Two-Person Fair Division of Indivisible Items: An Efficient, Envy-Free Algorithm

Brams, Steven J. and Kilgour, D. Marc and Klamler, Christian

New York University

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Steven J. Brams
Department of Politics
New York University
New York, NY 10012
USA
steven.brams@nyu.edu

D. Marc Kilgour
Department of Mathematics
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5
CANADA
mkilgour@wlu.ca

Christian Klamler
Institute of Public Economics
University of Graz
A-8010 Graz
AUSTRIA
christian.klamler@uni-graz.at

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Abstract

Many procedures have been suggested for the venerable problem of dividing a set of indivisible items between two players. We propose a new algorithm (AL), related to one proposed by Brams and Taylor (BT), which requires only that the players strictly rank items from best to worst. Unlike BT, in which any item named by both players in the same round goes into a "contested pile," AL may reduce, or even eliminate, the contested pile, allocating additional or more preferred items to the players. The allocation(s) that AL yields are Pareto-optimal, envy-free, and maximal; as the number of items (assumed even) increases, the probability that AL allocates all the items appears to approach infinity if all possible rankings are equiprobable. Although AL is potentially manipulable, strategizing under it would be difficult in practice.

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1. Introduction

In this paper, we present two simple algorithms for dividing indivisible items fairly between two players. Both assume that the players can *strictly* rank the items from best to worst (i.e., there are no ties in the rankings), and both use *only* these rankings to make allocations.

The first algorithm can be thought of as asking the players to make simultaneous and independent choices in sequence, starting with their most preferred item (if they are sincere) and progressively descending to less and less preferred items that have not already been allocated. The second algorithm requires the players to submit their complete rankings in advance to a referee (or computer).

The first algorithm was proposed by Brams and Taylor (1999) as a "query step" for allocating indivisible items fairly between two players, A and B. We call it BT, and it works as follows: At any point in the allocation process, if A and B name different items, BT allocates them immediately; if A and B name the same item, it goes into a "contested pile." BT does not allocate the items in the contested pile, which is a subject we will return to later.

The second algorithm, which we describe in section 3 and call AL, also allocates items sequentially, one at a time to each player, based on the players' rankings. Like BT, it does not necessarily allocate all the items—some may go into a contested pile.

But under AL, the contested pile is never larger, and may be smaller, than under BT. Furthermore, if the contested piles under AL and BT contain the same number of items, each player will never strictly prefer the items it receives under BT.

BT and AL share the property that, when they assign an item to one player, they simultaneously assign another item to the other player. Thus, A and B are allocated equal numbers of items.

The allocations given by both BT and AL are *envy-free (EF)*: A's items can be matched pairwise to B's items such that A prefers each of its items to the corresponding item of B; there is a similar pairwise matching of B's items to A's. But only the allocations given by AL are always efficient or *Pareto-optimal (PO*): There is no other allocation that is at least as good for A and B and better for one or both players, based on their rankings. Also, AL allocations are always *maximal*: There is no EF allocation that allocates additional items to the players.

Both BT and AL are *manipulable*: It is possible for a player to improve its allocation by ranking items insincerely (i.e., not according to its preferences). Practically speaking, however, successful manipulation of either algorithm would require that a player have essentially complete information about the preference ranking of its opponent, which is highly unlikely in most real-life situations.

In many disputes, including divorce and estate division, only an allocation in which the disputants receive about the same number of items will be perceived as fair. BT and AL work well for that purpose, especially when they allocate most, if not all, of the items. BT is a paragon of simplicity, and AL is not much harder to understand as we show in section 3, which should facilitate its acceptance as a practicable procedure.

The paper proceeds as follows. In section 2, we define envy-freeness formally, illustrate it with examples, provide a necessary and sufficient condition for an allocation to be envy-free, and give a condition on the players' preferences that is necessary for the existence of an envy-free allocation.

In section 3, we define and illustrate BT and AL, showing that AL generally allocates more or better items to the players than BT. Then we use AL to show that the necessary condition of section 2 is also sufficient for the existence of an envy-free allocation. In section 4 we prove that an AL allocation is Pareto-optimal and maximal, and then illustrate that it may be manipulable.

In section 5, we calculate the probability that an EF allocation of all the items exists when all possible preference rankings are equiprobable and offer a conjecture about this probability as the number of items, assumed to be even, approaches infinity. In section 6, we summarize our results and draw some conclusions.

2. Envy-Free Allocations

Consider the task of dividing a set of indivisible items between two players, A and B, so that each player receives an equal number of items. For example, if the items are numbered 1 – 6, the allocation might be {1, 3, 5} to A and {2, 4, 6} to B. We assume that each player can strictly rank the items from most to least preferred. Roughly speaking, an allocation is envy-free if each player prefers the subset of items it receives to the subset of items received by its opponent.

The precise definition of envy-freeness uses only the players' orderings to assess whether each player prefers its own subset of items to its opponent's subset. Denote the sets of items received by A and B by S_A and S_B, respectively. Recall that $|S_A| = |S_B|$, that is, S_A and S_B have the same cardinality (i.e., contain equally many items). An allocation (S_A, S_B) is *envy-free* iff there exists an injection $f_A: S_A \to S_B$ and an injection $f_B: S_B \to S_A$

such that, for each item x received by A, A prefers x to $f_A(x)$, and for each item y received by B, B prefers y to $f_B(y)$.[1] Thus, for each item received by a player in an envy-free allocation, it pairwise prefers its items to its opponent's items.

