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Barnett, William A. and Ghosh, Taniya

University of Kansas, Indira Gandhi Institute of Development Research, Reserve Bank of India

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William A. Barnett*
University of Kansas, Lawrence
and Center for Financial Stability, NY City

and

Taniya Ghosh
Indira Gandhi Institute of Development Research (IGIDR), Reserve Bank of India, Mumbai

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Abstract:

This paper analyzes, within its feasible parameter space, the dynamics of the Uzawa-Lucas endogenous growth model. The model is solved from a centralized social planner perspective as well as in the model's decentralized market economy form. We examine the stability properties of both versions of the model and locate Hopf and transcritical bifurcation boundaries. In an extended analysis, we investigate the existence of Andronov-Hopf bifurcation, branch point bifurcation, limit point cycle bifurcation, and period doubling bifurcations. While these all are local bifurcations, the presence of global bifurcation is confirmed as well. We find evidence that the model could produce chaotic dynamics, but our analysis cannot confirm that conjecture.

It is important to recognize that bifurcation boundaries do not necessarily separate stable from unstable solution domains. Bifurcation boundaries can separate one kind of unstable dynamics domain from another kind of unstable dynamics domain, or one kind of stable dynamics domain from another kind (called soft bifurcation), such as bifurcation from monotonic stability to damped periodic stability or from damped periodic to damped multiperiodic stability. There are not only an infinite number of kinds of unstable dynamics, some very close to stability in appearance, but also an infinite number of kinds of stable dynamics. Hence subjective prior views on whether the economy is or is not stable provide little guidance without mathematical analysis of model dynamics.

When a bifurcation boundary crosses the parameter estimates' confidence region, robustness of dynamical inferences from policy simulations are compromised, when conducted, in the usual manner, only at the parameters' point estimates.

Keywords: bifurcation, endogenous growth, Lucas-Uzawa model, Hopf, inference robustness, dynamics, stability.

**Corresponding author*: William A. Barnett, email: williamabarnett@gmail.com, tel: 1-785-832-1342, fax: 1-785-832-1527.

1. Introduction

The Uzawa-Lucas model (Uzawa (1965) and Lucas (1988)), upon which many others have been built, is among the most important endogenous growth models. The model has two sectors: the human capital production sector and the physical capital production sector producing human capital and physical capital, respectively. Individuals have the same level of work qualification and expertise (H). They allocate some of their time to producing final goods and dedicate the remaining time to training and studying.

The social planner solution for the Uzawa-Lucas model is saddle path stable, but the representative agent's equilibrium can be indeterminate, as shown by Benhabib and Perli (1994). As a result of the presence of externalities in human capital, the market solution is different from the social planner solution. The externality creates a distinction between return on capital, as perceived by the representative agent, to that perceived by a social planner.

We solve for the steady states and provide a detailed bifurcation analysis of the model. The task of this paper is to examine whether the dynamics of the model change within the feasible parameter space of the model. A system undergoes a bifurcation, if a small, smooth change in a parameter produces a sudden qualitative or topological change in the nature of singular points and trajectories of the system. The presence of bifurcation damages the inference robustness of the dynamics, when inferences are based on point estimates of the model. Hence, knowing the stability boundaries within the feasible region of the parameter space, especially near the point estimates, can lead to more robust dynamical inferences and more reliable policy conclusions.

Using Mathematica, we locate transcritical and Hopf bifurcation boundaries in two-dimension and three-dimension diagrams. The numerical continuation package, Matcont, is used to analyze further the stability properties of the limit cycles generated by Hopf bifurcations and the presence of other kinds of bifurcations. We show the existence of Hopf, branch-point, limit-point-of-cycles, and period-doubling bifurcations within the feasible parameters set of the model's parameter space. While these are all local bifurcations, presence of global bifurcation is also confirmed. There is some evidence of the possibility of chaotic dynamics through the detected series of period-doubling bifurcations, known to converge to chaos. Some of these results have not previously been demonstrated in the literature on endogenous growth models.

Benhabib and Perli (1994) analyzed the stability property of the long-run equilibrium in the Lucas (1988) model. Arnold (2000a,b) analyzed the stability of equilibrium in the Romer (1990) model. Arnold (2006) has done the same for the Jones (1995) model. Mondal (2008) examined the dynamics of the Grossman-Helpman (1991b) model of endogenous product cycles. The results derived in those papers provide important insights to researchers considering different policies. But, as in the Benhabib and Perli (1994) paper, a detailed bifurcation analysis has not been provided so far for many of these popular endogenous growth models. The current paper aims to fill this gap for the Uzawa-Lucas model.

As pointed out by Banerjee et al (2011): "Just as it is important to know for what parameter values a system is stable or unstable, it is equally important to know the nature of stability (e.g. monotonic convergence, damped single periodic convergence, or damped multi-periodic convergence) or instability (periodic, multi-periodic, or chaotic)." Barnett and his coauthors have made significant contribution in this area. Barnett and He (1999, 2002) examined the dynamics of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. Both transcritical bifurcation boundaries and the Hopf bifurcation boundaries for the model were found. Barnett and He (2008) have found singularity bifurcation boundaries within the parameter space for the Leeper and Sims (1994) model. Barnett and Duzhak (2010) found Hopf and period doubling bifurcations in a New Keynesian model. More recently, Banerjee et al (2011) examined the possibility of cyclical behavior in the Marshallian Macroeconomic Model and Barnett and Eryilmaz (2013a,b) have found bifurcation in open economy models.

2. The Uzawa-Lucas Model

The production function in the physical sector is defined as follows:

$$Y = AK^{\alpha}(\varepsilon hL)^{1-\alpha}h_a^{\zeta}, \qquad 0 < \alpha < 1,$$

where Y is output, A is constant technology level, K is physical capital, α is the share of physical capital, L is labor, and h is human capital per person. In addition, ε and $(1-\varepsilon)$ are the fraction of labor time devoted to producing output and human capital, respectively, where $0 < \varepsilon < 1$. Observe that εhL is the quantity of labor, measured in efficiency units, employed to produce output, and h_a^{ζ} measures the externality associated with average human capital of the work force, h_a, where ζ is the positive externality parameter in the production of human capital. In per capita terms, $y = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta}$.

The physical capital accumulation equation is $\dot{K} = AK^{\alpha}(\varepsilon hL)^{1-\alpha}h_a^{\zeta} - C - \delta K$. In per capita terms, $\dot{k} = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta} - c - (n+\delta)k$, and the human capital accumulation equation is $\dot{h} = \eta h(1-\varepsilon)$, where η is defined as schooling productivity.

The decision problem is

$$\max_{c_t,\varepsilon_t} \int_t^{\infty} e^{-(\rho-n)t} \; [c(\tau)^{1-\sigma} - 1] \,/\, (1-\sigma) \quad dt \tag{1}$$

subject to

$$\dot{k} = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta} - c - (n+\delta)k \tag{2}$$

and

$$\dot{h} = \eta(1-\varepsilon)h \tag{3}$$

where ρ $(\rho > n > 0)$ is the subjective discount rate, and σ (≥ 0) is the inverse of the intertemporal elasticity of substitution in consumption.

