# Global invertibility of excess demand functions

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**Abstract**

In this paper we provide necessary and sufficient conditions for the excess demand function of a pure exchange economy to be globally invertible so that there is a unique equilibrium. Indeed, we show that an excess demand function is globally invertible if and only if its Jacobian never vanishes and it is a proper map. Our result includes as special cases many partial results found in the literature that imply global uniqueness including Gale-Nikaido conditions and properties related to stability of equilibria. Furthermore, by showing that the condition is necessary, we are implicitly finding the weakest possible condition.

## 1 Introduction

One of the first results that need to be established in modeling the structure of markets is the existence of an equilibrium price system for an economy. The classical work of Arrow and Debreu (1954) showed that all economies have at least one equilibrium although maybe not a unique one; a simple Edgeworth box can illustrate economies with a continuum of them. Debreu (1983) pointed out that "the explanation of equilibrium given by a model of the economy would be complete if the equilibrium were unique, and the search for satisfactory conditions guaranteeing uniqueness has been actively pursued [...] However, the strength of the conditions that were proposed made it clear by the late sixties that global uniqueness was too demanding a requirement and that one would have to be satisfied with local uniqueness." In this paper we take a new look at the problem of global uniqueness

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and provide necessary and sufficient conditions that the equations describing when the equality of supply and demand admit exactly one solution.

Arrow and Hahn (1971, ch.9) wrote a survey containing *sufficient* conditions for a unique equilibrium dating as back as the mid fifties. Implicitly, two chapters of the Handbook of Mathematical Economics also contained a summary of the results of uniqueness known at the time (Hahn 1982, Shafer and Sonnenschein, 1982). Since then, there have been found ever weaker sufficient conditions although necessary conditions are not yet understood.

This paper establishes in Section 2 a necessary and sufficient condition for an excess demand function to be globally invertible. Thus, we are providing the *weakest* possible sufficient condition: the excess demand function is globally invertible *if and only if* it has a non vanishing Jacobian and it is a proper map. *Properness* is a mathematical term that will be explained below but has been interpreted as equivalent to desirability of goods. In Section 3 we compare our main theorem with many results found previously in the literature. In Section 4 we extend the main theorem to economies with an infinite number of goods. The conclusions will be the same: the excess demand function is globally invertible if and only if it is proper and the Jacobian never vanishes.

## 2 Invertibility of the excess demand function

### 2.1 Price sets

Consider an economy with $\ell+1$ goods so that the commodity space is $\mathbb{R}^{\ell+1}$. Throughout, we will use the notation $\mathbb{R}^{\ell}_{+} = \{x = (x_1, \ldots, x_\ell : x_j \geq 0, \forall j\}$ and $\mathbb{R}^{\ell}_{++} = \{x = (x_1, \ldots, x_\ell : x_j > 0, \forall j\}$. Thus, prices are naturally elements of $\mathbb{R}^{\ell+1}_{+}$. In the literature, several normalisations are usually chosen on the space of prices among which we find:

(i) Using a numeraire, say the $(\ell+1)$-th good, so that the price space is the set
$$\mathbb{P}^{\ell}_{+} = \left\{ p \in \mathbb{R}^{\ell+1}_{+} : p_{\ell+1} = 1 \right\};$$

(ii) The simplex

$$\Delta^\ell = \left\{ p \in \mathbb{R}_+^{\ell+1} : \sum_{i=1}^{\ell+1} p_i = 1 \right\};$$

(iii) The positive part of the unit sphere

$$S_+^\ell = \left\{ p \in \mathbb{R}_+^{\ell+1} : \|p\| = 1 \right\}.$$

Notice that these three examples give rise to topologically nonequivalent price sets. For example, the set $\mathbb{P}_+^\ell$ is equivalent to the set $\mathbb{R}_+^\ell$ which is neither closed nor bounded. On the other hand, the simplex $\Delta^\ell$ is a bounded, closed and convex set. Finally, the positive sphere, while closed and bounded, it is not convex. Further complexities could arise if the price set is initially asked to be $\mathbb{R}_{++}^\ell$ or $\mathbb{R}_+^\ell - \{0\}$. A common trait, however, is that locally all these sets away from their boundary have one dimension less than the original price space. This fact will be present in our approach.

In this paper we will allow many of these normalizations. Furthermore, one of the goals of this paper is to include widely different results found in the literature into a single and coherent framework that might clarify all those results. Thus, we give the following, rather general, definitions (see the Appendix).

