. If* $d_t = 0$*, the optimal bid and ask parameters are such that* $\beta = \delta$*.*

We can also put a lower bound on the bid-ask spread. The lower bound shows that in all situations, the spread is more than four times the nominal tax rate.

**Proposition 3** (Spread Exceeds 4× Nominal Tax)**.** *For any distribution* $F$ *of reservation values, the bid-ask spread* $\delta+\beta$ *is greater than* $4\tau$*:* i.e. $\delta+\beta > 4\tau$*. Put differently: the spread is more than four times the quoted tax rate, twice the taxes collected from any trader or at any time, and more than the total round-trip taxes for all involved traders.*

**Proposition 4** (Spread Upper Bound)**.** *For any distribution of reservation values symmetric about 0 with finite variance* $L^2$*, the bid-ask spread* $\delta + \beta$ *may be bounded above by* $\delta + \beta \leq \frac{L^2}{2G(\delta)^2 f(G(\delta))} + \frac{L^2}{2G(\beta)^2 f(G(\beta))} + 4\tau$*.*

With the lower bound, we can also show that the benefit of quoting a bid and ask for all players is always positive.

**Proposition 5** (Positive Quoting Benefit)**.** *For any player (with type* $d_t$*), the benefit of quoting is always positive:* i.e. $R_Q|d_t > 0$

Finally, with some very general assumptions, we may show the existence of a Markov Perfect equilibrium for the game between the current and next-period trader.

**Proposition 6** (Existence of Markov Perfect Equilibrium)**.** *Assume the preceding setup with bounded quotes and taxes. If the pdf of* $d_t$*'s,* $f$*, is such that* $d_t$ *has finite variance, the game played between Ilsa and Rick (or any two subsequent traders) has a Markov Perfect equilibrium.*

Showing that the Markov Perfect equilibrium is unique requires slightly more restrictive assumptions.

**Proposition 7** (Uniqueness of Markov Perfect Equilibrium: Continuous CDF)**.** *Assume the preceding setup with bounded quotes and taxes and that the cdf of $d_t$'s is continuous. Then the Markov Perfect equilibrium of the game played between Ilsa and Rick (or any two subsequent traders) is unique.*

## 3. Closed-Form Analysis

To confirm that the model behaves nicely, we examine model behavior for two simple distributions of reservation values. This enables us to find a closed-form solution and derive comparative statics.

The simplest distribution is that of the uniform on $[-L, L]$; using this allows us to gain intuition into the model's behavior. However, we are also interested in considering the effect of a tax on market makers. Therefore, the second distribution we consider is a mixture of the uniform with a point mass at $d_t = 0$ to represent the indifference of market makers between buying and selling. Examining this simple case lets us see how market maker behavior and participation is affected.

### 3.1. **Uniform Investors.**

We first consider a market with only investors, *i.e.* being indifferent between buying and selling is a measure-zero event. The reservation value offsets are uniformly-distributed: $d_t \overset{iid}{\sim} \text{Unif}(-L, L)$ or, equivalently,

$$f(x) = \begin{cases} \frac{1}{2L} & x \in [-L, L] \\ 0 & \text{else} \end{cases} \qquad F(x) = \begin{cases} 0 & x < -L \\ \frac{x+L}{2L} & x \in [-L, L) \\ 1 & x \geq L \end{cases} \tag{8}$$

From equations (6) and (8), we can solve for the bid and ask offsets explicitly:

$$\delta = \frac{1}{2}(L - R_Q^{0*} - d_t), \qquad \beta = \frac{1}{2}(L - R_Q^{0*} + d_t). \tag{9}$$

Differentiating with respect to tax $\tau$ or continuation probability $\rho$, we get

$$\frac{\partial \delta}{\partial \tau} = \frac{\partial \beta}{\partial \tau} = -\frac{1}{2}\frac{\partial R_Q^{0*}}{\partial \tau}, \qquad \frac{\partial \delta}{\partial \rho} = \frac{\partial \beta}{\partial \rho} = -\frac{1}{2}\frac{\partial R_Q^{0*}}{\partial \rho}. \tag{10}$$

Working with the constraints for an interior solution, we explicitly solve for the expected quoting benefit, $R_Q^{0*}$. (Details are in Appendix B.)

$$R_Q^{0*} = \frac{\rho}{24L^2}(2L - R_Q^{0*} - 4\tau)^3. \tag{11}$$

Differentiating with respect to tax $\tau$ or continuation probability $\rho$, we get

$$\frac{\partial R_Q^{0*}}{\partial \tau} = -\frac{\rho(2L - R_Q^{0*} - 4\tau)^2}{2L^2 + \frac{\rho}{4}(2L - R_Q^{0*} - 4\tau)^2} < 0, \tag{12}$$

$$\frac{\partial R_Q^{0*}}{\partial \rho} = \frac{(2L - R_Q^{0*} - 4\tau)^3}{24L^2 + 3\rho(2L - R_Q^{0*} - 4\tau)^2} > 0. \tag{13}$$

This implies that

$$\frac{\partial \delta}{\partial \tau} > 0, \qquad \frac{\partial \beta}{\partial \tau} > 0, \qquad \frac{\partial \delta}{\partial \rho} < 0, \qquad \frac{\partial \beta}{\partial \rho} < 0. \tag{14}$$

The comparative statics therefore suggest that in equilibrium:

(1) increasing taxes reduces the value of quoting (providing liquidity);
(2) increasing the likelihood the market stays open increases the value of quoting;
(3) increasing taxes increases bid-ask spreads; and,
(4) increasing the likelihood the market stays open decreases spreads.

3.2. **Uniform Investors with Market Makers.** We next consider the addition of market makers to the market by mixing the prior distribution of total mass $1-\mu$ (representing investors) with a point mass of $0<\mu<1$ at $d_t=0$ (representing the preferences of market makers).[15] This means that being indifferent between buying and selling is no longer a measure-zero event and that the $d_t$ CDF is no longer continuous. Therefore, we should expect the results to differ (and the work to be more involved). Supporting lemmas and proofs are in Appendix C.

If we keep the definition of $f$ and $F$ from equation (8), we have that:

$$d_t \overset{iid}{\sim} \begin{cases} 0 & w.p.\ \mu \\ \text{Unif}(-L,L) & w.p.\ 1-\mu. \end{cases} \tag{15}$$

We then define the quote revenue $R_Q|d_t$ as

$$R_Q|d_t = \rho V_1(\delta) + \rho V_2(\beta) \quad \text{where} \tag{16}$$

$$V_1(\delta) = [(1-\mu)F(G(\delta)) + \mu\mathbb{I}(G(\delta))](\delta + d_t - 2\tau) \quad \text{and} \tag{17}$$

$$V_2(\beta) = [(1-\mu)F(G(\beta)) + \mu\mathbb{I}(G(\beta))](\beta - d_t - 2\tau) \tag{18}$$

where $V_1(\delta)$ is the expected benefit of quoting a bid $v-\delta$ and $V_2(\beta)$ is the expected benefit of quoting an offer $v+\beta$.

Using these definitions, we can show that $\max_\delta V_1(\delta) \geq 0$ and $\max_\beta V_2(\beta) \geq 0$ with strict positivity holding iff $d_t \in (R_Q^{0*} + 4\tau - L, L]$ and $d_t \in [-L, L - R_Q^{0*} - 4\tau)$, respectively.

Now we note that

$$\lim_{\delta\to\infty} V_1(\delta) = 0, \qquad \lim_{\delta\to-\infty} V_1(\delta) = -\infty, \tag{19}$$

$$\lim_{\beta\to\infty} V_2(\beta) = 0, \qquad \lim_{\beta\to-\infty} V_2(\beta) = -\infty, \tag{20}$$

and that $V_1(\delta)$ and $V_2(\beta)$ are discontinuous at:

[15] Assigning $d_t = 0$ makes market makers indifferent between buying and selling; the effect of inventory risk is ignored. However, market makers with inventory risk can be thought of as belonging to the population of traders with non-zero $d_t$'s.

$$\delta_0 = -R_Q^{0*} - 2\tau, \quad \text{and} \qquad \beta_0 = -R_Q^{0*} - 2\tau. \tag{21}$$

Assume that $2\tau < L$ and $R_Q^{0*} < 2L - 4\tau$. We then have

$$\frac{\partial V_1(\delta)}{\partial \delta} = \begin{cases} (1-\mu) + \mu = 1 & G(\delta) > L \\ \frac{1+\mu}{2} - \frac{(1-\mu)(R_Q^{0*}+2\delta+d_t)}{2L} & 0 \le G(\delta) \le L \\ \frac{1-\mu}{2} - \frac{(1-\mu)(R_Q^{0*}+2\delta+d_t)}{2L} & -L \le G(\delta) \le 0 \\ 0 & G(\delta) < -L, \end{cases} \tag{22}$$

$$\frac{\partial V_2(\beta)}{\partial \beta} = \begin{cases} (1-\mu) + \mu = 1 & G(\beta) > L \\ \frac{1+\mu}{2} - \frac{(1-\mu)(R_Q^{0*}+2\beta+d_t)}{2L} & 0 \le G(\beta) \le L \\ \frac{1-\mu}{2} - \frac{(1-\mu)(R_Q^{0*}+2\beta+d_t)}{2L} & -L \le G(\beta) \le 0 \\ 0 & G(\beta) < -L. \end{cases} \tag{23}$$

We may then easily verify that the second derivatives are non-positive for all $\delta$ and $\beta$, $R_Q^{0*} > 0$, and $\tau \ge 0$. This implies we may find local maximizers $\delta_M$ and $\beta_M$ of $V_1(\delta)$ and $V_2(\beta)$.

We then have $V_1(\delta_M)$ defined on $R_Q^{0*} + 4\tau - L < d_t \le L$:

$$V_1(\delta_M) = (1-\mu)F(G(\delta_M))(\delta_M + d_t - 2\tau) \tag{24}$$

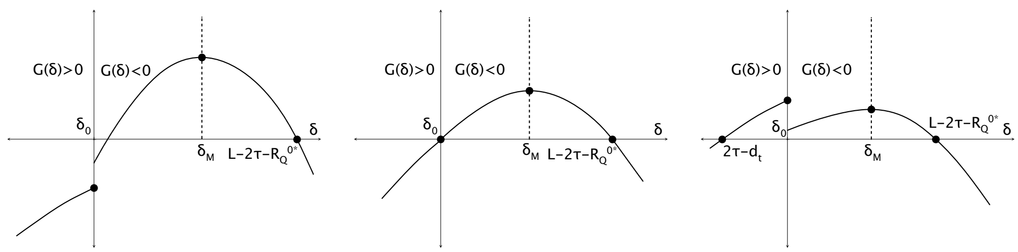
$$= \frac{1-\mu}{8L}(L - R_Q^{0*} + d_t - 4\tau)^2 > 0, \tag{25}$$

and $V_2(\beta_M)$ defined on $-L \le d_t < L - R_Q^{0*} - 4\tau$:

$$V_2(\beta_M) = (1-\mu)F(G(\beta_M))(\beta_M - d_t - 2\tau) \tag{26}$$

$$= \frac{1-\mu}{8L}(L - R_Q^{0*} - d_t - 4\tau)^2 > 0. \tag{27}$$

It follows that $\max_\delta V_1(\delta) = \max\{V_1(\delta_0), V_1(\delta_M)\}$ and $\max_\beta V_2(\beta) = \max\{V_2(\beta_0), V_2(\beta_M)\}$. Moreover, if $\mu \to 1$, $V_1(\delta_0) > V_1(\delta_M)$ and $V_2(\beta_0) > V_2(\beta_M)$. The various cases for the functional behavior of $V_1(\delta)$ are shown in Figure 2 to illustrate the above. The behavior for $V_2(\beta)$ is similar (albeit for $-d_t$).




$\{d_t < 4\tau + R_Q^{0*}\} \cup \{R_Q^{0*} + 4\tau - L < d_t\}$ $\qquad d_t = 4\tau + R_Q^{0*}$ $\qquad d_t > 4\tau + R_Q^{0*}$

FIGURE 2. Three possible cases for $V_1(\delta)$, the expected benefit of quoting a bid of $v - \delta$ vs $\delta$: $V_1(\delta)$ jumps up at $\delta_0$, $V_1(\delta)$ is continuous, or $V_1(\delta)$ jumps down at $\delta_0$. The cases are partitioned depending on the reservation price $v + d_t$ of the liquidity provider. The dot at $\delta_0$ denotes that $V_1(\delta)$ is left-continuous.

We also note that if the tax is excessively high, the value of quoting at the gain boundary ($G(\delta_0) = 0$) will be negative. In other words, a quote Ilsa expects Rick to be indifferent to taking does not have expected benefit to her. Specifically, if $4\tau \geq L$ then $V_1(\delta_0) \leq 0$ and $V_2(\beta_0) \leq 0$. If we instead assume $4\tau < L$ and $R_Q^{0*} < L - 4\tau$, we get that Ilsa expects positive benefit to quoting a bid and ask at the gain boundary for Rick. This is also the bid and ask most attractive to market makers; however, "most attractive" does not mean more attractive than quoting.