Suppose the players' preferences are as indicated below:

Example 1: A: $\underline{1}$ 2 $\underline{3}$ 4 $\underline{5}$ 6

B: $\underline{2}$ $\underline{4}$ $\underline{6}$ 1 3 5

Consider the underscored allocation, {1, 3, 5} to A and {2, 4, 6} to B. We show that this allocation is envy-free (EF) by exhibiting a 1-1 mapping from A's items to B's, and a 1-1 mapping from B's items to A's, such that each player prefers each of its own items to the opponent's item to which it is mapped. These mappings are, for A, $f_A(1) = 2$, $f_A(3) = 4$, $f_A(5) = 6$; and for B, $f_B(2) = 1$, $f_B(4) = 3$, and $f_B(6) = 5$. To simplify notation, we writing these mappings as $f_A(1, 3, 5) = (2, 4, 6)$ and $f_B(2, 4, 6) = (1, 3, 5)$.

We emphasize that each player pairwise prefers its own item to the corresponding item of its opponent. For example, A receives item 3 and prefers 3 to $f_A(3) = 4$, but A does not prefer item 3 to item 2, another item received by B. However, A prefers item 1, another item it receives, to item 2. In this example, the functions f_A and f_B are inverses, but this property is not essential and, indeed, cannot be achieved in some examples, as we will show later.

By comparison, the allocation {1, 2, 3} to A and {4, 5, 6} to B is not EF. This could be proven by checking exhaustively all possible injections from {1, 2, 3} to {4, 5, 6}, and from {4, 5, 6} to {1, 2, 3}, showing that no pair of them has the required property. But there is an easier proof, based on the following characterization.

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[1] An *injection* is a 1-1 mapping, that is, f_A maps each item in S_A to a different item in S_B.

Proposition 1. *An allocation is EF iff, for each item, x, received by a player (say, A), the number of items received by B that A prefers to x is not greater than the number of items received by A that A prefers to x.*

Proof. To show that the given property is necessary for an allocation to be EF, suppose that A receives x in an EF allocation, and it prefers r of its own items to x and s of B's items to x. We show that $r \geq s$. Consider the mapping f_A defined above. Suppose that for some item y received by A, $f_A(y)$ is preferred to x. Because A prefers y to $f_A(y)$, A must also prefer y to x. It follows that each of B's s items that A prefers to x must be the image under f_A of an item received by A that A also prefers to x. There are r such items, which implies that $r \geq s$. A similar argument beginning with an item received by B completes the proof of necessity.

To show sufficiency, suppose that an allocation satisfies the given property. We construct a 1-1 mapping, f_A, of A's items to B's items such that A always prefers the item it receives to the corresponding item received by B. To see that A must receive its most preferred item, x_1, assume otherwise. Then B receives at least one item that A prefers to the most preferred item it does receive, whereas A receives zero such items, contradicting the required property. Therefore, x_1 must have been assigned to A. Let x_k denote A's k^{th} most preferred of the items it receives, and define $f_A(x_k)$ to be A's k^{th} most preferred of the items B receives. Because the number of B's items that A prefers to x_k cannot exceed $k - 1$, it follows that A prefers x_k to $f_A(x_k)$. The mapping f_A thus defined, and the mapping f_B constructed analogously, show that the allocation is EF. ■

An alternative way to state Proposition 1 is as follows: If an allocation is EF, then whenever a player receives an item x, it must also receive at least half of all the allocated items that it strictly prefers to x. Assuming that all items are allocated, then if a player receives an item x that it ranked k^{th} in its original ranking, then it must also receive at least $(k - 1)/2$ items that it strictly prefers to x.

It is clear that the allocation {1, 2, 3} to A and {4, 5, 6} to B in Example 1 is EF for A: Because A receives its top three items, it cannot prefer any items that B receives.

But the story is different for B, as can be shown using Proposition 1. It receives item 5, which is 6^{th} in its ranking, and prefers only two of the items it receives, 4 and 6, to 5. The allocation cannot be EF for B, because it prefers to 5 more items in A's subset (three: 1, 2, and 3) than in its own subset (two: 4 and 6). (In general, in an EF allocation, a player must receive its most preferred item, and cannot receive its least preferred item.) Another proof can be based on the fact that B receives item 4, which it ranks 2^{nd}, but no item that it prefers to item 4 (A receives item 2, which B ranks 1^{st}).

We can now characterize all pairs of preference rankings for which EF allocations exist. Specifically, we present Condition D below, which we will show is necessary and sufficient for the existence of an EF allocation. The proof of necessity is given below; the proof of sufficiency will be given after we describe AL, which we show in section 3 always produces an EF allocation if Condition D is satisfied.

Observe that, for an EF allocation to be possible, the number of items to be allocated to the two players, n, must be even. This is because, for an allocation to be EF, there must exist a 1-1 mapping of A's items to B's (and of B's items to A's), which is not possible unless these two non-intersecting subsets have the same cardinality.

Let *n* be the number of items to be allocated. Fix the rankings of A and B. Before stating Condition D, we begin with a sequence of simpler conditions. We say that A's and B's rankings satisfy Condition C(*k*) iff

Condition C(*k*). *The set consisting of A's k most preferred items is equal to the set consisting of B's k most preferred items.*

Note that condition C(*k*) refers to an equality of sets—A's *ranking* of its first *k* items may or may not be the same as B's: What is required is that the same set of *k* items be most preferred by both A and B. We will be concerned about whether C(*k*) is true when *k* is an odd number. In Example 1, where $n = 6$, C(*k*) is *not* satisfied for any odd *k*:

$k = 1$: {1} for A is different from {2} for B.

$k = 3$: {1, 2, 3} for A is different from {2, 4, 6} for B.

$k = 5$: {1, 2, 3, 4, 5} for A is different from {2, 4, 6, 1, 3} for B.

We can now state Condition D in terms of Condition C(*k*):

Condition D. *Condition C(k) fails for all odd values of k,* $1 \leq k \leq n$.

