Funding Cost and a New Capital Model

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In asset and derivative pricing, funding costs and capital[1] costs are usually considered separately. A derivative will be funded at a given rate such as OIS, LIBOR or the bank's cost of borrowing, and a cost of capital will be added separately. This paper presents a model that combines the two, using funding attributions from a capital model based on the bank's Expected Loss (*EL*) rather than the market standard Probability of Default (*PD*).

The basic idea is: A bank[2] could fund a new asset with the combination of debt and equity that leaves its *EL* constant. The debt-equity mix gives a funding cost that reflects the risk of the asset rather than the bank, so is a more appropriate rate for assessing the asset than the bank's Weighted Average Cost of Capital (WACC). In this way, the model facilitates decisions consistent with the Modigliani and Miller theorem (i.e. decisions based on the risk of the asset rather than the bank's cost of funding).

A result of the model is that, in accordance with the view of Hull and White (2012), the cost of funding a derivative is given by its CVA-DVA adjusted price and does not require an additional Funding Value Adjustment (FVA).

Some of the funding ideas produced by the model have already been suggested by others, such as Piterbarg (2010) and Burgard and Kjaer (2011).

[1] The words capital, equity and shares are used interchangeably.
[2] This paper refers to a bank as the subject or investing entity, though it could equally be any entity funded by a combination of equity and debt.

EL Based Capital Model

Current practice in bank capital modelling is to base the capital number on the quantile of a loss distribution derived from a target *PD*. For example if a bank targets a AA rating which has a historical 1-year *PD* of say 0.05%, then the capital is given by the lower 0.05% tail of the 1-year loss distribution.[3]

This paper describes instead a capital model based on the percentage Expected Loss (*EL*) on the bank's debt; *EL* = *PD* × *LGD* where *LGD* = Loss Given Default. An *EL* model doesn't provide quite as simple a relationship between capital and target rating. However, it produces asset-level attributions of capital and debt that better reflect an asset's contribution the bank's risk, and produce funding rates that are better related to market rates of return.

EL and Borrowing Cost

The model's key assumption is a relation between the bank's *EL* and its borrowing cost. Specifically the assumption is:

> *A change to the portfolio will alter the borrowing cost unless it leaves the EL constant.*

So if a new asset alters the *EL*, its assigned funding cost is the cost of adjusting the bank's leverage to restore the *EL.* Put another way the cost of issuing new capital and debt so that the group (new asset, new capital, new debt) has a net neutral impact on the *EL*.

The assumption seems reasonable since the *EL* is the expected monetary loss to the bank's bond investors so should drive the bond price. Hull and White (2013) note that practitioners are wary of assigning a connection between a bank's risk and its borrowing cost, due to the limits of investor's ability to assess the bank's portfolio. However they make the point that though investors may be wrong, provided they do not systematically over or under-estimate the risk, the bank's best estimate is to assume they are getting it right.

In reality, a bank's bond price will be affected by the flow of information. A new asset will probably not affect the borrowing cost until investors learn of it in the next periodical report. Modelling this information flow would be an interesting extension to the model though for this paper I assume every asset affects the bond price from point of purchase.

[3] The Basel II advanced regulatory credit capital model uses a target PD of 0.1%, expressed as a survival probability of 0.999. See page 64 of http://www.bis.org/publ/bcbs128.pdf.

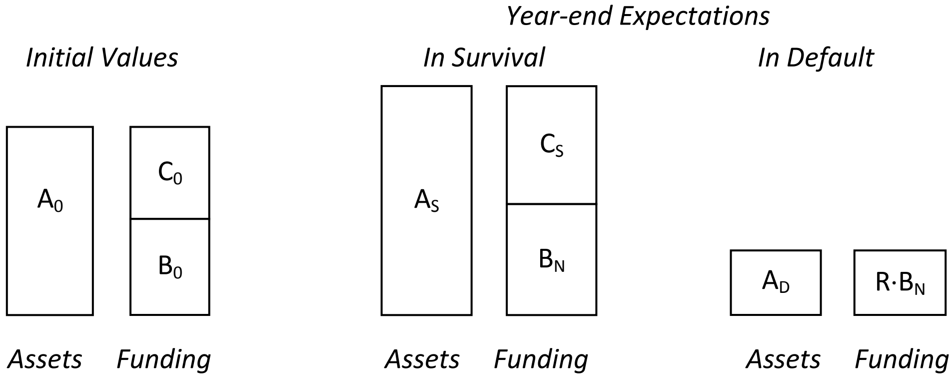
Negative Capital Attributions

An interesting aspect of the *EL* model is that assets less risky than the bank's own bonds receive a negative capital attribution. For example the bank could finance a $100 low risk asset by issuing $150 of bonds and using the remaining $50 to re-purchase its own shares.