For Ilsa to expect a market-making Rick to trade against Ilsa's quote, we require more restrictive conditions: the expected value of quoting and the tax must be sufficiently small; and, Ilsa's private reservation price must be sufficiently extreme. In particular, Lemma 7 in Appendix C shows that a market-making Rick will trade against Ilsa's quote if

(1) $0 \le \tau < \frac{\mu L}{\sqrt{2\mu+2\mu^2}+2\mu}$, (tax sufficiently small)
(2) $R_Q^{0*} \le \frac{4\mu L}{\sqrt{2\mu+2\mu^2}+2\mu} - 4\tau$, and (quote benefit sufficiently small)
(3) $|d_t| > x_1 + R_Q^{0*} + 4\tau$ (Ilsa's view sufficiently extreme)

where $x_1 = \frac{1+3\mu}{1-\mu}L - \frac{2\sqrt{2\mu+2\mu^2}}{1-\mu}L$.

We can also show that the comparative statics for this market are the same as for the uniform case without market makers.

**Proposition 8.** *Assume $2\tau < L$. There exists a unique $R_Q^{0*} \in (0, 2L - 4\tau)$ such that $E(R_Q|d_t) = R_Q^{0*}$, $\frac{\partial R_Q^{0*}}{\partial \tau} < 0$, and $\frac{\partial R_Q^{0*}}{\partial \rho} > 0$.*

In other words: taxes reduce the expected benefit of providing liquidity and a market more likely to continue trading increases the expected benefit of providing liquidity.

The comparative statics for the proportion of market makers are a bit more involved. We first restate the conditions for finding the sign of $\frac{\partial R_Q^{0*}}{\partial \mu}$:

**Proposition 9.** *Suppose $R_Q^{0*}$ is the solution to $E(R_Q|d_t) = R_Q^{0*}$. Then the sign of $\frac{\partial R_Q^{0*}}{\partial \mu}$ is the same as*

$$Z(\mu) = 3\left(1 - \frac{R_Q^{0*} + 4\tau}{L}\right)^2 + 1 - \alpha^2(\mu)\left(\frac{\sqrt{2\mu+2\mu^2}}{\mu(\sqrt{2\mu+2\mu^2}+2\mu)} + 1\right) \tag{28}$$

*where $\alpha(\mu) = \frac{4\mu}{\sqrt{2\mu+2\mu^2}+2\mu}$. Furthermore, this sign depends on $\mu$ and $\rho$.*

This suggest that the comparative statics need some qualification.

**Corollary 1.** *If $(1 - \frac{1}{\sqrt{6}})L < R_Q^{0*} + 4\tau < \alpha(\mu)L$, then $\frac{\partial R_Q^{0*}}{\partial \mu} < 0$.*

**Corollary 2.** *If $0 < R_Q^{0*} + 4\tau < (1 - \frac{1}{\sqrt{6}})L$, then $\exists \bar{\mu} > \mu_0$ such that when $\mu \ge \bar{\mu}$, $\frac{\partial R_Q^{0*}}{\partial \mu} > 0$.*

In other words: For high values of providing liquidity, increasing the proportion of market makers decreases the value of providing liquidity. However, for low values of providing liquidity, increasing the proportion of market makers increases the value of providing liquidity. These results suggest a future extension of this model: exploring the equilibrium proportion of market makers.

## 4. Numerical Analysis

To study how a transaction tax affects market quality for a setup closer to real capital markets, we analyze traders in a market with private reservation values distributed according to a mixture of: (i) a point mass of $0 < \mu < 1$ at $d_t = 0$ (representing market makers), and (ii) a normal distribution with mean zero, variance $L^2$, and total mass $1 - \mu$ (representing investors):

$$d_t \overset{iid}{\sim} f = \begin{cases} 0 & w.p.\ \mu \\ N(0, L^2) & w.p.\ 1 - \mu. \end{cases} \tag{29}$$

We examine the model for tax $\tau$ ranging from 0 to 50 bp. We also study the effect of taxes for different proportions $\mu$ of arriving traders being market makers. The mean reservation value, $v = \$20$, is calibrated to be close to the average stock price in the US equity market. The reservation value standard deviation of $L = \$0.50$ (2.5% of $v$) is calibrated to be the same as the daily volatility of a stock with a 40% annualized volatility; and, the market continuation probability is set to be $\rho = 0.9$.[16] We look at both the average effect across the market and the effect for market makers.

[16]Results are robust to changes in $\rho$ since $\rho$ falls out of the optimal bid and ask formula.

4.1. **Existence and Uniqueness of Equilibrium.** We can rewrite Ilsa's conditional expected quote revenue, $R_Q|d_t$, as:

$$\begin{aligned} R_Q|d_t = {} & \rho(1-\mu)\Phi\left(\frac{-R_Q^{0*}-\delta-2\tau}{L}\right)(\delta+d_t-2\tau) \\ & + \rho(1-\mu)\Phi\left(\frac{-R_Q^{0*}-\beta-2\tau}{L}\right)(\beta-d_t-2\tau) \\ & + \rho\mu\mathbb{I}(-R_Q^{0*}-\delta-2\tau\geq 0)(\delta+d_t-2\tau) \\ & + \rho\mu\mathbb{I}(-R_Q^{0*}-\beta-2\tau\geq 0)(\beta-d_t-2\tau) \end{aligned} \tag{30}$$

where $\Phi$ is the standard normal cdf. We then show that this distribution also admits a unique Markov Perfect equilibrium. (All supporting proofs are in Appendix D.)

**Proposition 10** (Uniqueness of Markov Perfect Equilibrium: Normal with Market Makers)**.** *Under the preceding setup and for bounded quotes and taxes, the game played between Ilsa and Rick (or any two subsequent traders) has a unique Markov Perfect equilibrium.*

Finally, these more specific assumptions about $d_t$ allow us to find a stronger upper bound for the bid-ask spread:

**Proposition 11** (Spread Upper Bound: Normal with Market Makers)**.** *If the distribution $F$ of reservation values is $N(0, L^2)$, the bid-ask spread $\delta+\beta$ is bounded above by: $\delta+\beta \leq \frac{L}{R_Q^{0*}+4\tau}+4\tau$.*

4.2. **Solving for Equilibrium.** Since the iid distribution of all trader's reservation values is common knowledge, Ilsa solves for *Rick's* optimal fixed-point quote revenue $R_Q^{0*}$ given the unconditional distribution of $d_{t+1}$ and maximizes $R_Q$ in equation (30).

In practical terms, solving for $R_Q^{0*}$ requires iterating over a sufficiently large range of values for $d_{t+1}$; computing each optimal $R_Q|d_{t+1}$; and then determining the expected $R_Q^{0*}$. We must make sure not to double-count the center of the distribution; and, we must handle the tails appropriately. We then repeat this until $R_Q^{0*}$ converges.

The one complication of our distribution of potential traders is the point mass of market makers at $d_t = 0$. Because of this, $R_Q$ contains two indicator functions to handle the possibility that Rick is a market maker: one for the possibility that Ilsa's bid appeals to a market maker and one for the possibility that her ask appeals to a market maker.[17] We then find the contrained maxima over three regions: where neither indicator function is active, where the first indicator function is active, and where the second indicator function is active. (We know both indicator functions are never simultaneously active by Proposition 1.)

4.3. **Metrics of Market Quality.** Figure 3 shows that the dynamics for a market with static $v$ and no taxes appear similar to that seen in real data. We can also note that having more market makers tends to stabilize the range in which a security trades.

While these plots may be reassuring that nothing is drastically wrong, visual examination is not a proper assessment of changes in market quality. We therefore examine the following metrics to assess the effects of the tax:

**Optimal Spread:** Optimal spread which may be quoted (or not).

**Quoted Spread:** Average spread that is quoted.

**Effective Spread:** Average difference between buy, sell prices.

**Fill Rate:** Fraction of orders which trade; analogous to volume.

**Search Cost:** Expected time until an order is filled.

**Expected Quote Revenue:** $R_Q^{0*}$; expected reward for quoting.

**Realized Volatility:** Volatility of trade price changes.

[17] Market makers have the same reservation value and so would not trade with one another.

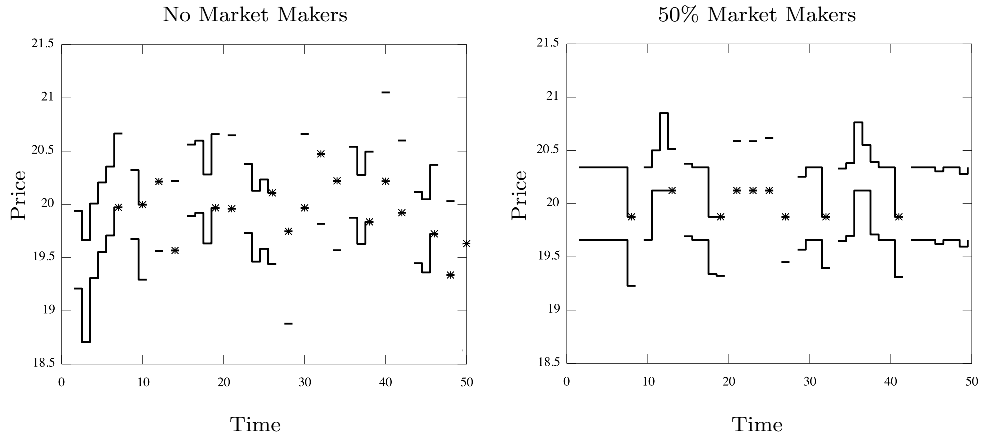



Figure 3. Simulated trades (stars) and quotes (lines) for a market with no transaction tax. The left plot is for a market with no market makers; the right plot is for a market with 50% of potential traders being market makers. The market with market makers trades in a tighter range: All trades occur between \$19.75 and \$20.25; trades occur outside this range for the market without market makers (left).

**Gains from Trade:** Equilibrium expected gains of make/take choice.

**Deadweight Loss:** Gains from trade + tax revenue vs without tax.

**Revenue-optimal Tax Rate:** Tax rate which maximizes tax revenue.

Not all of the comparative statics we care about are easily found by solving the game played by Ilsa and Rick. We find the quoted spread, effective spread, realized volatility, gains from trade, and deadweight loss via simulation.[18] For each level of tax explored, we generate a random sequence of 100,000 traders with iid $N(v, L)$ reservation value offsets. For each trader, we record their trading decision (make or take prices). If a trade occurs, we record the direction (buy/sell), the price, and the expected gain from trade.

4.4. **Optimal and Quoted Spread.** The left plot in Figure 4 shows that the optimal spread one would consider quoting increases with the tax: by

[18]While we could try to derive the volatility from the quoted spread using Roll (1984), that would ignore the endogeneity of when trade occurs and thus would be inaccurate.

270% without market makers, from 65 bp with no tax to 240 bp with a 50 bp ($0.10) tax; and, by 170% with half of potential traders being market makers, from 90 bp with no tax to 240 bp with a 50 bp tax. At $\tau = 50$ bp, the change in the optimal spread is 3.5× and 3× the change in the tax, for markets without market makers and with 50% market makers. In other words: traders pay 200%–250% more than the tax.

The right plot in Figure 4 shows the suoted spread — *i.e.* the optimal spread filtered by when traders choose to quote it instead of taking the prior quote. Regardless of the prevalence of market makers, adding a 50 bp tax approximately triples the quoted spread. The quoted spread without a tax is slightly higher for increasing proportions of market makers. This is because an increasing prevalence of market makers makes quoting less appealing for investors and often gives them better prices to take.

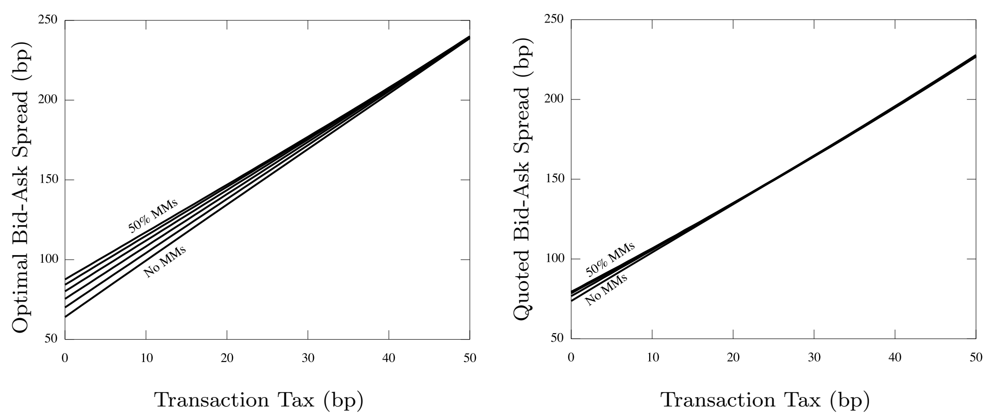



FIGURE 4. Optimal bid-ask spread (bp, left) and quoted bid-ask spread (bp, right) vs transaction tax rates (bp). Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). The marginal spread is always greater than the marginal tax, and investors always pay more than four times the nominal tax.