In other words, Condition D states that, for all odd *k*, at least one of A's top *k* items is not a top *k* item for B, which as we noted above is true in Example 1. Note that Condition D cannot be true if *n* is odd, because Condition C(*n*) must be true since the set of all items must be the same for both players. Thus, Condition D can be true only if the number of items to be allocated is even.

Proposition 2. *Let n be even. A pair of strict preference rankings of n items* *admits a complete EF allocation iff it satisfies Condition D.*

Proof (of Necessity). To show that Condition D must hold in order for an EF allocation of all n items to exist, we show that, if Condition D fails, then there can be no EF allocation. Now Condition D fails iff there is some odd value of k such that C(k) holds. Assume such a value of k, and let S be the subset consisting of A's (or B's) top k items.

Suppose that an EF allocation exists. Because S contains an odd number, k, of items, it follows that one of A and B, say A, must receive fewer than half of the items in S. Suppose that A receives $r < k/2$ items from S. Moreover, because each player must receive the same number of items in an EF allocation, A must receive at least one item that does not lie in S—that is, it is not among A's k most preferred items.

Let y be the item most preferred by A among the items that A receives that are not in S. If y is h^{th} ranked by A in A's original ranking, we must have $h \geq k + 1$. Moreover, A receives exactly r items that it prefers to y. According to Proposition 1, we must have $r \geq (h - 1)/2 \geq k/2$. But, as noted above, $r < k/2$. This contradiction shows that no EF allocation can exist, establishing that Condition D is necessary. ■

We postpone the proof of sufficiency, which depends on the performance of AL. We describe this algorithm and BT next.

3. The BT and AL Algorithms

In this section, we formally state the rules of BT and AL. Both algorithms allocate a set of indivisible items in a series of stages. In the case of BT, the players can be thought of as simultaneously and independently choosing the most preferred unallocated item at each stage, so a complete ranking is in fact not needed in advance. In the case of

AL, the players submit their complete rankings to a referee (or a computer), which makes choices solely on the basis of the rankings.

BT Rules

1. Players A and B name their most preferred item of those that have not yet been allocated.

2. If A and B name different items, each player receives the item it names; if they name the same item, it goes into the *contested pile* (CP).

3. If all items have been allocated to the players or put in CP, stop. Otherwise, go to step 1.

AL Rules

We begin with an informal description of AL, which also works by descending the preference rankings of the players. If the players have not yet been assigned any items, then if there is an item at the top of both preference orderings, it is put into CP, and this step is repeated until each player most prefers a different unallocated item. If the players most prefer different unallocated items, AL assigns each player its preferred item.

After the first assignment of items to the players is made, new assignments are made

(i) when the players prefer different items; or

(ii) when they prefer the same item, provided a new assignment—of the preferred item to one player and a less preferred item to the other—does not cause envy and so is *feasible*.

When there is a commonly preferred item, the feasibility of assigning it to either player is assessed, one player at a time. Only if there is no such assignment is the commonly preferred item put in CP.

Formally, we start AL at stage 0, which may be repeated. In stage t ($t = 0, 1, 2, \ldots$), exactly t items have already been assigned to each player. AL proceeds until there are no unallocated items.

Stage 0

Compare the most preferred unallocated items of players A and B. If they are identical, place the commonly preferred item in CP and repeat stage 0. If they are different, assign each player its most preferred item. Then go to stage $t = 1$.

Stage t

1. If one unallocated item remains, place it in CP and stop; if no unallocated items remain, stop. Otherwise, compare A's and B's most preferred unallocated items. If they are the same, go to step 2; if they are different, assign each player its most preferred item, and go to stage $t + 1$.

2. Determine whether the unallocated item that A and B both most prefer, say i, which we call the "tied" item, can be assigned to either A or B as follows: Let $j_{A1,}, j_{A2}, \ldots$ represent, in order of A's preference, the unallocated items that A finds less preferable than i. Let $j_{B1}, j_{B2}, \ldots$ represent, in order of B's preference, the unallocated items that B finds less preferable than i.

3. Consider all possible assignments of i to B and first j_{A1}, then j_{A2}, etc., to A. Such an assignment is *feasible* as long as the number of items assigned to B or unassigned, including i, that A prefers to the "compensation" item it receives, j_{A1} or j_{A2} or

…, is at most t. Stage $t + 1$ must be implemented for each feasible assignment of i to B. If the number of items assigned to B. or unassigned, that A prefers to j_{A1}, including i, is greater than t, then no assignment of i to B is feasible.

4. Consider all possible assignments of i to A and first j_{B1}, then j_{B2}, etc., to B. Such an assignment is feasible as long as the number of items assigned to A or unassigned, including i, that B prefers to its "compensation" item, j_{B1} or j_{B2} or …, is at most t. Stage $t + 1$ must be implemented for each feasible assignment of i to A. If the number of items assigned to B, or unassigned, that A prefers to j_{B1}, including i, is greater than t, then no assignment of i to A is feasible.

5. If the assignment of i to A is infeasible, and the assignment of i to B is infeasible, put i in CP. Then repeat stage t with the remaining unallocated items.

Whereas BT gives only one EF allocation, AL may give many, because for $t > 1$, there may be multiple ways to implement AL, as we will illustrate later. Although AL is more complex than BT, it is *not* so for the players, who only need submit their rankings of items.

The chief difference between BT and AL is in how CP is defined, as we next illustrate with two examples. In each example, we assume that the players are *sincere*, ranking each item according to their true preferences. Later we assume that the players may not be sincere—in particular, they may seek to manipulate BT or AL to their advantage.

Example 2: A: 1 2 3 4

B: 2 3 4 1

When BT is applied to Example 2, A indicates that its first choice is item 1, and B that its first choice is item 2; by BT rule 2, the players receive their preferred items because they are different. At stage 2, A and B both indicate that item 3 is their preferred item of those remaining, so it goes into CP, as does item 4 at stage 3, by BT rule 2. In summary, A receives item 1, B receives item 2, and CP = {3, 4}.

