An extension of the Sard-Smale Theorem to domains with an empty interior

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AN EXTENSION OF THE SARD-SMALE THEOREM TO DOMAINS WITH AN EMPTY INTERIOR

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Abstract. This work extends the Sard-Smale Theorem to maps between convex subsets of Banach spaces that may have an empty interior.

1. Introduction

The purpose of this note is to extend Sard's and Smale's Theorems to maps between subsets of Banach manifolds which may have an empty interior. Recall first the following theorems.

Theorem 1. ***(Sard Theorem,*** [2]***)*** *Let U be an open set of $\mathbb{R}^p$ and $f : U \to \mathbb{R}^q$ be a C^k map where $k > \max(p-q, 0)$. Then the set of critical values in $\mathbb{R}^q$ has measure zero.*

Theorem 2. ***(Smale Theorem,*** [3]***)*** *Let $f : M \to V$ be a C^s Fredholm map between differentiable manifolds locally like Banach spaces with $s > \max(index f, 0)$. Then the regular values of f are almost all of V.*

The proof of Theorem 2 is local, and requires for M to have a nonempty interior. However, there are instances that require a domain with an empty interior; these subsets will be called *star manifolds* and we will introduce its definition in the next section. The paradigmatic case is the positive cone of most of Banach spaces. For example, when the domain is the positive cone of an L^p or ℓ^p space, $1 \leq p < \infty$, as is commonly required in economic theory. Thus, the main issue of this paper is to extend Sard's Theorem to subsets of a Banach space locally isomorphic to convex sets with empty interior. To this end, we will prove the following result.

Theorem. ***(Main theorem)*** *Let $f : M \to V$ be a C^r star Fredholm map between star Banach manifolds, with $r > max(index\, f, 0)$. Suppose that*

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M and V are connected and have a countable basis. Furthermore, suppose that f is locally proper and that it has at least one regular value. Then, the regular values of f are almost all of V.

2. Analytical preliminaries

Let B be a Banach space, and let B_+ denote the positive cone of B which may have an empty interior. Notice that B_+ is a convex subset of B. The results of this paper can be generalized to any convex subset, not just the positive cone, but we restrict the analysis to this set because of an interest in economic applications.

Definition 1. ***(α-admissible directions)*** *We say that $h \in B$ is an α-*admissible direction for $x \in B_+$ *if and only if there exist $\alpha > 0$ such that $x + \alpha \frac{h}{\|h\|} \in B_+$. The set of $\alpha-$admissible directions at x will be denoted by $\mathcal{A}_\alpha(x)$.*

Note that since B_+ is a convex subset of B then, if y and x are points in B_+, it must be that $h = (y - x)$ is α-admissible for x. To see this, consider $z = \alpha' y + (1 - \alpha')x$, $0 \leq \alpha' \leq 1$. Then $z \in B_+$ and $z = x + \alpha'(y - x)$, $\forall\, 0 \leq \alpha$. Let $\alpha = \frac{\alpha'}{\|h\|}$. Similarly, it follows that if h is α-admissible, it is also β-admissible for all $0 < \beta \leq \alpha$. Note then that for $\alpha \geq \beta > 0$, we have $\mathcal{A}_\alpha(x) \subseteq \mathcal{A}_\beta(x)$.

Definition 2. ***(Star-differentiable functions)*** *Let $u : B_+ \to \mathbb{R}$ be a real function defined on B_+. We say that u is* star-differentiable at $x \in B^+$ *if the Gâteaux derivative of u at x exists for all $h \in \mathcal{A}_\alpha$.*

Definition 2, in other words, states that u is star-differentiable at x if and only if there exists a map $L_x \in L(B_+, \mathbb{R})$ such that

$$\lim_{\alpha \to 0} \frac{u(x + \alpha h) - u(x)}{\alpha} = \frac{d}{d\alpha}|_{\alpha=0} u(x + \alpha h) = L_x h$$

for all $h \in \mathcal{A}_\alpha(x)$; that is, if $u(x + \alpha h) - u(x) = \alpha L_x h + o(\alpha h)$.

