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Li, Minqiang and Mercurio, Fabio

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Minqiang Li*, Fabio Mercurio†

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Abstract

We develop an asymptotic expansion technique for pricing timer options under general stochastic volatility models around small volatility of variance. Closed-form approximation formulas have been obtained for the Heston model and the 3/2-model. The approximation has an easy-to-understand Black-Scholes-like form and many other attractive properties. Numerical analysis shows that the approximation formulas are very fast and accurate.

*Derivatives Research, Bloomberg LP. E-mail: minqiang.li@gmail.com.
†Derivatives Research, Bloomberg LP. E-mail: fmercurio@bloomberg.net.

1 Introduction

In 2007, Société Générale Corporate and Investment Banking introduced a new variance derivative to the market called "timer options" (See Sawyer (2007)). A timer call option is similar to a plain-vanilla call option, except that the expiry is not deterministic. Rather it is specified as the first time when the accumulated realized variance exceeds a given budget. By allowing a purchaser to custom choose the budget level, this product offers the purchaser a simple way to combine directional bet and volatility bet within a single product. Major banks such as Bank of America and UBS have been trading this product, sometimes under different names such as "mileage option" or "mileage warrant". Lehman Brothers at one point also published a product overview of timer options (Hawkins and Krol, 2008). The buyers of timer options are usually hedge funds, with the budget level often chosen below the accumulated variance level that would occur with a constant implied volatility and a fixed target expiration in order to lower the purchase price and to increase leverage. Sawyer (2008) also reported that Société Générale Corporate and Investment Banking started to sell other timer-style options such as "timer out-performance options" and "timer swaps".

Timer options were studied many years ago by Neuberger (1990) and Bick (1995) when such security did not even exist in the market place. We note the recent enlightening theoretical work of Carr and Lee (2010) and Lee (2012). The current paper deals with the computation of timer option prices under a general stochastic volatility model. Despite the relative simple payoff structure, pricing timer options turns out to be very challenging. One method to price the timer option is through Monte Carlo simulation, as was studied in detail by Li (2010), and Bernard and Cui (2011). Using the technique of time change, Bernard and Cui reduced the computational cost of a single timer option from many hours to a few minutes, which is a remarkable improvement. In addition to being time-consuming, there are other shortcomings of the Monte Carlo method. For example, sensitivity analysis on model parameters usually requires a complete re-run of the whole simulation. Also, often it requires large number of simulations to get a relatively accurate estimate of the greeks. Another method by Liang, Lemmens and Tempere (2011) involves multi-dimensional numerical integration of the Black-Scholes formula where the transition density is given by a complex Fourier inversion which involves special functions such as Bessel functions. This approach is very sophisticated in that it uses the path integral technique developed in quantum field theory. However, the result involves high-dimensional numerical integration, with one of the integration dimension involving complex Fourier inversion and special functions. The method gives exact result, but only works for very limited number of models where the transition density is known. It is also computationally expensive due to high-dimensional numerical integration and might have stability issues similar to the complex integration in Heston model for plain-vanilla options. A third method is to use techniques such as perturbation to get an accurate and fast approximation. Recently, Saunders (2010) considers an asymptotic expansion for stochastic volatility models under fast mean-reversion. The nice feature of this approximation is that it is in closed form. However, a key shortcoming of this approximation is that it requires extremely large mean-reversion coefficient to obtain satisfying accuracy. Table 1 of Saunders (2010) uses a value of 200 for this coefficient. In real life applications, it is usually in the order of 1 for equities or indices and much smaller for some other asset classes. This limits its potential application.

In this paper, we take a different asymptotic expansion approach. Instead of looking at the mean-reverting coefficient as in Saunders (2010), we consider an expansion on the volatility coefficient of the instantaneous variance process. The reason is that when this coefficient is zero, timer option has a closed-form solution. By proposing a simple Black-Scholes-like approximate formula, we decompose the second-order pricing PDE of the timer option into three PDEs. The first two PDEs are for the effective discounting maturities in the Black-Scholes formula for the stock and strike terms. The third PDE is for the effective variance budget used in the moneyness parameters in the Black-Scholes formula. It turns out that to low orders in the volatility coefficient, all three PDEs can be solved analytically to give exact closed-form formulas. Numerical analysis shows that our method is extremely fast and very accurate. Implemented rather naively in MATLAB on Intel E8400, we are able to price a thousand timer options within a couple of seconds. For the parameters used in Liang, Lemmens and Tempere (2011), we get percentage pricing errors around 0.03% under the Heston model, and around 0.20% under the 3/2 model across different strikes and different correlations.

Our method adds to the large literature of using asymptotic expansion in option pricing. Such an expansion involves a small parameter, often taken to be a small quantity in the problem such as volatility, time to maturity, mean-reverting strength or its reciprocal, etc. Oftentimes searching for such a small quantity is crucial for the approximation.[1] The current work is most closely related to Lewis (2000), where he develops a small volatility of variance expansion for plain-vanilla options under the Heston model. A big difference is that Lewis works under the Fourier-transformed momentum space, while our analysis is done completely with the original variables. We also note that small volatility expansion is also studied in Liption (2001). The idea of using Black-Scholes-like formula to approximate option prices is also not new, a notable example here being Kirk's approximation formula for spread options (see Kirk (1995)).

Our approximation has many attractive properties. It resembles the Black-Scholes formula

[1]For example, Li, Deng and Zhou (2008) developed an expansion based on the curvature of the exercise condition of two-asset spread option, and on the smallness of the Hessian matrix in the multi-dimensional spread option case.

which makes it easier to interpret. Because of this Black-Scholes-like form, greeks such as delta and gamma are given in very simple forms. The approximation is fast and accurate. It reduces to the known solvable cases, for example, when variance process is deterministic, or when interest rate and dividend rate are both zero. It always gives positive option prices and satisfies obvious arbitrage bounds. The approximation respects put-call parity, meaning that the approximate prices for timer cash contract, time share contract, and timer puts are consistent with the approximate timer call price. The method applies to generic stochastic volatility models. As long as the drift and diffusion functions of the variance process are simple, we can obtain a closed-form approximation. For all the papers on timer options cited in this introduction, dividend rate is assumed to be zero for easier analysis. This is a strong limitation in application. In this paper, we allow for nonzero dividend rate. An additional attractive feature of our approximation is that the approximation formula does not break down if the absolute value of the correlation coefficients of the two Brownian motions is around 1, or even is exactly 1. This contracts the Monte Carlo method in Bernard and Cui (2011) where a naive implementation with a perfect correlation would cause numerical problems. Finally, as byproducts, our approach also gives an approximation for the expected exercise time of the timer option and the expected stock price at the random exercise time. In fact, we are able to get an approximation for the joint moment generating function of the random exercise time and the stock price at the random exercise time.

Similar to the case of plain-vanilla options, we propose for the first time a concept of Black-Scholes implied volatility for timer options. In contrast to the plain-vanilla options, for timer options the implied volatility enters the Black-Scholes formula through the two discounting factors rather than through the total variance. We then define the Black-Scholes implied volatility surface for timer options. The availability of a fast closed-form formula allows us to easily examine these surfaces for the Heston model and the 3/2 model. Such surfaces are useful in practice when studying the effect of various parameters on timer option prices with different variance budgets and different strike prices in a collective way.

The rest of the paper is organized as follows. Section 2 discusses the pricing of timer options under the general stochastic volatility framework. We list three exactly solvable cases. We then discuss the general perturbation technique based on the pricing PDE. We also specialize the general technique to the Heston model and the 3/2 model. Section 3 makes several important remarks on our approximation. Section 4 contains numerical analysis of the proposed approximations. It also looks at the Black-Scholes implied volatility surfaces for timer options under the Heston model and the 3/2 model. Section 5 concludes. More detailed mathematical derivations are in Appendix.

2 Pricing Timer Options

2.1 The Pricing Problem

We consider a general stochastic volatility framework for the stock process S_t and instantaneous variance process V_t under the risk-neutral measure:

$$\mathrm{d}S_t = (r-\delta)S_t\,\mathrm{d}t + \sqrt{V_t}S_t\,\mathrm{d}W_t^S, \tag{1}$$

$$\mathrm{d}V_t = a(V_t)\,\mathrm{d}t + \eta b(V_t)\,\mathrm{d}W_t^V. \tag{2}$$

Here we assume that the interest rate r and dividend rate δ are both constants, and the two Brownian motions W_t^S and W_t^V have a constant correlation coefficient ρ. The drift and diffusion functions $a(V)$ and $b(V)$ are assumed to be functions of V only. This general framework incorporates the well-known Heston model (Heston, 1993) as a special case, as well as the 3/2 model (see, for example, Ahn and Gao (1999)) . Notice we introduced a parameter η in the diffusion function for V. We will call η the volatility of variance coefficient and assume that it is small.

Now define the accumulated variance process to be

$$\xi_t = \xi + \int_0^t V_u\,\mathrm{d}u. \tag{3}$$

Here $\xi_0 = \xi$ is the accumulated variance at time 0. Notice that

$$\mathrm{d}\xi_t = V_t\mathrm{d}t. \tag{4}$$

A timer call option pays $(S_\tau - K)^+$ with a random remaining maturity τ. Here τ is the random time remaining for a pre-specified variance budget B to be exceeded. Let the current time be 0 with initial stock price S, initial instantaneous variance V, and initial realized total variance ξ. If the contract is issued today, then $\xi = 0$, but we allow for the situation where the timer call was issued some time in the past, so that $\xi > 0$. Except for when considering the boundary conditions, we assume that $\xi < B$ so that the timer call has not expired. Since ξ_t is a continuous process, the random remaining maturity τ is given by

$$\tau \equiv \inf\{t > 0 : \xi_t = B\} = \inf\left\{t > 0 : \int_0^t V_u\,\mathrm{d}u = B - \xi\right\}. \tag{5}$$

The processes S_t, V_t and ξ_t form a Markovian system which is sufficient to model the timer option payoff. Therefore, we denote the price of the timer call today by $C(S, \xi, V)$. By risk-neutral pricing, we have

$$C(S,\xi,V) = \mathbb{E}_0\left[e^{-r\tau}(S_\tau - K)^+\right]. \tag{6}$$

The timer put option pays $(K - S_\tau)^+$ at random maturity τ, and we have

$$P(S, \xi, V) = \mathbb{E}_0 \left[e^{-r\tau} (K - S_\tau)^+ \right]. \tag{7}$$

They satisfy the put-call parity

$$C(S, \xi, V) - P(S, \xi, V) = \mathbb{E}_0 \left[e^{-r\tau} S_\tau \right] - K\mathbb{E}_0 \left[e^{-r\tau} \right]. \tag{8}$$

Here, the first term on the right-hand side is today's price of a timer share contract, and the second term is the price of a timer cash contract.