2.1. Social Planner Problem

In solving the maximization problem, (1), subject to the physical capital accumulation equation (2) and the human capital accumulation equation (3), the social planner takes into account the externality associated with human capital. From the first order conditions (see Appendix 2), we derive the equations describing the economy of the Uzawa-Lucas model from a social planner's perspective

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n+\delta),$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon),$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho+\delta)}{\sigma},$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(1-\alpha+\zeta)}{(1-\alpha)} \varepsilon + \eta \frac{(1-\alpha+\zeta)}{\alpha} - \frac{c}{k} + \frac{(1-\alpha)}{\alpha}(n+\delta),$$

$$\frac{\dot{L}}{L} = n.$$

Let $m = \frac{Y}{K} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta}$ & $g = \frac{c}{k}$. Taking logarithms of *m* and *g* and differentiating with respect to time, equations (4) and (5) define the dynamics of Uzawa-Lucas model

$$\frac{\dot{m}}{m} = -(1-\alpha)m + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha} \quad (4)$$

$$\frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right)m - \frac{\rho}{\sigma} - \delta\left(\frac{1}{\sigma} - 1\right) + g + n \quad (5)$$

The steady state (m^*, g^*) is given by $\dot{m} = \dot{g} = 0$ and derived to be

$$m^* = \eta\frac{(1-\alpha+\zeta)}{\alpha} + \frac{(n+\delta)}{\alpha},$$

$$g^* = \frac{\rho-n}{\sigma} + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha(1-\alpha)}\frac{(\sigma-\alpha)}{\sigma}.$$

A unique steady state exists, if

$$\Lambda = \frac{(1-\alpha+\zeta)}{\alpha}(\sigma-1)\eta(1-\varepsilon) + \rho > 0.$$

In addition, Λ provides the necessary and sufficient for the transversality condition to hold for the consumer's utility maximization problem (see Appendix 1)**.** Following the footsteps of Barro and Sala-i-Martín (2003) and Mattana (2004), it can be shown that social planner solution is saddle path stable. We linearize around the steady state, $s^* = (m^*, g^*)$, to analyze the local stability properties of the system defined by equations (4) and (5). The result is

$$\begin{bmatrix}\dot{m}\\ \dot{g}\end{bmatrix} = \underbrace{\begin{bmatrix}\left.\frac{\partial \dot{m}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{m}}{\partial g}\right|_{s^*}\\ \left.\frac{\partial \dot{g}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{g}}{\partial g}\right|_{s^*}\end{bmatrix}}_{J_s}\begin{bmatrix}m_t - m^*\\ g_t - g^*\end{bmatrix},$$

where

$$J_s = \begin{bmatrix}-(1-\alpha)m^* & 0\\ \left(\frac{\alpha}{\sigma} - 1\right)g^* & g^*\end{bmatrix}.$$

As $m^*, g^* > 0,$ it follows that $\mathrm{Det}(J) = -(1-\alpha)m^* g^* < 0,$ hence saddle path stable.

2.2. Representative Agent Problem

From the first order conditions (see Appendix 3) and setting $h = h_a$,we derive the equations describing the economy of the Uzawa-Lucas model from a decentralized economy's perspective.

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n+\delta),$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon),$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho+\delta)}{\sigma},$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \frac{c}{k} + \frac{(1-\alpha)}{\alpha}(n+\delta),$$

$$\frac{\dot{L}}{L} = n.$$

Let $m = \frac{Y}{K}$ and $g = \frac{c}{k}$. Taking logarithms of *m* and *g* and differentiating with respect to time, the following three equations define the dynamics of Uzawa Lucas model

$$\frac{\dot{m}}{m} = -(1-\alpha)m + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \eta\frac{\zeta}{\alpha}\varepsilon \qquad (6)$$

$$\frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right)m - \frac{\rho}{\sigma} - \delta\left(\frac{1}{\sigma} - 1\right) + g + n \tag{7}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(\alpha - \zeta)}{\alpha}\varepsilon + \eta\frac{(1 - \alpha + \zeta)}{\alpha} - g + \frac{(1 - \alpha)}{\alpha}(n + \delta) \tag{8}$$

The steady state $(m^*, g^*, \varepsilon^*)$ is given by $\dot{m} = \dot{g} = \dot{\varepsilon} = 0$ and derived to be

$$\varepsilon^* = 1 - \frac{(1 - \alpha)(\rho - n - \eta)}{\eta[\zeta - \sigma(1 - \alpha + \zeta)]},$$

$$m^* = \eta\frac{[1 - \alpha + \zeta(1 - \varepsilon^*)]}{\alpha(1 - \alpha)} + \frac{n}{\alpha},$$

$$g^* = \eta\frac{[1 - \alpha + \zeta(1 - \varepsilon^*) + \alpha\varepsilon^*]}{\alpha} + \frac{n(1 - \alpha)}{\alpha}.$$

Note that as shown by Benhabib and Perli (1994) $0 < \frac{(1-\alpha)(\rho-n-\eta)}{\eta[\zeta-\sigma(1-\alpha+\zeta)]} < 1$ is necessary for $0 < \varepsilon^* < 1$.

A unique steady state exists if

$$\Lambda = \frac{(1 - \alpha + \zeta)}{\alpha}(\sigma - 1)\eta(1 - \varepsilon) + \rho > 0,$$

$$0 < \varepsilon < 1.$$

In addition, Λ is the necessary and sufficient for the transversality condition to hold for the consumer's utility maximization problem (appendix 1), and $0 < \varepsilon^* < 1$ is necessary for $m^*, g^* > 0$. We linearize the system (6), (7) and (8) around the steady state, $s^* = (m^*, g^*, \varepsilon^*)$, to acquire

$$\begin{bmatrix} \dot{m} \\ \dot{g} \\ \dot{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial \dot{m}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{m}}{\partial g}\right|_{s^*} & \left.\frac{\partial \dot{m}}{\partial \varepsilon}\right|_{s^*} \\ \left.\frac{\partial \dot{g}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{g}}{\partial g}\right|_{s^*} & \left.\frac{\partial \dot{g}}{\partial \varepsilon}\right|_{s^*} \\ \left.\frac{\partial \dot{\varepsilon}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{\varepsilon}}{\partial g}\right|_{s^*} & \left.\frac{\partial \dot{\varepsilon}}{\partial \varepsilon}\right|_{s^*} \end{bmatrix}}_{J_m} \begin{bmatrix} m_t - m^* \\ g_t - g^* \\ \varepsilon_t - \varepsilon^* \end{bmatrix},$$

where, $J_m = \begin{bmatrix} -(1-\alpha)m^* & 0 & -\eta\frac{\zeta}{\alpha}m^* \\ \left(\frac{\alpha}{\sigma}-1\right)g^* & g^* & 0 \\ 0 & -\varepsilon^* & \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon^* \end{bmatrix}$.

The characteristic equation associated with J_m is $q^3 + c_2q^2 + c_1q + c_0 = 0$, where

$$c_0 = \eta\frac{[\sigma(1-\alpha+\zeta)-\zeta]}{\sigma}m^*g^*\varepsilon^*,$$

$$c_1 = \eta^2\frac{(\alpha-\zeta)}{\alpha}\varepsilon^{*2} - (1-\alpha)m^*g^*,$$

$$c_2 = -\eta\frac{(2\alpha-\zeta)}{\alpha}\varepsilon^*.$$

3. Bifurcation Analysis of Uzawa-Lucas Model

In this section, we examine the existence of codimension 1 and 2, transcritical, and Hopf bifurcations in the system (6), (7), and (8). The codimension, as defined by Kuznetsov (1998), is the number of independent conditions determining the bifurcation boundary. Varying a single parameter helps to identify codimension-1 bifurcation, and varying 2 parameters helps to identify codimension-2 bifurcation.

An equilibrium point, $s*$, of the system is called hyperbolic, if the coefficient matrix, J_m, has no eigenvalues with zero real parts. For small perturbations of parameters, there are no structural changes in the stability of a hyperbolic equilibrium, provided that the perturbations are sufficiently small. Therefore, bifurcations occur at nonhyperbolic equilibria only.

A transcritical bifurcation occurs, when a system has a nonhyperbolic equilibrium at the bifurcation point with a geometrically simple zero eigenvalue. Also additional transversality conditions must be satisfied, as given by Sotomayor's Theorem [Barnett and He (1999)]. The first condition we are going to use to find the bifurcation boundary is $c_0 = \det(J_m) = 0$. The result is the following.