**Definition 1.** Consider a pure exchange economy with $\ell + 1$ goods. An $\ell$-*normalised price set* is any $C^1$ connected manifold (possibly with boundary) of dimension $\ell$. When obvious, we will drop the explicit mention of $\ell$.

For example, $\mathbb{P}_+^\ell$, $\Delta^\ell$ and $S_+^\ell$ are all normalised price sets. From now on, to include these examples of price sets, we will let $S$ be an $\ell$-normalised price set.

## 2.2 Global invertibility

Having defined the rather general category of price sets that we will consider, the next step is to focus on properties of excess demand functions, which are maps $\hat{Z} : S \to \mathbb{R}^{\ell+1}$. By Walras law, we can delete, say, the last component so the *reduced excess demand function* is the map $Z : S \to \mathbb{R}^{\ell}$. Abusing notation, we will refer to $Z$ simply as the excess demand function from now on. The strategy is to understand when $Z$ is a bijection, such that both $Z$ and its inverse are differentiable. This gives rise to the following definition.

**Definition 2.** A differentiable map $f$ is said to be a *diffeomorphism* if the map is a bijection and its inverse is also differentiable.

The Implicit Function Theorem will tell us that if the Jacobian of $f$ is non vanishing then $f$ is a local homeomorphism. Of course this in itself does not guarantee that $f$ will be a bijection. The surprising fact is that if $f$ is proper then it can be shown that it will be. First recall that a sequence of points $\{x_i\}$ in X is said to *escape to infinity* if for every compact set $K \subset X$ there are at most finitely many values of $i$ for which $x_i \in K$.

**Definition 3.** A continuous map $f$ is said to be *proper* if any of the following two equivalent conditions is satisfied:

- $f^{-1}(K)$ is compact whenever $K$ is compact;
- $\|f(x)\| \to \infty$ as $\|x\| \to \infty$;
- for every sequence $\{x_i\}$ in $K$ that escapes to infinity, $\{f(x_i)\}$ escapes to infinity.

The third condition is particularly useful for economic interpretation. We will see in the next section that many papers that study the problem of global uniqueness of equilibria have specified in one way or another this condition, in the form of "if a price goes to zero, then excess demand goes to infinity". This condition can be interpreted as a form of desirability of goods. At this point, we are ready to propose the main theorem.

**Theorem 4.** ***(Main Theorem)*** *Let $S$ be an $\ell$-normalised price set. Then, the excess demand function $Z : S \to \mathbb{R}^{\ell}$ is a diffeomorphism if and only if $Z$ has a nonvanishing Jacobian and it is a proper map.*

*Proof.* First note that $\mathbb{R}^\ell$ is simply connected. Now, we suppose that $Z : S \to \mathbb{R}^\ell$ is a diffeomorphism. We wish to show that it has a non vanishing Jacobian and that it is a proper map. First, since it is a diffeomorphism, then the Jacobian of $Z$ will always be different to zero. Additionally, since $Z$ is a diffeomorphism, its inverse $Z^{-1}$ is continuous and so must map closed sets into closed sets. Therefore, $Z$ is a proper map.

Conversely, assume now that $Z$ has a nonvanishing Jacobian and it is a proper map. We wish to show that it is a diffeomorphism. The implicit function theorem guarantees that the everywhere the inverse is differentiable. It only remains to be shown that it is a bijection. Palais (1970) shows that a proper map send closed sets into closed sets and so $Z(S)$ is closed. But also, since the Jacobian of $Z$ never vanishes, it is a local homeomorphism so it also sends open sets onto open sets so that $Z(S)$ is also open. Since $Z(S)$ is an open, close and nonempty subset of $\mathbb{R}^\ell$ it must be that $Z(S) = \mathbb{R}^\ell$ and so it is surjective.

Finally, consider two price systems $p_1$ and $p_2$ in $S$ such that $Z(p_1) = Z(p_2) = x$. Since $S$ is topologically a disk, there is a path $\alpha(t)$ connecting $p_1$ and $p_2$ in $S$. And so $Z \circ \alpha(t)$ is a loop in $\mathbb{R}^{\ell-1}$ based in $x$. We may use a homotopy $F(s,t)$ such that $F(0,t) = Z \circ \alpha(t)$ and $F(1,t) = x$. Since we have seen that $Z$ is surjective, proper and a local homeomorphism from $S$ to $\mathbb{R}^\ell$, then by a result of Ho (1975, p.239), $Z$ must be a covering projection. And every covering projection has the homotopy lifting property property (Hatcher, 2002, p.60). So there has to be a unique lifting $\tilde{F}(s,t)$ of $F(s,t)$ with $\tilde{F}(0,t) = \alpha(t)$. The lift of $\tilde{F}(1,t)$ must be a connected set containing both $p_1$ and $p2$. But $Z^{-1}(x)$ is discrete, so $p_1 = p_2$.