It is difficult to perform the expectation for timer call options because the formula involves the expectation of a function of two dependent random variables τ and S_τ. Because of this difficulty, we take a different route. Our starting point is the following pricing PDE for $C = C(S, \xi, V)$ under the general stochastic volatility model:

$$VC_\xi + a(V)C_V + (r - \delta)SC_S + \frac{1}{2}\eta^2 b^2(V)C_{VV} + \frac{1}{2}S^2 VC_{SS} + \rho\eta S\sqrt{V} b(V)C_{SV} - rC = 0. \tag{9}$$

The subscripts here denote partial derivatives. The boundary condition is

$$C(S, B, V) = (S - K)^+. \tag{10}$$

The boundary conditions for S and V are similar to those for European style plain-vanilla options and usually do not need to be specified explicitly. The above PDE is valid for any European style derivative whose payoff is determined by the state variables S, ξ and V. In particular, timer share contract, time cash contract, and time puts also satisfy this PDE.

2.2 Exactly Solvable Cases

Below we will mainly focus on the timer call. There are a few simple cases where the pricing PDE can be solved exactly. These exact solutions provide clues on developing an approximation technique. They are also useful for sanity checks once an approximation formula is obtained. A good approximation should reduce to the exactly solvable cases when the conditions are met.

2.2.1 Case 1: $K = 0$ and $\delta = 0$

In this limiting case, the timer call option becomes a timer share contract with a random maturity. Assuming that under the general stochastic volatility model the quantity $e^{-rt}S_t$ is a true martingale,[2] the solution is then given by

$$C(S, \xi, V) = \mathbb{E}_0 \left[e^{-r\tau} S_\tau \right] = S. \tag{11}$$

[2] Note this put restrictions on the drift and diffusion functions $a(V)$ and $b(V)$. See Section 1.2 of Zeliade (2011) and the references therein for the discussion of this issue for the Heston model.

It is easy to see that this solution satisfies the pricing PDE since C only depends linearly on S and

$$rSC_S - rC = 0. \tag{12}$$

Notice also that this solution is model independent. That is, in this case the dynamics of the instantaneous variance process does not affect the timer call option price. Notice that if δ is not zero, then in general we do not have $C(S, \xi, V) = S$.

2.2.2 Case 2: $r = \delta = 0$

In this case, the exact solution is given by (see Bernard and Cui (2011), or Lee (2012))

$$C(S, \xi, V) = SN(d^+) - KN(d^-), \tag{13}$$

where

$$d^\pm = \frac{\log(S/K)}{\sqrt{B-\xi}} \pm \frac{1}{2}\sqrt{B-\xi}. \tag{14}$$

Here $N(\cdot)$ is the cumulative normal distribution function. Notice that the solution does not depend on V. It does not depend on ρ or η. It is easy to check that the above solution satisfies the simplified PDE since

$$C_\xi + \frac{1}{2}S^2 C_{SS} = 0. \tag{15}$$

Notice also that the solution for this $r = \delta = 0$ case is model independent. That is, it does not depend on the dynamics of the instantaneous variance.

2.2.3 Case 3: $\eta = 0$

When $\eta = 0$, the instantaneous variance process is deterministic, so we know exactly when we are going to exercise the timer call. The solution $C(S, \xi, V)$ in this case reduces to the Black-Scholes formula for plain-vanilla options:

$$C(S, \xi, V) = Se^{-\delta T}N(d^+) - Ke^{-rT}N(d^-), \tag{16}$$

with

$$d^\pm = \frac{\log(Se^{(r-\delta)T}/K)}{\sqrt{B-\xi}} \pm \frac{1}{2}\sqrt{B-\xi}. \tag{17}$$

Here $T = T(\xi, V)$ is the solution of the first-order PDE

$$VT_\xi + a(V)T_V + 1 = 0, \tag{18}$$

with the boundary condition

$$T(B, V) = 0. \tag{19}$$

Unlike the previous two solutions, this solution is model dependent as the solution of T depends on $a(V)$.

It's a little bit tedious but still quite simple to check that with $T = T(\xi, V)$, the above solution satisfies the simplified PDE

$$VC_\xi + a(V)C_V + (r - \delta)SC_S + \frac{1}{2}S^2VC_{SS} - rC = 0. \tag{20}$$

Instead of verifying that $C(S, \xi, V)$ with variance budget exceeding time $T(\xi, V)$ given above satisfies the pricing PDE, we could also directly prove that $T(\xi, V)$ satisfying the above first-order PDE is the variance budget exceeding time. See Appendix for details.

2.3 The Approximation Technique

2.3.1 A Particular Solution Form

We propose the following solution form for the timer call price in a general stochastic volatility model:

$$C(S, \xi, V) = Se^{-\delta T'}N(d^+) - Ke^{-rT}N(d^-), \tag{21}$$

where

$$d^\pm \equiv d^\pm(S, T, T', \Sigma) = \frac{\log(S/K) + rT - \delta T'}{\Sigma} \pm \frac{1}{2}\Sigma. \tag{22}$$

Here we assume $T = T(\xi, V)$, $T' = T'(\xi, V)$, and $\Sigma = \Sigma(\xi, V)$. That is, they have no dependence on S. We will comment more on this assumption later.

This particular solution form is very attractive for many reasons. It is homogeneous of degree 1 in S and K, as it should be. It is easy to verify that this solution form will reduce to the three exactly solvable cases above if the total variance Σ^2 reduces to $B - \xi$. Also, as long as Σ is positive, we always have $C(S, \xi, V) > 0$. This can be seen from the fact that $C(0^+, \xi, V) = 0$ and $C_S > 0$ for all $S > 0$. The formula has sensible limits when S or K goes to 0 or $+\infty$. If $\Sigma \to 0$ when $\xi \to B$, and $\Sigma \to +\infty$ if $B \to +\infty$, the formula will also have sensible limiting behavior for B. Also, it is easy to check that we have kept the attractive property in the Black-Scholes formula

$$Se^{-\delta T'}n(d^+) - Ke^{-rT}n(d^-) = 0. \tag{23}$$

Here $n(\cdot)$ is the standard normal probability density function. This property is very attractive because it simplifies the derivatives of $C(S,\xi,V)$, especially for the Greeks delta and gamma. They are given by

$$\Delta \equiv C_S = e^{-\delta T'} N(-d^+), \tag{24}$$

$$\Gamma \equiv C_{SS} = \frac{1}{S\Sigma} e^{-\delta T'} n(d^+). \tag{25}$$

Finally, as we discuss later, the put-call parity for timer options has a simple form closely resembling that of the plain-vanilla options:

$$C(S,\xi,V) - P(S,\xi,V) = Se^{-\delta T'} - Ke^{-rT}. \tag{26}$$

The proposed form is nonetheless more complicated than the small vol expansion for plain-vanilla options under the Heston model as was done in Lewis (2000). In that case, T and T' are both constants, and the only approximation is through an expansion for Σ. In our case, because the exercise time is random, T and T' will in general be different. They are differen for two reasons. First, if we write $\mathbb{E}_0 e^{\lambda\tau} = e^{\lambda T_{\text{eff}}}$, then in general T_{eff} will depend on λ. The second reason is that in the expectation $\mathbb{E}_0 S_\tau e^{-r\tau}$, S_τ and $e^{-r\tau}$ are dependent. Therefore, we should expect that T does not depend ρ, but T' does. We could also have defined quantities $D(\xi,V) = e^{-rT}$ and $D'(\xi,V) = e^{-\delta T'}$ and perform expansion on these two quantities. We prefer the current approach of performing the expansion inside the exponents. This has the slight benefit of guaranteeing the positivity of the two discount factors.

One possible alternative is to use the following expansion

$$C(S,\xi,V) \approx C_0(S,\xi,V) + \eta C_1(S,\xi,V) + \eta^2 C_2(S,\xi,V) + \cdots, \tag{27}$$

where $C_0(S,\xi,V)$ is the exact solution for $\eta = 0$ in equation (16). We do not take this route for several reasons. First, In Lewis (2000), it is shown that this form of expansion for given order of η is not as accurate as the expansion using Σ, especially for away-from-the-money options. Expansion in Σ works better because the two cumulative normal functions act as some sort of regularizer. Second, the above expansion can violate positivity in some cases, and obscure the put-call parity. As we will see later, the expansion we choose allows a very intuitive interpretation. For example, the term e^{-rT} gives an approximation to second-order in η for the price of a timer cash contract where cash is to be received at the random variance budget exceeding time. Finally, our expansion form allows us to decompose the problem of a complicated PDE for $C(S,\xi,V)$ into three PDEs for T, T' and Σ, which as we show later, are often exactly solvable to low orders in η.

2.3.2 The PDEs

Our starting point is the pricing PDE for a timer call, which we rewrite here

$$VC_\xi + a(V)C_V + (r-\delta)SC_S + \frac{1}{2}\eta^2 b^2(V)C_{VV} + \frac{1}{2}S^2VC_{SS} + \eta\rho S\sqrt{V}b(V)C_{SV} - rC = 0. \tag{28}$$

The idea of our approximation is very simple. We take the derivatives of the trial solution and put them into the PDE and try to satisfy the PDE as best as we can. This technique of satisfying the PDE asymptotically was very fruitfully utilized in Aït-Sahalia (1999), Aït-Sahalia (2002), and Aït-Sahalia (2008) to derive transition density approximations for diffusion processes. See also Li (2010) for the exact sense in which the backward and forward equations of the transition density are satisfied.