This may seem strange though purchasing the low risk asset reduces the risk to the bank's debt holders by diluting the possible losses from the bank's other assets. Re-purchasing the bank's shares increases its leverage so restores the risk level. The negative capital attribution means the asset's funding cost will be below the bank's cost of borrowing. This is useful since being less risky, its expected return implied by its market price will probably also be lower than the bank's borrowing cost.

Model

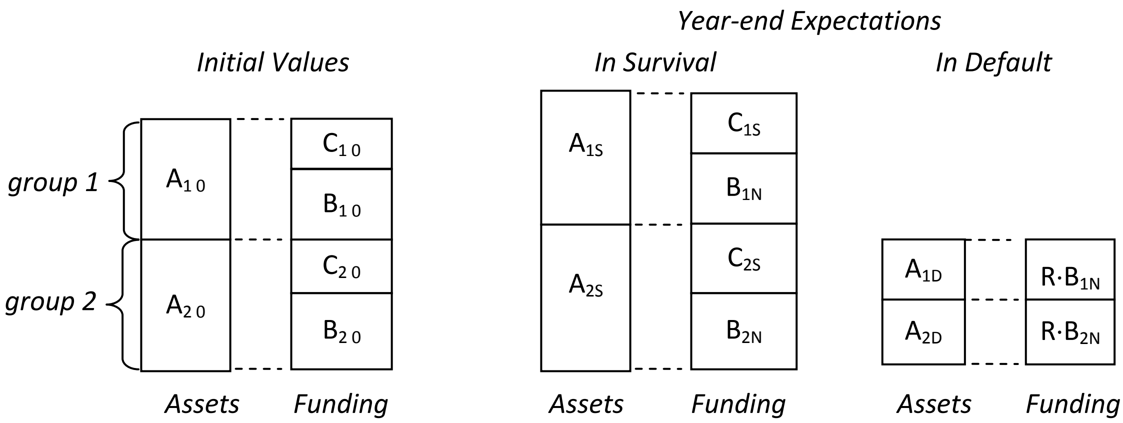
Consider a one-period (1-year) model of a bank balance sheet where A = B + C are the bank's assets (A), debt (B) and capital (C). All debt is equally senior zero-coupon bonds (B for bonds) maturing at year-end. So at year-end the bank is either in default (A<B) or survival (A>=B). Let subscripts $_{0}$, $_{S}$, $_{D}$ denote initial values ($_0$) and expectations in the event of survival ($_S$) and default ($_D$). In the case of debt, the expected value given survival is the bond notional denoted B_N, and in default is the notional times the expected recovery rate R. The value of capital in default is zero.



Attributions of Capital and Debt

Now let $\{A_i\}$ denote the individual assets so that $A = \Sigma A_i$ and divide the capital and debt into asset level attributions: $B = \Sigma B_i$, $C = \Sigma C_i$, and let *group i* denote the combination of an asset and its funding attributions (A_i, B_i, C_i). Suppose the attributions are defined so that their value equals the asset value initially and in expectation in survival and default:

$$\begin{aligned} A_{i0} &= B_{i0} + C_{i0} \\ A_{iS} &= B_{iN} + C_{iS} \\ A_{iD} &= R\cdot B_{iN} \end{aligned} \qquad (1)$$



These attribution formulas are the key to this model. In particular $A_{iD} = R{\cdot}B_{iN}$ defines a funding attribution that neutralises an asset's effect on the *EL*. This means the weighting of a group could be increased or decreased by a small amount without affecting the *EL.* For example, if a $2m asset has attributions of $1m debt and $1m equity, then the bank could purchase another $2 of the asset by issuing $1 debt and $1 equity without affecting the *EL*.

A proof of this is given in Appendix B. A rough explanation is that although the expected loss is given by *EL* = *PD·LGD* = *PD·*(1-R), the effect of a small change (like the above) is captured though its effect on R only[4]. So, to leave R constant a new asset's funding must equal its expected value in default: $A_{iD} = R{\cdot}B_{iN}$.

Pricing

Suppose we know (or have an assumption about) the year-end expectations of a given asset A_{iS} and A_{iD}. We can solve equations (1) for the year-end survival attributions.

$$\begin{aligned} B_{iN} &= A_{iD} / R \\ C_{iS} &= A_{iS} - A_{iD} / R \end{aligned} \qquad (2)$$

Now define relations between the initial debt and capital values and the expected year-end survival values:

$$\begin{aligned} B_0 &= v_B\, B_N \\ C_0 &= v_C\, C_S \end{aligned} \qquad (3)$$

where v_B is the initial price of $1 notional of the bank's bonds and v_C is defined by the bank's expected or desired future capital value C_S. In other words, v_C is a discount factor based on the bank's cost of capital. Since debt and capital are homogeneous, the same relations apply to the attributions:

$$\begin{aligned} B_{i0} &= v_B\, B_{iN} \\ C_{i0} &= v_C\, C_{iS} \end{aligned} \qquad (4)$$