Theorem 1: J_m has zero eigenvalues, if

$$\eta \frac{[\sigma(1-\alpha+\zeta)-\zeta]}{\sigma} m^* g^* \varepsilon^* = 0 \tag{9}$$

Hopf bifurcations occur at points at which the system has a nonhyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. Also additional transversality conditions must be satisfied. We use the following theorem, based upon the version of the Hopf Bifurcation Theorem in Guckenheimer and Holmes (1983): J_m has precisely one pair of pure imaginary eigen values, if $c_0 - c_1 c_2 = 0$ and $c_1 > 0$. If $c_0 - c_1 c_2 \neq 0$ and $c_1 > 0$, then J has no pure imaginary eigenvalues. The result is:

Theorem 2: The matrix J_m has precisely one pair of pure imaginary eigenvalues, if

$$\left.\begin{array}{c} \alpha m^* g^*\big((\alpha-1)\alpha\sigma + \zeta(\sigma-\alpha)\big) + \eta^2\sigma\, \varepsilon^{*2}(2\alpha-\zeta)(\alpha-\zeta) = 0, \\ \text{and} \\ \dfrac{\eta^2}{\alpha}\varepsilon^{*2}(\alpha-\zeta) - (1-\alpha)m^* g^* > 0. \end{array}\right\} \tag{10}$$

3.1 . Case Studies

To be able to display the boundaries, we consider two or three parameters. But the procedure is applicable to any number of parameters.

Let $\vartheta^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.65, 0.0505, 0.15, 0, 0)$ and

$\omega^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.75, 0.0505, 0.15, 0, 0)$.

Case I: Free parameters, α, η.

Assume that free parameters vary at fixed ϑ^*(values based on Benhabib and Perli (1994)). The result is illustrated in Figure 1. The boundary in Figure 1 called Hopf gives a range of α and η satisfying the Hopf bifurcation conditions, while the one named Transcritical depicts the value of α and η satisfying conditions for a transcritical bifurcation boundary.

Similarly, the following cases gives the range of parameter values satisfying conditions (9) and (10), represented in the graphs by Transcritical and Hopf, respectively, while the rest of the parameters are set at ϑ^*.

Case II: Free parameters, ζ, α (figure 2).

Case III: Free parameters, σ, α (figure 3).

Case IV: Free parameters, ζ, ρ (figure 4). Notice that for case IV, we do not have a Hopf bifurcation boundary.

We now add another parameter as a free parameter and continue with the analysis. The following cases give the range of parameter values satisfying conditions (9) or (10), represented in the graphs (5)-(9), while the rest of the parameters are assumed to be at ω^*.

Case V: Free parameters, α, ζ, ρ (figure 5).

Case VI: Free parameters, η, ζ, σ (figure 6).

Case VII: Free parameters, α, η, ρ (figure 7). Notice that for case IV, we do not have a Hopf bifurcation boundary.

Case VIII: Free parameters, α, σ, ρ (figure 8).

Case IX: Free parameters,α, η, σ (figure 9).

The following is an approach to exploring cyclical behavior in the model. Suppose the externality parameter ζ increases. This causes the savings rate to increase. This is because when consumers are willing to cut current consumption in exchange for higher future consumption; that is, when intertemporal elasticity of substitution for consumption is high (σ is low), people start saving more. Hence there is a movement of labor from output production to human capital production. Human capital begins increasing. This implies faster accumulation of physical capital, if sufficient externality to human capital in production of physical capital is present. If people care about the future more (subjective discount rate ρ is lower), consumption starts rising gradually with faster capital accumulation, leading to greater consumption-goods production in the future. This will eventually lead to a decline in savings rate. Two opposing effects come into play when savings rate is different from the equilibrium rate, causing a cyclical convergence to equilibrium. Hence, interaction between different parameters can cause cyclical convergence to equilibrium (figure 10) or may cause instability; and for some parameter values we may have convergence to cycles (figure 11).

Using the numerical continuation package Matcont, we further investigate the stability properties of cycles generated by different combinations of parameters. While some of the limit cycles generated by Andronov-Hopf bifurcation are stable (supercritical bifurcation), there could be some unstable limit cycles (subcritical bifurcation) created as well. When Hopf bifurcations are generated, Table 1 reports the values of the share of capital (α), externality in production of human capital (ζ), and the inverse of intertemporal elasticity of substitution in consumption (σ). A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Thus for each of the cases reported in Table 1, an unstable limit cycle (periodic orbit) bifurcates from the equilibrium. All of these cases also produce branch points (pitchfork/transcritical bifurcations).

Continuation of limit cycles from the Hopf point, when α is the free parameter, gives rise to limit point (Fold/ Saddle Node) bifurcation of cycles. From the family of limit cycles bifurcating from the Hopf point, limit point cycle (LPC) is a fold bifurcation, where two limit cycles with different periods are present near the LPC point at $\alpha = 0.738$.

Continuing computation further from a Hopf point gives rise to a series of period doubling (flip) bifurcations. Period doubling bifurcation is defined as a situation in which a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. The first period doubling bifurcation, at $\alpha = 0.7132369$, has positive normal form coefficients, indicating existence of unstable double-period cycles. The rest of the period doubling bifurcations have negative normal-form coefficients, giving rise to stable double-period cycles.

The period of the cycle rapidly increases for very small perturbation in parameter α, as is evident in figure 12(C). The limit cycle approaches a global homoclinic orbit. A homoclinic orbit is a dynamical system trajectory, which joins a saddle equilibrium point to itself. In other words, a homoclinic orbit lies in the intersection of equilibrium's stable manifold and unstable manifold. There exists the possibility of reaching chaotic dynamics in the decentralized version of Uzawa-Lucas model through a series of period doubling bifurcations.

For the cases in which ζ and σ are free parameters, we carry out the continuation of limit cycle from the first Hopf point. Both cases give rise to limit point cycles with a nonzero normal-form coefficient, indicating the limit cycle manifold has a fold at the LPC point. Similar results are found, if we carry out the continuation of limit cycles from the second Hopf point for each of these cases, and hence we do not report those results.

Table 1

Parameters Varied	**Equilibrium Bifurcation**	**Bifurcation of Limit Cycle**
$\boldsymbol{\alpha}$ (Figure 12) Other parameters set at ϑ^*	Figure 12 (A)	Figure 12 (B)
	Hopf (H) First Lyapunov coefficient = 0.00242, α =0.738207	**Limit point cycle (LPC)** period = 231.206, $\boldsymbol{\alpha}$ = 0.7382042 Normal form coefficient = 0.007
		Period Doubling (PD) period = 584.064, $\boldsymbol{\alpha}$ = 0.7132369 Normal form coefficient = 0.910
		Period Doubling (PD) period = 664.005, $\boldsymbol{\alpha}$ = 0.7132002 Normal form coefficient = -0.576
		Period Doubling (PD) period = 693.988, $\boldsymbol{\alpha}$ = 0.7131958 Normal form coefficient = -0.469
		Period Doubling (PD) period = 713.978, $\boldsymbol{\alpha}$ = 0.7131940 Normal form coefficient = -0.368

		Period Doubling (PD) period = 725.667, $\boldsymbol{\alpha}$ = 0.7131932 Normal form coefficient = -0.314
		Period Doubling (PD) period = 784.104, $\boldsymbol{\alpha}$ = 0.7131912 Normal form coefficient = -0.119
	Branch Point (BP)	
$\boldsymbol{\zeta}$	Figure 13 (A)	Figure 13 (B)
(Figure 13) Other parameters set at ω^*	**Hopf (H)** First Lyapunov coefficient = 0.00250, $\boldsymbol{\zeta}$=0.107315	**Limit point cycle (LPC)** period = 215.751, $\boldsymbol{\zeta}$= 0.1073147 Normal form coefficient = 0.009
	Hopf (H) First Lyapunov coefficient = 0.00246 $\boldsymbol{\zeta}$ =0.052623	
	Branch Point (BP) $\boldsymbol{\zeta} = 0.047059$	
$\boldsymbol{\sigma}$	Figure 14 (A)	Figure 14 (B)
(Figure 14) Other parameters set at ω^*	**Hopf (H)** First Lyapunov coefficient = 0.00264 $\boldsymbol{\sigma}$ =0.278571	**Limit point cycle (LPC)** period = 213.83, $\boldsymbol{\sigma}$ = 0.1394026 Normal form coefficient = 0.009
	Hopf (H) First Lyapunov coefficient = 0.00249 $\boldsymbol{\sigma}$=0.139394	
	Branch Point (BP) $\boldsymbol{\sigma} = 0.278571$	