By utilizing equation (23) and the fact that

$$d^+ - d^- = \Sigma, \tag{29}$$

and

$$d^\pm_\Sigma = -\frac{d^\mp}{\Sigma}, \quad d^+_\Sigma - d^-_\Sigma = 1, \tag{30}$$

the partial derivatives we need can be readily computed as

$$\begin{aligned}
C_\xi &= -\,\delta T'_\xi S e^{-\delta T'} N(d^+) + rT_\xi K e^{-rT} N(d^-) + \Sigma_\xi S e^{-\delta T'} n(d^+), && (31)\\
C_V &= -\,\delta T'_V S e^{-\delta T'} N(d^+) + rT_V K e^{-rT} N(d^-) + \Sigma_V S e^{-\delta T'} n(d^+), && (32)\\
C_S &= e^{-\delta T'} N(d^+), && (33)\\
C_{SS} &= \frac{1}{S\Sigma} e^{-\delta T'} n(d^+), && (34)\\
C_{SV} &= -\,\delta T'_V e^{-\delta T'} N(d^+) + d^+_V e^{-\delta T'} n(d^+), && (35)\\
C_{VV} &= \delta\left[\delta(T'_V)^2 - T'_{VV}\right] S e^{-\delta T'} N(d^+) - r\left[r(T_V)^2 - T_{VV}\right] K e^{-rT} N(d^-)\\
&\quad + C^n_{VV} S e^{-\delta T'} n(d^+), && (36)
\end{aligned}$$

where the derivatives of $d^\pm$ with respect to V are given by

$$\begin{aligned}
d^+_V &\equiv \frac{rT_V - \delta T'_V - d^-\Sigma_V}{\Sigma}, && (37)\\
d^-_V &\equiv \frac{rT_V - \delta T'_V - d^+\Sigma_V}{\Sigma}, && (38)
\end{aligned}$$

and we define

$$C^n_{VV} \equiv -\delta T'_V d^+_V + rT_V d^-_V + \left[\Sigma_{VV} - \delta T'_V \Sigma_V - \Sigma_V d^+ d^+_V\right]. \tag{39}$$

In order for the PDE to be satisfied for all possible values of K, we postulate that the terms proportional to $N(d^+)$, $N(d^-)$, and $n(d^+)$ should all be zero. Collecting the $N(d^+)$ terms and simplifying gives a PDE for $T'(\xi, V)$ as follows

$$VT'_\xi + a'(V)T'_V + \frac{1}{2}\eta^2 b^2(V)\left[T'_{VV} - \delta(T'_V)^2\right] + 1 = 0, \tag{40}$$

with the boundary condition

$$T'(B, V) = 0. \tag{41}$$

Here for notational convenience we have also defined

$$a'(V) \equiv a(V) + \eta\rho\sqrt{V}b(V). \tag{42}$$

Collecting the $N(d^-)$ terms gives a PDE for $T(\xi, V)$ as follows

$$VT_\xi + a(V)T_V + \frac{1}{2}\eta^2 b^2(V)\left[T_{VV} - r(T_V)^2\right] + 1 = 0, \tag{43}$$

with the boundary condition

$$T(B, V) = 0. \tag{44}$$

The $n(d^+)$ terms are more complicated, and we have a PDE for $\Sigma(\xi, V)$ as follows

$$V\Sigma_\xi + a(V)\Sigma_V + \frac{V}{2\Sigma} + \eta\rho\sqrt{V}b(V)d^+_V + \frac{1}{2}\eta^2 b^2(V)C^n_{VV} = 0, \tag{45}$$

with the boundary condition

$$\Sigma(B, V) = 0. \tag{46}$$

The boundary conditions on T, T' and Σ are chosen such that when $\xi = B$, the timer call option price reduces to $(S - K)^+$ for any $K \geq 0$.

By looking at the PDEs, we notice that the assumptions that T and T' do not depend on S are consistent with their PDEs. The assumption that Σ does not depend on S is not consistent with its PDE as the PDE involves terms such as d^+_V which is a function of S. However, as we will see later, for any stochastic volatility model, if we only solve Σ to first order in η, then Σ will not be a function of S. This is what we do in this paper since we find that first-order approximation in Σ is very satisfying in terms of accuracy. If one wants to solve Σ to higher orders in η, equation (45) would have to be modified to include terms proportional to Σ_S, Σ_{SS}, Σ_{SV}, etc.

When $r = 0$, we do not need to consider the PDE for T and should just set $rT = 0$ in the approximation formula. Similarly, when $\delta = 0$, we do not need to consider the PDE for T'.

Notice also that when $\delta = 0$, the PDE for T does not involve ρ, so the effect of ρ is only captured in the PDE for Σ. In particular, this means that the simple treatment of just taking $\Sigma^2 = B - \xi$ would not be sufficient for nonzero ρ.

So far, no approximation has been made except that we postulate a particular form of solution. The PDEs for T, T' and Σ are quite complicated in that they are all nonlinear and interrelated. Below we describe our approach to solve them asymptotically to lowest orders in η.

2.3.3 The Perturbation Approach

Motivated by the small volatility of variance expansion for stochastic volatility models in Lewis (2000), we also seek an expansion in η for T, T' and Σ. The solution strategy of the above interrelated PDEs is to solve them to zeroth order in η first, and then to perturb around these zeroth-order solutions.

For T and T', we first aim to solve the following two first-order PDEs:

$$VT_{0,\xi} + a(V)T_{0,V} + 1 = 0, \tag{47}$$

$$VT'_{0,\xi} + a'(V)T'_{0,V} + 1 = 0. \tag{48}$$

These two equations can often be solved exactly in time-homogeneous stochastic volatility models. We then solve the following two first-order PDEs

$$VT_{\xi} + a(V)T_V + \frac{1}{2}\eta^2 b^2(V)\left[T_{0,VV} - r(T_{0,V})^2\right] + 1 = o(\eta^2), \tag{49}$$

$$VT'_{\xi} + a'(V)T'_V + \frac{1}{2}\eta^2 b^2(V)\left[T'_{0,VV} - \delta(T'_{0,V})^2\right] + 1 = o(\eta^2). \tag{50}$$

Notice that in the above equations, we put the error $o(\eta^2)$ on the right-hand side. It is easy to see that the solutions of the above equations satisfy the original PDEs to second-order in η.

We want to make an important remark for the treatment of T' above. Because $a'(V)$ is linear in η, the solution T'_0 rigorously speaking is the approximate solution of T' to first order in η. However, we will treat $a'(V)$ as raw model input and absorb the η dependence inside $a'(V)$. Therefore, we use the notation T'_0 instead of T'_1. In equation (50), the difference of using the zeroth-order or first-order approximations in the η^2 term will not affect the PDE to order η^2. The reason we choose to do this is that in many models, $a(V)$ and $a'(V)$ will formally be identical. Therefore, the current treatment allows us to only solve one PDE instead of two PDEs.

The solution for Σ when $\eta = 0$ is simply $\Sigma = \sqrt{B - \xi}$. Despite a simple zeroth-order solution, the PDE for Σ is more complicated than those for T and T' when η is not zero. One of the reasons is that the PDE for Σ involves derivatives of T and T'. It turns out that in many cases we can solve the PDE for Σ exactly to first order in η. Therefore, we aim to solve

$$V\Sigma_\xi + a(V)\Sigma_V + \frac{V}{2\Sigma} + \eta\rho\sqrt{V}b(V)\frac{rT_V - \delta T'_V - d^-\Sigma_V}{\Sigma} = \mathcal{O}(\eta^2). \tag{51}$$

Approximating Σ to second order in η is much more difficult. First, we will be forced to introduce S dependency in Σ. Second, the PDE for Σ to second order gets more complicated. Fortunately, as we will see later, approximating Σ to first order in η gives very satisfying accuracy in the models we consider.

The above PDE can be further simplified. Notice that $T_V - T_{0,V} = \mathcal{O}(\eta^2)$. It can also be shown that the solutions from the first two equations satisfy $T'_V - T_V = \mathcal{O}(\eta)$. In addition, we postulate that $\Sigma_V = \mathcal{O}(\eta)$, which is fairly reasonable to expect since Σ_0 is not a function of V. These two statements are easy to verify in the two concrete models we consider later. Collecting only the leading linear terms in η and multiplying the PDE above by 2Σ, the PDE for Σ becomes a PDE for Σ^2

$$V(\Sigma^2)_\xi + a(V)(\Sigma^2)_V + V + 2\eta\rho(r-\delta)\sqrt{V}b(V)T_{0,V} = \mathcal{O}(\eta^2). \tag{52}$$

We could have used $T'_{0,V}$ in place of $T_{0,V}$ in the equation above while still keeping $\mathcal{O}(\eta^2)$ accuracy, but because the term $(\Sigma^2)_V$ involves $a(V)$, it is most convenient to use $T_{0,V}$ here.

The solution $\Sigma_0^2(\xi, V) = B - \xi$ solves the PDE to zeroth order in η with the boundary condition $\Sigma_0^2(B, V) = 0$. That is, we have

$$V(\Sigma_0^2)_\xi + a(V)(\Sigma_0^2)_V + V = 0. \tag{53}$$

Notice that this zeroth-order solution is model-independent. Now assume a first-order expansion of Σ^2 in η given by

$$\Sigma^2(\xi, V) = B - \xi + 2\eta\rho(r-\delta)G(\xi, V). \tag{54}$$

In order to satisfy equation (52) to first-order in η, we need $G(\xi, V)$ to solve the following PDE

$$VG_\xi + a(V)G_V + \sqrt{V}b(V)T_{0,V} = 0, \tag{55}$$

with the boundary condition

$$G(B, V) = 0. \tag{56}$$

Equations (49), (50) and (55) are the main results of this section. Once we obtain their solutions, the final approximate timer option price is then given by the simple approximation form in equation (21). These three equations look a little bit complicated on first sight, but formally they can all be solved exactly. Notice that all three equations have the form of

$$VQ_\xi + A(V)Q_V = q(\xi, V) \tag{57}$$

for some functions $A(V)$ and $q(\xi, V)$ with the boundary condition $Q(B, V) = 0$. This equation can be solved in general by switching to the characteristic coordinates and changing the PDE to an ODE. In Appendix, we explain this approach in more detail.

2.4 Specializing to the Heston Model

2.4.1 The PDEs

We now specialize the general approach in the last section to specific stochastic volatility models. We first consider the well-known Heston model. In this model, $a(V) = \kappa(\theta - V)$, and $b(V) = \sqrt{V}$. So that we have

$$\mathrm{d}S = (r - \delta)S\mathrm{d}t + \sqrt{V}S\ \mathrm{d}W^S, \tag{58}$$

$$\mathrm{d}V = \kappa(\theta - V)\mathrm{d}t + \eta\sqrt{V}\ \mathrm{d}W^V, \tag{59}$$

where $\mathrm{d}W^S \cdot \mathrm{d}W^V = \rho\mathrm{d}t$. We assume that the usual Feller condition $2\kappa\theta > \eta^2$ is satisfied for the variance process, and that $\kappa - \rho\eta > 0$. The Feller condition assures that the origin is unattainable. The quantity $\kappa - \rho\eta$ comes out from performing a measure change on the Heston model by using stock as the numeraire. See Section 1.4 of Zeliade (2011) and the references therein for more discussions on this.