Combining equations (2) and (4) we can calculate the initial price of an asset in terms of its year-end expectations:

$$\begin{aligned} A_{i0} &= B_{i0} && + C_{i0} \\ &= v_B\, B_{iN} && + v_C\, C_{iS} \\ &= v_B\, A_{iD} / R && + v_C\, (A_{iS} - A_{iD} / R) \end{aligned} \qquad (5)$$

[4] This is similar to the Euler allocation of Expected Shortfall (ES) being the expected value of a given trade in the event of the bank's default. See for example:
http://www.greta.it/credit/credit2007/thursday/1_Tasche.pdf

Bonds & CDS

Now consider two other year-end-maturing zero-coupon bonds, one risk-free and one issued by a risky entity referred to as the *counterparty*. Consider also CDS protection on the bank and counterparty bonds purchased from a risk-free entity for a single premium at year-start, and define the following terms:

Values per $1 Notional

v_R v_F	Initial prices of the risky counterparty bond (v_R) and risk-free bond (v_F)
CDS_R CDS_B	Initial prices of CDS protection on the counterparty bond (CDS_R) and the bank's bond (CDS_B)
L_S L_D	Expected loss on the <u>counterparty</u> bond in the event of the <u>bank's</u> survival (L_S) and default (L_D)

This paper does not include an explicit model for the bank-counterparty correlation, but it is captured implicitly by L_S and L_D. For example, a positive correlation means the counterparty's expected loss will be higher if the bank defaults than survives, so that $L_D > L_S$. Possible combinations of L_S and L_D are:

$L_S=L_D=0$	Counterparty is risk-free
$L_S=L_D$	Counterparty – bank correlation is zero
$L_D>L_S$	Counterparty – bank correlation is positive
$L_D<L_S$	Counterparty – bank correlation is negative
$L_S=0$, $L_D=LGD$	Counterparty – bank correlation is high and the two entities are equally risky

Under the model, a risky bond and CDS combination is equivalent to the risk-free bond so has the same price. That is:

$$v_F = v_R + CDS_R \qquad \textit{counterparty bond plus CDS protection}$$
$$v_F = v_B + CDS_B \qquad \textit{bank bond plus CDS protection} \qquad (6)$$

Combining these terms with the attribution and pricing formulas we can derive initial values in terms of the expected loss rates.

Values per $1 Notional

Asset	Expectations given the bank's Survival and Default		Funding Attributions		Initial Value A_{i0}
	A_{iS}	A_{iD}	B_{iN}	C_{iS}	
Counterparty Bond	$1-L_S$	$1-L_D$	$(1-L_D)/R$	$(1-L_S)-(1-L_D)/R$	$v_R = v_B(1-L_D)/R + v_C((1-L_S)-(1-L_D)/R)$
Risk-Free Bond	1	1	1/R	1-1/R	$v_F = v_B/R + v_C(1-1/R)$
Bank's Bond	1	R	1	0	$v_B = v_B$
Counterparty CDS	L_S	L_D	L_D/R	$L_S - L_D/R$	$CDS_R = v_B L_D/R + v_C(L_S - L_D/R)$
Bank CDS	0	1-R	1/R-1	1-1/R	$CDS_B = v_B(1/R-1) + v_C(1-1/R)$

Now consider an example where $v_B = 0.9$, $v_C = 0.85$, R = 50%, $L_S = 0$ and $L_D = 75\%$. (L_S should really be >0 since there is always a chance the counterparty will default. However setting L_S=0 does not affect the logic of the model and simplifies the calculations in the following examples.)

Example – Values per $1 notional

Asset	Expectations given the bank's Survival and Default		Funding Attributions		Initial Value A_{i0}
	A_{iS}	A_{iD}	B_{iN}	C_{iS}	
Counterparty Bond	1	0.25	0.50	0.50	$v_R = 0.9\times0.5 + 0.85\times0.5 = 0.875$
Risk-Free Bond	1	1	2	-1	$v_F = 0.9\times2 - 0.85\times1 = 0.95$
Bank's Bond	1	0.5	1	0	$v_B = 0.9\times1 = 0.9$
Counterparty CDS	0	0.75	1.5	-1.5	$CDS_R = 0.9\times1.5 - 0.85\times1.5 = 0.075$
Bank CDS	0	0.5	1	-1	$CDS_B = 0.9\times1 - 0.85\times1 = 0.05$