□

## 3 Comparison with other results in the literature

We now aim to compare the results of Theorem 4 with others found in the literature. We have not aimed to be comprehensive and instead have chosen some results that reflect the progress in this area in chronological order. The reference list below provides a longer list of results in this direction.

### 3.1 Gale-Nikaido conditions

Gale (1962), Nikaido (1962) and Gale an Nikaido (1965) sequence of results led to the following definitions. An $n$ by $n$ real matrix $A = (a_{ij})$ is said to be a *weak P-matrix*, if $\|A\| > 0$ and all other principal submatrices of order less than $n$ have nonnegative determinants. With this definition, they prove the following result. Similarly, an $n$ by $n$ real matrix $A = (a_{ij})$ is said to be a *weakly positive quasi-definite*, if $\|A\| > 0$ and $\frac{1}{2}(A+A')$ is positive definite.

**Theorem.** *(Gale and Nikaido, 1965) Suppose prices are in an open rectangular region of $\mathbb{R}^\ell$. If the excess demand is a differentiable mapping such that the Jacobian is a weak P-matrix for all prices, then $Z$ is univalent in its domain.*

**Theorem.** *(Gale and Nikaido, 1965) Suppose prices are in an open convex region of $\mathbb{R}^\ell$. If the excess demand is a differentiable mapping such that the Jacobian is a weakly positive quasidefinite matrix for all prices, then $Z$ is univalent in its domain.*

Both set of prices proposed by these authors are examples of an $\ell$-normalised price set as we defined above. Also, notice that a nonvanishing Jacobian is a weaker requirement than both being a weak P-matrix and a quasi definite matrix. Now, Mas-Colell (1979) provided a refinement of the Gale-Nikaido conditions in the form of the following result.

**Theorem.** *(Mas-Colell, 1979) Suppose prices are in a compact, convex polyhedron set $S$ of full dimension and the excess demand of class $C^1$. If for every price $p \in S$, and subspace $L \subset \mathbb{R}^{\ell-1}$ spanned by a face of $S$ which includes $p$, the map $\Pi_L \cdot DZ(p) : L \to L$ has a positive determinant, then $Z$ is one-to-one and so, a homeomorphism.*

### 3.2 Jacobian conditions

There are other results that have explicitly mentioned the Jacobian as a necessary condition for global invertibility of the excess demand function. These results are hence closer in spirit to our paper. The first result was provided by Dierker (1972).

**Theorem.** *(Dierker, 1972) Suppose prices are in the positive simplex* $\Delta$ *and that if* $p^j \to p \in \partial\Delta$*, there exists an* $h \in \{1, \ldots, \ell\}$ *such that* $(p_h^j)$ *converges to zero and* $\lim_{j\to\infty} Z_h(p^j) = +\infty$*. Let* $\dot{p} = v(p)$ *be a price adjustment process for an economy such that* $v : S \to \mathbb{R}^\ell$ *is continuously differentiable,* $v(p) = 0$ *if and only if* $Z(p) = 0$ *and* $g := id + v$ *satisfies that there is a homotopy* $g_t : \Delta \to \mathbb{R}^{\ell-1}$*,* $0 \leq t \leq 1$*, between* $g_0 := g$ *and the constant mapping* $g_1$*, defined by* $g_1(p) := (1/\ell, \ldots, 1/\ell) \in \mathbb{R}^{\ell-1}$ *for all* $p \in \Delta$*, such that* $\Phi := \cup_{t=0}^{1}\{p \in \Delta : g_t(p) = p\}$ *is compact. If all equilibrium price systems are locally stable with respect to* $v$*, then there exists exactly one.*

**Theorem.** *(Varian, 1975) Suppose that as the price of a good goes to 0, its excess demand becomes positive. If the Jacobian of the excess supply function is positive at all Walras equilibria, there is exactly one equilibrium.*

**Theorem.** *(Mas-Colell, 1979) Suppose prices are in a compact, convex set* $S$ *of full dimension with a* $C^1$ *boundary* $\partial S$ *and the excess demand of class* $C^1$*. If for every price* $p \in S$*, the Jacobian has a positive determinant and if for all* $p \in \partial S$*, the Jacobian is positive quasidefinite on* $T_p$ *(i.e., v'DZ(P)v ¿ 0 for* $v \in T_p, v \neq 0$*), then* $Z$ *is one-to-one and so, a homeomorphism.*