The PDEs for T and T' become

$$VT_\xi + \kappa(\theta - V)T_V + \frac{1}{2}\eta^2 V\left[T_{VV} - r(T_V)^2\right] + 1 = 0, \tag{60}$$

$$VT'_\xi + \kappa'(\theta' - V)T'_V + \frac{1}{2}\eta^2 V\left[T'_{VV} - \delta(T'_V)^2\right] + 1 = 0. \tag{61}$$

Here $\kappa' = \kappa - \rho\eta$, and $\kappa'\theta' = \kappa\theta$. The change to κ' and θ' is due to the term $\eta\rho b(V)\sqrt{V}$. This adjusts for the fact that in $S_\tau e^{-\delta\tau}$, the two terms S_τ and $e^{-\delta\tau}$ are dependent. As commented in the last section, we absorb the η dependence in $a'(V)$ and treat κ' and θ' as raw model inputs. Notice that these two equations are now formally identical except that in the equation for T', we replace κ, θ and r with κ', θ' and δ. This allows us to only solve the PDE for T and the solution for T' can be obtained by just replacing κ, θ and r with κ', θ' and δ.

The PDE for Σ becomes

$$V\Sigma_\xi + \kappa(\theta - V)\Sigma_V + \frac{V}{2\Sigma} + \eta\rho V d_V^+ + \frac{1}{2}\eta^2 V C_{VV}^n = 0. \tag{62}$$

2.4.2 Solving T and T'

To zeroth-order in η, we need to solve

$$VT_{0,\xi} + \kappa(\theta - V)T_{0,V} + 1 = 0, \tag{63}$$

with the boundary condition

$$T_0(B, V) = 0. \tag{64}$$

It turns out that $T_0(\xi, V)$ is the implicit solution of

$$\theta T_0 + (V - \theta)\frac{1 - e^{-\kappa T_0}}{\kappa} = B - \xi. \tag{65}$$

Appendix contains a proof of the above equation.

The implicit solution in equation (65) is not very convenient to use later on when looking for higher-order solutions. It turns out that we can express T_0 explicitly using Lambert's product log function $W(x)$ which is defined as the unique solution of

$$W(x)e^{W(x)} = x. \tag{66}$$

The solution for T is then given by

$$T_0(\xi, V) = \frac{1}{\kappa}\log R, \tag{67}$$

where

$$R \equiv \frac{z_0}{z} = e^{z - z_0 + \kappa\frac{B-\xi}{\theta}}, \tag{68}$$

$$z_0 \equiv \frac{V - \theta}{\theta}, \tag{69}$$

$$z \equiv W\left(z_0 e^{z_0} \cdot e^{-\kappa\frac{B-\xi}{\theta}}\right). \tag{70}$$

Notice that we give two expressions for R. While the definition of z_0/z is easier to understand, it has a removable singularity at $V = \theta$, where both z and z_0 are zero. On the other hand, the second expression is always well-defined.

Using the second expression for R, an equivalent way to write T which does not have the apparent singularity at $V = \theta$ is

$$T_0(\xi, V) = \frac{z - z_0}{\kappa} + \frac{B - \xi}{\theta}. \tag{71}$$

This formula in preferred in the actual computation rather than equation (67).[3] Appendix contains a proof of equations (67) and (71).

[3] The apparent singularity in equation (67) at $V = \theta$ is removable since near $x = 0$, we have $W(x) \approx x - x^2 + \mathcal{O}(x^3)$. Indeed, in this case, we have the result $T_0 = (B - \xi)/\theta$ from the alternative expression.

The zero-order solution for T' is similar. We have

$$T'_0(\xi, V) = \frac{1}{\kappa'} \log R' = \frac{z' - z'_0}{\kappa'} + \frac{B - \xi}{\theta'}, \tag{72}$$

where

$$R' \equiv \frac{z'_0}{z'} = e^{z' - z'_0 + \kappa' \frac{B-\xi}{\theta'}}, \tag{73}$$

$$z'_0 \equiv \frac{V - \theta'}{\theta'}, \tag{74}$$

$$z' \equiv W\left(z'_0 e^{z'_0} \cdot e^{-\kappa' \frac{B-\xi}{\theta'}} \right). \tag{75}$$

To second order in η, we need to solve the following first-order PDEs:

$$VT'_\xi + \kappa'(\theta' - V)T'_V + \frac{1}{2}\eta^2 V \left[T'_{0,VV} - \delta (T'_{0,V})^2 \right] + 1 = 0, \tag{76}$$

$$VT_\xi + \kappa(\theta - V)T_V + \frac{1}{2}\eta^2 V \left[T_{0,VV} - r (T_{0,V})^2 \right] + 1 = 0. \tag{77}$$

Let us focus on the equation for T first. Having obtained the solution for the $\eta = 0$ case, we seek a solution of the form

$$T(\xi, V) \approx T_0(\xi, V) + \eta^2 H(\xi, V; \kappa, \theta, r) \tag{78}$$

for some function $H(\xi, V)$. Plugging this solution into the PDE for T and collecting terms proportional to η^2, we need to solve a first-order PDE

$$VH_\xi + \kappa(\theta - V)H_V + \frac{1}{2} V \left[T_{0,VV} - r (T_{0,V})^2 \right] = 0, \tag{79}$$

with the boundary condition

$$H(B, V) = 0. \tag{80}$$

Although the above PDE involves the complicated terms $T_{0,VV}$ and $(T_{0,V})^2$, it turns out that this equation can be solved exactly to give

$$\begin{aligned} H(\xi, V; \kappa, \theta, r) =& \frac{(R-1)\Big[-r(1+z)(1+2R^2 z + R(2z-3)) + \kappa(2R^2 z^2 + R(2-5z-2z^2) - 2 - z) \Big]}{4\kappa^3 R^2 (1+z)^3 \theta} \\ &+ \frac{\Big[3kz + r(2z^2 + z - 1) \Big] \log R}{2\kappa^3 (1+z)^3 \theta}. \end{aligned} \tag{81}$$

Notice that z and R are functions of ξ and V. A proof is given in Appendix.

Similarly, we have

$$T'(\xi, V) \approx T'_0(\xi, V) + \eta^2 H(\xi, V; \kappa', \theta', \delta). \tag{82}$$

Notice in $H(\xi, V; \kappa', \theta', \delta)$ we also need to replace z and R by z' and R' in addition to replacing κ, θ and r with κ', θ' and δ.

2.4.3 Solving Σ

To first order in η, we need to solve the following first-order PDE

$$V(\Sigma^2)_\xi + \kappa(\theta - V)(\Sigma^2)_V + V + 2\eta\rho V(r-\delta)T_{0,V} = 0. \tag{83}$$

with the boundary condition

$$\Sigma^2(B, V) = 0. \tag{84}$$

In Appendix, we show that to first order in η, the approximation for Σ^2 is given by

$$\Sigma^2 = B - \xi + \frac{2\eta\rho(r-\delta)}{\kappa^2} \cdot \frac{(1-R)(Rz-1) + R(z-1)\log R}{R(1+z)}. \tag{85}$$

The above formula is very interesting because it allows us to examine the effect of ρ qualitatively. In particular, it's not necessarily true that a negative ρ will make the timer option more expensive. Notice that $|\rho| \approx 1$ does not impose any computational problem in our approximation.

Compared with the Black-Scholes formula, the slight additional cost for timer call options is to compute the product log z. Once z is computed, T, T' and Σ are simple functions of z given in equations (78), (82), and (85).

2.5 Specializing to the 3/2 Model

2.6 The PDEs

In this model, the variance process is specified as an inverse Feller process as follows

$$\mathrm{d}V = \kappa V(\theta - V)\,\mathrm{d}t + \eta V^{\frac{3}{2}}\,\mathrm{d}W^V. \tag{86}$$

Therefore, $a(V) = \kappa V(\theta - V)$ and $b(V) = V^{\frac{3}{2}}$. Same as in the Heston model, $\mathrm{d}W^S \cdot \mathrm{d}W^V = \rho \mathrm{d}t$. Using Ito's lemma, it can be verified that the reciprocal of V follows a Cox-Ingersoll-Ross process which is the case for the Heston model.

The PDEs we need to solve are

$$VT'_\xi + \kappa' V(\theta' - V)T'_V + \frac{1}{2}\eta^2 V^3 \left[T'_{VV} - \delta(T'_V)^2\right] + 1 = 0, \tag{87}$$

$$VT_\xi + \kappa V(\theta - V)T_V + \frac{1}{2}\eta^2 V^3 \left[T_{VV} - r(T_V)^2\right] + 1 = 0, \tag{88}$$

$$V\Sigma_\xi + \kappa V(\theta - V)\Sigma_V + \frac{V}{2\Sigma} + \eta\rho V^2 d_V^+ + \frac{1}{2}\eta^2 V^3 C_{VV}^n = 0. \tag{89}$$

Here, we have

$$\kappa' = \kappa - \rho\eta, \quad \theta' = \frac{\kappa\theta}{\kappa'}. \tag{90}$$

Notice that the equation for T' has a similar modification as in the Heston model due to the correlation between the two Brownian motions driving the stock process and the instantaneous variance process.

2.6.1 Solving T and T'

In this 3/2 model, T_0 can be worked out explicitly to give

$$T_0(\xi, V) \equiv \frac{1}{\kappa\theta} \log\left(\frac{V + \theta(e^{\kappa(B-\xi)} - 1)}{V}\right). \tag{91}$$

It is easy to check that $T_0(\xi, V)$ satisfies the zeroth-order PDE

$$VT_{0,\xi} + \kappa V(\theta - V)T_{0,V} + 1 = 0. \tag{92}$$

Similarly, the zeroth-order solution for T' is

$$T_0'(\xi, V) \equiv \frac{1}{\kappa'\theta'} \log\left(\frac{V + \theta'(e^{\kappa'(B-\xi)} - 1)}{V}\right). \tag{93}$$

To second order in η, we need to solve

$$VT_\xi + \kappa V(\theta - V)T_V + \frac{1}{2}\eta^2 V^3 \left[T_{0,VV} - r(T_{0,V})^2\right] + 1 = 0. \tag{94}$$

In Appendix, we show that the solution is

$$T \approx T_0(\xi, V) + \eta^2 H(\xi, V; \kappa, \theta, r), \tag{95}$$

with

$$\begin{aligned} H(\xi, V; \kappa, \theta, r) = & \frac{1 - 4R + [3 - 2\log R]\, R^2}{4\kappa^3 \left[V + \theta(R-1)\right]^2} r \\ & + \frac{4V\left[1 + \left(\log R - 1\right)R\right] + \theta\left[-3 + (4 - 4\log R)R + (2\log R - 1)R^2\right]}{4\kappa^2 \left[V + \theta(R-1)\right]^2}, \end{aligned} \tag{96}$$

where to simplify the expression a little bit we have defined

$$R = e^{\kappa(B-\xi)}. \tag{97}$$

It is easy to check that $H(B, V; \kappa, \theta, r) = 0$, so the approximate solution $T(\xi, V)$ satisfies the boundary condition $T(B, V) = 0$.