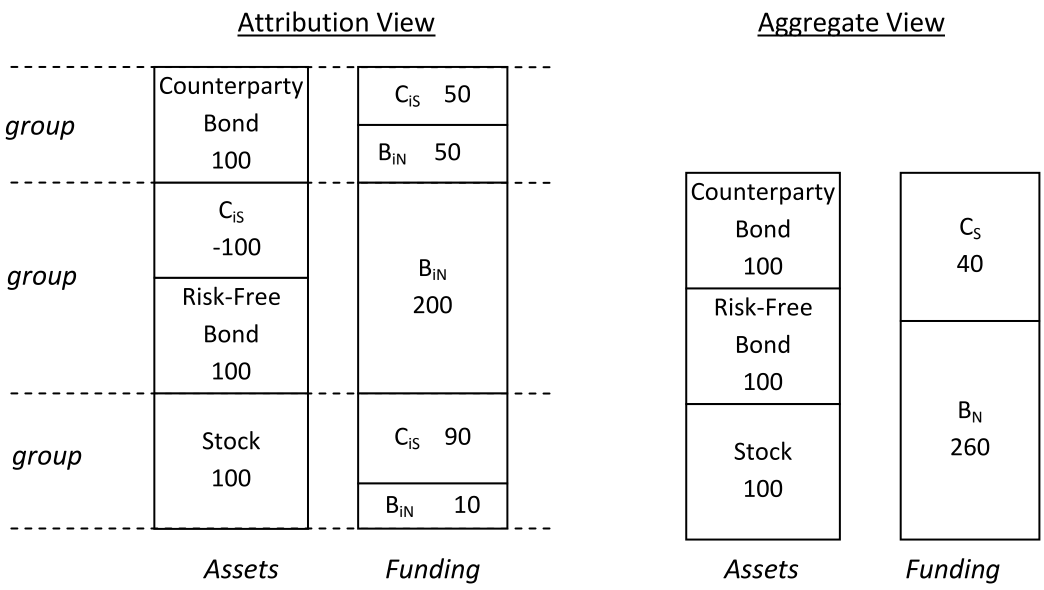
Example with $100 Notionals

The below diagrams illustrate the above example with notionals of $100. Negative funding attributions are shown on the assets side. For example, the risk-free bond's capital attribution of C_{i0} = -$85 means re-purchasing $85 of the bank's equity, so is shown as an asset. In this way, the *EL* neutrality of the asset-funding group is shown by the two sides being equal in all three cases.

			Year-end Expectations			
	Initial Values		*In Survival*		*In Default*	
	Assets	*Funding*	*Assets*	*Funding*	*Assets*	*Funding*
Counterparty Bond	A_{i0} 87.5	C_{i0} 42.5 B_{i0} 45	A_{iS} 100	C_{iS} 50 B_{iN} 50	A_{iD} 25	$R{\cdot}B_{iN}$ 25
Risk-Free Bond	C_{i0} -85 A_{i0} 95	B_{i0} 180	C_{iS} -100 A_{i0} 100	B_{iN} 200	A_{iD} 100	$R{\cdot}B_{iN}$ 100
Bank's Bond	A_{i0} 90	B_{i0} 90	A_{iS} 100	B_{iN} 100	A_{iD} 50	$R{\cdot}B_{iN}$ 50
CDS on Counterparty Bond	C_{i0} -135 A_{i0} 7.5	B_{i0} 142.5	C_{iS} -150	B_{iN} 150	A_{iD} 75	$R{\cdot}B_{iN}$ 75
CDS on Bank's Bond	C_{i0} -85 A_{i0} 5	B_{i0} 90	C_{iS} -100	B_{iN} 100	A_{iD} 50	$R{\cdot}B_{iN}$ 50

Negative Attributions

As shown above, individual assets can have negative attributions of debt or capital. However, the aggregate debt and capital are always positive, being the sum of the positive and negative attributions. The below illustration shows an aggregate and attribution view of the year-end survival expectations, for a balance sheet comprising the counterparty and risk-free, as well as a stock financed mostly by equity.



Bank's Own Bond

The funding attribution for the bank's own bond is entirely debt. That is, the bank must fund the purchase of \$100 of its own bonds by issuing \$100 of new bonds – effectively cancelling out the transaction. This makes sense since any capital in the funding mix would alter the bank's leverage and therefore its *EL*.

Market Prices and Private Values

Each investor will have its own estimates of L_S and L_D based on its assessment of the counterparty's risk and correlation to itself. So each investor will have a private value of a bond or other asset, and the market price will be that which matches supply and demand.

However taking market prices as inputs, the equations can be back-solved for R, v_C, L_S and L_D. That is, we can set these parameters so that that v_F matches the market price of a risk-free bond and v_R matches the price of the counterparty bond.

In the following sections we assume the parameters have been calibrated in this way, and derive funding attributions for derivatives with the counterparty. We see that the value of the funding attributions equals the CVA-DVA adjusted price, with no FVA.

So, the CVA adjusted price is equivalent to the market price of bond. A bank can have a different private value but this is due to its assessment of the counterparty risk or its desired return on capital, not due to its borrowing cost.

Derivative – Counterparty owes Bank

Consider a derivative contract between the bank and counterparty where all cashflows occur at year-end and have equal seniority to the bond payments for both entities.