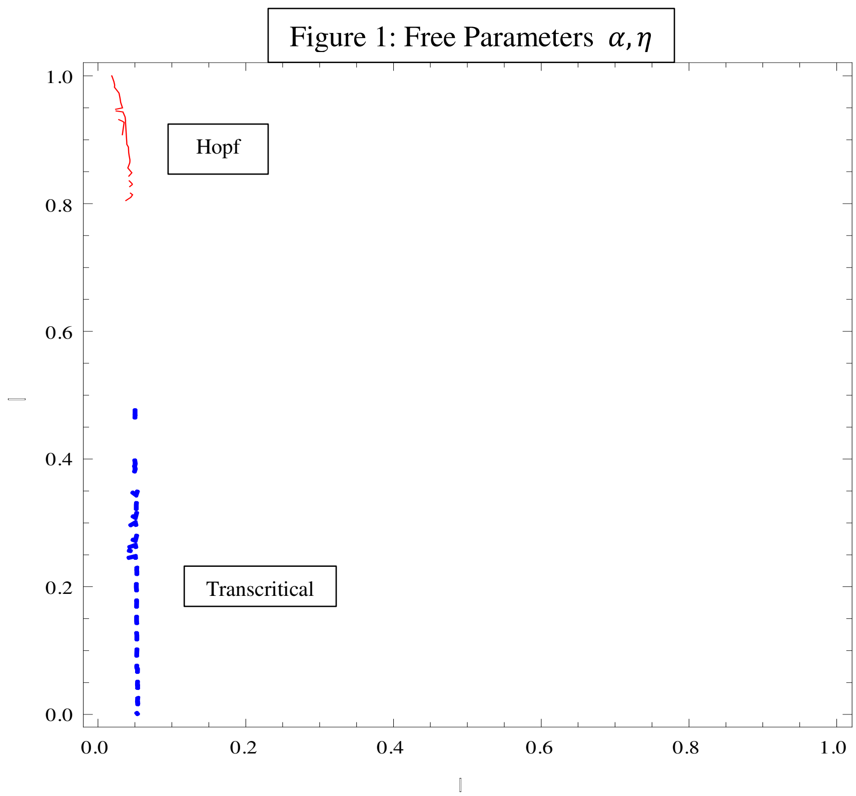


Figure 1: Free Parameters α, η
Hopf
Transcritical
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

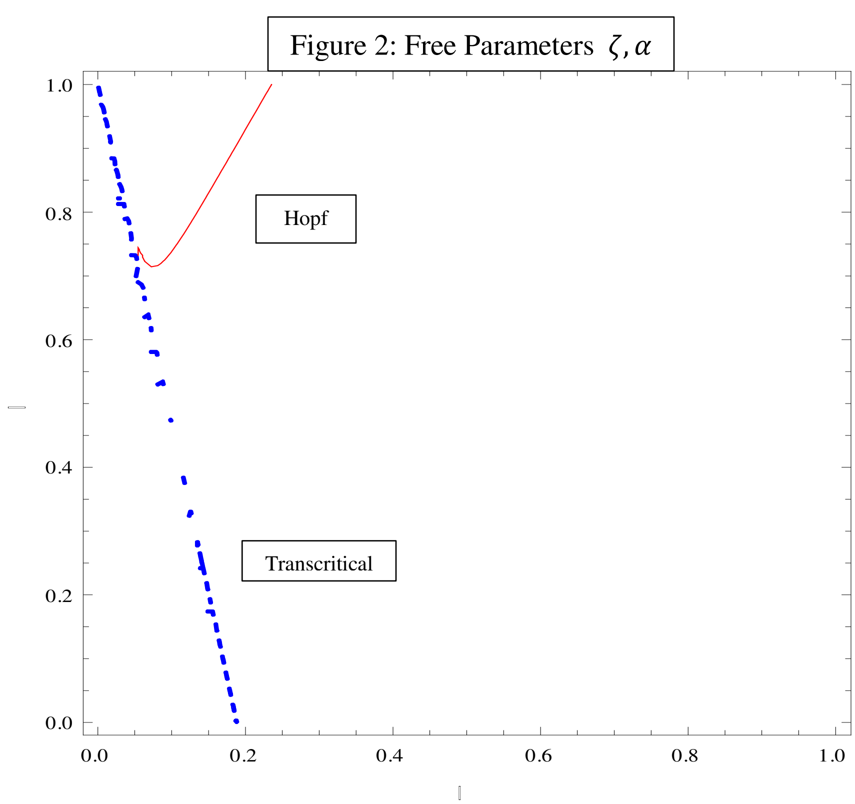


Figure 2: Free Parameters ζ, α
Hopf
Transcritical
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Figure 3: Free Parameters σ, α

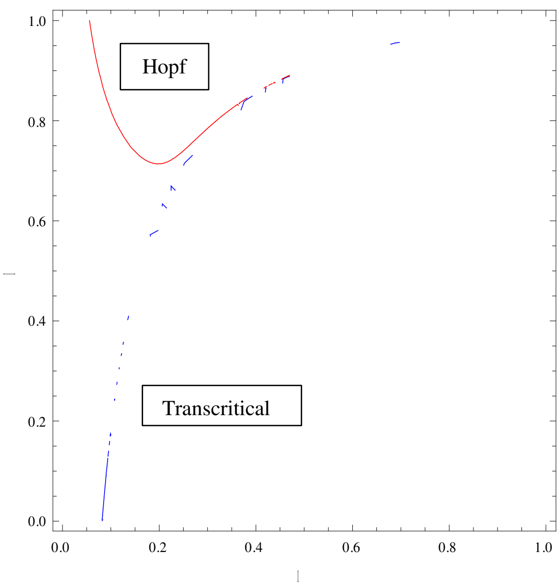


Figure 4: Free Parameters ζ, ρ

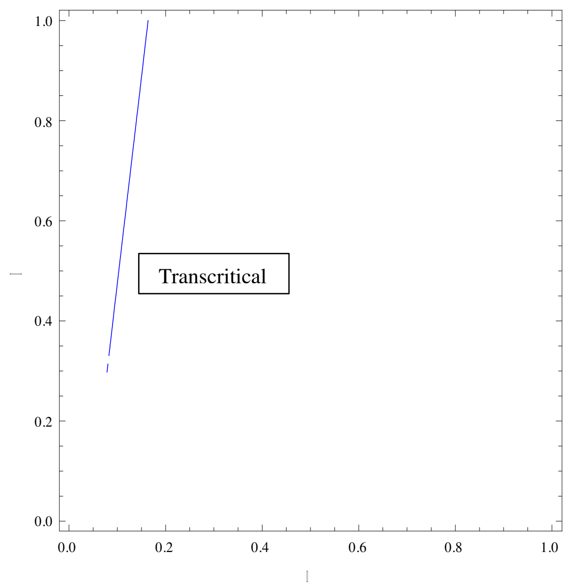


Figure 5: α, ζ, ρ are free parameters
(Hopf Bifurcation Boundary)

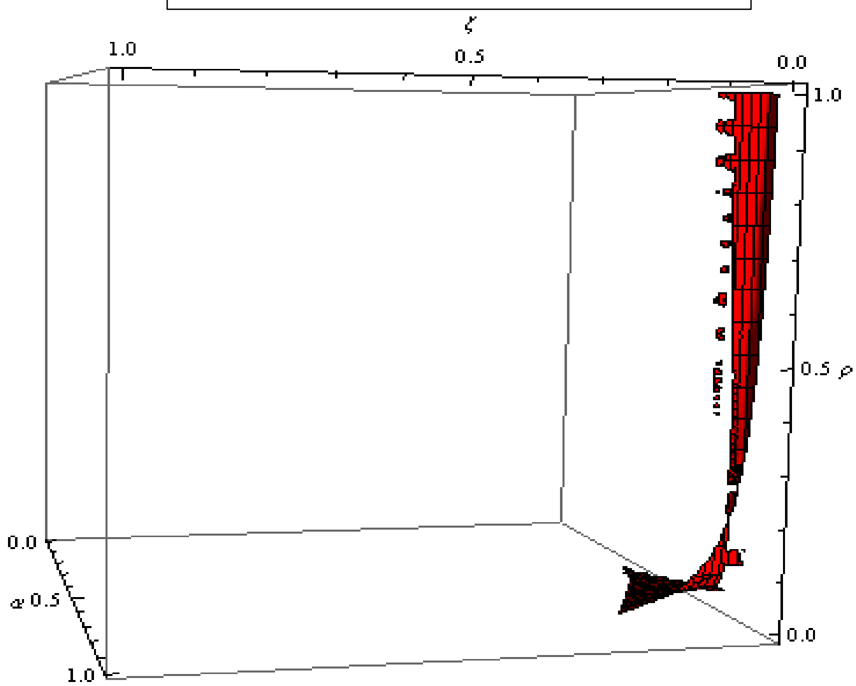


Figure 6: Free Parameters η, ζ, σ
(Transcritical Bifurcation Boundary)