4.5. **Effective Spread.** Figure 5 shows that regardless of the prevalence of market makers, adding a 50 bp tax approximately triples the effective spread. (The effective spread without a tax is slightly lower for increasing proportions of market makers.) This differs from the results for the optimal spread in that the optimal spread increased with higher proportions of market makers. This difference is because an increasing prevalence of market makers makes quoting less appealing for investors and often gives them better prices to take.

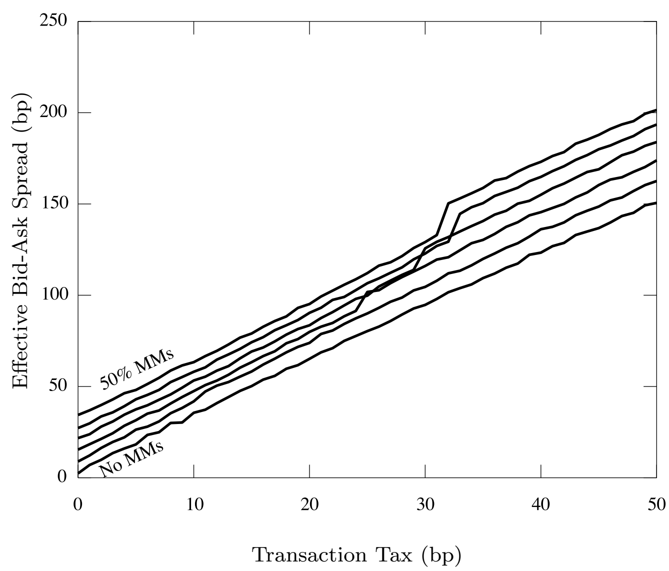



FIGURE 5. Effective spreads (spreads at trade times) vs transaction tax rate (bp). Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). More market makers reduce effective spreads slightly and higher taxes increase effective spreads.

4.6. **Fill Rate and Search Cost.** Figure 6 shows two plots: the left plot shows the fill rate and the right plot shows the fill rate adjusted for the fact that market makers do not trade with one another. In both plots we see a 50% drop in the fill rate (volume) at a tax of 50 bp regardless of the prevalence of market makers. The lower fill rates for increasing proportions of market makers is largely due to the fact that market makers do not trade with one another in this model. For example, were we to ignore the times when one market maker follows another, the untaxed fill rate for half the arriving traders being market makers would be 72% — much closer to the 75% fill rate without market makers.[19]

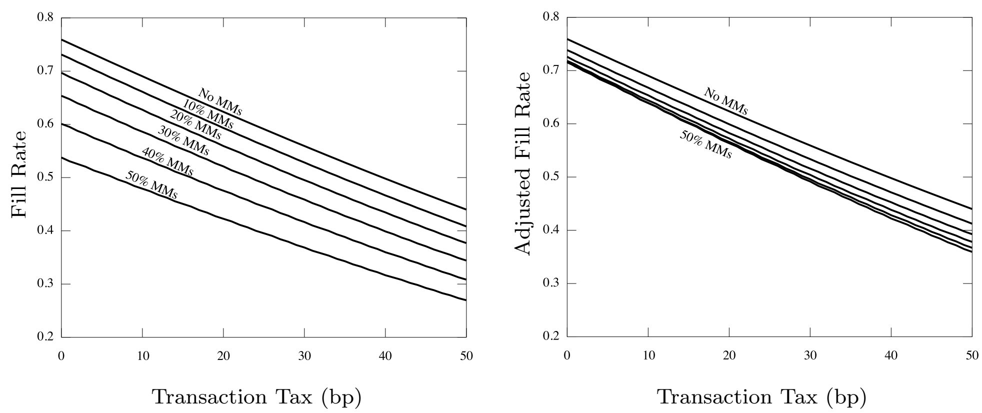



FIGURE 6. Fill rate (left) and adjusted fill rate (right) vs transaction tax rates (bp). The adjusted fill rate corrects for the fact that market makers do not trade with one another. Curves are shown for 10% increments of market makers, from none to half of arriving traders being market makers. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). The fill rate drops by about half for a tax of 50 bp.

Harris (2002) says, "Trading is a search problem. [. . . ] Sellers seek buyers willing to pay high prices. Buyers seek sellers willing to sell at low prices." While search models often take the form of sequential search and bargain models, our model can yield insight into search costs if we focus only on how long it takes *in toto* until a trade occurs. Since Lippman and McCall

[19]The 72% figure is found by dividing the untaxed fill rate of 54% by the probability of not having two market makers in a row (0.75).

(1986) view liquidity as "the time until an asset is exchanged for money", we study how a transaction tax affects the average time between trades. If the probability of a fill is $P_f$ (*i.e.* the fill rate), we can infer relative search costs (waiting times), $t_w$, by inverting the fill rate: $t_w \propto {P_f}^{-1}$. Search costs are shown in Figure 7.

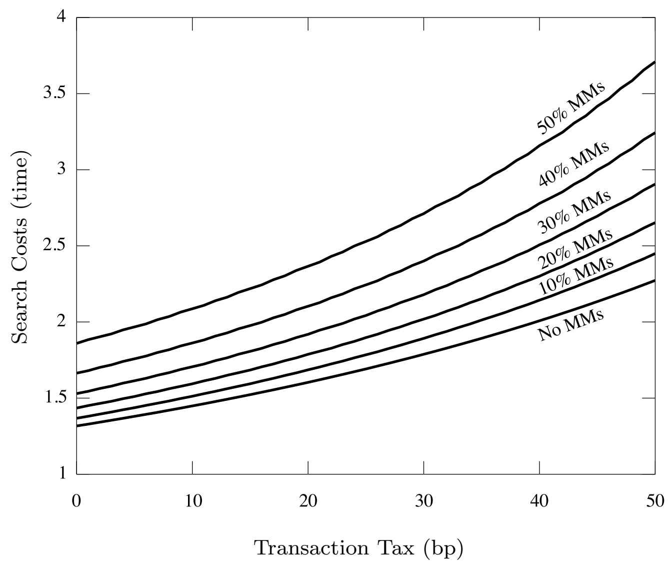



FIGURE 7. Search costs (expected waiting times to trade) vs transaction tax rate (bp). Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors).

As taxes increase, the waiting time between trades increases. The effect of a transaction tax on search costs is the inverse of the results for fill rates: a 50 bp tax roughly doubles search costs.

4.7. **Expected Quote Revenue.** The average traders' expected quote revenue/share without market makers decreases from \$0.064 (no tax) to \$0.032 (50 bp tax) — a 50% decrease. Without a tax, the expected quote revenue increases with the proportion of market makers up to about 40% of arriving traders being market makers. For 50% of arriving traders being market makers, the expected quote revenue falls slightly. In all of these cases, however, the expected quote revenue drops faster as we increase the proportion of market makers. Thus when half the arriving traders are market makers, the effects of a tax are much greater: the expected quote revenue/share decreases from \$0.074 to \$0.027 — a 64% decrease. Thus the expected benefit of providing liquidity falls by about half to two-thirds for a 50 bp tax. To the extent that market makers have costs of doing business comparable to expected quote revenues, we could expect market makers to exit the market more than this model suggests.[20]

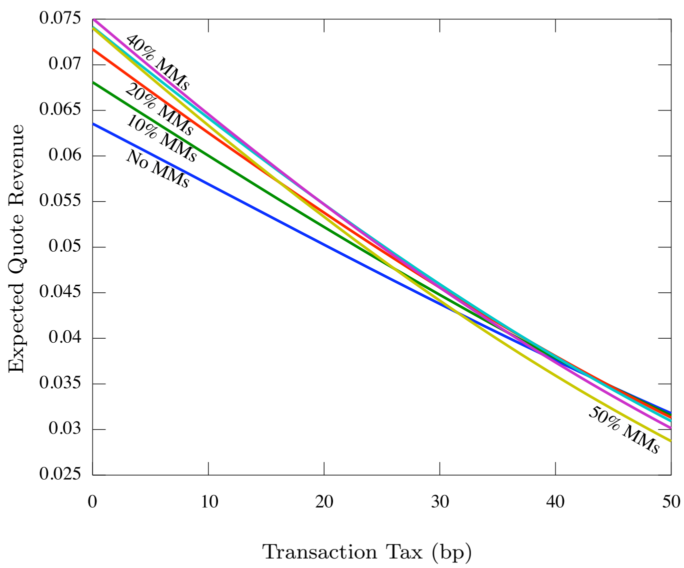



FIGURE 8. Expected quote revenues vs transaction tax (bp). Curves show 10% increments of market makers, from none to half of potential traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). Without market makers, the expected quote revenues drop by about half for a 50 bp tax; with half the arriving traders being market makers, the expected quote revenues drop by about two-thirds for a 50 bp tax. This model does not include costs which might imply even greater reductions in expected revenues.

[20]While this model does not allow market makers to exit the market, they can publish quotes which greatly reduce their probability of trading.

4.8. **Realized Volatility.** The plot of realized volatility in Figure 9 shows that the effects of a transaction tax are not always obvious.

For markets where 30% or more of the arriving traders are market makers, a transaction tax increases the realized volatility monotonically. Furthermore, markets with more market makers have lower realized volatility for taxes up to 50 bp and are more sensitive to a transactions tax. For example, a market where half of the arriving traders are market makers has an untaxed realized volatility of about \$0.11 and a realized volatility of about \$0.22 for a tax of 50 bp — a doubling of volatility.

However, for markets where 20% or less of the arriving traders are market makers, a transaction tax initially lowers volatility. The decrease is small, but the most extreme decrease is for market without market makers. Without market makers, the volatility decreases from \$0.193 without a tax to \$0.186 for a 12 bp tax (a 4% decrease). The initial decrease in volatility is because an increase in taxes lowers fill rates which makes taking a worse price (*i.e.* a price closer to $v$) more likely. However, this is a second-order effect which is dominated by the increase in spreads as taxes increase.

4.9. **Gains from Trade.** Since market makers are intermediaries, they are unlikely to take a directional bet. Therefore we might expect that they will trade closer to the mean reservation value and thus for lower expected revenue than most other traders. This is indeed the case: increasing the fraction of market makers lowers the average gains from trade. (One could also view this as market makers reducing the dispersion in beliefs about prices.)

While market makers intermediate for lower returns, they also (in our model) lower volume because they compete for liquidity. Therefore we might expect market makers to lower total gains from trade. Figure 10 shows that

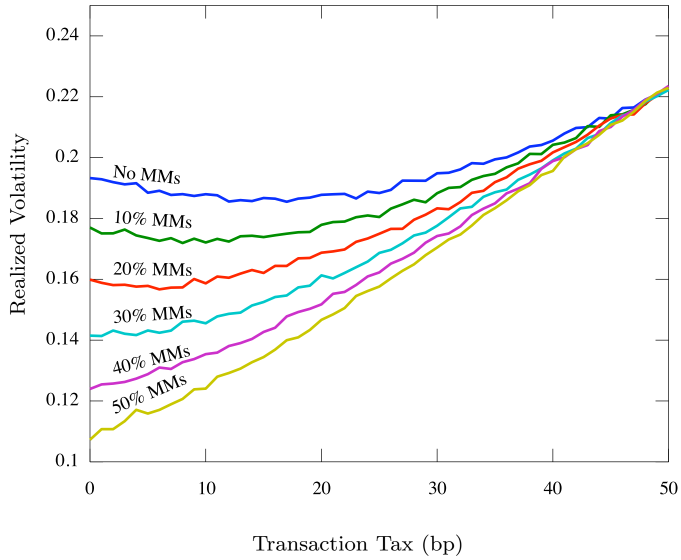



FIGURE 9. Volatility (of trade prices) vs transaction tax rate (bp). Curves show 10% increments of market makers, from none to half of potential traders. Market makers have a reservation value offset of 0 while investors have an offset with volatility of 2.5%. More market makers leads to lower volatility yet makes market volatility more sensitive to taxes: a 50 bp tax doubles volatility. Taxes up to about 15 bp decrease volatility for markets without market makers but increase volatility by about one-fifth for a 50 bp tax.

to be true; total gains from trade are higher without market makers. For all markets, regardless of the proportion of arriving traders who are market makers, the total gains from trade drop about 60% for a 50 bp tax. Were we to correct the gains for trade by only examining investors, we would see that the gains for trade for investors are higher with market makers.

4.10. **Deadweight Loss.** Since we are considering gains from trade and tax revenue, it makes sense to think about the costs and benefits of the tax. We therefore look at the deadweight loss: the reduction in gains from trade less tax revenues.[21] If there were an externality that we could price, we

[21]Thanks to Martin Šuster for this idea.