Similarly, the solution for T' is

$$T' \approx T_0'(\xi, V) + \eta^2 H(\xi, V; \kappa', \theta', \delta). \tag{98}$$

Notice in the above expression for $H(\xi, V; \kappa', \theta', \delta)$, we also need to use $R' = e^{\kappa'(B-\xi)}$ instead of R.

2.6.2 Solving Σ

The PDE we need to solve in the 3/2 model is

$$V(\Sigma^2)_\xi + \kappa V(\theta - V)(\Sigma^2)_V + V + 2\eta\rho(r-\delta)V^2 T_{0,V} = \mathcal{O}(\eta^2). \tag{99}$$

It turns out that this equation can be solved exactly to order η, yielding

$$\Sigma^2 = (B - \xi) - \frac{2\eta\rho(r-\delta)}{\kappa^2} \cdot \frac{1 + (\log R - 1)R}{V + \theta(R-1)}. \tag{100}$$

It is easy to check that Σ satisfies the boundary condition $\Sigma = 0$ when $\xi = B$. The proof of the above solution is given in Appendix. We note also that $|\rho| \approx 1$ does not impose any computational problem in the above approximation.

3 Remarks

3.1 Put-Call Parity

Before we turn to numerical analysis, let us make a few important observations. Readers who are interested in numerical results can skip to the next section directly.

So far we only considered timer call options. The timer put option price can be worked out similarly, and is given by the approximation

$$P(S,\xi,V) \approx Ke^{-rT}N(-d^-) - Se^{-\delta T'}N(-d^+) \tag{101}$$

with exactly the same d^- and d^+ as those in the approximation for timer calls. Therefore, the put and call prices satisfy the put-call parity in our approximation

$$C(S,\xi,V) - P(S,\xi,V) = Se^{-\delta T'} - Ke^{-rT}. \tag{102}$$

Here $Se^{-\delta T'}$ is the approximate time 0 price of receiving one share of stock at a random future time τ, and Ke^{-rT} is the approximate time 0 price of receiving cash K at time τ. By repeating the analysis we have just done, it can be shown that both approximations $Se^{-\delta T'}$ and Ke^{-rT} satisfy the pricing PDE to order η^2. In terms of risk-neutral expectations, we have

$$\mathbb{E}_0 e^{-r\tau} S_\tau = S_0 e^{-\delta T'} + o(\eta^2), \tag{103}$$

$$\mathbb{E}_0 e^{-r\tau} = e^{-rT} + o(\eta^2). \tag{104}$$

3.2 Forward Price at Random Exercise Time τ

Some quantities of interest for timer options come out as byproducts of our approximation. One quantity of interest is the forward price at the random exercise time. Denote it by

$$F = F(S, \xi, V) = \mathbb{E}_0 S_\tau. \tag{105}$$

Then it is easy to show that it satisfies the following PDE

$$VF_\xi + a(V)F_V + (r-\delta)SF_S + \frac{1}{2}\eta^2 b^2(V)F_{VV} + \frac{1}{2}S^2 V F_{SS} + \rho\eta\sqrt{V}b(V)SF_{SV} = 0, \tag{106}$$

with the boundary condition

$$F(S, B, V) = S. \tag{107}$$

Notice that equations (103) and (104) do not necessarily imply that $F = Se^{rT-\delta T'} + o(\eta^2)$. If we postulate an approximate form

$$F(S, \xi, V) \approx Se^{(r-\delta)T^F(\xi,V)}, \tag{108}$$

then similar algebra as before shows that in order to satisfy the PDE of F to second order in η, we need T^F to solve

$$VT^F_\xi + a'(V)T^F_V + \frac{1}{2}\eta^2 b^2(V)\left[T^F_{VV} - (r-\delta)(T^F_V)^2\right] + 1 = 0, \tag{109}$$

with the boundary condition

$$T^F(S, B, V) = 0. \tag{110}$$

This is the same approximate PDE for T' but with the dividend rate δ replaced by $\delta - r$. Therefore, in our approximation, we automatically get an approximation for F.

In both the Heston model and 3/2 model, we have that

$$F = \mathbb{E}_0 S_\tau = Se^{(r-\delta)\left[T'_0(\xi,V)+\eta^2 H(\xi,V;\kappa',\theta',\delta-r)\right]} + o(\eta^2), \tag{111}$$

where we should use the expressions of T'_0 and H for the respective models. For the Heston model, they are given in equations (71) and (81). For the 3/2 model, they are given in equations (91) and (96). Notice that the above equation is exact when $\eta = 0$.

3.3 Moments of Random Exercise Time τ

We first consider the expected time to exercise. A holder of the timer option may be interested in this for risk management purposes. Denote this quantity by

$$T^E(\xi, V) \equiv \mathbb{E}_0 \tau. \tag{112}$$

From diffusion process theory (see, for example, Karlin and Taylor (1981)), T^E satisfies the following PDE exactly

$$VT^E_\xi + a(V)T^E_V + \frac{1}{2}\eta^2 b^2(V)T^E_{VV} + 1 = 0, \tag{113}$$

with the boundary condition

$$T^E(B, V) = 0. \tag{114}$$

This is exactly the same PDE for T but with $r = 0$. Therefore, setting the solution for T with $r = 0$ will also give us an approximation for T^E to order η^2. Specifically, for the Heston model, we have

$$T^E = \mathbb{E}_0\tau = T_0(\xi, V) + \eta^2 H(\xi, V; \kappa, \theta, 0) + o(\eta^2), \tag{115}$$

where T_0 is given in equation (71) and H is given in equation (81). The same equation is true in the 3/2 model if we replace T_0 and H with their counterparts in equations (91) and (96).

An easier way to derive the result above in these two models is to notice that equation (104) actually gives us an approximation for the moment generating function $M_\tau(\mu)$ because the dynamics of V_t has no relation with the interest rate r. Notice that for any real number μ, the function $H(\xi, V; \kappa, \theta, -\mu)$ can be decomposed into a sum of two terms under both models

$$H(\xi, V; \kappa, \theta, -\mu) = H_0(\xi, V; \kappa, \theta) + \mu H_1(\xi, V; \kappa, \theta). \tag{116}$$

Here H_0 and H_1 are not functions of μ. Equation (104) then says that

$$M_\tau(\mu) = \mathbb{E}_0 e^{\mu\tau} = e^{\mu T^E + \mu^2\eta^2 H_1(\xi, V; \kappa, \theta)}. \tag{117}$$

This is precisely the moment generating function of a normal random variable. The mean can be read off to be T^E and the variance of τ is approximated as

$$\mathrm{Var}(\tau) = 2\eta^2 H_1(\xi, V; \kappa, \theta) + o(\eta^2). \tag{118}$$

That is, for both the Heston model and the 3/2 model, our approximation effectively treats τ as a normal random variable with mean and variance approximated to second order in η.

When $\rho = 0$, timer option price can be written as a mixing of Black-Scholes prices where the mixing is an expectation on the random exercise time τ. See for example, Theorem 3.4 in Bernard and Cui (2011). Assuming normality for τ, we can actually perform this expectation in closed form to get an approximation formula which is only very slightly different from the current one. Unreported numerical analysis shows that this approximation is also very accurate. However, the current PDE approach has the advantage of being applicable for nonzero ρ.

3.4 The Joint Moment Generating Function of $\log S_\tau$ and τ

3.4.1 The Perturbation Approach

For two real numbers λ and μ, the joint moment generating function of $\log S_\tau$ and τ is defined as

$$M_{\log S_\tau,\tau}(\lambda,\mu) \equiv \mathbb{E}_0\, e^{\lambda \log S_\tau + \mu\tau}. \tag{119}$$

Our perturbation method can be used to get an approximation for $M_{\log S_\tau,\tau}(\lambda,\mu)$ to order η^2. This could be useful to value other exotic derivatives with a timer feature.

The moment generating function can be viewed as the expectation of a discounted payoff of a power contract with discount rate $-\mu$. Define this expectation as

$$\Pi(S,\xi,V;\lambda,\mu) = \mathbb{E}_0\, S_\tau^\lambda e^{\mu\tau}. \tag{120}$$

Then $M_{\log S_\tau,\tau}(\lambda,\mu) = \Pi(S,\xi,V;\lambda,\mu)$. We will often just write $\Pi(S,\xi,V)$ when there is no ambiguity. By Feynman-Kac theorem, Π satisfies the following PDE

$$V\Pi_\xi + a(V)\Pi_V + (r-\delta)S\Pi_S + \frac{1}{2}\eta^2 b^2(V)\Pi_{VV} + \frac{1}{2}S^2 V\Pi_{SS} + \rho\eta S\sqrt{V}b(V)\Pi_{SV} + \mu\Pi = 0, \tag{121}$$

with the boundary condition

$$\Pi(S,B,V) = S^\lambda. \tag{122}$$

Same as before, we postulate a particular solution form

$$\Pi(S,\xi,V) = S^\lambda e^{p(\xi,V)}. \tag{123}$$

Notice here $p(\xi,V)$ has no S dependence. This actually can be justified on price homogeneity ground. Plugging this solution into the PDE for Π, we get a nonlinear PDE for $p(\xi,V)$:

$$Vp_\xi + a_\lambda(V)p_V + \alpha V + \beta + \frac{1}{2}\eta^2 b^2(V)\left[p_{VV} + (p_V)^2\right] = 0, \tag{124}$$

with the boundary condition

$$p(B,V) = 0. \tag{125}$$

Here we have defined

$$\alpha \equiv \frac{1}{2}\lambda(\lambda-1), \tag{126}$$

$$\beta \equiv (r-\delta)\lambda + \mu, \tag{127}$$

$$a_\lambda(V) \equiv a(V) + \lambda\rho\eta\sqrt{V}b(V). \tag{128}$$

Same as before, we will treat $a_\lambda(V)$ as raw model input and absorb the η dependence. The PDE for $p(\xi, V)$ can be solved exactly to second order in η. Write

$$p(\xi, V) \approx p_0(\xi, V) + \eta^2 J(\xi, V). \tag{129}$$

Then the zeroth-order solution (actually first-order in η since $a_\lambda(V)$ is linear in η) p_0 satisfies the following first-order PDE

$$V p_{0,\xi} + a_\lambda(V) p_{0,V} + \alpha V + \beta = 0 \tag{130}$$

with the boundary condition

$$p_0(B, V) = 0. \tag{131}$$

After we solve for p_0, the first-order PDE for J is given by

$$V J_\xi + a_\lambda(V) J_V + \frac{1}{2} V \left[p_{0,VV} + (p_{0,V})^2 \right] = 0 \tag{132}$$

with the boundary condition

$$J_0(B, V) = 0. \tag{133}$$

The equations for p_0 and J can be solved by using the method of characteristic coordinates, as we did before. Notice that equations (103) and (104) are special cases of the above general solution by setting $\lambda = 0$ or $\lambda = 1$.