Under AL, the players' top-ranked items—1 for A and 2 for B—are different, so item 1 goes to A and item 2 goes to B at stage 0. Now proceed to stage $t = 1$. Of the unallocated items, both players most prefer $i = 3$. For A, one unallocated item, $j_{A1} = 4$ is less preferred than $i = 3$. We consider assigning item 3 to B and $j_{A1} = 4$ to A. But then B will be assigned two items, namely 2 and 3, that A prefers to $j_{A1} = 4$, which exceeds $t = 1$, so i cannot be assigned to B.

For B, too, the only unallocated item less preferred than $i = 3$ is $j_{B1} = 4$. We consider assigning $i = 3$ to A and $j_{B1} = 4$ to B. This assignment is feasible, because only one item that is preferred by B to $j_{B1} = 4$ is allocated to A. In summary, AL produces the allocation in which A receives {1, 3}, B receives {2, 4}, and CP = ϕ in Example 2.

In general, we refer to an allocation in which CP = ϕ as *complete*. Example 2 shows that AL may sometimes produce a complete allocation when BT does not.

Example 2 also shows that under AL, the 1-1 mappings, f_A and f_B, of A's items onto B's and B's onto A's need not be inverse functions. In particular, the allocation given by AL is EF for A because $f_A(1, 3) = (2, 4)$, and it is EF for B because $f_B(2, 4) = (3, 1)$.[2]

[2] The mappings f_A and f_B are inverses iff $f_B(f_A(x)) = x$ for all x in A's subset. When an EF allocation exists despite a common preference, such as both players preferring item 2 to item 3, it can be shown that the mappings f_A and f_B cannot be inverses. Thus in Example 2, $f_A(1) = 2$, so $f_B(f_A(1)) = 3 \neq 1$.

Example 2 also shows that AL may not only reduce the CP given by BT—containing items {3, 4}—but may eliminate it entirely, allocating all four items to the players. However, as we illustrate next, this will not always be the case.

Example 3: A: 1 2 3 4 5 6

B: 2 3 5 4 1 6

When BT is applied to Example 3, A and B initially receive their most preferred items, 1 and 2, respectively. Next, because both players name item 3, it goes into CP. Then A and B receive the items they name, 4 and 5, respectively. Finally, both players name item 6, so it goes into CP. Altogether, A receives {1, 4}, B receives {2, 5}, and CP = {3, 6}. This allocation is EF, using $f_A(1,4) = (2, 5)$, and $f_B(2, 5) = (1, 4)$ or $(4, 1)$.

Under AL, because the players' top-ranked items are different, item 1 goes to A and item 2 goes to B in stage 0. In stage 1, both players prefer $i = 3$. For A, the most preferred unallocated item less preferred than $i = 3$ is $j_{A1} = 4$. But we cannot assign $i = 3$ to B and $j_{A1} = 4$ to A, because B would be assigned more than $t = 1$ item (namely, 2 and 3) that A prefers to $j_{A1} = 4$.

For B, the first unallocated item less preferred than $i = 3$ is $j_{B1} = 5$. We can assign $i = 3$ to A and $j_{B1} = 5$ to B, because only one item assigned to A is preferred by B to $j_{B1} = 5$. But we cannot proceed further, because after $j_{B1} = 5$ the next unallocated item in B's preference ranking is $j_{B2} = 4$. However, assigning $i = 3$ to A and $j_{B2} = 4$ to B is infeasible, because more than $t = 1$ item—in fact, two items (namely, 2 and 3)—that B prefers to $j_{B2} = 4$ would be unallocated or assigned to A. Therefore, there is only one way to proceed to stage 2, namely with A assigned items 1 and 3, and B assigned items 2 and 4.

In stage 2, A and B both prefer item 4 and next most prefer item 6. As already noted, in an EF allocation, neither player can be assigned item 6, the common last choice. Consequently, both 4 and 6 are put in CP. In summary, AL produces exactly one allocation in Example 3: {1, 3} to A, {2, 5} to B, and {4, 6} to CP.

Example 3 illustrates another possible difference between BT and AL. Neither algorithm produces a complete allocation. Both yield a CP that contains two items. In both cases, CP contains item 6; under BT it also contains item 3, whereas under AL it contains item 4. The AL and BT allocations also differ: A receives {1, 4} under BT and {1, 3} under AL, and B receives {2, 5} under both BT and AL.

Consider two allocations, (S_A, S_B) and (S_A', S_B'), where all four subsets are of equal cardinality. We say that (S_A, S_B) *Pareto-dominates* (S_A', S_B') iff there are injections g_A:$S_A \rightarrow S_A'$ and g_B:$S_B \rightarrow S_B'$ such that A finds x at least as preferable as $g_A(x)$ for all $x \in S_A$, B finds y at least as preferable as $g_B(y)$ for all $y \in S_B$, and for at least one x or y this preference is strict. Thus one allocation Pareto-dominates another if the first allocation is at least as good for both players and better for at least one of them, based on pairwise comparisons.

Note that the Pareto-comparison of (S_A, S_B) and (S_A', S_B') depends only on the assumptions that the four subsets have equal cardinality, that S_A does not overlap S_B, and that S_A' does not overlap S_B'. In particular, the sets of items allocated, $S_A \cup S_B$ and $S_A' \cup S_B'$, need not be identical, making it possible to Pareto-compare two allocations when unallocated items remain, or when the CPs are different.

In Example 3, A prefers its AL allocation, {1, 3}, to its BT allocation {1, 4}, because while both allocations contain item 1, A prefers item 3 to item 4; B is indifferent between its BT and AL allocations, which are both {2, 5}.