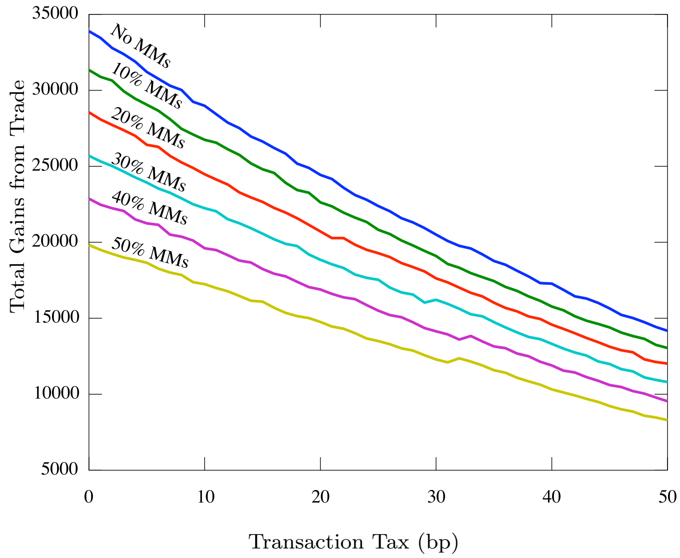



FIGURE 10. Total gains from trade vs transaction tax rate (bp) for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). A 50 bp tax decreases the total gains from trade by about 60%. Market makers lower total gains from trade since they trade for small margins.

could also consider that.[22] Figure 11 shows the deadweight loss per order. We look at the loss per order because considering the loss per trade would ignore the damaging effects of reduced volume. The plot shows that for no positive tax level is the deadweight loss negative — implying that the socially optimal tax is none.

4.11. **Revenue-Optimal Tax Rates.** Since one argument for transaction taxes is their revenue-generating capabilities, we consider the maximal government revenue possible in our model. The Laffer curves in Figure 12 show that the maximum government revenue of 14–24 bp per order would

[22]Volatility could be one such externality; however, the prior analysis of volatility suggests that the tax does little to mitigate volatility.

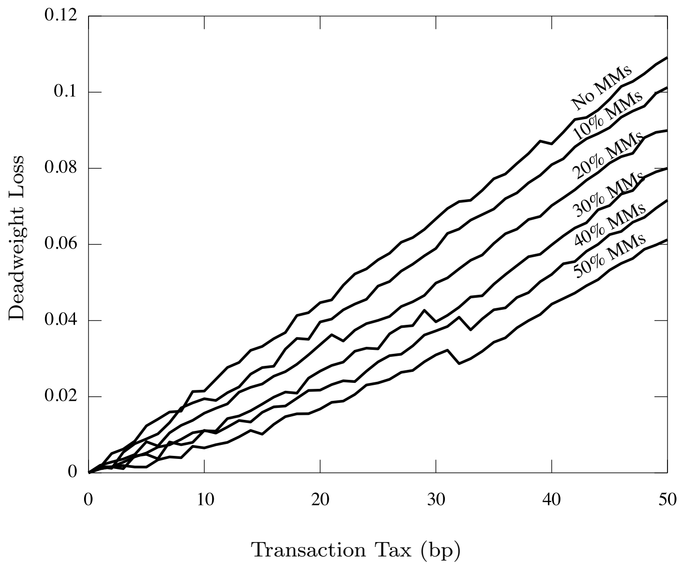



FIGURE 11. Deadweight loss per order vs transaction tax rate (bp) for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). At no tax level is the deadweight loss negative, suggesting that the socially optimal tax is 0 bp.

be generated at tax rates of 57–69 bp per trade with lower taxes being revenue-optimal for greater proportions of market makers.

At the revenue-optimal tax rates, however, other measures reveal a great decrease in market quality. Table 2 shows the various measures of market quality for the different revenue-optimal tax rates. In general, the revenue-optimal tax rates would increase effective spreads (direct trading costs) by 200%–300%, decrease fill rates (volume) by more than half, reduce the reward for providing liquidity by two-thirds, and increase volatility by between one-third and more than double. All of these results suggest conditions that would likely induce trading to move elsewhere as documented in empirical studies. Finally, the revenue raised falls by over 40% between markets with no intermediation and those with half of arriving traders being market

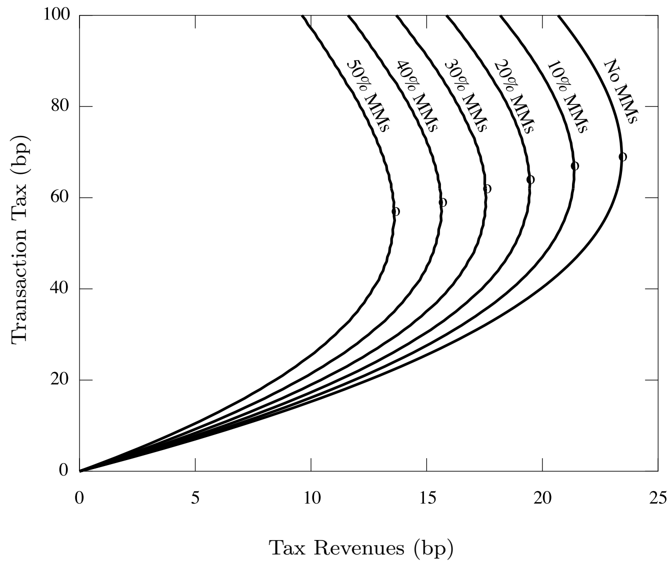



FIGURE 12. Laffer curve: Transaction tax rates vs equilibrium tax revenues (bp) with revenue-optimal rates indicated by a circle. Curves show 10% increments of market makers, from none to half of potential traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). The revenue-optimal tax ranges from 57 bp to 69 bp. Tax revenue falls by about 40% for half of arriving traders being market makers; more developed markets (more intermediation) yield less expected revenue per order.

makers. This suggests that more developed markets are more sensitive to taxation and may yield lower expected revenue per order.

4.12. **Positive-Feedback Traders.** One idea mentioned earlier was that a transaction tax might reduce distortions created by "destabilizing speculators." While there could be many formulations for such an idea, we investigate the positive-feedback traders of De Long et al. (1990). In that work, destabilizing speculators impart "positive feedback" to the market; in other words, they mimic the trade that preceded them. This pushes markets further from equilibrium and should yield a loss in allocative efficiency.

| % Market Makers | 0% | 10% | 20% | 30% | 40% | 50% |
|---|---|---|---|---|---|---|
| Rev.-optimal Tax (bp) | 69 | 67 | 64 | 62 | 59 | 57 |
| Revenue/order (bp) | 23 | 21 | 19 | 18 | 16 | 14 |
| Optimal Spread (bp) | 305 | 298 | 287 | 280 | 269 | 262 |
| *vs untaxed* | *+377%* | *+326%* | *+281%* | *+249%* | *+219%* | *+199%* |
| Effective Spread (bp) | 292 | 285 | 274 | 267 | 257 | 250 |
| *vs untaxed* | *+296%* | *+271%* | *+249%* | *+237%* | *+223%* | *+215%* |
| Fill Rate | 34% | 32% | 30% | 28% | 27% | 24% |
| *vs untaxed* | *-55%* | *-56%* | *-56%* | *-57%* | *-56%* | *-56%* |
| E(Quote Revenue) $ | 0.022 | 0.022 | 0.023 | 0.024 | 0.025 | 0.024 |
| *vs untaxed* | *-65%* | *-67%* | *-68%* | *-68%* | *-67%* | *-67%* |
| Realized Volatility $ | 0.260 | 0.259 | 0.256 | 0.253 | 0.247 | 0.244 |
| *vs untaxed* | *+34%* | *+46%* | *+60%* | *+79%* | *+99%* | *+128%* |
| Gains from Trade $/k | 97.71 | 91.78 | 88.01 | 83.86 | 77.30 | 70.15 |
| *vs untaxed* | *-71%* | *-71%* | *-69%* | *-68%* | *-66%* | *-65%* |
| Deadweight Loss $/k | 147.5 | 136.1 | 119.8 | 104.1 | 88.8 | 73.6 |

TABLE 2. Measures of market quality for revenue-optimal transaction tax rates. Investors have private reserve values for the asset with a mean of $20 and volatility of 2.5%; market makers have reserve values equal to the average of $20. Gains from trade and deadweight loss are displayed in dollars per 1000 orders.

In our constant-asset-value setup, this is equivalent to having traders whose reservation value depends in part on the reservation value of the trader preceding them. We structure this as an AR(1) process to maintain covariance stationarity. The dynamics would thus look like this:

$$d_t|d_{t-1} \overset{iid}{\sim} f = \begin{cases} 0 & w.p.\ \mu \\ N\left(0, L^2\left(1+\frac{\lambda\gamma^2}{1-\gamma^2}\right)\right) & w.p.\ 1-\mu. \end{cases} \tag{31}$$

Therefore, the preceding results still hold: Even in the presence of positive-feedback traders, we can expect a tax to widen spreads and increase volatility. Intuitively, this makes sense: positive-feedback traders take more extreme views than normal (non-positive-feedback) investors; therefore, they are less likely to be dissuaded from trading by a comparatively-small tax.

In comparison, normal investors are more likely to be dissuaded by such a tax and, thus, the influence of positive-feedback traders is likely to be proportionally *greater* with a tax.

For intuition, we appeal to the idea of taxing alcohol: The tax does little to change the drinking habits of alcoholics but does dissuade moderate drinkers. One problem with this analogy is that moderate drinkers can switch to being alcoholics, and so a tax dissuading moderate drinkers from imbibing might still have social benefit. Here, however, we believe there is no danger of addiction. Therefore, a transactions tax would seem to be welfare-reducing since it does nothing to reduce the externalities of positive-feedback traders.

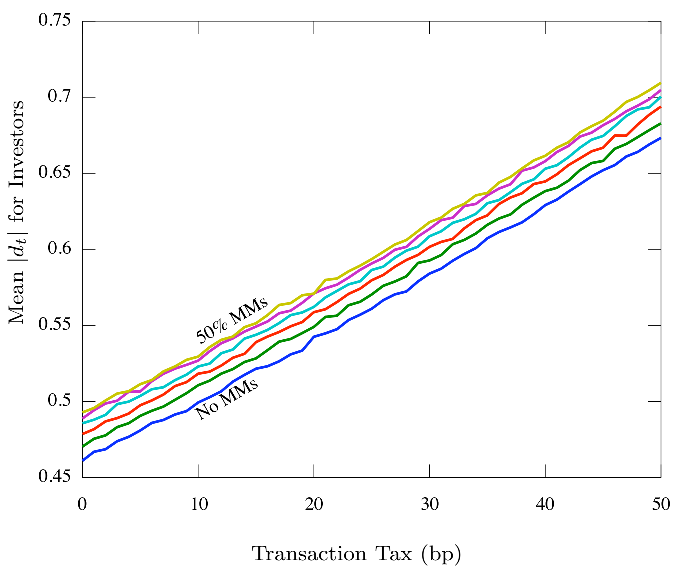



FIGURE 13. Dispersion of reservation prices for investors who are price takers for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). Note that taxes increase the dispersion of investors who trade — by chasing away investors with less extreme views. The plot for investors who are price makers (*i.e.* who quote) is the same.

## 5. Conclusion

Following the recent financial crisis, regulators have tried to assure the public that such a crisis would not recur, that those responsible would be penalized, and that the financial industry would pay for being bailed out. Some policy makers have proposed a securities transaction tax to meet these goals with the idea that a tax would reduce price volatility; encourage long-term investing; raise large amounts of revenue from a very small tax; and, push harmful speculators out of the market. At the time of this writing, European politicians are the most vocal proponents of a Europe/Eurozone financial transaction tax.[23] Many of their speeches cite these very goals for the implementation of a tax.

Opponents of transaction taxes, however, have argued that a transaction tax will reduce liquidity and increase trading costs making trading too expensive for some investors. They have also said a tax such as the broad-based one proposed by DeFazio will be difficult to implement (especially across asset classes); will distort the market by reducing market efficiency; and, will push traders to other venues or countries.

Policy makers cannot easily experiment with their markets; and, it is not clear how informative empirical studies are to proposed taxes on other markets. We developed this model to guide policy makers and to help academics understand how different aspects of market quality may be related or affected by a tax. The resulting sequential trader model is very clean in its assumptions — a contribution to the market microstructure literature

[23]Peel (2010) notes that German Chancellor Angela Merkel is "pushing for a Europe-wide financial transaction tax by 2012" which she expects to raise "an extra €2B per year for the German budget." In France, Wall Street Journal (2012) quotes President Françoise Hollande saying he will "put production before speculation, investment before immediate gratification" and notes that his government doubled France's proposed tax to 20 bp.

in its own right. However, the model also has much to say about the effects of intermediation in thin/fast markets as well as the effects of financial transaction taxes.