Below we give the solutions for both the Heston model and the 3/2 model.

3.4.2 Solution for the Heston Model

The joint moment generating function above under the Heston model might not be defined for all $\lambda \in \mathbb{R}$. The condition for $\mathbb{E}_0 S_t^\lambda < \infty$ for all $t > 0$ in the Heston model has been analyzed in Andersen and Piterbarg (2007) which puts a restriction on ρ. To our knowledge, the well-definedness of the joint moment generating function above has not been analyzed. Intuitively, because τ is the random time the cumulated variance reaches $B - \xi$, it should require a weaker condition to exist than in Andersen and Piterbarg (2007). In what follows, we assume that η is small and $\kappa - \lambda\rho\eta > 0$.

Define the following three quantitites

$$R_\lambda \equiv \frac{z_{\lambda 0}}{z_\lambda} = e^{z_\lambda - z_{\lambda 0} + \kappa_\lambda \frac{B-\xi}{\theta_\lambda}}, \tag{134}$$

$$z_{\lambda 0} \equiv \frac{V - \theta_\lambda}{\theta_\lambda}, \tag{135}$$

$$z_\lambda \equiv W\left(z_{\lambda 0} e^{z_{\lambda 0}} \cdot e^{-\kappa_\lambda \frac{B-\xi}{\theta_\lambda}} \right), \tag{136}$$

where the parameters κ_λ and θ_λ are defined by

$$\kappa_\lambda = \kappa - \lambda\rho\eta, \tag{137}$$

$$\theta_\lambda = \kappa_\lambda\theta_\lambda/\kappa. \tag{138}$$

Notice that we assume that $\kappa_\lambda > 0$. In this case, we also have $\theta_\lambda > 0$. These two parameters come out from $a_\lambda(V) = \kappa_\lambda(\theta_\lambda - V)$.

The solution of $p_0(\xi, V)$ is given by

$$p_0(\xi, V) = \frac{\alpha\theta_\lambda + \beta}{\theta_\lambda}(B - \xi) + \frac{\beta}{\kappa_\lambda}(z_\lambda - z_{\lambda 0}). \tag{139}$$

Notice that when $\lambda = 0$, $\alpha = 0$ and $\beta = 1$, the solution reduces to that of T_0. If $\lambda = 1$, $\alpha = 0$ and $\beta = 1$, the solution reduces to that of T_0'. If $\alpha = 1$ and $\beta = 0$, the solution reduces to Σ_0.

The solution for $J(\xi, V)$ is given by

$$J(\xi, V) = \beta H(\xi, V; \kappa_\lambda, \theta_\lambda, -\beta), \tag{140}$$

with the function H exactly the same as in equation (81). This is easy to see since the PDE for J is formally identical to the PDE for H because the first term in p_0 has no V dependence.

Putting everything together, we have

$$M_{\log S_\tau, \tau}(\lambda, \mu) = S^\lambda \exp\big(p_0(\xi, V) + \beta H(\xi, V; \kappa_\lambda, \theta_\lambda, -\beta)\big) + o(\eta^2). \tag{141}$$

The results for $\mathbb{E}_0 S_\tau$, $\mathbb{E}_0\tau$, $\mathrm{Var}(\tau)$, $\mathbb{E}_0 e^{-r\tau}$, $\mathbb{E}_0 S_\tau e^{-r\tau}$ and $M_\tau(\mu)$ we have obtained earlier all come out from the above result easily. For example, to evaluate $\mathbb{E}_0 S_\tau e^{-r\tau}$, we just need to set $\lambda = 1$ and $\beta = (r - \delta) - r = -\delta$ in the above formula for $M_{\log S_\tau, \tau}(\lambda, \mu)$.

We remark that besides these quantities, the above result on the joint moment generating function also allows us to get approximations for useful quantities such as $\mathbb{E}_0 \log S_\tau$, $\mathrm{Var}(S_\tau)$, $\mathrm{Corr}(S_\tau, e^{-r\tau})$, etc.

3.4.3 Solution for the 3/2 Model

We will only give the result here without details. Define κ_λ and θ_λ same as in the Heston model above. The solution of $p_0(\xi, V)$ is given by

$$p_0(\xi, V) = \alpha(B - \xi) + \frac{\beta}{\kappa_\lambda\theta_\lambda}\log\left(\frac{V + \theta_\lambda(e^{\kappa_\lambda(B-\xi)} - 1)}{V}\right). \tag{142}$$

Same as in the Heston model, the solution for $J(\xi, V)$ is given by

$$J(\xi, V) = \beta H(\xi, V; \kappa_\lambda, \theta_\lambda, -\beta), \tag{143}$$

with the function H exactly the same as in equation (96).

4 Numerical Analysis

4.1 Accuracy Comparison

Before we look at the accuracy performance, we emphasize that the approximation we develop in earlier sections is extremely fast. When implemented in MATLAB on a PC with an Intel E8400 CPU, we are able to price a thousand timer options in a couple of seconds. It is possible to have additional time saving when pricing a basket of timer options. For example, notice that T, T' and G does not depend on K or ρ. Therefore, if we need to price otherwise similar options with different K or different ρ, we do not need to recompute them.

Table 1 reports the accuracy comparison for the Heston model. Parameters used here are the same ones as in Liang, Lemmens and Tempere (2011): $V_0 = 0.087$, $\kappa = 2$, $\theta = 0.09$, $\eta = 0.375$, $B = 0.087$, $\xi = 0$, $r = 0.015$, $S_0 = 100$, and $\delta = 0$. This corresponds to a target expected exercise time of roughly about 1 year. Analytical results are taken from Liang, Lemmens and Tempere (2011), which are obtained through multi-dimensional numerical integration. All errors reported are percentage errors with respect to the analytic results.

In Panel A, we give the timer option prices assuming that $r = 0$ and $\eta = 0$, where we have exact pricing formulas. These exact solutions do not depend on ρ. As we see, even with a relatively low interest rate of 1.5%, the naive approximation of assuming $r = 0$ gives very inaccurate results, with the relative pricing errors ranging from 4.48% to 8.17%. On the other hand, for this set of parameters, the $\eta = 0$ approximation performs much better than the $r = 0$ approximation, especially for positive ρ. The relative errors exhibit strong ρ dependence, indicating that the effect of ρ is important.

In Panel B, we compare the analytic results with the approximation in this paper. `Approx1` uses prices obtained if we solve T, T' and Σ all to first order in η. In `Approx1-2`, Σ is solved exactly to first order in η, while T and T' are solved exactly to second order in η. As we see, `Approx1` sufficiently captures the effect of ρ on timer options. The relative errors for different ρ are now fairly close and are around 0.70%. The accuracy is significantly improved if we use the second-order approximation for T and T'. The relative errors for `Approx1-2` are now around 0.03%. This should be satisfying for most real-life applications.

Table 2 reports the accuracy comparison for the 3/2 model. Parameters used here are the same ones as in Liang, Lemmens and Tempere (2011): $V_0 = 0.295^2$, $\kappa = 22.84$, $\theta = 0.4669^2$, $\eta = 8.56$, $B = V_0$, $\xi = 0$, $r = 0.015$, $S_0 = 100$, and $\delta = 0$. Again, this roughly corresponds to an expected exercise time of 1 year. Analytical results are taken from Liang, Lemmens and Tempere (2011). Comparing Tables 1 and 2, we see that with the chosen parameters, the analytic call option prices are very close for these two different models.

Panel A indicates that the $r = 0$ approximation is still very inaccurate. It also shows that the $\eta = 0$ approximation is now less accurate than in the Heston model in Table 1. It is difficult to have a fair comparison of η under two different models. One method is to look at the long-run variance. If exists, a diffusion process with drift function $\mu(V)$ and diffusion function $\sigma(V)$ has a stationary density given by

$$\pi(V) = \frac{M(v)}{\sigma^2(V)} \exp\left(\int_v^V \frac{2\mu(u)}{\sigma^2(u)} \mathrm{d}u\right). \tag{144}$$

Here v is an arbitrary lower integration limit, and $M(v)$ is a constant depending on v so that the density $\pi(V)$ integrates to 1. With the parameters in the two tables, the long-run means under the Heston model and the 3/2 model are 0.084 and 0.090, respectively. The long-run variances are given by 0.0032 and 0.0099, respectively. Therefore, the variance process is more volatile in the 3/2 model than in the Heston model with the given parameters.

In Panel B, we compare the analytic results with the approximation in this paper. We see that although `Approx1` corrects for the effect of ρ in the right direction, it does not improve the accuracy over the $\eta = 0$ approximation much. However, `Approx1-2` improves the accuracy significantly. Despite a much larger η, our approximation gives relative errors of around 0.2%.

The accurate analytic approximate formula allows us to quickly examine the effect of various parameters on the timer call option price. Such a sensitivity analysis also provides a sanity check for our approximation formula. Table 3 illustrates such an analysis for the Heston model. The base case uses the same parameters as in Table 1. We also fix $K = 110$ and $\rho = -0.5$, as out-of-the-money options with negative correlation is more relevant in real-life applications. In each sensitivity analysis, one parameter is increased or decreased by 10% from its base value. We look at the effect of the parameter change on the following quantities: effective discounting e^{-rT}, effective total variance Σ, deterministic variance budget exceeding time T_0, risk-neutral expected time to exercise T^E, the timer call price C, as well as the delta Δ.

Table 3 provides much information and we will only make a few comments. First, a plain-vanilla option is usually more expensive with a longer maturity. Therefore, if we think of a timer option as the average of a string of plain-vanilla options with different fixed maturities and an average exercise time T^E, then generally speaking the larger T^E is, the more expensive is the timer option. The expected exercise time intuitively is larger for smaller κ (since we have $V_0 < \theta$), smaller θ, smaller V_0, and larger budget B. These are all confirmed in the table. Second, except for the case of changing B, the effective variance budget Σ changes slowly with the parameters, explaining why a first-order approximation in η for Σ gives very accurate results. Third, the higher η is, the more optionality the timer call has, and the more expensive the price is. Increasing volatility also increases the expected exercise time. Finally, the behavior of the delta is similar to that of a plain-vanilla option. The delta gets higher with higher call option price.