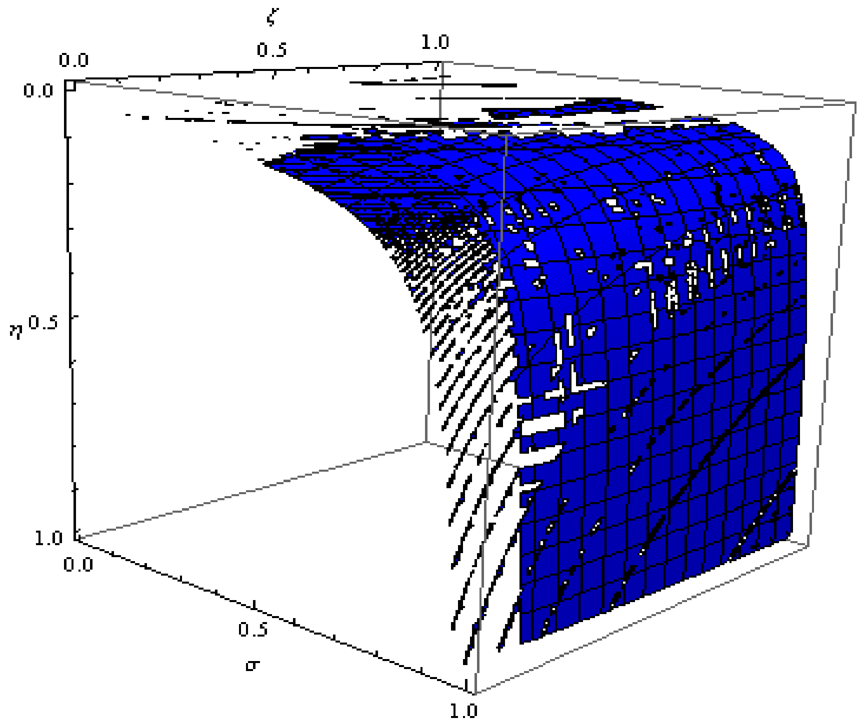


Figure 7: α, η, ρ are free parameters
(Hopf Bifurcation Boundary)

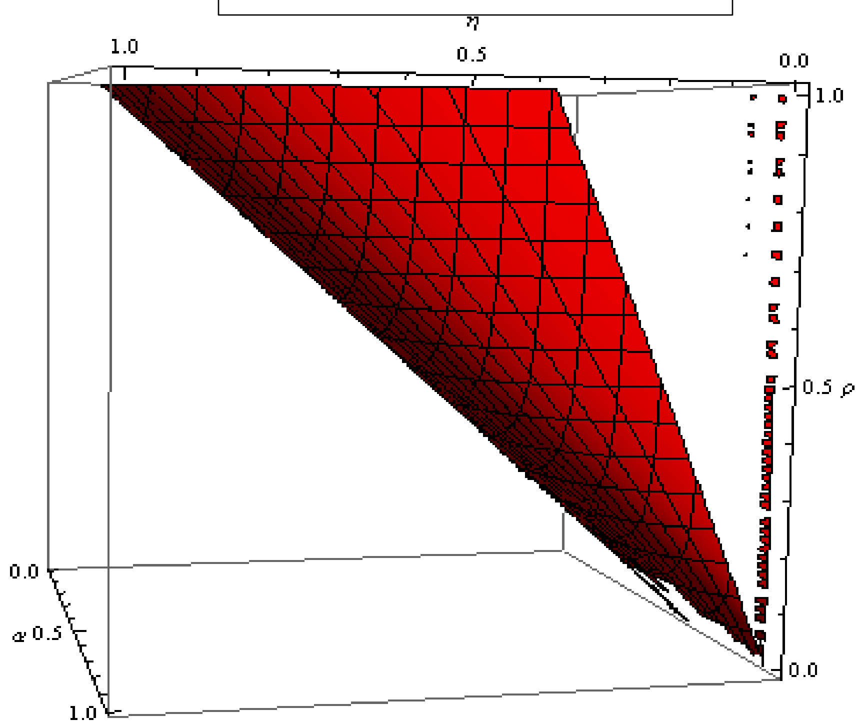


Figure 8: α, σ, ρ are free parameters
(Hopf Bifurcation Boundary)

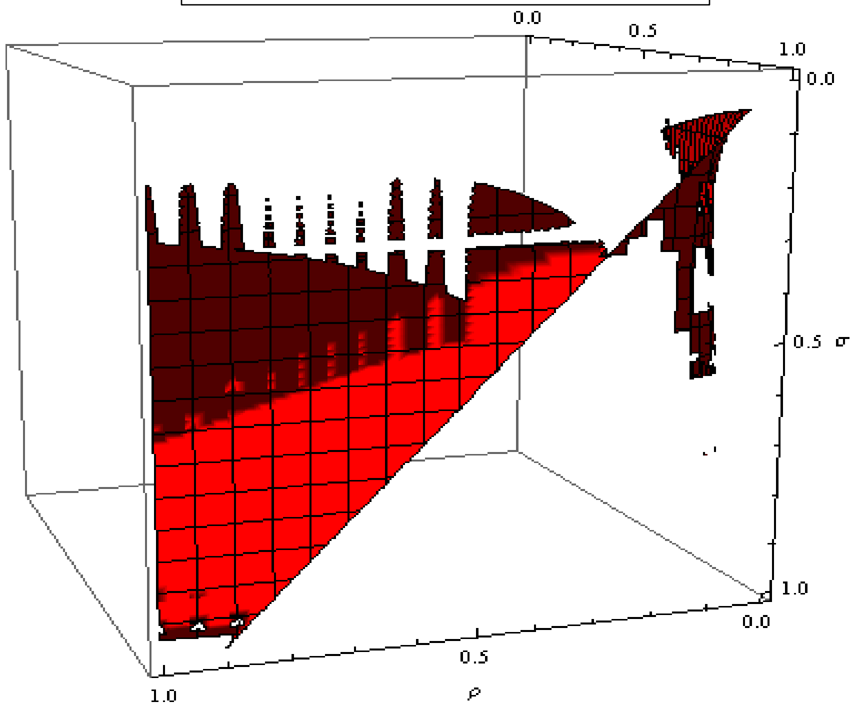
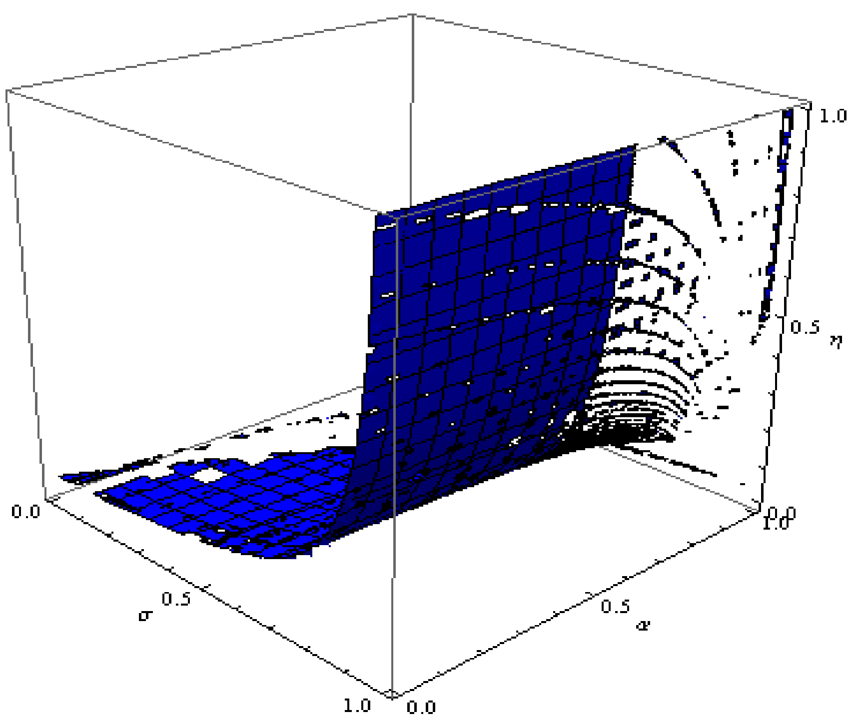