What the model suggests should give pause to the hasty imposition of transaction taxes. We find the tax increases the effects of destabilizing speculators; it does not chase them out of the market. We also find that optimal and effective spreads widen; volatility may slightly decrease (while other metrics get much worse) or, more often, increases greatly; and, both the benefits of providing liquidity and gains from trade decrease. All of these effects are strong enough to be highly economically significant. Depending on their urgency, investors trading in this lower liquidity environment might be subject to higher adverse selection. Any one of these effects could be harmful; in concert, they could hobble traders with genuine needs to trade. In particular, we suspect the effects will be borne disproportionately by retail traders and others lacking superior information. These changes in the secondary market would likely reduce the ability of companies to raise funds in the primary market; thus a tax could reduce job and wealth creation. Furthermore, our analysis suggests that the revenue-optimal tax would be very high, between 55 bp and 70 bp. These rates would be very harmful to market quality.

Because the model is one of sequential traders, it assumes potential traders come to the market, make one small trade, and then leave. Thus the model is especially applicable to markets where there are close substitutes (*e.g.* investors who buy a small amount of the cheapest corporate bond meeting certain criteria) or where liquidity is thin (inside bid-ask quotes are of small size). As many have noted, and Black (1971) predicted, this includes markets where high-frequency trading is prevalent since those markets can be effectively thin. In thin markets, market makers can reduce liquidity for other traders if they compete to trade. This would be particularly relevant for high-frequency traders who are not only intermediaries but also engage in proprietary trading. Since this describes a large portion of current securities markets, we believe this analysis is important for policy makers considering implementing a financial transaction tax.

While not our primary goal, the model also yields information about the effects of intermediation in thin/fast markets. Increasing the proportion of actors who are market makers may increase optimal bid-ask spreads while having little effect on effective (realized) spreads. This may help disentangle seemingly conflicting findings between these two measures of liquidity. Increasing the fraction of market makers reduces volume and gains from trade overall, increases the gains from trade for investors, and may increase or decrease the value of quoting. The strongest effect of increasing the prevalence of market makers is a large reduction in the realized volatility. While some of these could be viewed negatively, taken together they likely reflect that markets are more efficient and thus there are fewer opportunities to make outsized profits.

Increasing the fraction of market makers causes markets to be more sensitive to transaction taxes. Under a tax, volatility increases much faster that of a market lacking intermediaries. Volume also drops by a greater fraction as do the benefits of providing liquidity and gains from trade. Thus markets with high levels of intermediation may respond differently to a tax than markets with little or no intermediation. This is crucial for policy makers because it suggests that the findings of empirical studies may not be applicable to markets with differing levels of intermediation.

Finally, there are many issues which have been left for further work. For example, we do not address the effects of quote taxes (recently implemented in France). However, we believe that prior theory (by one of the co-authors) will help us conduct a similar study of those effects. Taken with the work here, that would then allow us to examine the French policy to tease apart the effects of the two taxes. We also suspect this would help us determine the fraction of people somehow escaping the tax as well as the propensity of traders to leave for other (less taxed) markets.

## Appendix A. Proofs

### A.1. **Clear Course of Action.**

*Proof of Proposition 1.* In Table 1, Rick takes Ilsa's prices if $d_{t+1} < -R_Q^{0*} - \delta - 2\tau$ or $d_{t+1} > R_Q^{0*} + \beta + 2\tau$. Suppose both could be satisfied. Then we would have that:

$$R_Q^{0*} + \beta + 2\tau < -R_Q^{0*} - \delta - 2\tau, \text{ which implies} \tag{32}$$

$$\beta + \delta < -2R_Q^{0*} - 4\tau. \tag{33}$$

Since $R_Q^{0*}$ and $\tau$ are non-negative, this implies a negative bid-ask spread — and we have a contradiction. □

### A.2. **Characterizing the Bid-Ask Spread.**

*Proof of Proposition 2.* We prove the case for $d_t > 0$; the case for $d_t < 0$ proceeds similarly.

Suppose $\delta \geq \beta$. The optimal revenue is given by:

$$R_Q = \rho F(-R_Q^{0*} - \delta - 2\tau)(\delta + d_t - 2\tau) + \rho F(-R_Q^{0*} - \beta - 2\tau)(\beta - d_t - 2\tau). \tag{34}$$

We define a sub-optimal revenue $R'_Q$ by switching $\delta$ and $\beta$:

$$R'_Q = \rho F(-R_Q^{0*} - \beta - 2\tau)(\beta + d_t - 2\tau) + \rho F(-R_Q^{0*} - \delta - 2\tau)(\delta - d_t - 2\tau). \tag{35}$$

Since $R_Q$ is optimal, we have that $R_Q - R'_Q > 0$. However,

$$R_Q - R'_Q = \rho 2d_t \underbrace{\left(F(-R_Q^{0*} - \delta - 2\tau) - F(-R_Q^{0*} - \beta - 2\tau)\right)}_{<0} \tag{36}$$

is negative by the properties of a cdf. This implies $R_Q < R'_Q$ and that $\delta$ and $\beta$ are not optimal — a contradiction.

Finally, if $d_t = 0$, the optimal revenue is given by:

$$R_Q = \rho F(-R_Q^{0*} - \delta - 2\tau)(\delta - 2\tau) + \rho F(-R_Q^{0*} - \beta - 2\tau)(\beta - 2\tau). \tag{37}$$

These two summands must have the same maximizer, so $\delta = \beta$. □

*Proof of Proposition 3.* Define $H(x) = F(G(x))/f(G(x))$, then we can rewrite the equilibrium $\delta$ and $\beta$ as,

$$\delta = H(\delta) - d_t + 2\tau, \tag{38}$$

$$\beta = H(\beta) + d_t + 2\tau, \text{ and thus,} \tag{39}$$

$$\delta + \beta = H(\delta) + H(\beta) + 4\tau \geq 4\tau. \tag{40}$$

Since we assumed $\tau$/share tax is collected at both position entry and exit (a total of $2\tau$), the bid-ask spread $\delta+\beta$ is more than twice the tax collected. □

*Proof of Proposition 4.* We first note that

$$\delta + \beta = \frac{F(G(\delta))}{f(G(\delta))} + \frac{F(G(\beta))}{f(G(\beta))} + 4\tau. \tag{41}$$

Since $f$ is symmetric about 0, $F(x) = 1 - F(-x)$, letting us rewrite the above. Since $d_t \sim F$ have $\mathrm{Var}(d_t) = L^2 < \infty$, we may apply the Chebyshev inequality:

$$\delta + \beta = \frac{1 - F(-G(\delta))}{f(G(\delta))} + \frac{1 - F(-G(\beta))}{f(G(\beta))} + 4\tau \tag{42}$$

$$\leq \frac{L^2}{2G(\delta)^2 f(G(\delta))} + \frac{L^2}{2G(\beta)^2 f(G(\beta))} + 4\tau \tag{43}$$

with the factor of 2 arising from the symmetry of $f$. □

### A.3. **Quoting Benefit is Positive.**

*Proof of Proposition 5.* Rearranging the terms for $R_Q|d_t$, we have

$$R_Q|d_t = \rho F(G(\delta)(\delta + d_t - 2\tau) + \rho F(G(\beta))(\beta - d_t - 2\tau) \tag{44}$$

$$\begin{aligned} &= \rho[(F(G(\delta)) - F(G(\beta))]d_t + \\ &\quad \rho F(G(\delta))(\delta - 2\tau) + \rho F(G(\beta))(\beta - 2\tau). \end{aligned} \tag{45}$$

We prove the case for $d_t > 0$; the case for $d_t < 0$ proceeds similarly. If $d_t > 0$, we have $\beta > \delta$ from Proposition 2. Therefore, we have $G(\delta) > G(\beta)$ and $F(G(\delta)) > F(G(\beta))$, as $G(\cdot)$ is a decreasing function and $F(\cdot)$ is an increasing function by definition.

It follows that,

$$F(G(\delta))(\delta - 2\tau) + F(G(\beta))(\beta - 2\tau) \tag{46}$$

$$> F(G(\beta))(\delta - 2\tau) + F(G(\beta))(\beta - 2\tau) \tag{47}$$

$$= F(G(\beta))(\delta + \beta - 4\tau) > 0 \tag{48}$$

where the last inequality follows from Proposition 3 □

### A.4. **Existence of Equilibrium.**

*Proof of Proposition 6.* We handle two cases together (with qualifications as needed) for both a pdf $f$ with finite support on $[-L, L]$ and for $f$ with support over the entire real line. The pdf with finite support is defined so that $f(x) = 0 \ \forall \ x \in \mathbb{R} \backslash [-L, L]$.

Rewrite the expected quote profit $R_Q|d_t$ as,

$$R_Q|d_t \equiv \rho w(\delta,\beta) = \rho w_1(\delta) + \rho w_2(\beta) \quad \text{where} \tag{49}$$

$$w_1(\delta) = F(G(\delta))(\delta + d_t - 2\tau), \quad \text{and} \tag{50}$$

$$w_2(\beta) = F(G(\beta))(\beta - d_t - 2\tau). \tag{51}$$

Holding $R_Q^{0*}$, $\tau$, and $d_t$ fixed, we have that

$$\lim_{\delta\to-\infty} F(G(\delta))(\delta + d_t - 2\tau) = -\infty; \quad \text{and,} \tag{52}$$

$$\lim_{\delta\to\infty} F(G(\delta))(\delta + d_t - 2\tau) = \lim_{\delta\to\infty} F(-\delta)\delta = -\lim_{\delta\to-\infty} F(\delta)\delta = 0 \tag{53}$$

with the last equality justified by finite variance: $\mathrm{Var}(x) < \infty$ implies the cdf converges to 0 faster than $x$.

We then note that $\delta + d_t - 2\tau > 0$ implies $\delta > 2\tau - d_t$, in which case:

$$w_1(\delta) = F(G(\delta))(\delta + d_t - 2\tau) > 0. \tag{54}$$

If $f$ has only finite support, then $G(\delta) > -L$ and $G(\beta) > -L$ imply lower bounds $\delta, \beta < L - R_Q - 2\tau$. From these bounds, we can show that $w_1(\delta) > 0$ iff $d_t > R_Q + 4\tau - L$ and $w_2(\beta) > 0$ iff $d_t < L - R_Q - 4\tau$; $w_1$ and $w_2$ are otherwise 0. Furthermore, it is easy to verify that if $R_Q + 4\tau - L \geq L$ then $\nexists\, \delta$ s.t. $w_1(\delta) > 0$ and, similarly, that if $L - R_Q - 4\tau - L \leq -L$ then $\nexists\, \beta$ s.t. $w_2(\beta) > 0$.

Therefore, since $R_Q^{0*}$ and $\tau$ are bounded, we have that:

$$\max_\delta w_1(\delta) = \max_\delta F(G(\delta))(\delta + d_t - 2\tau) = F(G(\delta_M))(\delta_M + d_t - 2\tau) \tag{55}$$

for some $\delta_M > 2\tau - d_t$. We can also show likewise for $w_2(\beta)$: That there exists a maximizing $\beta_M > 2\tau + d_t$.

Therefore, we have that

$$\max_{\delta,\beta}(w_1(\delta) + w_2(\beta)) = w_1(\delta_M) + w_2(\beta_M) > 0. \tag{56}$$

Since we attain a bounded maximum at some $\delta_M, \beta_M$ for all $d_t$ over the support of the distribution, there is an expected best strategy and thus a Markov Perfect equilibrium exists. □

A.5. **Uniqueness of Equilibrium.**

*Proof of Proposition 7.* Continuing with the notation in the preceding proof, we find the optimal $\delta_M$ and $\beta_M$ by differentiating $w_1$ and $w_2$. This gives us our first-order conditions:

$$\left.\frac{\partial w_1}{\partial \delta}\right|_{\delta=\delta_M} = -f(G(\delta_M))(\delta_M + d_t - 2\tau) + F(G(\delta_M)) = 0 \tag{57}$$

$$\left.\frac{\partial w_2}{\partial \beta}\right|_{\beta=\beta_M} = -f(G(\beta_M))(\beta_M - d_t - 2\tau) + F(G(\beta_M)) = 0. \tag{58}$$

The equilibrium expected quote revenue, $R_Q^{0*}$, is then given by

$$R_Q^{0*} = E(R_Q|d_t) = \int_{-\infty}^{\infty} (w_1(\delta_M) + w_2(\beta_M))dF(d_t). \tag{59}$$

Since $F$ is continuous, the pdf $f$ is dominated by some function and Fatou's Lemma allows us to interchange integration and limit operations. This lets us differentiate under the integral sign, yielding

$$\frac{\partial R_Q^{0*}}{R_Q} = \int_{-\infty}^{\infty} \left(\frac{\partial w_1(\delta_M)}{\partial R_Q} + \frac{\partial w_2(\beta_M)}{\partial R_Q}\right) dF(d_t) \tag{60}$$

for all $R_Q^{0*} \in [0, \infty)$.