Definition 3. ***(Star-neighborhoods)*** *Let $x \in B_+$. We define a* star-neighborhood of x *by*

$$V_x^*(\alpha) = \left\{ y \in B_+ : y = x + \beta \frac{h}{\|h\|}, \forall\, h \in \mathcal{A}_\alpha(x),\ \textit{where } 0 \leq \beta \leq \alpha \right\}.$$

We will say that O is a star-open subset of B_+ *if for each $x \in O$ there exist $V_x^*(\alpha) \subset O$.*

One can check that star neighborhoods form a base of a topology, called the *star-topology.*

Remark 1. *From now on to represent admissible directions we consider vectors h such that $\|h\| = 1$.*

Definition 4. ***(Star-charts)*** *Let Γ be a Hausdorff topological space. A* star-chart *on Γ is a pair (U^*, ϕ^*) where the set U^* is an open set in Γ and $\phi : V^* \to U^*$ is an homeomorphism from the star-neighborhood $V^* \subset B^+$ onto U^*. We call ϕ a parametrization. In such case, we say that the parametrization is of class C^k if the function ϕ is k times star-differentiable.*

Definition 5. ***(Star-manifold)*** *Let Γ be a Hausdorff topological space. We say that that Γ is a C^k* star-manifold *if for every $p \in \Gamma$, there exists an open star-neighborhood of B_+, denoted $V_a^*(\alpha)$, and a C^k parametrization, $\phi : V_a(\alpha) \to V_p$, where $V_p \subset \Gamma$ is an open neighborhood of p.*

To highlight the structure of Γ as a star manifold, we will use the notation Γ^*. We wish to remark in Definition 5 that, since we consider $\Gamma \subset B$ where B is a Banach space, then the open set in question can be considered to be $V_p^* = V_p \cap \Gamma$ where V_p is an open neighborhood of p in the topology of the norm.

Definition 6. ***(Star-atlas)*** *A C^k* star-atlas *is a collection of star charts $(V_{p_i} \cup M, \phi_i)$, $i \in I$, that satisfies the following properties:*

- (i) *The collection $V_{p_i} \cup M$, $i \in I$, covers Γ.*
- (ii) *Any two charts are compatible.*
- [iii) *The map $\phi : V_{a_i}(\alpha) \to V_{p_i}$ is C^k star differentiable.*
- (iv) *The set $\phi_i^{-1}(V_{p_i} \cup M)$ is a star-open subset of B_+.*

Definition 7. ***(Star-submanifolds)*** *Let Γ^* a C^k Banach manifold, $k \geq 0$. A subset S of Γ^* is called a* star submanifold of Γ^* *if and only if for each point $x \in S$ there exists an admissible chart in Γ^* such that*

- (i) $\phi_i^{-1}(S \cap V_{p_i}) \subset V_{a_i}^*(\alpha)$.

(ii) *The admissible directions* $\mathcal{A}_a$ *contain a closed subset* $\mathcal{B}_a$ *which splits* $\mathcal{A}_a$*.*

(iii) *The star chart image* $\phi^{-1}(V_p \cap S)$ *is an open star set* $V^* = V_a^*(\alpha) \cap \mathcal{B}_a$*.*

Definition 8. ***(Tangent spaces)*** *Let* M^* *be a star manifold. The* tangent set *at* $p \in M^*$ *is the subset* T_pM^* *that can be described in the following way. Let* $\phi : V_a^*(\alpha) \to V_p$ *with* $p = \phi(a)$*. We write* $T_pM^* = \phi'(a)(\mathcal{A}_\alpha(a))$*. That is,* $y \in T_pM^*$ *if and only if* $y = \phi'(a)h,\ h \in \mathcal{A}_\alpha(x)$*.*

Definition 9. ***(Submersions)*** *Let* $f : D(f) \to Y$ *be a mapping between* $B_+ = D(f)$ *and the Banach space* Y*. Then,* f *is called a* submersion *at the point* x *if and only if*

(1) f *is a* C^1 *mapping on a star-neighborhood* $V^*(x)$ *of* x*;*
(2) $f'(x) : B \to Y$ *is surjective; and,*
(3) *the null space* $N(f'(x))$ *splits* B*.*

We also say f *is a* submersion *on the set* M *if it is a submersion at each* $x \in M$*.*

Definition 10. ***(Regular points and regular values)*** *Let* $f : D(f) \to Y$ *be a mapping between* $B_+ = D(f)$ *and the Banach space* Y*. Then, the point* $x \in D(f)$ *is called a* regular point *of* f *if and only if* f *is a submersion at* x*. The point* $y \in Y$ *is called a* regular value *if and only if* f^{-1} *is empty or it consists solely of regular points. Otherwise* y *is called a* singular value*, i.e.* $f^{-1}(y)$ *contains at least one singular point.*