In unreported work, we also perform a sensitivity analysis for the 3/2 model. The results are similar to those in the Heston model discussed above.

4.2 The Black-Scholes Implied Volatility Surface

The Black-Scholes implied volatility surface for plain-vanilla options provides a quick way to appreciate the pricing behavior of a general stochastic volatility model. It also allows easier comparison of different models. We can define such a surface for timer options as well. We assume that interest rate r and dividend rate δ are not simultaneously zero. For any fixed variance budget B and strike K, we compute the time T_{eff} such that the timer call option can be viewed as a plain-vanilla call option with a fixed maturity T_{eff}. That is, T_{eff} solves

$$C(S, \xi, V) = Se^{-\delta T_{\text{eff}}} N(d_{\text{eff}}^{+}) - Ke^{-rT_{\text{eff}}} N(d_{\text{eff}}^{-}), \tag{145}$$

with

$$d_{\text{eff}}^{\pm} = \frac{\log(S/K) + (r-\delta)T_{\text{eff}}}{\sqrt{B-\xi}} \pm \frac{1}{2}\sqrt{B-\xi}. \tag{146}$$

Here we will not be concerned with the general existence or uniqueness properties of T_{eff}. For the zero dividend rate case we examine here, it is easy to show that T_{eff} is always uniquely defined.

Once we get T_{eff}, we define the Black-Scholes implied volatility $\sigma_{\text{imp}}(B-\xi, K)$ as

$$\sigma_{\text{imp}}(B-\xi, K) \equiv \sqrt{\frac{B-\xi}{T_{\text{eff}}}}. \tag{147}$$

The motivation is as follows. In the Black-Scholes framework, volatility is constant, so the exercise time of the timer call option is a constant determined by the remaining variance budget $B - \xi$. Therefore, $\sigma_{\text{imp}}(B-\xi, K)$ under the Black-Scholes framework gives us the same timer option price $C(S, \xi, V)$ in a stochastic volatility model. Such a quantity is very useful for practical trading and quoting purposes because it makes price comparison across different variance budgets, strikes and underlyings on more equal footing.

For simplicity, we will assume that currently $\xi = 0$. If it is not zero, we just modify B to be the remaining variance budget $B - \xi$. The function $\sigma_{\text{imp}}(B, K)$ gives us an implied volatility surface for timer options. This implied volatility surface is different from the usual implied volatility surface for plain-vanilla options due to the different nature of timer and plain-vanilla options. In particular, we see that the implied volatility here enters into the pricing equation (145) through the two discounting factors $e^{-\delta T_{\text{eff}}}$ and $e^{-rT_{\text{eff}}}$, rather than through the total variance, as is the case for plain-vanilla options.

The extremely fast and accurate approximation we have established allows us to plot such implied volatility surface for timer options under stochastic volatility model with little cost in computing time. In Figure 1, we plot the implied volatility surface for timer options under the Heston model. The left and right subplots are for $\rho = -0.5$ and $\rho = 0.5$, respectively. The other parameters used are the same as those in Table 1. Figure 2 plots the implied volatility surface under the 3/2 model with parameters the same as those in Table 2. In both figures, the range for strike is from 80 to 120, and the variance budget range goes from 0.05 to 0.5, roughly corresponding to a target expected exercise time of 5.5 years.

The timer call prices in Tables 1 and 2 with the chosen parameters are very close for the three values of K and three values of ρ reported. Figures 1 and 2 show that the implied volatility surfaces from these two models also share some similarities. Both models exhibit negative skew for negative ρ and positive skew for positive ρ. Also, the skew in both models when plotted against strike is more prominent for small variance budget. However, these two models also exhibit some remarkable differences when we consider the term structure of implied volatility as a function of variance budget with fixed strike level. The Heston model has a dip around $B = 0.1$ while the term structure in the 3/2 model has a more simple monotonic structure. In the positive correlation case, the 3/2 model has a downward sloping term structure for out-of-the-money timer calls, while in the Heston model, the term structure is U-shaped and eventually increasing. As stochastic volatility models are often calibrated using plain-vanilla option prices with maturities up to about two years, the above result shows that there might be considerable model risk when valuing timer options with large variance budget, especially when the parameter calibration process uses few maturities.

5 Conclusion

We have developed a perturbation technique for pricing timer options under general stochastic volatility models. For the special cases of the Heston model and the 3/2 model, we obtain very intuitive Black-Scholes-like closed-form formulas. Numerical analysis shows that these formulas are very accurate and extremely fast. This offers a considerable advantage over computationally expensive methods such as high-dimensional numerical integration or Monte Carlo. Besides being fast and accurate, our method also has many other attractive features.

There are many research directions one can take following our approach. First, it is useful to consider other stochastic volatility models. Because of the general solution technique in this paper, we should be able to get analytic formula for the timer options if the drift and diffusion functions of the variance process are sufficiently simple. Second, our method requires that the volatility coefficient η of the variance process be small. This is an intrinsic shortcoming of the perturbation method. It is therefore useful to study the behavior of the approximation for relatively large η in order to improve the current approximation or to redesign a better approximation. Third, it is useful to design an approximation for timer options with a maximum mandated expiry. This is an important feature for some timer options in practice. However, it seems difficult to extend the current PDE perturbation technique to this case. We leave these to future exploration.

6 Appendix

6.1 Verifying Equation (18)

For this purpose, let us denote the deterministic variance process at time t with initial value V at initial time 0 by $v(t, V)$, where $v(0, V) = V$. Notice that we have

$$\frac{\mathrm{d}v(t, V)}{\mathrm{d}t} = a(v(t, V)). \tag{148}$$

Integrating, we get

$$t = \int_V^{v(t,V)} \frac{1}{a(u)}\,\mathrm{d}u. \tag{149}$$

The above equation also verifies that v is a function of t and V. Differentiating with respect to V, we get

$$\frac{\partial v(t, V)}{\partial V} = \frac{a(v(t, V))}{a(V)}. \tag{150}$$

Differentiating the defining equation for $T = T(\xi, V)$

$$\int_0^T v(s, V)\,\mathrm{d}s = B - \xi \tag{151}$$

with respect to V and ξ, we get by using equation (150) that

$$T_V \equiv \frac{\partial T}{\partial V} = -\frac{1}{v(T, V)} \int_0^T \frac{\partial}{\partial V} v(s, V)\,\mathrm{d}s = -\frac{1}{v(T, V)} \cdot \frac{v(T, V) - V}{a(V)}, \tag{152}$$

$$T_\xi \equiv \frac{\partial T}{\partial \xi} = -\frac{1}{v(T, V)}. \tag{153}$$

It is now easy to verify that $VT_\xi + a(V)T_V + 1 = 0$.

6.2 Method of Characterisitcs for Solving Equation (57)

Pick any constant $V^* > 0$ and any smooth function $\Phi(\cdot)$. Define

$$z_0 = \frac{V - V^*}{V^*}, \tag{154}$$

$$z = \Phi\left(B - \xi + \int_{V^*}^V \frac{u}{A(u)}\,\mathrm{d}u\right). \tag{155}$$

Any constant V^* will work, but in practice stochastic volatility models are often mean-reverting and it is often convenient to choose V^* to be the long-run mean so that we do not introduce an unnecessary constant. We will always choose the function Φ such that when $B = \xi$, we have $z_0 = z$. This will usually pick up a unique form for Φ. Now switch to the new variables by defining

$$\widetilde{Q}(z, z_0) = Q(\xi, V). \tag{156}$$

Because $Vz_\xi + A(V)z_V = 0$, it can be worked out that the PDE for $Q(\xi, V)$ now reduces to an ODE for $\widetilde{Q}(z, z_0)$:

$$\widetilde{Q}_{z_0} = \frac{V^*}{A(V^*(1+z_0))}\widetilde{q}(z, z_0), \tag{157}$$

where $\widetilde{q}(z, z_0) = q(\xi, V)$. With the specific choice of Φ, the boundary condition becomes

$$\widetilde{Q}(z_0, z_0) \ = \ 0. \tag{158}$$

Therefore, the solution of Q is then simply

$$Q(\xi, V) = \widetilde{Q}(z(\xi, V), z_0(V)) = \int_z^{z_0} \frac{V^*}{A(V^*(1+z_0))}\widetilde{q}(z, z_0)\,\mathrm{d}z_0. \tag{159}$$

The only potential difficulty in the above method is that the function Φ might be given as an implicit function. However, in the Heston model, Φ can be written explicitly using the product log function, and in the 3/2 model, Φ is explicit.

6.3 Solving the PDEs in the Heston Model

6.3.1 Solving T_0 in the Heston Model

By definition, $T_0(\xi, V)$ is the first time the variance budget B is exceeded with initial variance V and initial accumulated variance ξ. In the Heston model when $\eta = 0$, variance process $v(t)$ is deterministic

$$v(t) = V + (V - \theta)e^{-\kappa t}, \tag{160}$$

with initial variance $v(0) = V$. Therefore, the integrated variance process ξ_t with initial value ξ is given by

$$\xi_t = \xi + \int_0^t v(s)\ \mathrm{d}s = \xi + \theta t + (V - \theta)\frac{1 - e^{-\kappa t}}{\kappa}. \tag{161}$$

Letting $\xi_t = B$ then gives us the solution in equation (65).

By the implicit function theorem, the derivatives of T_0 are given by

$$T_{0,V} = -\frac{1 - e^{-\kappa T_0}}{\kappa}\frac{1}{\theta + (V - \theta)e^{-\kappa T_0}}, \tag{162}$$

$$T_{0,\xi} = -\frac{1}{\theta + (V - \theta)e^{-\kappa T_0}}. \tag{163}$$

It's straightforward to verify that this implicit solution satisfies the PDE in equation (63).

The same implicit solution could also be obtained using Lagrange's method of characteristics for first-order PDEs. The characteristics are given by the solution of

$$\frac{\mathrm{d}\xi}{V} = \frac{\mathrm{d}V}{\kappa(\theta - V)} = \frac{\mathrm{d}T_0}{-1}. \tag{164}$$

Solving these equations will give us two integral constants which when coupled with the boundary condition gives us the same implicit solution in the main text. We omit the details here.