Figure 9: Free Parameters α, η, σ
(Transcritical Bifurcation Boundary)



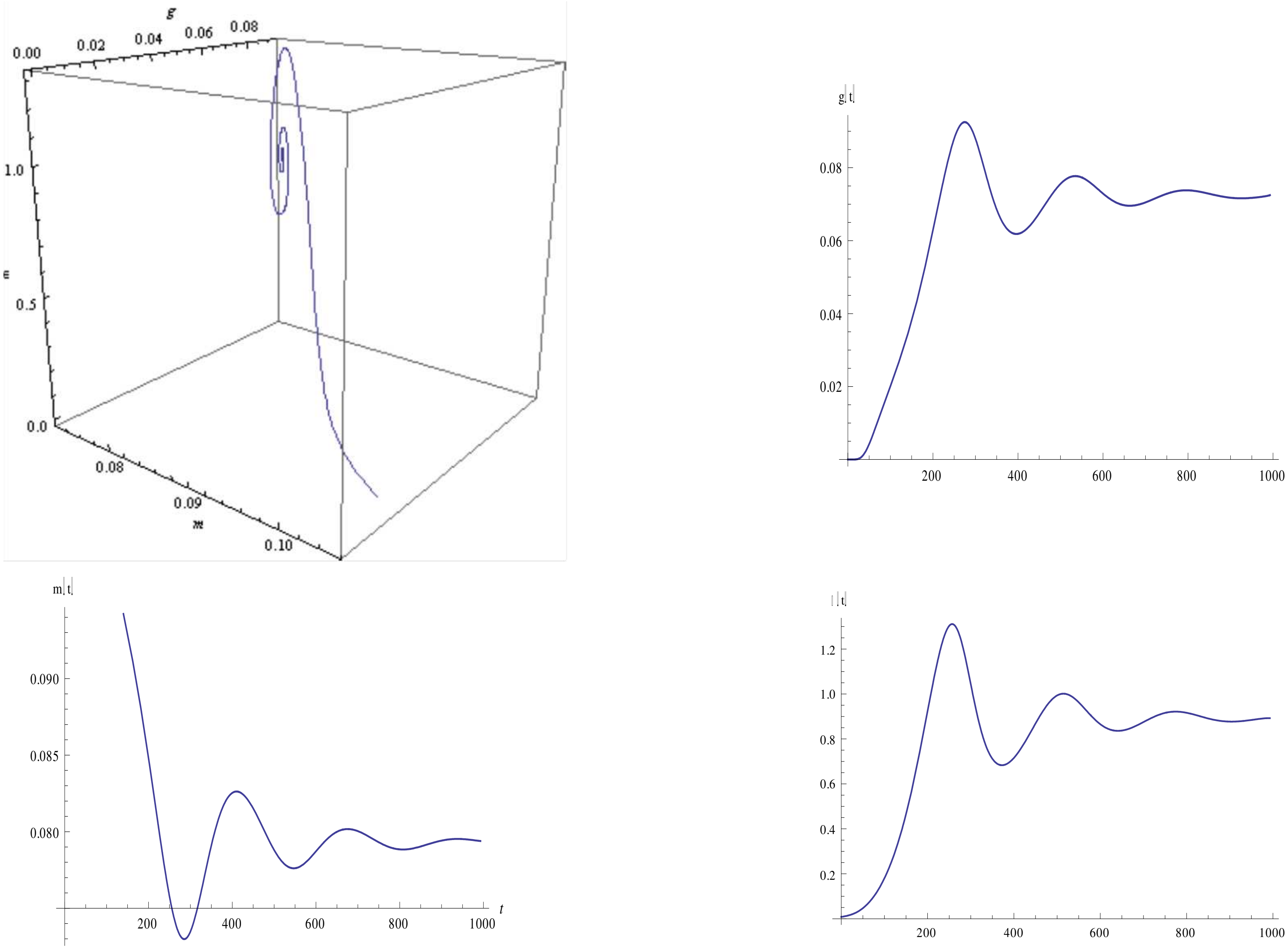


Figure 10: Parameters in the Stable Region

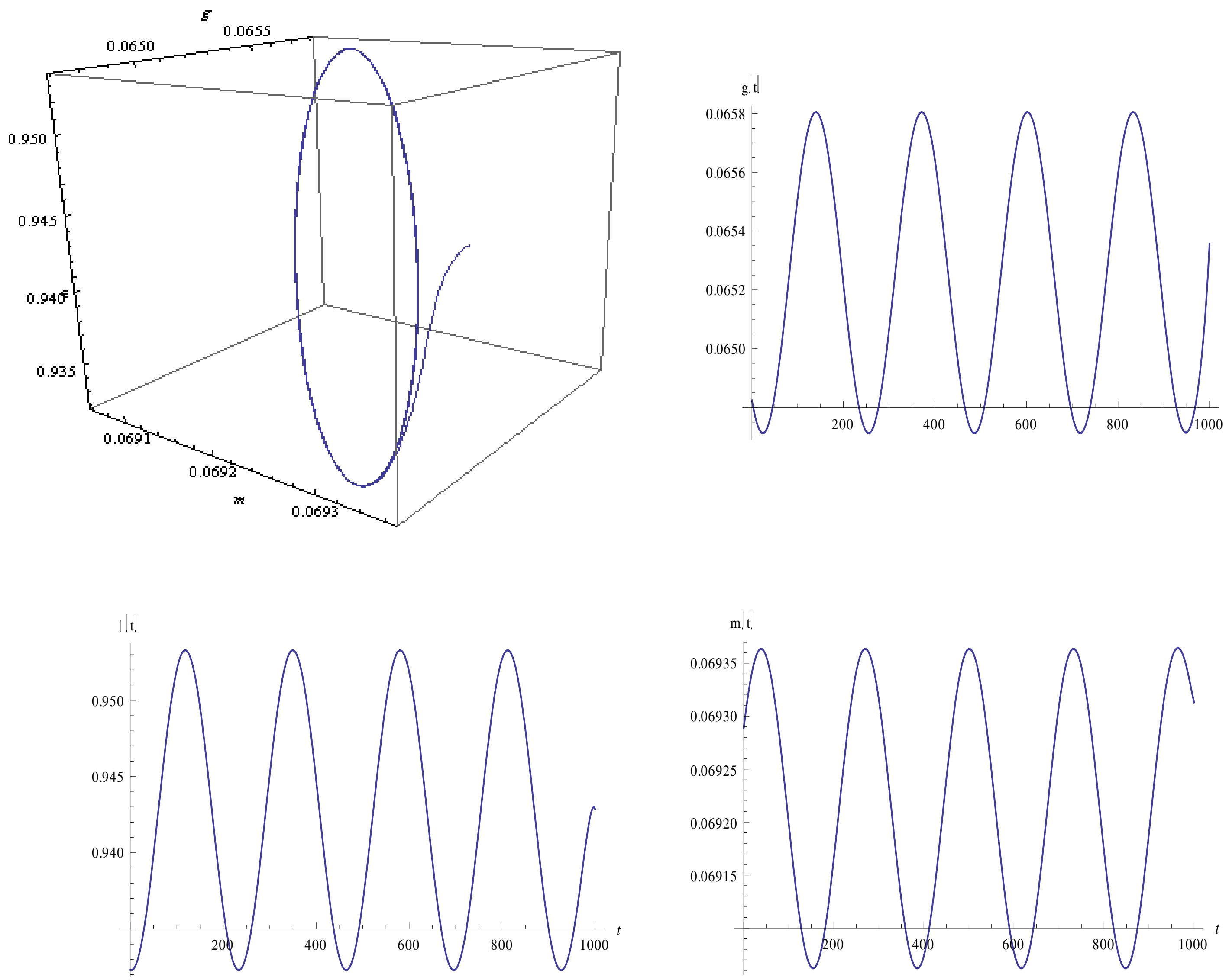


Figure 11: Parameters on the Hopf Bifurcation Boundary

Figure 12

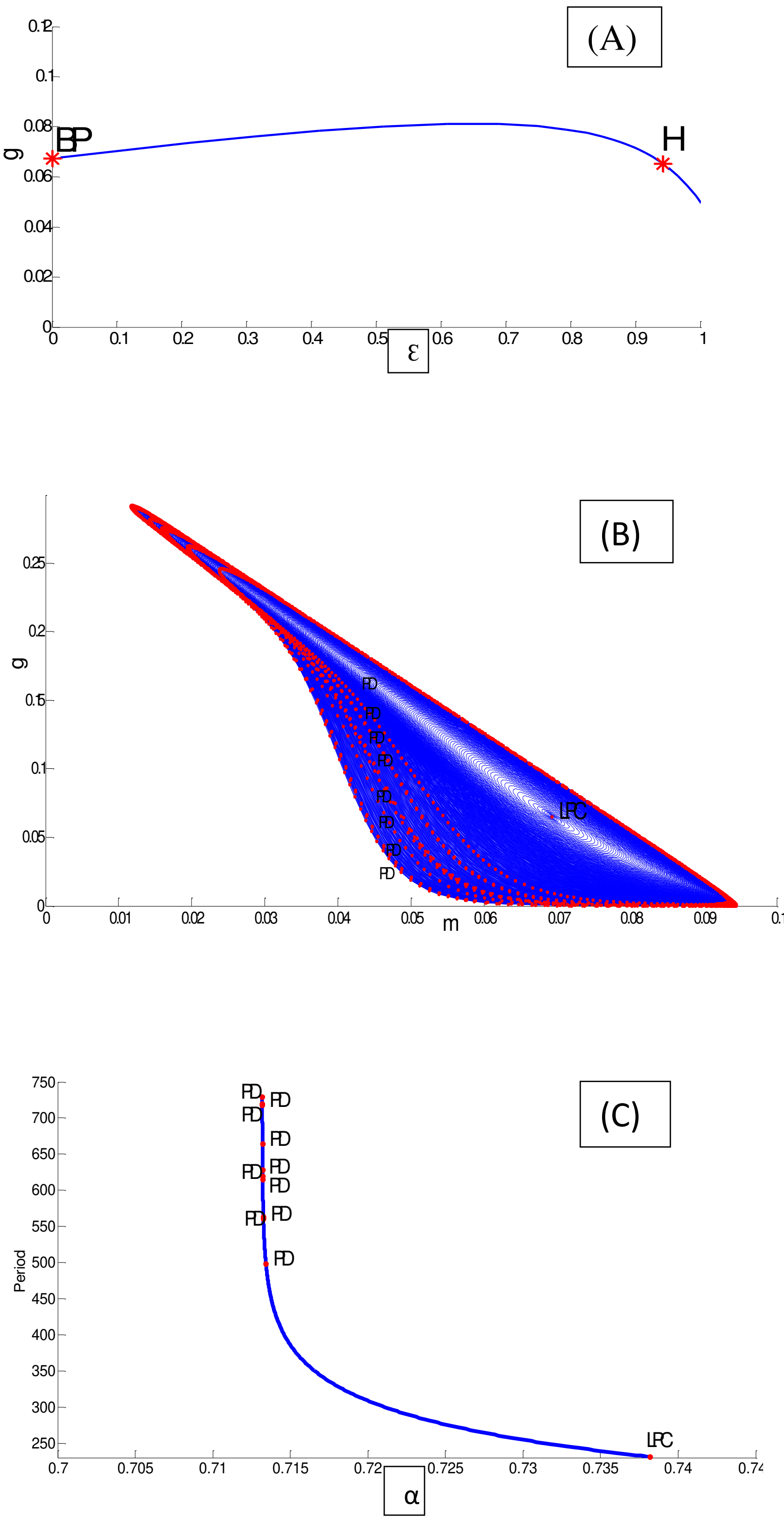


Figure 13

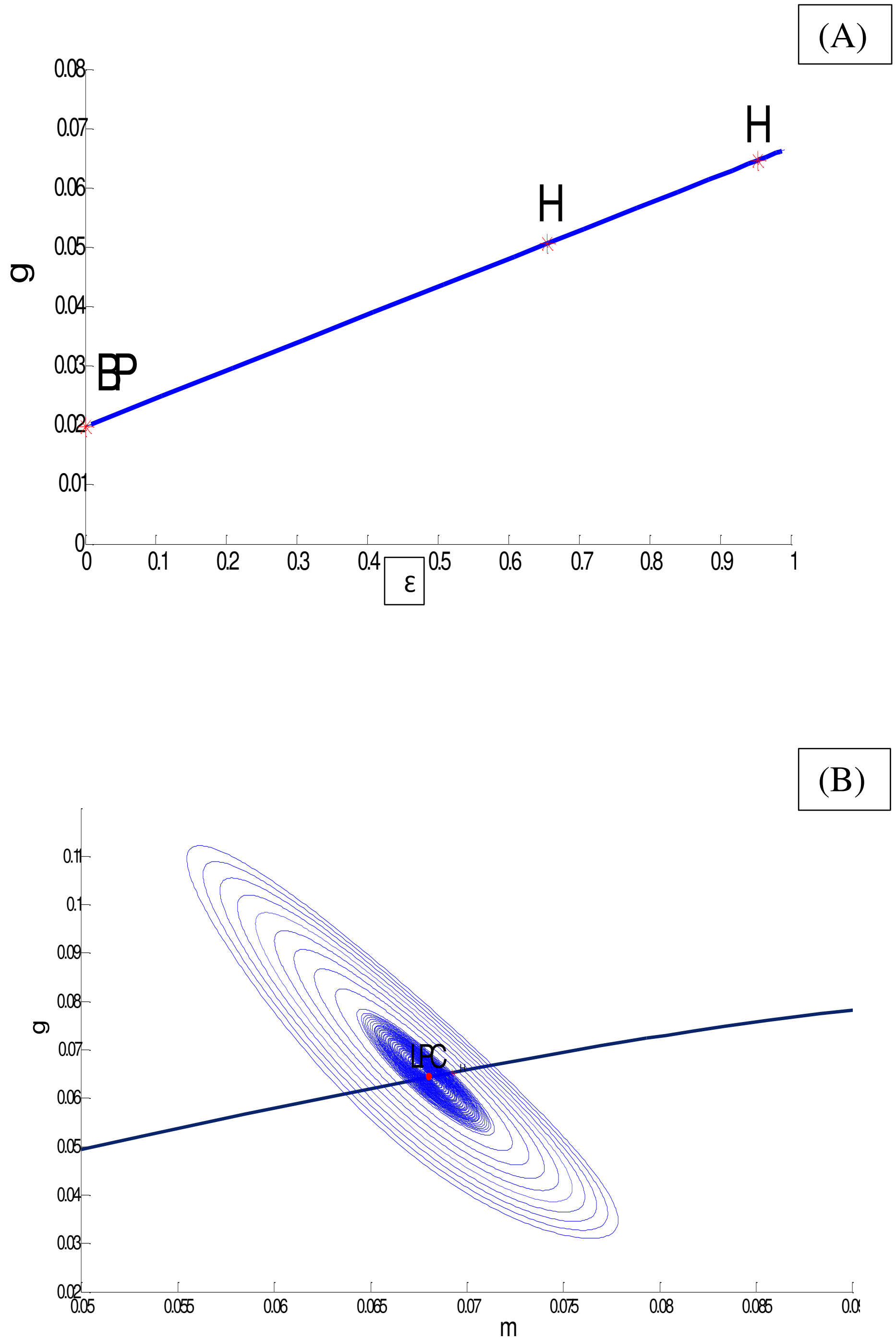
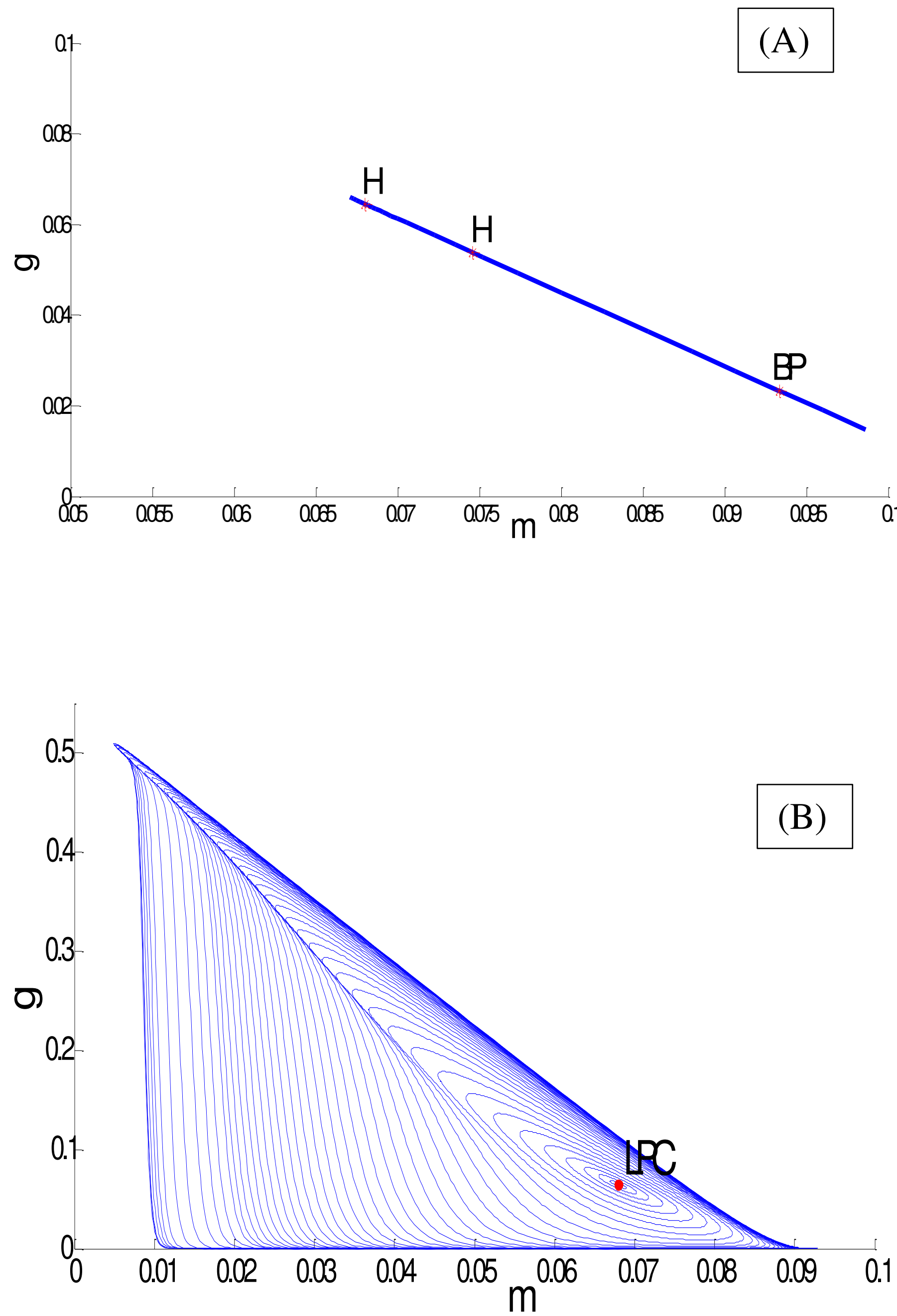

(A)
H
H
BP
g
ε
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(B)
LPC
g
m
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.055
0.06
0.065
0.07
0.075
0.08
0.085

Figure 14


(A)
0.1
0.08
0.06
0.04
0.02
0
g
H
H
BP
0.05
0.055
0.06
0.065
0.07
0.075
0.08
0.085
0.09
0.095
0.1
m
(B)
0.5
0.4
0.3
0.2
0.1
0
g
LPC
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
m

4. Conclusion

This paper provides a detailed stability and bifurcation analysis of the Uzawa-Lucas model. Transcritical bifurcation and Hopf bifurcation boundaries, corresponding to different combinations of parameters, are located for the decentralized version of the model. Examination of the stability properties of the limit cycles generated from various Hopf bifurcations in the model depicts occurrence of limit point-of-cycles bifurcations and period-doubling bifurcations within the model's feasible parameter set. The series of period-doubling bifurcations confirms the presence