Thus, the AL allocation Pareto-dominates the BT allocation. Note also that both players agree that CP = {3, 6}, given by BT, is preferable to CP = {4, 6} given by AL, reflecting the fact that one player (A) prefers its AL allocation to its BT allocation, whereas the other player (B) is indifferent.

Examples 2 and 3 illustrate the following proposition:

Proposition 3. *The number of items allocated to the players under AL is never less, and may be more, than under BT. If the number of items allocated to the players is the same under BT and AL but some items are different, then the AL allocation Pareto-dominates the BT allocation.*

Proof. A commonly preferred item, *i*, which we will call a *tied item,* may be assigned to a player under AL but is never assigned under BT. Thus, one or more tied items may go into CP under BT that would not under AL, so the number of items allocated under AL may be greater, and will never be less, than the number allocated under BT.

When a tied item is allocated under AL, the consequence may be the creation of later tied items, which would not have occurred if the tied item had been put in CP, as it would have under BT. Thus, the total number of items in CP may be the same as under BT. But ties that occur later involve less preferred items, so an AL allocation—even if it does not reduce the cardinality of CP—Pareto-dominates the corresponding BT allocation if they differ. ■

Proposition 4. *An AL allocation is a maximal EF allocation: There is no other EF allocation that allocates more items to the players*.

Proof. The AL algorithm continues until all items are either assigned or in CP. Thus any EF allocation that contains an AL allocation must allocate items from CP to the players. But AL puts an item, *i*, in CP only if it is tied and the assignment of *i* to either player, and any less preferred item to its opponent, cannot preclude the opponent from being envious. ■

This is not to say that AL finds all maximal EF allocations. In Example 3, we found two maximal EF allocations of two items to each player—one by AL and one by BT—but the AL allocation Pareto-dominates the BT allocation. Indeed, Proposition 3 shows that such dominance must be the case when these two allocations are the same size but not identical.

So far we have shown that, for any pair of strict rankings of *n* items,

1. An AL allocation may give each player more items than the BT allocation;
2. An AL allocation may give each player the same number of items as the BT allocation, but the sets may not be the same, in which case the AL allocation Pareto-dominates the BT allocation;
3. The AL and BT allocations may be exactly the same.

Possibility 3 occurs in Example 1, wherein both procedures give {1, 3, 5} to A and {2, 4, 6} to B. It also occurs in two extreme cases: when the players rank all items exactly the same (in which case all items go into CP); and when their rankings are diametrically opposed and n is even (in which case each player will obtain its more preferred half of the items, and CP will be empty).

It is obvious that BT always gives an EF allocation, because it allocates items to players only when they prefer different ones at the same time. This implies that f_A and f_B can be inverses. But recall that Example 2 showed that the no-envy mappings of an AL allocation may not be inverses. Examples 2 and 3 also showed that AL may give larger or more preferred EF allocations than BT.

Earlier we proved the necessity part of Proposition 2: That Condition D—for each odd k, $1 \leq k \leq n$, at least one of A's top k items is not a top k item of B—is necessary for the existence of an EF allocation of all n items, i.e., a *complete EF allocation*. We next show that Condition D is also sufficient by adding the proof of sufficiency to Proposition 2, which we repeat below:

Proposition 2. *Let n be even. A pair of strict preference rankings of n items admits a complete EF allocation iff it satisfies Condition D.*

Proof (of Sufficiency). We show that Condition D is sufficient for the existence of a complete EF allocation by proving that AL produces a complete EF allocation unless Condition D fails. Specifically, we show that, if AL puts any item in CP, then for some odd k, the subset comprising A's k most preferred items must equal the subset comprising B's k most preferred items.

Suppose that we are applying AL to find an EF allocation. At stage 0, if A's and B's top-ranked items are the same, AL will put this item in CP. Thus, if AL puts an item in CP at stage 0, then Condition $C(k)$ must be satisfied for $k = 1$, i.e., A's and B's most preferred items are identical.

Next suppose that A's and B's top-ranked items are different, and that AL has reached stage $t > 0$, so that both players have received t items without violating envy-freeness. For an item to be added to CP, it must be the case that (i) both players prefer it to all other unallocated items (i.e., it is a tied item), and (ii) allocation of the tied item to either player must cause the opponent to be envious.

Assume the tied item is i. If it is possible to assign i to B and j_{A1}—A's most preferred unallocated item after i—to A, while preserving envy-freeness, the number of items assigned to B, including i, that A prefers to j_{A1} must be at most t. If it is not possible to assign i to B and j_{A1} to A, then the number of items assigned to B that A prefers to j_{A1} must exceed t. Because only t items were assigned to each player prior to i, then the number of items assigned to B, including i, that A prefers to j_{A1} must equal exactly $t + 1$. In particular, i itself, together with the items previously assigned to A or to B, must be the first $2t + 1$ items in A's preference ranking.

An analogous argument can be made for B. If it is not possible to assign i to A and j_{B1} to B, then it must be the case that the subset consisting of i, the items previously assigned to A, and the items previously assigned to B must be the first $2t + 1$ items in B's preference ranking.

When the players have the same $2t + 1$ items in their preference rankings—no matter which player receives tied item i—Condition C(k) holds for $k = 2t + 1$, so Condition D fails. To conclude, AL puts an item in CP when Condition D fails, which means that Condition C(k) must hold for some odd k. When Condition D holds, a complete EF allocation must exist, because AL never puts an item in CP. ■

Although Condition D is both necessary and sufficient for the existence of an EF allocation, it does not say what the EF allocation(s) are.[3] For that purpose, we need AL.

As noted previously, both AL and BT always allocate to each player the same number of items, although AL may allocate more items *in toto* (Example 2). Therefore, the number of items allocated to CP—if it is not empty—will be even or odd, depending on whether the total number of items to be allocated is even or odd. In particular, if n is odd, then CP must contain at least one item.