3. Results

Theorem 3. ***(The preimage theorem)*** *Let* $f : M^* \to N$ *a* C^k *mapping from a star manifold* M^* *to a Banach space* N*. If* y *is a regular value of* f*, then* $S = f^{-1}(y)$ *is a star-submanifold of* M^*

Proof. It suffices to study the local problem. Let $V_a^*(\epsilon)$ be a star neighborhood of $a \in B^+$ and consider the C^k star differentiable map $\phi : B_+ \to M^*$ such that $\phi(a) = p$. Without loss of generality, let $f(p) = 0$. Let $h \in \mathcal{A}_\epsilon(a)$ and let V_p be a neighborhood of p. Then $V_p \cap M^* = \phi(V_a^*(\epsilon))$. Thus, $\phi(a + \alpha h) \in V_p \cap M^*$. From the local submersion theorem if f is a submersion, there exists a parametrization ϕ such that, $\phi(a) = p,\ \phi'(a) = I$.

From Definition 5, for all $p' \in V_p \cap M^*$, there exists h and α such that $h \in \mathcal{A}_a(\alpha)$. Since *ker* $f'(p)$ splits B, there exists a projection $P : B \to N$. Let $P^\perp = I - P$ and $N^\perp = P^\perp$. Thus, we obtain that $B = N \oplus N^\perp$ and that $f'(p) : N^\perp \to Y$ is bijective. Denote its inverse by $A : Y \to N^\perp$. So let

$$a = P(p) + Af(p) \ \ and \ \ a + \alpha h = [P(p') + Af(p')],$$

$$\phi^{-1}(x) = Px + Af(x)$$

where $A = f'(p)^{-1}$.

Multiplying both sides of this equation we obtain: $f'(p)\phi(x) = f(x)$. So, from the local submersion theorem, given that $f(\phi(a)) = 0$, the equality

$$f(\phi(a + \alpha h)) = f'(\phi(a))(\alpha h) + y,$$

implies that, the solution of the equation $f(z) = y$ in a star-neighborhood V_p^* of p, corresponds to the solution of the equation $f'(\phi(a))h = 0$. Therefore, S is the set $h \in \mathcal{A}_a$ such that $f'(\phi(a))h = 0$ for some $h \in \mathcal{A}_\alpha(a)$. Hence, S is a star-submanifold of M^*. □

Definition 11. ***(Star Fredholm maps)*** *A* star Fredholm operator *is a star continuous linear map* $L : E_1 \to E_2$ *such that*

(i) *dim* $kerL < \infty$*;*
(ii) *range* L *is closed;*
(iii) *dim* $cokerL < \infty$.

The index *of* L *is* A star Fredholm map *is a star continuous map between star manifolds such that at each point in the domain, its star Gateaux derivative is a star Fredholm operator. The index of a star Fredholm map is the index of its linearization.*

Definition 12. ***(Locally star proper maps)*** *A star Fredholm map* $F : M \to V$ *is said to be* locally star-proper *if for every* $x \in M$ *there is a star neighborhood* U *of* x *such that* f *restricted to* U *is proper.*

Theorem 4. ***(Main theorem)*** *Let* $f : M \to V$ *be a* C^r *star Fredholm map between star Banach manifolds, with* $r > max(index\, f, 0)$*. Suppose that* M *and* V *are connected and have a countable basis. Furthermore, suppose that* f *is locally proper and that it has at least one regular value. Then, the regular values of* f *are almost all of* V*.*

Proof. The proof follows closely [3]. The theorem is proved locally, since we assume M has a countable base and first category. Thus, let U be a star neighborhood of $x_0 \in M$. In this case, U is a subset of some Banach space E. Then, $A = Df(x_0) : E \to E'$ for some Banach space E'. Since A is a star Fredholm operator, we can write $x_0 = (p_0, q_0) \in E_1 \times kerA = E$. Thus, the Gateaux-star derivative $D_1 f(p,q) : E_1 \to E$ maps E_1 injectively onto a closed subspace of E for all (p,q) sufficiently close to (p_0, q_0). From the generalized implicit function theorem of [1], we know there is a star neighborhood $U_1 \times U_2 \subset E_1 \times kerA$ of (p_0, q_0) such that D_2 is compact and if $q \in U_2$, f restricted to $U_1 \times q$ is a homeomorphism onto its image. Since f is is locally proper by assumption, the critical points of f (which by assumption there is at least one) are closed. □

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