of global bifurcation. Period-doubling bifurcations also reveal the possibility of chaotic dynamics, occurring in the converged limit of the sequence of period doubling. In contrast, the centralized social planner solution is always saddle path stable, with no possibility of bifurcation within the feasible parameter set, but with no decentralized informational privacy. Thus the externality of the human capital parameter plays an important role in determining the dynamics of the decentralized Uzawa-Lucas model. Our result emphasizes the need for simulations of decentralized macro econometric models at settings throughout the parameter-estimates' confidence regions, rather than at the point estimates alone, since dynamical inferences otherwise can produce oversimplified conclusions subject to robustness problems.

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Appendix 1:

In the steady state, the constancy of $m, g,$ and ε implies

$$\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{k}}{k} + n = \frac{(1-\alpha+\zeta)}{(1-\alpha)}\eta(1-\varepsilon) + n.$$

Transversality conditions require $\lim_{t\to\infty} \lambda_t K_t = 0$ and $\lim_{t\to\infty} \mu_t h_t = 0$, which in turn imply

$$\rho > (1-\sigma)\frac{(1-\alpha+\zeta)}{(1-\alpha)}\eta(1-\varepsilon),$$

where λ and μ are costate variables (appendix 2 and appendix 3)

Appendix 2:

Social Planner Problem:

$$\mathcal{H} = \frac{[c(\tau)^{1-\sigma}-1]}{(1-\sigma)} + \lambda[Ak^{\alpha}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - c - (n+\delta)K] + \mu[\eta h(1-\varepsilon)].$$

Lucas (1988) showed that the first order conditions are

(1) c: $c^{-\sigma}e^{-(\rho-n)} = \lambda,$

(2) ε: $\lambda(1-\alpha)Ak^{\alpha}\varepsilon^{-\alpha}h^{1-\alpha+\zeta} = \mu\eta h,$

(3) k: $\lambda\left[\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (n+\delta)\right] = -\dot{\lambda},$

(4) h: $\lambda(1-\alpha+\zeta)Ak^{\alpha}\varepsilon^{1-\alpha}h^{-\alpha+\zeta} + \mu\eta(1-\varepsilon) = -\dot{\mu}.$

Appendix 3:

Decentralized or Market Solution:

$$\mathcal{H} = \frac{[c(\tau)^{1-\sigma}-1]}{(1-\sigma)} + \lambda[Ak^{\alpha}\varepsilon^{1-\alpha}h^{1-\alpha}h_a^{\zeta} - c - (n+\delta)K] + \mu[\eta h(1-\varepsilon)].$$

Lucas (1988) showed that the first order conditions are

(1) c: $c^{-\sigma}e^{-(\rho-n)} = \lambda,$

(2) ε: $\lambda(1-\alpha)Ak^{\alpha}\varepsilon^{-\alpha}h^{1-\alpha}h_a^{\zeta} = \mu\eta h,$

(3) k: $\lambda\left[\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha}h_a^{\zeta} - (n+\delta)\right] = -\dot{\lambda},$

(4) h: $\lambda(1-\alpha)Ak^{\alpha}\varepsilon^{1-\alpha}h^{-\alpha}h_a^{\zeta} + \mu\eta(1-\varepsilon) = -\dot{\mu}.$

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