Suppose under the contract the counterparty must pay the bank \$100 at year-end. This is equivalent to the bank owning \$100 notional of the counterparty's bonds. Under CVA adjusted pricing these should have the same price and using equations (6) we see this is the case. In this one-period model the CVA is the expected exposure (\$100) times the risk-free discounted risk-neutral expected loss which is conveniently given by the counterparty CDS price CDS_R.

$$
\begin{aligned}
A_{i0} &= 100v_F - \text{CVA} && \textit{CVA adjusted derivative price} \\
&= 100v_F - 100\,CDS_R \\
&= 100v_F - 100\,(v_F - v_R) && \textit{by equation (6)} \\
&= 100v_R && \textit{Counterparty Bond Price} \qquad (7)
\end{aligned}
$$

This equation just expresses the economic equivalence of the derivative and bond, so does not require the *EL* capital model. However, what the *EL* capital model adds is that being economically equivalent, the derivative will have the same funding attributions as the bond. So as with the bond, the derivative can be funded with a combination of debt and equity equal in value to this price, so an additional FVA is not necessary.

Derivative – Bank owes Counterparty

Now consider a contract where the bank pays the counterparty \$100 at year-end. This is equivalent to the counterparty owning \$100 notional of the bank's bonds. As above we see the equivalence of the derivative and bond price.

$$
\begin{aligned}
A_{i0} &= -100v_F + DVA && \textit{DVA adjusted derivative price} \\
&= -100v_F + 100\,CDS_B && \\
&= -100v_F + 100\,(v_F - v_B) && \textit{by equation (6)} \\
&= -100v_B && \textit{Bank Bond Price} \qquad (8)
\end{aligned}
$$

So the counterparty must pay the bank $100v_B$ to enter the agreement. The expected value of the derivative in default is $A_{iD} = R \cdot 100 = \$50$, so by equation (2) the funding attributions are $C_{iS} = 0$, $B_{iN} = -\$100$. Given this, we consider two ways the bank could construct an *EL* neutral group.

1) ***Repurchase own Bonds***
   Use the $100v_B$ to repurchase \$100 notional of its own bonds[5].

2) ***Purchase Risk-Free Bond and Sell CDS on Itself*** [6]
   Sell \$100 notional CDS protection on itself and use the combined premiums $100v_B + 100\,CDS_D = 100v_F$ to purchase a \$100 notional risk-free bond.

Both groups effectively monetise and hedge the DVA. In 2) the DVA can be equated to the CDS premium received, in 1) the DVA can be equated to the price difference between the bank and risk-free bonds.

Booking a DVA ***and FVA*** would amount to combining 1) and 2) (repurchasing the bank's own bond ***and*** selling CDS protection on itself). As shown below, this would mean doubling up on the bank's credit risk, so the resulting group would not be *EL* neutral. That is, would not have equal-opposing expectations in default.

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[5] In practice this could mean investing the money with the treasury desk, which offsets it against other funding requirements so pays the bank's cost of borrowing.

[6] In practice, some banks sell CDS protection on names highly correlated to themselves to monetise and hedge DVA.

The CDS price would actually be different from CDS_B because the bank is both the issuer and underlying, the recovery rate R would hit the payout twice. However, the payout and price would reduce in the same proportion, so the notional could be adjusted accordingly.

Possible Funding / Hedging Combinations for a fixed derivative with DVA

		Year-end Expectations	
	Initial Values	In Survival	In Default

1) *Repurchase own Bonds*

Initial Values		In Survival		In Default	
Bank's Bond 90	Derivative -90	Bank's Bond 100	Derivative -100	Bank's Bond R·100 =50	Derivative -R·100 =-50

2) *Purchase Risk-Free Bond and Sell CDS on Itself*

Initial Values		In Survival		In Default	
Risk-Free Bond 95	CDS -5	Risk-Free Bond 100	Derivative -100	Risk-Free Bond 100	CDS -R·100 =-50
	Derivative -90				Derivative -R·100 =-50

3) *Book DVA and FVA*
(The equivalent of repurchasing own bonds and Selling CDS on Itself)

Initial Values		In Survival		In Default	
Bank's Bond 90	CDS_D -5	Bank's Bond 100	Derivative -100	Bank's Bond R·100 =50	CDS -R·100 =-50
	Derivative with FVA -85				Derivative -R·100 =-50

Expectations not equal in default

General Equity Derivative

Let X be a derivative contract on a non-dividend-paying stock S with a single payment at year-end. Let X_N be the contractual obligation of the contract. If $X_N > 0$, the counterparty pays the bank, if $X_N < 0$ the bank pays the counterparty. Suppose the bank holds the sub-portfolio P comprising the derivative and a holding of S that perfectly hedges the contractual obligation so that at maturity $X_N + S = K$, a constant. The expected year-end values of P in survival and default are:

$$P_S = K - X^+_N L_S$$
$$P_D = K - X^+_N L_D + X^-_N(1-R)$$

where

$$X^+_N = E[\max(X_N,0) \mid \textit{counterparty default}]^7 \qquad (9)$$
$$X^-_N = E[\min(X_N,0) \mid \textit{bank default}]$$

That is, the no default value K minus the expected counterparty credit losses, plus the expected bank credit losses. In survival the bank's loss rate is 0 and in default it is *LGD* = 1-R. The initial value of P given by the CVA-DVA adjusted price is:

$$\begin{aligned} P_0 &= v_F K \quad - \text{CVA} \quad + \text{DVA} \\ &= v_F K \quad - X^+_N\, CDS_R \quad + X^-_N\, CDS_B \end{aligned} \qquad (10)$$

As shown below, a portfolio can be constructed of risk-free bond and bank and counterparty CDS with the same initial price and the same survival/default expectations as P, meaning P could be financed in the same way. So as with the other examples, the portfolio (and therefore the derivative) could be funded with a combination of debt and equity equal in value to the CVA-DVA adjusted price.