Differentiating $w_1$ and $w_2$ with respect to $R_Q$, we get

$$\frac{\partial w_1(\delta_M)}{\partial R_Q} = \overbrace{(F(G(\delta_M)) - f(G(\delta_M))(\delta_M + d_t - 2\tau))}^{=0 \text{ by FOC in (57)}} \frac{\partial \delta_M}{\partial R_Q} - f(G(\delta_M))(\delta_M + d_t - 2\tau), \tag{61}$$

$$\frac{\partial w_2(\beta_M)}{\partial R_Q} = \overbrace{(F(G(\beta_M)) - f(G(\beta_M))(\beta_M - d_t - 2\tau))}^{=0 \text{ by FOC in (58)}} \frac{\partial \beta_M}{\partial R_Q} - f(G(\beta_M))(\beta_M - d_t - 2\tau). \tag{62}$$

The first-order conditions on $w_1$ and $w_2$ then reduce these to

$$\frac{\partial w_1(\delta_M)}{\partial R_Q} = -f(G(\delta_M))(\delta_M + d_t - 2\tau) < 0, \tag{63}$$

$$\frac{\partial w_2(\beta_M)}{\partial R_Q} = -f(G(\beta_M)(\beta_M - d_t - 2\tau) < 0. \tag{64}$$

Combining these and invoking the continuity of $F$, we then have that $R_Q^{0*}(R_Q)$ is a strictly decreasing continuous function. Therefore, for all $\rho \in (0, 1]$, there must be a unique $R_Q^*$ such that $R_Q^{0*}(R_Q^*) = R_Q^*/\rho$. Thus the Markov Perfect equilibrium is unique. □

## Appendix B. Solution of the Uniform Case

Using $\delta$ and $\beta$ from equation (6), an interior solution requires that

$$R_Q^{0*} + \delta + 2\tau \in [-L, L] \iff -L + R_Q^{0*} + 4\tau \leq d_t \leq 3L + R_Q^{0*} + 4\tau, \tag{65}$$

$$R_Q^{0*} + \beta + 2\tau \in [-L, L] \iff -3L - R_Q^{0*} - 4\tau \leq d_t \leq L - R_Q^{0*} - 4\tau. \tag{66}$$

Both the $\delta$ and the $\beta$ result imply that

$$L - R_Q^{0*} - 4\tau > -L \iff R_Q^{0*} < 2L - 4\tau. \tag{67}$$

We can then compute the expected $R_Q|d_t$:

$$\frac{R_Q^{0*}}{\rho} = E(R_Q|d_T) \tag{68}$$

$$= \frac{\int_{-L+R_Q^{0*}+4\tau}^{L}(\delta + d_t - 2\tau)^2 dd_t + \int_{-L}^{L-R_Q^{0*}-4\tau}(\beta + d_t - 2\tau)^2 dd_t}{4L^2} \tag{69}$$

$$\begin{aligned} &= (d_t - R_Q^{0*} + L - 4\tau)^3/(3 \cdot 16L^2)\Big|_{-L+R_Q^{0*}+4\tau}^{L} \\ &\quad + (d_t + R_Q^{0*} - L + 4\tau)^3/(3 \cdot 16L^2)\Big|_{-L}^{L-R_Q^{0*}-4\tau} \end{aligned} \tag{70}$$

$$= (2L - R_Q^{0*} - 4\tau)^3/(24L^2). \tag{71}$$

## Appendix C. Solution of Uniform with Market Makers Case

Using the definitions in equations (15)–(18), we first prove a number of lemmas necessary for the main result.

**Lemma 1.** $\max_\delta V_1(\delta) \geq 0$ *and* $\max_\beta V_2(\beta) \geq 0$*.*

*Proof.* By the nature of CDFs and indicator functions, we know that

$$(1-\mu)F(G(\delta)) + \mu\mathbb{I}(G(\delta)) \geq 0 \quad \text{and} \tag{72}$$

$$(1-\mu)F(G(\beta)) + \mu\mathbb{I}(G(\beta)) \geq 0. \tag{73}$$

For any $\delta, \beta > 2\tau - d_t$, $\delta + d_t - 2\tau > 0$ and $\beta - d_t - 2\tau > 0$. Since $V_1$ and $V_2$ are products of non-negative entities, they are themselves non-negative. □

**Lemma 2.** $\max_\delta V_1(\delta) > 0$ *iff* $d_t \in (R_Q^{0*} + 4\tau - L, L]$*.*

*Proof.* "$\Longleftarrow$" Since $d_t > R_Q^{0*} + 4\tau - L$, we have $2\tau - d_t < L - R_Q^{0*} - 2\tau$. For any $\delta \in (2\tau - d_t, L - R_Q^{0*} - 2\tau)$, we have: $\delta > 2\tau - d_t$, *i.e.* $\delta + d_t - 2\tau > 0$ and $\delta < L - R_Q^{0*} - 2\tau$, *i.e.* $G(\delta) = -R_Q^{0*} - \delta - 2\tau > -L$. Therefore, we know that $F(G(\delta)) > 0$ and $(1-\mu)F(G(\delta)) + \mu\mathbb{I}(G(\delta)) > 0$. Thus $V_1(\delta) > 0$ and so $\max_\delta V_1(\delta) > 0$.

" $\Longrightarrow$ " $\max_\delta V_1(\delta) > 0$ implies that $\exists\, \delta$ s.t. $V_1(\delta) > 0$. Hence $G(\delta) > -L$ and $\delta + d_t - 2\tau > 0$. Therefore, $2\tau - d_t < \delta < L - R_Q^{0*} - 2\tau$ which implies that $d_t > R_Q^{0*} + 4\tau - L$. □

**Lemma 3.** $\max_\beta V_2(\beta) > 0$ *iff* $d_t \in [-L, L - R_Q^{0*} - 4\tau)$.

*Proof.* The proof proceeds as that for Lemma 2. □

**Lemma 4.** *For all* $\delta \in \mathbb{R}$, $\beta \in \mathbb{R}$, $d_t \in \mathbb{R}$, $\mu \in (0,1]$, $R_Q^{0*} > 0$, *and* $\tau \geq 0$*:*

$$\frac{\partial^2 V_1(\delta)}{\partial \delta^2} \leq 0, \ \textit{and}\ \frac{\partial^2 V_2(\beta)}{\partial \beta^2} \leq 0. \tag{74}$$

*Proof.* By inspection of equations (22) and (23). □

**Corollary 3.** *For* $d_t \in (R_Q^{0*} + 4\tau - L, L]$, *if* $\delta \leq -R_Q^{0*} - 2\tau$, *then* $\frac{\partial V_1(\delta)}{\partial \delta} > 0$. *For* $d_t \in [-L, L - R_Q^{0*} - 4\tau)$, *if* $\beta \leq -R_Q^{0*} - 2\tau$, *then* $\frac{\partial V_2(\beta)}{\partial \beta} > 0$.

*Proof.* Let $\delta_0 = -R_Q^{0*} - 2\tau$ and $\beta_0 = -R_Q^{0*} - 2\tau$. For $\delta \leq \delta_0$:

$$\frac{\partial V_1(\delta)}{\partial \delta} \geq \left.\frac{\partial V_1(\delta)}{\partial \delta}\right|_{\delta=\delta_0} = \frac{1-\mu}{2L}(-L - R_Q^{0*} - 2\delta_0 - d_t + \frac{2L}{1-\mu}) \tag{75}$$

$$= \frac{1-\mu}{2L}(R_Q^{0*} + 4\tau - L - d_t) + 1 \tag{76}$$

$$> \frac{1-\mu}{2L}(0 - 2L) + 1 = -(1-\mu) + 1 = \mu > 0. \tag{77}$$

Similarly, for $\beta \leq \beta_0$ $\frac{\partial V_2(\beta)}{\partial \beta} \geq \left.\frac{\partial V_2(\beta)}{\partial \beta}\right|_{\beta=\beta_0} > 0$. □

**Corollary 4.** *There exists a unique* $\delta_M > \delta_0$ *such that:*

$$\left.\frac{\partial V_1(\delta)}{\partial \delta}\right|_{\delta=\delta_M} = 0 \ \textit{where}\ \delta_M = \frac{1}{2}(L - R_Q^{0*} - d_t) \tag{78}$$

*for* $d_t \in (R_Q^{0*} + 4\tau - L, L]$. *There also exists a unique* $\beta_M > \beta_0$ *such that:*

$$\left.\frac{\partial V_2(\beta)}{\partial \beta}\right|_{\beta=\beta_M} = 0 \ \textit{where}\ \beta_M = \frac{1}{2}(L - R_Q^{0*} - d_t) \tag{79}$$

*for* $d_t \in [-L, L - R_Q^{0*} - 4\tau)$.

**Lemma 5.** *If $4\tau \geq L$, then $V_1(\delta_0) \leq 0$ and $V_2(\beta_0) \leq 0$.*

*Proof.* If $4\tau \geq L$, then

$$V_1(\delta_0) = [(1-\mu)F(G(\delta_0)) + \mu\mathbb{I}(G(\delta_0) \geq 0)](\delta_0 + d_t - 2\tau) \tag{80}$$

$$= \frac{1+\mu}{2}(-R_Q^{0*} + d_t - 4\tau) \leq \frac{1+\mu}{2}(-R_Q^{0*}) \leq 0 \tag{81}$$

since $d_t \leq L$. Likewise

$$V_2(\beta_0) = [(1-\mu)F(G(\beta_0)) + \mu\mathbb{I}(G(\beta_0) \geq 0)](\beta_0 - d_t - 2\tau) \leq 0 \tag{82}$$

since $d_t \geq -L$. □

**Lemma 6.** *If $4\tau < L$ and $R_Q^{0*} < L - 4\tau$, then $V_1(\delta_0) > 0$ and $V_2(\beta_0) > 0$.*

*Proof.* Under the assumptions,

$$\delta_0 + d_t - 2\tau = -R_Q^{0*} - 4\tau + d_t > 0 \iff d_t > R_Q^{0*} + 4\tau, \tag{83}$$

$$\beta_0 - d_t - 2\tau = -R_Q^{0*} - 4\tau - d_t > 0 \iff d_t < -R_Q^{0*} - 4\tau. \tag{84}$$

Since $(1-\mu)F(G(\delta)) + \mu\mathbb{I}(G(\delta) \geq 0) > 0$ also, $V_1(\delta) > 0$ and $V_2(\beta) > 0$. □

**Corollary 5.** *If $4\tau < L$ and $R_Q^{0*} \geq L - 4\tau$, then $V_1(\delta_0)$ and $V_2(\beta_0)$ may be positive or negative, depending on $d_t$.*

**Lemma 7.** *If $0 \leq \tau < \frac{\mu L}{\sqrt{2\mu+2\mu^2}+2\mu}$ and $R_Q^{0*} \leq \frac{4\mu L}{\sqrt{2\mu+2\mu^2}+2\mu} - 4\tau$, then:*

(1) *For any $d_t \in (x_1 + R_Q^{0*} + 4\tau, L]$, $\max_\delta V_1(\delta) = V_1(\delta_0)$; and,*

(2) *For any $d_t \in [-L, -x_1 - R_Q^{0*} - 4\tau)$, $\max_\beta V_2(\beta) = V_2(\beta_0)$*

*where $x_1 = \frac{1+3\mu}{1-\mu}L - \frac{2\sqrt{2\mu+2\mu^2}}{1-\mu}L$.*

*Proof.*

$$V_1(\delta_0) > V_1(\delta_M) \iff \tag{85}$$

$$\frac{1+\mu}{2}(-R_Q^{0*}-4\tau+d_t)-\frac{1-\mu}{8L}(L-R_Q^{0*}-4\tau+d_t)^2>0, \tag{86}$$

$$V_2(\beta_0)>V_2(\beta_M) \iff \tag{87}$$

$$\frac{1+\mu}{2}(-R_Q^{0*}-4\tau-d_t)-\frac{1-\mu}{8L}(L-R_Q^{0*}-4\tau-d_t)^2>0. \tag{88}$$

For $d_t \in (R_Q^{0*}+4\tau, L]$, let $x = d_t - R_Q^{0*} - 4\tau$. Then equation (86) $\iff$

$$\frac{1+\mu}{2}x-\frac{1-\mu}{8L}(L+x)^2>0 \tag{89}$$

$$\iff (1-\mu)x^2-2L(1+3\mu)x+(1-\mu)L^2<0. \tag{90}$$

By the quadratic formula, we define:

$$x_1=\frac{1+3\mu}{1-\mu}L-\frac{2\sqrt{2\mu+2\mu^2}}{1-\mu}L, \quad x_2=\frac{1+3\mu}{1-\mu}L+\frac{2\sqrt{2\mu+2\mu^2}}{1-\mu}L. \tag{91}$$

Equation (90) holds iff $x_1 < x < x_2$ and, in that case, $x_1 < L < x_2$. (An aside: If $\mu = 0$, $x_1 = x_2 = x = L$ and equation (86) cannot hold.)