**Theorem.** *(Mukherji,1997) Suppose prices are in the positive sphere and that if* $p^j \to p \in \partial S_{++}$*, then* $\sum_i Z_i(p^j) \to \infty$*. Suppose also the excess demand function is of class* $C^1$*. Let* $J(p)$ *be the Jacobian matrix. If for all* $p \in S$*,* $Z(p)^t \cdot J(p = 0)$ *implies* $p$ *is an equilibrium, then the equilibrium must be unique.*

**Theorem.** *(Chichilnisky, 1998) Suppose prices are in the positive sphere* $S_{++}$ *and that if* $p^j \to p \in \partial S_{++}$ *then* $\|Z(p^j)\| \to \infty$*. Then, if the excess demand* $Z : S_{++} \to \mathbb{R}^\ell$ *has a non vanishing Jacobian, then* $Z$ *is globally invertible.*

Again, all these sets of prices are examples of an $\ell$-normalised price set as defined in this paper. Similarly, all of these papers choose a particular condition of properness on the excess demand function. Finally, a nonvanishing Jacobian is weaker than the requirements in these theorems.

## 4 Extensions

The result found in the previous section provided the weakest condition on the excess demand function of a pure exchange economy to be globally invertible. As it turns out, the Main Theorem can be generalized in other directions. We provide an extensions in this section for economies with an infinite number of goods. First, consider the following generalisation of an $\ell$ normalised price set to infinite dimensions.

**Definition 5.** A consider a pure exchange economy with a simply-connected commodity space $X$. An $\infty$-*normalised price set* $S$ is any set for which there is a homeomorphism between every open set $U$ of $S$ and some neighbourhood $V$ in $X$.

Two remarks are in order. The first is that we now require the commodity space $X$ to be simply connected. This was not an issue in finite dimensions, since $\mathbb{R}^\ell$ is simply connected indeed. The second point is that we require for there to be a local homeomorphism everywhere between the price set and the commodity space. This is not always the case since the price space is naturally the dual of the commodity space and this homeomorphism does not always exist. Even with these drawbacks, we can provide the following theorem.

**Theorem 6.** *Let $S$ be an $\infty$-normalised price set and suppose the commodity space $X$ is simply connected. Then, the excess demand function $Z : S \to X$ is a diffeomorphism if and only if $Z$ has a nonvanishing Jacobian and it is a proper map.*

The proof of this theorem follows line by line the proof of the Main Theorem in there previous section. Comparing this results with those found previously in the literature seems straightforward since there seems to be only two results in this direction.

**Theorem.** *(Chichilnisky, 1998) Suppose prices are in $\ell_2^{++}$ and that if $p^j \to p \in \partial\ell_2^{++}$ then $\|Z(p^j)\| \to \infty$. Then, if the Frechet derivative of the excess demand $Z : S_{++} \to \mathbb{R}^{\ell-1}$ is an invertible operator, then $Z$ is globally invertible.*

**Theorem.** *(Covarrubias, 2013) Suppose prices are in $S = C(M, \mathbb{R}^{\ell-1}_{++})$ with $M$ compact and that if $(p_n) \to p \in \partial S$ then $\|Z(p_n)\| \to \infty$. Then, if the sign of the derivative of the excess supply function is positive at all equilibria, then there is a unique equilibrium.*

Again, in both results, the price set is a specific case of an $\infty$-normalised price set, and in both cases the assumption on $Z$ is an example or properness.

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## Appendix

**Definition.** A topological space $T$ is called a *Hausdorff space* if for each pair $t_1$, $t_2$ of distinct points of $T$, there exists neighbourhoods $U_1$ and $U_2$ of $t_1$ and $t_2$ respectively that are disjoint.

**Definition.** Let $T$ be a topological space. A *separation* of $T$ is a pair $U$, $V$ of disjoint nonempty open subsets of $T$ whose union if $T$. The space $T$ is said to be *connected* if there does not exist a separation of $T$.

**Definition.** Given points $t_1$ and $t_2$ of the space $T$, a *path* in $T$ from $t_1$ to $t_2$ is a continuous map $f : [a, b] \to T$ of some closed interval in the real line into $T$, such that $f(a) = t_1$ and $f(b) = t_2$. A space $T$ is said to be *pathconnected* if every pair of points in $T$ can be joined by a path in $T$.

**Definition.** A space $T$ is said to be *simply connected* if it is a path-connected space and if $\pi_1(T, t_0)$ the fundamental group of $T$ relative to the base point $t_0$, is the trivial (one-element) group for some $t_o \in T$, and hence for every $t_0 \in T$.