Constructing CVA-DVA Adjusted Price

Price Component	Equivalent Instrument	Notional	Expectations in Survival and Default		Initial Value
			A_{iS}	A_{iD}	A_{i0}
Risk-Free	Risk-Free Bond	K	K	K	$v_F K$
CVA	Counterparty CDS	$-X^+_N$	$-X^+_N\ L_S$	$-X^+_N\ L_D$	$-X^+_N\ CDS_R$
DVA	Bank CDS	X^-_N	0	$X^-_N(1-R)$	$X^-_N\ CDS_B$
Total			P_S	P_D	P_0

[7] X^+_N should also depend on the bank's survival / default. That is there should be two values:
$E[\max(X_N,0) \mid$ *counterparty default & bank survival*$]$
$E[\max(X_N,0) \mid$ *counterparty default & bank default*$]$
I use one value of X^+_N to simplify the calculation.

Example – Equity Forward Contract

Suppose X is an ATM forward contract on S, S_0=\$95 and K=\$100 (the risk-free accumulation of S). Suppose also that X^+_N = \$10 and X^-_N = -\$10. We can calculate the value and funding attributions for P as the sum of the components discussed above.

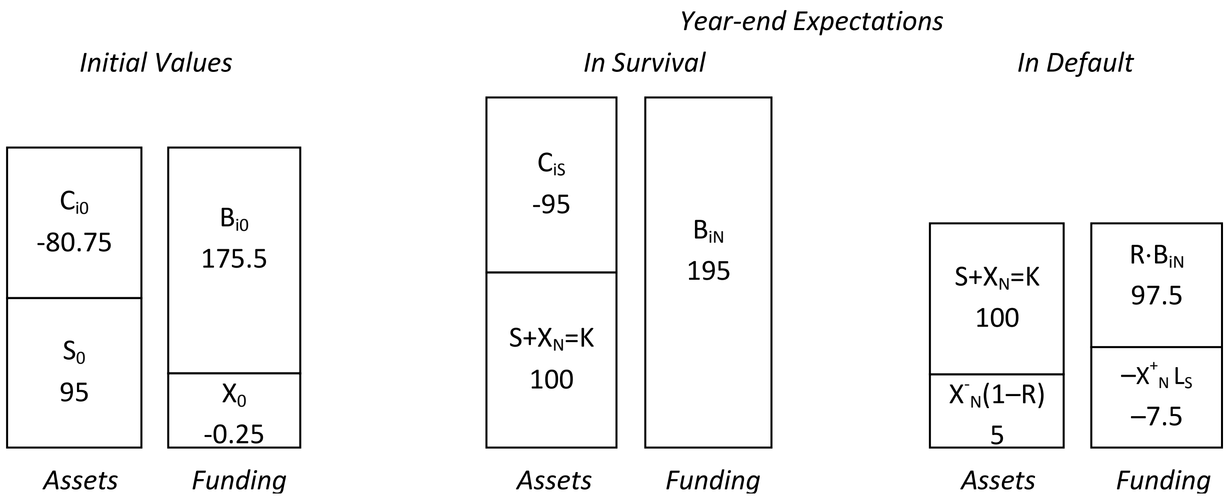
Portfolio Value

Price Component	Equivalent Instrument	Notional	Unit Price	Init Value	Funding Attributions	
				A_{i0}	B_{iN}	C_{iS}
Risk-Free	Risk-Free Bond	100	0.95	95	200	-100
CVA	Counterparty CDS	-10	0.075	-0.75	-15	15
DVA	Bank CDS	10	0.05	0.5	10	-10
Total				**94.75**	195	-95

We can check the funding attribution's initial value equals that of the components:
$v_B B_{iN} + v_C C_{i0} = 0.9\times195 - 0.85\times95 = \$94.75$.

So, the initial forward contract value is $X_0 = P_0 - S_0 = 94.75 - 95 = -\$0.25$. The value is slightly negative because the counterparty is more risky than the bank (has a higher CDS price). So the counterparty should pay the bank \$0.25 to enter the trade.

The below diagram shows the complete *EL* neutral group: derivative, stock, debt attribution and capital attribution.[8]



[8] From the assumption on page 6 that L_S=0, there is no counterparty credit loss in the event of the bank's survival. This is not an accurate assumption but makes the calculations simpler.