Since $d_t \le L$, $x \le L - R_Q^{0*} - 4\tau$; if $x_1 > L - R_Q^{0*} - 4\tau$, then $d_t > L$ which is not possible — so equation (86) cannot be satisfied. Thus we require

$$L-R_Q^{0*}-4\tau \ge x_1 \iff L-x_1 \ge R_Q^{0*}+4\tau \tag{92}$$

$$\iff \frac{2L}{1-\mu}(\sqrt{2\mu+2\mu^2}-2\mu) \ge R_Q^{0*}+4\tau \tag{93}$$

$$\iff R_Q^{0*}+4\tau \le \frac{4\mu L}{\sqrt{2\mu+2\mu^2}+2\mu}. \tag{94}$$

Similarly, for equation (88) to hold, we require that

$$R_Q^{0*}+4\tau \le \frac{4\mu L}{\sqrt{2\mu+2\mu^2}+2\mu}. \qquad \square$$

*Proof of Proposition 8.* First consider the case $\frac{2\mu L}{\sqrt{2\mu+2\mu^2}+2\mu} \leq 2\tau < L$. Assume that $R_Q^{0*} \in (0, 2L-4\tau)$. From Lemmas 2 and 3,

$$\begin{aligned}\frac{E(R_Q|d_t)}{\rho} &= \frac{1-\mu}{48L^2}(d_t + R_Q^{0*} + 4\tau - L)^3\Big|_{-L}^{L-R_Q^{0*}-4\tau} \\ &\quad + \frac{1-\mu}{48L^2}(d_t - R_Q^{0*} - 4\tau + L)^3\Big|_{R_Q^{0*}+4\tau-L}^{L} \qquad (95)\\ &= \frac{1-\mu}{24L^2}(2L - R_Q^{0*} - 4\tau)^3. \qquad (96)\end{aligned}$$

Since this is a decreasing cubic with a triple root at $R_Q^{0*} = 2L - 4\tau$, it decreases to 0 on $R_Q^{0*} \in (0, 2L-4\tau)$, implying there is a unique equilibrium.

Now consider the case $0 \leq \tau < \frac{\mu L}{\sqrt{2\mu+2\mu^2}+2\mu}$. Assume $R_Q^{0*} \in (0, L + \frac{2L}{1+\mu} - 4\tau]$ and recall $x_1 = \frac{1+3\mu-2\sqrt{2\mu+2\mu^2}}{1-\mu}L$. From Lemmas 2, 3, and 7,

$$\begin{aligned}\frac{E(R_Q|d_t)}{\rho} &= -\frac{1+\mu}{8L}(d_t + R_Q^{0*} + 4\tau)^2\Big|_{-L}^{-x_1-R_Q^{0*}-4\tau} \\ &\quad + \frac{1-\mu}{48L^2}(d_t + R_Q^{0*} + 4\tau - L)^3\Big|_{-x_1-R_Q^{0*}-4\tau}^{L-R_Q^{0*}-4\tau} \\ &\quad + \frac{1-\mu}{48L^2}(d_t + L - R_Q^{0*} - 4\tau)^3\Big|_{R_Q^{0*}+4\tau-L}^{x_1+R_Q^{0*}+4\tau} \\ &\quad + \frac{1+\mu}{8L}(d_t - R_Q^{0*} - 4\tau)^2\Big|_{x_1+R_Q^{0*}+4\tau}^{L} \qquad (97)\\ &= \frac{6(1+\mu)L((L-R_Q^{0*}-4\tau)^2 - x_1^2) + (1-\mu)(L+x_1)^3}{24L^2} \qquad (98)\\ &= \frac{(1+\mu)(3(L-R_Q^{0*}-4\tau)^2 - 3x_1^2 + 2x_1(L+x_1))}{12L} \qquad (99)\\ &= \frac{(1+\mu)(3(L-R_Q^{0*}-4\tau)^2 + 2x_1L - x_1^2)}{12L} > 0. \qquad (100)\end{aligned}$$

where equation (89) equalling zero for $x = x_1$ yields the penultimate step and $x_1 \leq L$ yields the last step. Note that

$$\frac{E(R_Q|d_t)|_{R_Q^{0*}=0}}{\rho} = \frac{1+\mu}{12L}(3(L-4\tau)^2 + x_1(2L-x_1)) > 0 \tag{101}$$

since $0 < x_1 < L$ and $4\tau < L$. Now consider the difference $\Delta$ between $E(R_Q|d_t)$ and $R_Q^{0*}$ to see if a fixed point exists:

$$\Delta(R_Q^{0*}) = \frac{E(R_Q|d_t)|_{R_Q^{0*}}}{\rho} - \frac{R_Q^{0*}}{\rho}; \tag{102}$$

$$\frac{1}{\rho}\frac{\partial\Delta(R_Q^{0*})}{\partial R_Q^{0*}} = \frac{1+\mu}{12L}6(L - R_Q^{0*} - 4\tau)(-1) - 1 = 0 \tag{103}$$

Since $\frac{\partial\Delta}{\partial R_Q^{0*}} < 0$ on $R_Q^{0*} \in (0, L + \frac{2L}{1+\mu} - 4\tau)$, this implies $\Delta$ is minimized at $R_Q^{0*} = L + \frac{2L}{1+\mu} - 4\tau$. For any $\rho \in (0, 1]$:

$$\exists \text{ a solution } \frac{R_Q^{0*}}{\rho} = \frac{E(R_Q^{0*})}{\rho} \text{ on } R_Q^{0*} \in (0, L + \frac{2L}{1+\mu} - 4\tau] \tag{104}$$

$$\iff \Delta(R_Q^{0*})\Big|_{L+\frac{2L}{1+\mu}-4\tau} = [E(R_Q^{0*}) - R_Q^{0*}]\Big|_{L+\frac{2L}{1+\mu}-4\tau} < 0 \tag{105}$$

$$\iff \frac{1+\mu}{12L}\left(3\left(\frac{2L}{1+\mu}\right)^2 + 2x_1L - x_1^2\right) - \frac{L}{\rho} - \frac{2L/\rho}{1+\mu} + \frac{4\tau}{\rho} < 0. \tag{106}$$

We next examine $\left.\frac{\Delta(R_Q^{0*})}{\rho}\right|_{L+\frac{2L}{1+\mu}-4\tau}$ to see if the last inequality holds:

$$\frac{1+\mu}{12L}\left(3\left(\frac{2L}{1+\mu}\right)^2 + 2x_1L - x_1^2\right) - \frac{L}{\rho} - \frac{2L/\rho}{1+\mu} + \frac{4\tau}{\rho} \tag{107}$$

$$= \frac{L}{1+\mu} + \frac{(1-\mu)x_1}{6} - \frac{(1+\mu)x_1^2}{12L} - \frac{L}{\rho} - \frac{2L/\rho}{1+\mu} + \frac{4\tau}{\rho} \tag{108}$$

$$= -\frac{L}{\rho}\left(1 + \frac{2-\rho}{1+\mu}\right) + \frac{(1+\mu)x_1}{6}\left(1 - \frac{x_1}{2L}\right) + \frac{4\tau}{\rho} \tag{109}$$

$$< \frac{4\tau - L}{\rho} - \frac{L}{1+\mu} + \frac{(1+\mu)x_1}{6} < 4\tau - L - \frac{L}{1+\mu} + \frac{1+\mu}{6}L \tag{110}$$

$$< \underbrace{4\tau - L}_{<0} + L\left(-\frac{1}{2} + \frac{1}{3}\right) < 0. \tag{111}$$

Thus there is a unique solution. □

**Lemma 8.** *For $\mu_0 \leq \mu \leq 1$,*

$$\frac{\partial Z_1(\mu)}{\partial \mu} < 0 \quad \textit{where} \tag{112}$$

$$Z_1(\mu) = \alpha^2(\mu)\left(\frac{\sqrt{2\mu + 2\mu^2}}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)} + 1\right) \quad \textit{and} \tag{113}$$

$$\mu_0 = \frac{2\tau^2}{(L - 2\tau)^2 - 2\tau^2}. \tag{114}$$

*Proof.*

$$\frac{\partial}{\partial \mu}\left[\alpha^2(\mu)\left[1 + \frac{1}{\mu} - \frac{2}{\sqrt{2\mu + 2\mu^2} + 2\mu}\right]\right] \tag{115}$$

$$\begin{aligned} &= \frac{2\alpha(\mu)4\mu}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2}\frac{1 + \frac{1}{\mu} - \frac{2}{\sqrt{2\mu+2\mu^2}+2\mu}}{\sqrt{2\mu + 2\mu^2}} \\ &\quad + \alpha^2(\mu)\left[\frac{\frac{2+4\mu}{\sqrt{2\mu+2\mu^2}} + 4}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2} - \frac{1}{\mu^2}\right] \end{aligned} \tag{116}$$

$$\begin{aligned} &= \frac{\alpha(\mu)}{\sqrt{2\mu + 2\mu^2} + 2\mu}\left[\frac{8(1 + \mu)}{(\sqrt{2\mu + 2\mu^2} + 2\mu)\sqrt{2\mu + 2\mu^2}} - \frac{4}{\mu}\right. \\ &\quad \left. - \frac{8\mu - 16\mu^2}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2\sqrt{2\mu + 2\mu^2}} + \frac{16\mu}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2}\right] \end{aligned} \tag{117}$$

$$\begin{aligned} &= \frac{\alpha^2(\mu)}{4\mu}\left[\frac{8(1 + \mu)\mu(\sqrt{2\mu + 2\mu^2} + 2\mu) - 8\mu^2 + 16\mu^3}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)^2\sqrt{2\mu + 2\mu^2}}\right. \\ &\quad \left. - \frac{4(\sqrt{2\mu + 2\mu^2} + 2\mu)^2\sqrt{2\mu + 2\mu^2} + 16\mu^2 + \sqrt{2\mu + 2\mu^2}}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)^2\sqrt{2\mu + 2\mu^2}}\right] \end{aligned} \tag{118}$$

$$\begin{aligned} &= \frac{\alpha^2(\mu)}{16\mu}\left[\frac{2\mu\sqrt{2\mu + 2\mu^2} - 4\mu^2 + 2\mu^2\sqrt{2\mu + 2\mu^2} + 8\mu^3}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)^2\sqrt{2\mu + 2\mu^2}}\right. \\ &\quad \left. - \frac{2\mu\sqrt{2\mu + 2\mu^2} + 2\mu^2\sqrt{2\mu + 2\mu^2} + 4\mu(2\mu + 2\mu^2)}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)^2\sqrt{2\mu + 2\mu^2}}\right] \end{aligned} \tag{119}$$

$$= -12\mu^2 < 0 \ \forall \ \mu > 0. \tag{120}$$

□

*Proof of Proposition 9.*

$$\frac{E(R_Q|d_t)}{\rho} = \frac{R_Q^{0*}}{\rho} \tag{121}$$

$$\iff \frac{1+\mu}{12L}\left[3(L - R_Q^{0*} - 4\tau)^2 + L^2 - \alpha^2(\mu)L^2\right] = \frac{R_Q^{0*}}{\rho}. \tag{122}$$

Differentiating both sides wrt $\mu$ yields

$$\begin{aligned} &\frac{1}{12L}\left[3(L - R_Q^{0*} - 4\tau)^2 + L^2 - \alpha^2(\mu)L^2\right] \\ &\quad + \frac{1+\mu}{12L}\left[6(L - R_Q^{0*} - 4\tau)\left(-\frac{\partial R_Q^{0*}}{\partial \mu}\right) - 2\alpha(\mu)\frac{\partial \alpha(\mu)}{\partial \mu}L^2\right] = \frac{\partial R_Q^{0*}}{\rho \partial \mu} \end{aligned} \tag{123}$$

$$\iff \left(\frac{1}{\rho} + \frac{1+\mu}{2L}(L - R_Q^{0*} - 4\tau)\right)\frac{\partial R_Q^{0*}}{\partial \mu} \tag{124}$$

$$= \frac{L\left[3\left(1 - \frac{R_Q^{0*}+4\tau}{L}\right)^2 + 1 - \alpha^2(\mu) - 2(1+\mu)\alpha(\mu)\frac{\partial \alpha(\mu)}{\partial \mu}\right]}{12} \tag{125}$$

$$= \frac{L\left[3\left(1 - \frac{R_Q^{0*}+4\tau}{L}\right)^2 + 1 - \alpha^2(\mu)\left(\frac{\sqrt{2\mu+2\mu^2}}{\mu(\sqrt{2\mu+2\mu^2}+2\mu)} + 1\right)\right]}{12}. \tag{126}$$

Now we note that final term of this (involving $\alpha^2(\mu)$ is the same as $Z_1(\mu)$ from Lemma 8. Since $\frac{\partial Z_1(\mu)}{\partial \mu} < 0$ for $\mu_0 \le \mu \le 1$, we then note that $Z_1(\mu_0) \ge Z_1(\mu) \ge Z_1(1) = \frac{3}{2}$ where $Z(\mu_0) = \frac{4\tau}{L}$. Then

$$Z_1(\mu_0) = \alpha^2(\mu_0)\left(\frac{1}{\mu_0(1 + \frac{2}{\sqrt{2/\mu_0+2}})} + 1\right) \tag{127}$$

$$= \frac{16\tau^2}{L^2}\left(\frac{(L-2\tau)^2 - 2\tau^2}{2\tau^2\left(1 + \frac{2\tau}{L-2\tau}\right)} + 1\right) \tag{128}$$

$$= 8\left(1 - 6\frac{\tau}{L} + 12\left(\frac{\tau}{L}\right)^2 - 4\left(\frac{\tau}{L}\right)^3\right) \tag{129}$$

makes it easy to see that $\frac{3}{2} \le Z_1(\mu_0) \le 8$.