We showed earlier (Proposition 3) that if AL and BT give different EF allocations to the players, AL's allocation must include more, or more preferred, items; furthermore, it gives a maximal EF allocation (Proposition 4). We next assess how well AL and BT do according to other properties.

4. Other Properties of EF Allocations

We begin with an example that illustrates how AL may produce more than one complete EF allocation:

Example 4:

A: 1 2 3 4 5 6 7 8

B: 3 4 5 6 7 8 1 2

In stage 0, AL assigns item 1 to A and item 3 to B. In stage 1, AL assigns item 2 to A and item 4 to B. Then, in stage 2, there is a tie on item 5. The tie cannot be resolved by assigning $i = 5$ to B, because $j_{A1} = 6$, and the assignment of items 3, 4, and 5 to B would mean that of the items A prefers to 6, fewer than half (i.e., only items 1 and 2) are assigned to A. But the tie can be resolved by assigning $i = 5$ to A, in which case B can

[3] We postpone until section 4 examples showing that AL may produce multiple EF allocations.

receive either $j_{B1} = 6$ or $j_{B2} = 7$. Thus, stage 3 can begin with A assigned {1, 2, 5} and B assigned {3, 4, 6}, or with A assigned {1, 2, 5} and B assigned {3, 4, 7}. In the first case, A is assigned item 7 and B item 8 in stage 3; in the second case, A is assigned item 6 and B item 8 in stage 3.

The two resulting EF allocations are underscored below:

(i) A: $\underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ 6\ \underline{7}\ 8$ (ii) A: $\underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ \underline{6}\ 7\ 8$

B: $\underline{3}\ \underline{4}\ 5\ \underline{6}\ 7\ \underline{8}\ 1\ 2$ B: $\underline{3}\ \underline{4}\ 5\ 6\ \underline{7}\ \underline{8}\ 1\ 2$

In (i), a player's minimal ranking for an item it receives is 7th (item 7 for A), whereas in (ii) this minimal ranking is 6th (item 6 for A and item 8 for B). We call (ii) the *maximin* complete allocation—it maximizes the minimum rank of the players—which may be desirable in certain situations.

A complete allocation is called *locally Pareto-optimal* if there is no other allocation of the same items that Pareto-dominates it, i.e., the items could not be re-distributed between the players so that each player is at least as well off, and some player is better off, where comparisons are always pairwise. For example, if there are $n = 2$ items and A prefers item 1 to item 2 and B prefers item 2 to item 1, then the allocation of 2 to A and 1 to B is not Pareto-optimal, because both players would be better off if 1 were assigned to A and 2 to B. Note that in section 3 we defined the Pareto-optimality of allocations that were not constrained by the "same items" condition.

Call an allocation *sequential* if it assigns each player its most preferred item when it is that player's turn according to some sequence (e.g., ABAB or AABB). Note that the players need not alternate in the sequence, though each player must have the same number of turns. The resulting allocation of items, called a *sincere sequence of choices*, clearly depends on the sequence.

Proposition 5 (Brams and King, 2005). *An allocation of a fixed set of items is locally Pareto-optimal (LPO) iff it is the product of a sincere sequence of choices.*

Strictly speaking, BT and AL are not sequential algorithms, because items are assigned to the players simultaneously. But if at some stage the players' first choices are different, then the players can be considered to receive items in either order, AB or BA, because the items received by A and B would be the same. Therefore, when A and B most prefer (and receive) different items, the assignment can be considered as part of a sincere sequence of choices.

Now suppose that, at some stage, the players' first choices are the same. Under BT, this item always goes into CP and, hence, will not be part of the allocation to A and B. Under AL, by comparison, this item will go into CP if and only if the item cannot be assigned to either player so as to maintain envy-freeness. Nonetheless, the resulting allocation is certain to be PO in the sense that no re-allocation of items can Pareto-dominate what the algorithm yields.

Proposition 6. *Both BT and AL produce LPO allocations.*

Proof. We have already noted that the BT allocation of items that do not end up in CP is a sincere sequence of choices. To show that the same is true of the AL allocations, we need only check that it is true at any point when both players prefer the same item. Suppose that the tied item, i, is assigned to B while some compensating item, j_{A1} or j_{A2} or … is assigned to A. Recall that j_{A1}, j_{A2}, … represent, in order of A's preference, the unallocated items that A finds less preferable than i. Clearly, an allocation in which B receives i and A receives j_{A1} is the result of a sincere choice sequence, in the order BA. If B receives i and A receives, say, j_{Ah} where $h > 1$, then the allocation is the result of a sincere choice sequence, B … A, where the missing entries are determined by the eventual allocation of the unallocated items, including $j_{A1}, j_{A2}, \ldots, j_{Ah-1}$. (This may be considered an "out-of-order" assignment in that it does not make assignments strictly according the players' preferences.)

In an AL allocation, every item that is not assigned to a player who most prefers it among all unallocated items is a deferred-compensation item, such as j_{Ah}. When all items that precede j_{Ah} in A's order have been allocated, it will be possible to identify a sincere choice sequence, containing equally many A's and B's, that corresponds to any AL allocation. ■

Complete allocations of items under BT or AL are both EF and LPO. Similarly, incomplete allocations, under BT or AL, satisfy both properties only for the items that are allocated to the players (i.e., that do not go into CP). Moreover, as Proposition 3 shows, when a BT allocation produces the same number of items as an AL allocation but the items are different, then the BT allocation is Pareto-dominated by the AL allocation.

The reason that the AL allocation in Example 3 Pareto-dominates the BT allocation is that, while each algorithm allocates four of the six items to A and B, AL assigns a preferred item (3) to A and BT does not, which puts this item in CP before assigning item 4 to A. This enables A to do better under AL than it does under BT, without changing the allocation to B (but changing the contents of CP).