Conclusion

The *EL* based capital model presented here shows how a bank can fund an asset with a mix of debt and equity so that the funding cost reflects the risk of the asset, and can be calibrated to match the asset's market price.

Under the *EL* model an asset with a lower risk and lower expected return than the bank's debt can be an economically sensible investment because its risk-reducing effect is reflected in a lower funding rate. This is different from a standard *PD* based model, where the capital can only be positive so the implied funding rate is always greater than the bank's borrowing cost.

If the model is calibrated to bond prices it can be used to derive funding attributions for derivatives with a value equal to the CVA-DVA adjusted price (no FVA). Investor's private value can be different to this price, but this is equivalent to investors having different private values for a bond. That is, the difference comes from assessments of risk and correlation, and required return on capital, not from borrowing cost.

Many extensions to the model are possible including:

- A multi-period model, discussed in Appendix C.
- P&L attribution, discussed in Appendix D.
- Making the cost of capital dependant on the shareholder's risk.
- Relating the cost of borrowing to the bank's correlation to the market as well as the *EL* (incorporating CAPM concepts).
- Modelling the flow of information, so that an asset doesn't affect the borrowing cost until it is announced to the market.
- Multiple tiers of debt.

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Appendix A — Attribution Equations

As stated in the paper, the attribution equation $A_{iD} = R \cdot B_{iN}$ allows the weighting of a group to be increased or decreased by a small amount without affecting the bank's *EL.* To prove this, we can derive the equation by setting the derivative of the *EL* with respect to the weighting of a group, to zero.

Let $\{w_i\}$ be a set of weights where $w_i = 1$ for all i, and re-express A, B and C as $A = \Sigma w_i A_i$, $B = \Sigma w_i B_i$, $C = \Sigma w_i C_i$, so that a derivative with respect to w_i is a derivative with respect to the weighting of group i. Note that $\partial A/\partial w_i = A_i$ and $\partial B_N/\partial w_i = B_{iN}$. Now let the *EL* be defined as the expected percentage loss on the debt notional.

$$
\begin{aligned}
EL &= PD \cdot LGD \\
&= \Pr(A < B_N)\,E\left[\frac{A}{B_N} - 1 \,\middle|\, A < B_N\right] \\
&= E\left[I_{A<B_N}\left(\frac{A}{B_N} - 1\right)\right]
\end{aligned}
$$

Where $I_{A<B_N}$ equals 1 if $A<B_N$ (default) and 0 otherwise. Now, take the derivative with respect to w_i:

$$
\begin{aligned}
\frac{\partial EL}{\partial w_i} &= \frac{\partial}{\partial w_i} E\left[I_{A<B_N}\left(\frac{A}{B_N} - 1\right)\right] \\
&= E\left[I_{A<B_N}\frac{\partial}{\partial w_i}\left(\frac{A}{B_N} - 1\right)\right] \qquad \text{(see Appendix B)} \\
&= \frac{1}{B_N^2} E[I_{A<B_N}(B_N A_i - A B_{iN})] \\
&= \frac{1}{B_N^2}(B_N A_{iD} - A_D B_{iN})
\end{aligned}
$$

Setting equal to zero gives:

$$
\begin{aligned}
B_N A_{iD} &= A_D B_{iN} \\
A_{iD} &= (A_D/B_N) B_{iN} \\
&= R \cdot B_{iN}
\end{aligned}
$$

Appendix B — Derivative of Partial Expectation

Appendix A includes the step:

$$\frac{\partial}{\partial \mathrm{w_i}}\mathrm{E}\left[I_{\mathrm{A<B_N}}\left(\frac{\mathrm{A}}{\mathrm{B_N}}-1\right)\right]=\mathrm{E}\left[I_{\mathrm{A<B_N}}\frac{\partial}{\partial \mathrm{w_i}}\left(\frac{\mathrm{A}}{\mathrm{B_N}}-1\right)\right]$$

To prove this I let Y = A/$\mathrm{B_N}$-1 and x = $\mathrm{w_i}$ and re-write as:

$$\frac{\partial}{\partial x}\mathrm{E}[I_{Y<0}Y]=\mathrm{E}\left[I_{Y<0}\frac{\partial}{\partial x}Y\right]$$

where Y is a random variable that depends on x. Provided Y is suitably smooth and continuous, we can express the expectation of Y as:

$$\mathrm{E}[Y]=\int_{-\infty}^{\infty} yf(y,x)dy$$

Where $f(y,x)$ is the density function of y and depends on x. Also, we can express the expectation of any function of Y, $G(Y)$ as:

$$\mathrm{E}[G(Y)]=\int_{-\infty}^{\infty} G(y)f(y,x)dy$$

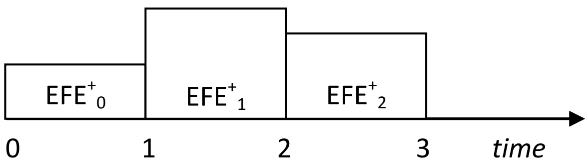
Combining the above we can write:

$$\begin{aligned}\frac{\partial}{\partial x}\mathrm{E}[I_{Y>0}Y] &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} I_{y>0}yf(y,x)dy\\ &= \frac{\partial}{\partial x}\int_{0}^{\infty} yf(y,x)dy\\ &= \int_{0}^{\infty}\frac{\partial}{\partial x}yf(y,x)dy\\ &= \int_{-\infty}^{\infty} I_{y>0}\frac{\partial}{\partial x}yf(y,x)dy\\ &= \mathrm{E}\left[I_{Y>0}\frac{\partial}{\partial x}Y\right]\end{aligned}$$