Note that $3\left(1-\frac{R_Q^{0*}-4\tau}{L}\right)^2+1\in(1,4)$. Also note that $R_Q^{0*}$ depends on both $\mu$ and $\rho$. In particular, when $\rho$ is small enough, $R_Q^{0*}$ is very close to zero. Therefore, the sign of $\frac{\partial R_Q^{0*}}{\partial\mu}$ is not the same for all $\mu$ (*i.e.* there is no uniform conclusion). □

*Proof of Corollary 1.* The proof is simple: Note that

$$Z(\mu)<3\left(\frac{1}{\sqrt{6}}\right)^2+1-Z_1(1)=\frac{3}{2}-\frac{3}{2}=0. \tag{130}$$
□

*Proof of Corollary 2.*

$$R_Q^{0*}+4\tau<\left(1-\frac{1}{\sqrt{6}}L\right)\implies 3\left(1-\frac{R_Q^{0*}+4\tau}{L}\right)+1>\frac{3}{2} \tag{131}$$

Since $\min_\mu Z_1(\mu)=Z_1(1)=\frac{3}{2}$, when $\mu$ is close to 1:

$$Z_1(\mu)<3\left(1-\frac{R_Q^{0*}+4\tau}{L}\right)+1. \tag{132}$$
□

## Appendix D. Solution of Normal with Market Makers Case

**Lemma 9.** *The function $f(x)=(x+c)(1-\Phi(ax+b))+d$ for $a>0$ has a unique global maximum on $\mathbb{R}$.*

*Proof.*

$$\frac{\partial f}{\partial x}=-a(x+c)\phi(ax+b)+1-\Phi(ax+b) \tag{133}$$

$$\frac{\partial^2 f}{\partial x^2}=\phi(ax+b)(-a+a^2(x+c)(ax+b)-a) \tag{134}$$

$$=a\phi(ax+b)(-2+a^2x^2+a(b+ac)x+abc). \tag{135}$$

$\frac{\partial^2 f}{\partial x^2}$ has roots at

$$x_0=\frac{-ab-a^2c\pm\sqrt{a^2(b+ac)^2-4a^2(abc-2)}}{2a^2} \tag{136}$$

$$=\frac{-b}{2a}-\frac{c}{2}\pm\frac{\sqrt{(b-ac)^2+8}}{2a}. \tag{137}$$

The two roots are real (one positive, one negative) since the interior of the square root is positive. For $x < c$, $f$ is negative; for $x > c$, $f$ is positive. However, for $x$ greater than the largest root $x > \frac{-b}{2a} - \frac{c}{2} + \frac{\sqrt{(b-ac)^2+8}}{2a}$, $\frac{\partial f}{\partial x} < 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$. Thus $f$ descends to an asymptote. Since $f$ is continuous and $\frac{\partial^2 f}{\partial x^2} < 0$ for $x$ between the two roots, $f$ attains a global maximum on $(\frac{-b-ac}{2a} - \frac{a\sqrt{(b-ac)^2+8}}{2a^2}, \frac{-b-ac}{2a} + \frac{a\sqrt{(b-ac)^2+8}}{2a^2})$. □

*Proof of Proposition 10.* Since $R_Q|d_t$ contains two indicator functions, we consider three regions: where neither is active, where the first indicator is active, and where the second indicator is active. By Proposition 1, we know the indicator functions are never both active.

By Lemma 9, we can find a global maximum on an unconstrained interval. We find maxima for each constrained interval and thus the max of $R_Q|d_t$ over the three regions. We then compute the expected value of $R_Q|d_t$ over the $d_t$ distribution (common knowledge), yielding $E(R_Q|d_t) = R_Q^{0*}$. Since the maximum is unique, we have a unique Markov Perfect equilibrium. □

*Proof of Proposition 11.* If the distribution of reservation values $F$ is Gaussian, then the bid-ask spread is given by

$$\delta + \beta = \frac{\Phi((-R_Q^{0*} - \delta - 2\tau)/L)}{\phi((-R_Q^{0*} - \delta - 2\tau)/L)} + \frac{\Phi((-R_Q^{0*} - \beta - 2\tau)/L)}{\phi((-R_Q^{0*} - \beta - 2\tau)/L)} + 4\tau. \tag{138}$$

From Lemma 2 in Feller I (p. 175), we get that:

$$\delta + \beta \leq \frac{L}{|R_Q^{0*} + \delta + 2\tau|} + \frac{L}{|R_Q^{0*} + \beta + 2\tau|} + 4\tau. \tag{139}$$

Using Jensen's inequality, we get

$$\delta + \beta \leq \frac{2L}{2R_Q^{0*} + \delta + \beta + 4\tau} + 4\tau \leq \frac{L}{R_Q^{0*} + 4\tau} + 4\tau.$$ □

## References


Anand, A., S. Chakravarty, and T. Martell (2005). Empirical evidence on the evolution of liquidity: Choice of market versus limit orders by informed and uninformed traders. *Journal of Financial Markets 8*(3), 288–308.

Baker, D. (2000). The case for a unilateral speculation tax in the united states. Briefing paper, Center for Economic and Policy Research, Washington, DC.

Binmore, K. G. and M. J. Herrero (1988). Matching and bargaining in dynamic markets. *Review of Economic Studies 55*(1), 17–31.

Black, F. (1971). Towards a fully automated exchange, part 1. *Financial Analyst's Journal 27*(4), 28–35,44.

Campbell, J. Y. and K. A. Froot (1994). Securities transaction taxes: What about international experiences migrating markets? In *Transaction Taxes: False Hopes and Unintended Consequences*. Chicago: Irwin Professional Publishing.

Cipriani, M. and A. Guarino (2008). Transaction costs and informational cascades in financial markets. *Journal of Economic Behavior and Organization 68*(3–4), 581–582.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann (1990). Positive feedback investment strategies and destabilizing rational speculation. *Journal of Finance 45*(2), 379–395.

Demary, M. (2010). Transaction taxes, greed and risk aversion in an agent-based financial market model. *Journal of Economic Interaction and Coordination*, 1–28.

Diamond, P. A. (1982). Aggregate demand management in search equilibrium. *Journal of Political Economy 90*(5), 881–894.

Duffie, D., N. Gârleanu, and L. H. Pedersen (2005). Over-the-counter markets. *Econometrica 73*(6), 1815–1847.

Duffie, D., G. Giroux, and G. Manso (2010). Information percolation. *American Economic Journal: Microeconomics 2*(1), 100–111.

Dupont, D. Y. and G. S. Lee (2007). Effects of securities transaction taxes on depth and bid-ask spread. *Economic Theory 31*, 393–400.

Eichengreen, B., J. Tobin, and C. Wyplosz (1995). Two cases for sand in the wheels of international finance. *Economic Journal 105*(428), 162–172.

Felix, D. and R. Sau (1996). On the revenue raising potential and phasing in of the Tobin tax. In M. ul Haq, I. Grunberg, and I. Karl (Eds.), *The Tobin Tax: Coping with Financial Volatility*. Oxford: Oxford University Press.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications* (third ed.), Volume I. New York: John Wiley and Sons.

Forbes, K. J. (2001). Comment on "securities transaction taxes and financial markets". *IMF Staff Papers* (50), 181–186.

Foucault, T. (1999). Order flow composition and trading costs in a dynamic limit order market. *Journal of Financial Markets 2*(2), 99–134.

Frankel, J. (1996). How well do foreign exchange markets work: Might a Tobin tax help? In M. ul Haq, I. Grunberg, and I. Karl (Eds.), *The Tobin Tax: Coping with Financial Volatility*. Oxford: Oxford University Press.

Friedman, M. (1953). *Essays in Postive Economics*. Chicago: University of Chicago Press.

Gale, D. (1987a). Bargaining and competition part i: Characterization. *Econometrica 54*, 785–806.

Gale, D. (1987b). Bargaining and competition part ii: Existence. *Econometrica 54*, 807–818.

Glosten, L. R. and P. R. Milgrom (1985). Ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics 14*, 71–100.

Grundfest, J. A. and J. B. Shoven (1991). Adverse implications of a securities transactions excise tax. *Journal of Accounting, Auditing and Finance 6*, 409–442.

Habermeier, K. and A. A. Kirilenko (2001). Securities transaction taxes and financial markets. *IMF Staff Papers* (50), 165–180.

Harris, L. (2002). *Trading and Exchanges: Market Microstructure for Practitioners*. Oxford: Oxford University Press.

Hasbrouck, J. and G. Saar (2009). Technology and liquidity provision: The blurring of traditional definitions. *Journal of Financial Markets 12*(2), 143–172.

Jones, C. and P. Seguin (1997). Transaction costs and price volatility: Evidence from commission deregulations. *American Economic Review 82*, 728–737.

Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money*. London: Macmillan.

Kupiec, P. H. (1995). A securities transactions tax and capital market efficiency. *Contemporary Economic Policy 93*, 101–112.

Kupiec, P. H. (1996). Noise traders, excess volatility, and a securities transactions tax. *Journal of Financial Services Research 10*(2), 115–129.

Lippman, S. A. and J. J. McCall (1986). A operational measure of liquidity. *American Economic Review 76*(1), 43–55.

Liu, S. and Z. Zhu (2009). Transaction costs and price volatility: New evidence from the tokyo stock exchange. *Journal of Financial Services Research 36*, 65–83.

Lo, A. W. and A. C. MacKinlay (1990). An econometric analysis of non-synchronous trading. *Journal of Econometrics 45*(1–2), 181–211.

Lu, X. and R. P. McAfee (1996). The evolutionary stability of auctions over bargaining. *Games and Economic Behavior 15*(2), 228–254.

Mannaro, K., M. Marchesi, and A. Setzu (2008). Using an artificial financial market for assessing the impact of Tobin-like transaction taxes. *Journal of Economic Behavior and Organization 67*(2), 445–462.

Mortensen, D. T. and R. Wright (2002). Competitive pricing and efficiency in search equilibrium. *International Economic Review 43*(1), 1–20.

Palley, T. I. (1999). Speculation and Tobin taxes: Why sand in the wheels can increase economic efficiency. *Journal of Economics 69*(2), 113–126.

Peel, Q. (2010, June 8). Berlin reveals drastic €80bn cuts. *Financial Times*.

Pellizzari, P. and F. Westerhoff (2009). Some effects of transaction taxes under different microstructures. *Journal of Economic Behavior and Organization 72*(3), 850–863.

Pollin, R., D. Baker, and M. Schaberg (2003). Securities transaction taxes for U.S. financial markets. *Eastern Economic Journal 29*, 527–558.

Roll, R. (1984). A simple implicit measure of the effective bid-ask spread in an efficient market. *Journal of Finance 39*(4), 1127–1139.

Rubenstein, A. and A. Wolinsky (1985). Equilibrium in a market with sequential bargaining. *Econometrica 53*, 1133–1150.

Schwert, G. W. and P. J. Seguin (1993). Securities transaction taxes: An overview of costs, benefits and unresolved questions. *Financial Analyst's Journal 49*(2), 27–35.

Spahn, P. B. (2002). On the feasibility of a tax on foreign exchange transactions. Report to the federal ministry for economic cooperation and development, Federal Republic of Germany.

Stiglitz, J. (1989). Using tax policy to curb speculative short-term trading. *Journal of Financial Services Research 3*(2–3), 101–115.

Summers, L. H. and V. P. Summers (1989). When financial markets work too well: A cautious case for a securities transactions tax. *Journal of Financial Services Research 3*(2), 261–286.

Tobin, J. (1974). *The New Economics a Decade Older*. Princeton: Princeton University Press.

Tobin, J. (1978). A proposal for international monetary reform. *Eastern Economic Review 4*, 153–159.

Tobin, J. (1984). On the efficiency of the financial system. *Lloyd's Bank Review 153*, 1–15.

ul Haq, M., I. Grunberg, and I. Karl (1996). *The Tobin Tax: Coping with Financial Volatility*. Oxford: Oxford University Press.

Umlauf, S. R. (1993). Transaction taxes and the behavior of the swedish stock market. *Journal of Financial Economics 33*, 227–240.

Wall Street Journal (2012, July 31). What's french for boomerang? Staff editorial.