We note that LPO allocations need not be EF. In Example 2, for instance, the allocation of {1, 2} to A and {3, 4} to B is LPO in that any other allocation is less preferred by A. But B might envy A (for receiving items that bracket its two middle items), so we call such an allocation *envy-possible* (it does not ensure envy). In contrast, allocating {2, 4} to A and {1, 3} to B would be *envy-ensuring* (Brams and King, 2005).

In Example 4, both EF allocations are LPO, because they can be produced by sincere sequences. A sincere sequence that produces (i) is ABABABAB, whereas a sincere sequence that produces (ii) is ABABAABB (there are several other sincere sequences that give each allocation).

In all examples so far in which there is a complete EF allocation (Examples 1, 2, and 4), A and B rank all the items differently (they also do so in Example 3 for the four items that do not go into CP). By contrast, if they ranked all items the same, there would be no EF allocation, because all items would go into CP. It seems plausible, therefore, that different rankings by the players might be a sufficient condition for there to be a complete EF allocation. However, we have the following:

Proposition 7. *Even if the players rank all items differently, there may be no complete EF allocation.*

Proof. Consider the following example, in which the premise of Proposition 7 is satisfied:

Example 5: A: 1 2 3 4 5 6

B: 2 3 1 5 6 4

Condition C(3) holds, because the top $k = 3$ items, {1, 2, 3}, are the same for both A and B. Condition D, therefore, fails. By Proposition 2, there can be no complete EF allocation. ■

Condition D does not tell us what *partial* EF allocation is possible in Example 5. For this purpose, we apply AL, which in stage 0 assigns item 1 to A and item 2 to B.

In stage 1, there is a tie on item 3. The tie cannot be resolved by assigning $i = 3$ to B, because $j_{A1} = 4$, and the assignment of two items (2 and 3) to B would mean that of the items A prefers to 4, fewer than half (only item 1) are assigned to A. Likewise, if $i = 3$ is assigned to A, because $j_{B1} = 5$, the assignment of two items (1 and 3) to A would mean that of the items B prefers to 5, fewer than half (only item 2) are assigned to B.

Therefore, we must put item 3 into CP, after which AL allocates item 4 to A and item 5 to B. Because item 6 is then the only remaining unallocated item, it also must go into CP. In summary, for Example 5, AL gives {1, 4} to A and {2, 5} to B, with CP = {3, 6}. Coincidentally, BT produces the same EF allocation.

Our next example shows that AL may give several complete EF allocations, all of which are maximin (unlike Example 4).

Example 6: A: 1 2 3 4 5 6 7 8

B: 7 8 3 4 5 6 1 2

In stage 0, AL assigns item 1 to A and item 7 to B. In stage 1, AL assigns item 2 to A and item 8 to B. Now notice that both players have exactly the same ranking of the unallocated items (3, 4, 5, and 6). It is not hard to show that AL allows the assignment of any two of these items to one player and the other two to the other player, which makes for a total of $\binom{4}{2} = 6$ EF allocations. Because all can be generated by the 6 sincere sequences (AABB, ABAB, ABBA, BAAB, BABA, BBAA), they are LPO. They are also maximin, because in every one, either A or B receives item 6, which is ranked 6^{th} by both players.

Up to now we have assumed that the players rank items sincerely.[4] Call an algorithm *manipulable* if a player, by submitting an insincere preference ranking, can obtain a preferred allocation.

Proposition 8. *AL and BT are manipulable.*

Proof. We begin with AL, for which there are two allocations in Example 7:

Example 7: (i) A: $\underline{1}$ 2 $\underline{3}$ $\underline{4}$ 5 6 (ii) A: $\underline{1}$ 2 $\underline{3}$ 4 $\underline{5}$ 6

B: $\underline{2}$ $\underline{6}$ 4 $\underline{5}$ 3 1 B: $\underline{2}$ $\underline{6}$ $\underline{4}$ 5 3 1

Allocation (i) is minimax (lowest rank of a player is 4^{th}), whereas allocation (ii) is not (lowest rank of a player is 5^{th}). BT comes close to this allocation, assigning {1, 3} to A and {2, 6} to B, but putting {4, 5} in CP. This is presumably unsatisfying for the players compared to what AL yields, which gives two complete PO and EF allocations, one of which, (i), is maximin.

Now assume that instead of reporting its sincere preferences in Example 7, B reports its preferences to be B'—interchanging items 4 and 6—whereas A continues to be sincere. This yields the following unique AL allocation:

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[4] The implications of insincere behavior are studied in Vetschera and Kilgour (2013). Variations on the rules for making fair allocations, such as claiming or rejecting one or more items in a round, are analyzed in Vetschera and Kilgour (2014).

Example 7 (manipulated by B): A: <u>1</u> 2 <u>3</u> 4 <u>5</u> 6

B': <u>2</u> <u>4</u> <u>6</u> 5 3 1

Thereby B obtains its top three items, whereas without manipulation B's allocation of these items was only one of two possibilities—and not the minimax one (had this property been used to choose between the two AL allocations without manipulation). BT gives exactly the same result, so B's misrepresentation helps it under BT, compared with obtaining only its top two items when it is sincere. ■

We conclude that both AL and BT are manipulable if one player (B in Example 7) knows its adversary's (A's) sincere ranking and exploits its knowledge. But such manipulation seems improbable, short of A's having complete information about B's ranking of items, and A's being in the dark about the possibility of B's misrepresentation. Furthermore, the determination of an optimal misrepresentation strategy, especially when the number of items is large, is far from trivial, particularly in the case of AL because of its greater complexity. It is further complicated if there is a random selection from multiple EF and PO assignments.