Appendix C — Possible Extension: Multi-Period Model

This paper uses a single period model where default either occurs or not at year-end. In reality default can occur at any time, so CVA and DVA are generally calculated using bucketed profiles of positive and negative expected future exposure EFE^+ and EFE^- multiplied by CDS implied risk-neutral loss rates for each time bucket.

Example EFE^+ Profile



The general formulas could be written as:

$$CVA = \Sigma_t \, EFE^+_t \, CDS_{Rt}$$
$$DVA = \Sigma_t \, EFE^-_t \, CDS_{Bt}$$

Where EFE^+_t is the EFE for time bucket (t, t+1) and CDS_{Rt} is the current value (as a single premium) of CDS protection on the counterparty bond for the same period. We can create multi-period versions of equations (6) as:

$$CDS_{Rt} = (v_{F\,t+1} - v_{F\,t}) \quad - \quad (v_{R\,t+1} - v_{R\,t})$$
$$CDS_{Bt} = (v_{F\,t+1} - v_{F\,t}) \quad - \quad (v_{B\,t+1} - v_{B\,t})$$

That is, the CDS protection can be replicated by a long-short position across tenors of risks-free bonds and bonds of the underlying. Funding attributions for risky and risk-free bonds of various tenors could be created using capital and the bank's own bonds of the same tenor. Putting all this together, CVA-DVA funding attributions could be constructed from capital and the bank's bonds of various tenors over the life of the trade.

Appendix D — Possible Extension: P&L Attribution

Another possible extension is a functional relationship between the bank's *EL* and credit spread. That is putting; *Credit Spread* = *f*(*EL*) for some function *f*(). This would allow calculation of through-time metrics such as an attribution of P&L. The bank's debt value B depends on interest rates and time to maturity as well as credit spread, but if we let α denote the credit sensitivity (CS01) of B, we can define the relation:

$$\frac{\partial \mathrm{B}}{\partial EL} = \alpha\, f^{'}(EL)$$

Now let subscript $_t$ define a value at time t[9] and using the $\{w_i\}$ weights from Appendix A we can define a profit attribution to asset i as the derivative of the profit with respect to w_i.

$$\begin{aligned}
\mathrm{Profit}_{i} &= \frac{\partial \mathrm{Profit}}{\partial \mathrm{w_i}} \\
&= \frac{\partial}{\partial \mathrm{w_i}} \mathrm{C_t} - \mathrm{C_0} \\
&= \mathrm{C_{it}} - \mathrm{C_{i0}} - \left(\alpha_t f^{'}(EL_t) \frac{\partial EL_t}{\partial \mathrm{w_i}} - \alpha_0 f^{'}(EL_t) \frac{\partial EL_0}{\partial \mathrm{w_i}} \right) \\
&= \mathrm{C_{it}} - \mathrm{C_{i0}} - \alpha_t f^{'}(EL_t) \frac{\partial EL_t}{\partial \mathrm{w_i}}
\end{aligned}$$

The last step comes from the attribution definitions given in the paper ($\partial EL_0/\partial w_i = 0$). This is a comprehensive profit measure. It measures the incremental profit from investing in an extra $1 of the asset, accounting for its funding attributions and effect on the bank's debt value given by its effect on the *EL*. For example if the bank invests in the counterparty bond and the counterparty becomes more risky, there will be a loss (C_{it}–C_{i0}) as the bond loses value, but this will be partly offset by a reduction in the bank's debt value as the bond increases the *EL*. In a similar way we can define an asset level attribution of Return on Capital (ROC).

$$\begin{aligned}
\mathrm{ROC_i} &= \frac{\partial \mathrm{ROC}}{\partial \mathrm{w_i}} \\
&= \frac{\partial}{\partial \mathrm{w_i}} \frac{\mathrm{C_t}}{\mathrm{C_0}} \\
&= \frac{1}{\mathrm{C_0^2}} \left(\mathrm{C_0} \left(\mathrm{C_{it}} - \alpha_t f^{'}(EL_t) \frac{\partial EL_t}{\partial \mathrm{w_i}} \right) - \mathrm{C_t C_{i0}} \right)
\end{aligned}$$

[9] This is different from the definition of subscript $_t$ in Appendix C