In the face of these difficulties, we think that A and B, especially when using AL, are likely to be sincere in submitting preference rankings to a referee. This presumption is reinforced by the fact that if the players are sincere, they can ensure themselves of an EF, LPO, and maximal allocation, though it may not be complete.

5. The Probability of Envy-Free Allocations

There are many pairs of preference rankings (of an even number of items) for which there is no EF allocation. This is certainly true if both players rank all items the same, but it is in fact true if both players agree only on their top-ranked item, because whoever does not get that item will envy the other player (according to our definition of envy in section 2). Similarly, no EF allocation is possible if the two players agree on only the least-preferred item, because whoever gets it will envy the other.

On the other hand, if an EF allocation exists, it need not be unique. Examples 5, 6, and 7 have multiple EF allocations, which raises the question of which one is fairest (in some sense), a question we will turn to later.

To calculate the probability of an EF allocation, fix A's preference ranking as 1 2 3 …, and assume all preference rankings of B are equiprobable. If $n = 2$ items, and A's ranking is 1 2, then B's ranking can be 1 2 or 2 1. In the former case, there will be envy if A receives item 1 and B receives item 2, whereas in the latter there will not be envy, making the probability of an EF allocation ½.

If $n = 4$, B can have any of $4! = 24$ possible preference rankings. To calculate the probability of an EF allocation, we use Condition D. It requires that (i) the first choices of A and B be different and (ii) the top three choices of A and B be different.

Let us instead count the number of ways that Condition D can fail. For (i) to fail, B's first choice must be 1; there are $3! = 6$ orderings in which B's first choice is 1. For (ii) to fail, B's fourth choice must be 4; there are $3! = 6$ orderings that satisfy this condition. But Condition D fails if either (i) fails or (ii) fails, and both may fail simultaneously.

We have counted the number of ways that (i) can fail without regard to (ii), and that (ii) can fail without regard to (i). In so doing, we have double-counted the cases in which both (i) and (ii) fail, which requires that B's first choice be 1 and B's last choice be

4; there are 2 such rankings. We conclude that Condition D can fail in 6 + 6 − 2 = 10 ways. Thus, there are 24 − 10 = 14 preference rankings for B for which Condition D holds.

We have shown that when there are $n = 4$ items, 14/24 = 0.5833 of the possible allocations admit a complete EF allocation. Similarly, it can be shown that when $n = 6$, the probability of a complete EF allocation is 488/720 = 0.6778; when $n = 8$, the probability is 30,224/40,320 = 0.7496. Unfortunately, a generalization of this calculation appears quite difficult, but we conjecture that, for even values of n, the probability of an EF allocation approaches $\frac{n-2}{n}$, which approaches 1 as n approaches infinity.

6. Summary and Conclusions

Given two players can rank a set of indivisible items from best to worst, the main algorithm we have analyzed (AL) finds an allocation, giving the players the same number of items, that is LPO, EF, and maximal—and complete if such an allocation exists. A simpler algorithm (BT), which is also EF, may allocate fewer or less preferred items to the players and so may not be maximal or, if it is maximal, will be Pareto-dominated by an AL allocation if the BT allocation is different.

An advantage of BT, however, is that because the players make sequential decisions, they can decide, based on the items they have already acquired, which of the remaining items to try to obtain next. By contrast, AL requires that the players rank all items in advance, so if the players' valuations are interdependent (i.e., the acquisition of one item affects the value of others that may be acquired), they cannot take advantage of possible synergies among the items.

Because AL and BT are manipulable, players can sometimes do better by misrepresenting their preferences. But without complete information about an adversary's preferences, AL, especially, because of its greater complexity, would be difficult to exploit.

Indeed, trying but failing to do so could result in an allocation that is neither LPO nor EF. Thus, players would seem to have good reason to be sincere in using AL.

At least one, but not necessarily all, allocations produced by AL will be minimax. This seems an important property to ensure balanced allocations—one player does not suffer because it receives an especially low-ranked item. If there is more than one minimax allocation, a random choice could be made from among them.

It would also be desirable to investigate further how the size and distribution of LPO and EF allocations, and CP, depend on the number of items, assuming the preferences of the players are randomly chosen. Of course, a complete allocation is impossible if the number of items of odd. Our preliminary calculations suggest that the probability that a complete EF allocation exists increases rapidly with the number of items and approaches 1 as the number approaches infinity. But we have not attempted any more realistic computation, for example by assuming that the players' preferences are correlated, especially for more preferred items. We do not know how such conditions would affect the size and distribution of such allocations, and of CP.

Brams, Kilgour, and Klamler (2012) have proposed the "undercut procedure" to allocate items in CP, based on players' relative preferences for bundles of items.[5] The allocations it produces are EF (based on a different definition of envy) but not necessarily

[5] The extent to which this procedure is vulnerable to strategic manipulation is analyzed in Vetschera and Kilgour (2013, 2014).

LPO. Combined with AL, it can be used to find an allocation of all the items, including those that AL puts into CP.

Alternative two-person procedures—including "adjusted winner" (Brams and Taylor, 1996, 1999), in which players allocate points to items, and a swapping procedure in which player can make trades after an initial allocation (Brams and Kaplan, 2004)—also lead to fair division of items with desirable properties.[6] However, both procedures require that the players provide more information than a simple ranking of the items and, in the case of adjusted winner, that one item, which is not known in advance, be divisible. The fact that AL only requires that the players indicate their preferences is surely an advantage in some applications, but in others it may be desirable to elicit and use information on the strength of their preferences.

[6] One such property is equitability, in which all players perceive that they received the same proportion of the total value. Procedures for finding equitable as well as envy-free allocations of indivisible items are analyzed in Herreiner and Puppe (2002). Unlike BT and AL, they require that players specify preferred bundles of items rather than individual items, which makes them more akin to the undercut procedure for dividing CP of Brams, Kilgour, and Klamler (2012).

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