We now express T_0 in terms of the product log function. From the implicit solution for T_0 and the definition for z_0, we have

$$\kappa T_0 + z_0\left(1 - e^{-\kappa T_0}\right) = \kappa\frac{B-\xi}{\theta}. \tag{165}$$

Rearranging, we get

$$-\kappa T_0 + z_0 e^{-\kappa T_0} = z_0 - \kappa\frac{B-\xi}{\theta}. \tag{166}$$

Taking exponentials of both sides and multiplying by z_0, we get

$$z_0 e^{-\kappa T_0} e^{z_0 e^{-\kappa T_0}} = z_0 e^{z_0} e^{-\kappa\frac{B-\xi}{\theta}}. \tag{167}$$

If we define

$$z = z_0 e^{-\kappa T_0}, \tag{168}$$

we get from equation (167) that

$$z = W\left(z_0 e^{z_0} e^{-\kappa\frac{B-\xi}{\theta}}\right). \tag{169}$$

The solution T_0 is then given by

$$T_0 = \frac{1}{\kappa}\log\frac{z_0}{z}. \tag{170}$$

This solution has the shortcoming of a removable singularity at $z_0 = z = 0$. From equation (167), we have

$$\frac{z_0}{z} = e^{z - z_0} e^{\kappa\frac{B-\xi}{\theta}}. \tag{171}$$

Taking log of both sides, we get the alternative expression for T_0

$$T_0 = \frac{z - z_0}{\kappa} + \frac{B-\xi}{\theta}. \tag{172}$$

We can easily verify that when $\xi = B$, we get $z = z_0$ so $T_0 = 0$. Notice also that since $\kappa > 0$, $\theta > 0$, and $B \geq \xi$, z is always real. It can also be seen that z and z_0 will always have the same sign. In addition, we always have $|z_0| \geq |z|$ since the function xe^x is an increasing function on the real line. That is, we have $R \geq 1$. When $V > \theta$, we have $z_0 \geq z > 0$. When $V < \theta$, we have $z_0 \leq z < 0$. The above facts imply that T is always well-defined and nonnegative. The only case for $T_0 = 0$ is when $\xi = B$.

The quantity z has the meaning of percentage deviation of instantaneous variance at time T_0 from the long-run mean θ, in particular, we always have $1 + z > 0$. The variable z_0 is the percentage deviation of the current V from the long-run mean θ. In the Heston model, regardless of whether η is 0 or not, this percentage deviation decays at the rate κ, which is precisely what equation (67) is saying.

6.3.2 Solving T to Second Order in η in the Heston Model

We need to solve the first-order PDE

$$H_\xi + \kappa \frac{\theta - V}{V} H_V + \frac{1}{2}\left[T_{0,VV} - r(T_{0,V})^2\right] = 0. \tag{173}$$

First we need the expression for the second-order derivative $T_{0,VV}$. Using the fact that

$$\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{W(x)}{x(1 + W(x))}, \tag{174}$$

we can express the derivatives of T_0 in terms of z and z_0 as

$$T_{0,\xi} = -\frac{1}{\theta}\frac{1}{1+z}, \tag{175}$$

$$T_{0,V} = \frac{z - z_0}{\theta\kappa(1+z)z_0}, \tag{176}$$

$$T_{0,\xi\xi} = \frac{\kappa z}{(1+z)^3\theta^2}, \tag{177}$$

$$T_{0,\xi V} = \frac{z(1+z_0)}{(1+z)^3 z_0 \theta^2}, \tag{178}$$

$$T_{0,VV} = \frac{z(2z - 2z_0 + z^2 - z_0^2)}{\kappa(1+z)^3 z_0^2 \theta^2}. \tag{179}$$

This suggests a change of coordinates from (ξ, V) to (z, z_0). Simple calculation shows that

$$\frac{\partial z}{\partial \xi} = \frac{\kappa z}{\theta(1+z)}, \tag{180}$$

$$\frac{\partial z}{\partial V} = \frac{z(1+z_0)}{\theta z_0(1+z)}, \tag{181}$$

$$\frac{\partial z_0}{\partial \xi} = 0, \tag{182}$$

$$\frac{\partial z_0}{\partial V} = \frac{1}{\theta}. \tag{183}$$

Now define

$$\widetilde{H}(z, z_0) = H(\xi, V). \tag{184}$$

By the chain rule,

$$H_\xi = \frac{\kappa z}{\theta(1+z)} \widetilde{H}_z, \tag{185}$$

$$H_V = \frac{1}{\theta} \widetilde{H}_{z_0} + \frac{z(1+z_0)}{\theta z_0 (1+z)} \widetilde{H}_z. \tag{186}$$

Notice that

$$\frac{\theta - V}{V} = -\frac{z_0}{1+z_0}. \tag{187}$$

Simple calculation then shows that

$$H_\xi + \kappa \frac{\theta - V}{V} H_V = -\frac{\kappa z_0}{\theta(1+z_0)} \widetilde{H}_{z_0}. \tag{188}$$

The PDE for H then becomes an ODE for $\widetilde{H}$:

$$\widetilde{H}_{z_0} = \frac{\theta(1+z_0)}{\kappa z_0} \frac{T_{0,VV} - r(T_{0,V})^2}{2}. \tag{189}$$

Using the boundary condition $\widetilde{H}(z_0, z_0) = 0$, we get

$$\widetilde{H}(z, z_0) = \int_z^{z_0} \frac{\theta(1+u)}{\kappa u} \frac{T_{0,VV}(z,u) - r(T_{0,V}(z,u))^2}{2} \, \mathrm{d}u. \tag{190}$$

It is now a simple matter of integration to get the expression in equation (81).

The above method of solving the first-order PDE is actually very general. For any source term which can be written as $Vf(z, z_0)$, the solution to

$$VF_\xi + \kappa(\theta - V)F_V + Vf(z, z_0) = 0 \tag{191}$$

with boundary condition $F(B, V) = 0$ is given by

$$\widetilde{F} = \int_z^{z_0} \frac{\theta(1+u)}{\kappa u} f(z, u) \, \mathrm{d}u. \tag{192}$$

In particular, since $V = \theta(1+z_0)$, by taking $f(z, z_0) = 1/V$, we can also verify that

$$\widetilde{T}_0(z, z_0) = \int_z^{z_0} \frac{\theta(1+u)}{\kappa u} \frac{1}{\theta(1+u)} \, \mathrm{d}u = \frac{1}{\kappa} \log \frac{z_0}{z} \tag{193}$$

gives us the solution for T_0.

6.3.3 Solving Σ^2 to First Order in η in the Heston Model

The zeroth-order solution is simply

$$\Sigma^2 = \Sigma_0^2(\xi, V) = B - \xi. \tag{194}$$

For nonzero η, we postulate the following solution form

$$\Sigma^2 = B - \xi + 2\eta\rho(r - \delta)G(\xi, V). \tag{195}$$

Plugging this trial solution in, we get a Cauchy problem for $G = G(\xi, V)$:

$$VG_\xi + \kappa(\theta - V)G_V + VT_{0,V} = 0, \tag{196}$$

with the boundary condition

$$G(B, V) = 0. \tag{197}$$

Notice that the technique in solving $\widetilde{H}$ above is valid for any source term. Therefore, if we define $\widetilde{G}(z, z_0) = G(\xi, V)$, we can immediately write the solution as

$$\widetilde{G}(z, z_0) = \int_z^{z_0} \frac{\theta(1+u)}{\kappa u} T_{0,V}(z, u)\ \mathrm{d}u \tag{198}$$

$$= \frac{1}{\kappa^2(1+z)} \int_z^{z_0} \frac{(1+u)(z-u)}{u^2}\ \mathrm{d}u. \tag{199}$$

Integrating, we get

$$G(\xi, V) = \frac{(1-R)(Rz-1) + R(z-1)\log R}{\kappa^2 R(1+z)}. \tag{200}$$

6.4 Solving the PDEs in the 3/2 Model

6.4.1 Solving T_0 in the 3/2 Model

When $\eta = 0$, the variance process $v(t, V)$ with initial value V at time 0 is deterministic

$$v(t, V) = \frac{V\theta}{(\theta - V)e^{-\kappa\theta t} + V}. \tag{201}$$

A more intuitive way to understand the above equation is

$$\frac{1}{v(t, V)} = \left(\frac{1}{V} - \frac{1}{\theta}\right) e^{-\kappa\theta t} + \frac{1}{\theta}. \tag{202}$$

This equation shows that when $\eta = 0$, the reciprocal of instantaneous variance decays exponentially to its long-run level $1/\theta$ with decaying rate $\kappa\theta$.

The deterministic time $T_0(\xi, V)$ to hit the remaining variance budget $B - \xi$ is given by the solution of

$$\int_0^T v(t, V)\ \mathrm{d}t = B - \xi. \tag{203}$$

The above gives the solution for T_0 in the main text.

Below, we provide here a more general method by switching to two new variables z and z_0. This approach is useful for solving the higher-order PDEs for T, T' and Σ^2.

Similar to the situation in the Heston model, we define

$$z_0 \equiv \frac{V - \theta}{\theta}, \tag{204}$$

$$z \equiv \frac{V - \theta}{\theta} e^{-\kappa(B-\xi)}, \tag{205}$$

$$R \equiv \frac{z_0}{z} = e^{\kappa(B-\xi)}. \tag{206}$$

Notice that we always have $R \geq 1$.

The following derivatives are needed when we switch from the variables (ξ, V) to (z, z_0):

$$\frac{\partial z}{\partial \xi} = \kappa z, \tag{207}$$

$$\frac{\partial z}{\partial V} = \frac{z}{\theta z_0}, \tag{208}$$

$$\frac{\partial z_0}{\partial \xi} = 0, \tag{209}$$

$$\frac{\partial z_0}{\partial V} = \frac{1}{\theta}. \tag{210}$$

For any function $F(\xi, V)$, define $\widetilde{F}(z, z_0) = F(\xi, V)$. It can be easily worked out that

$$VF_\xi + \kappa V(\theta - V)F_V = -\kappa\theta z_0(1 + z_0)\widetilde{F}_{z_0}. \tag{211}$$

Therefore, for any function $f(z, z_0)$, the solution to

$$VF_\xi + \kappa V(\theta - V)F_V + f(z, z_0) = 0 \tag{212}$$

with boundary condition $F(B, V) = 0$ is given by a simple integration

$$F(\xi, V) = \widetilde{F}(z, z_0) = \int_z^{z_0} \frac{1}{\kappa\theta u(1+u)} f(z, u)\ \mathrm{d}u. \tag{213}$$

Taking $f(z, z_0) = 1$ and integrating, we get

$$T_0 = \frac{1}{\kappa\theta} \log\left(\frac{1+z}{z}\frac{z_0}{1+z_0}\right) = \frac{1}{\kappa\theta}\log\left(\frac{R(1+z)}{1+Rz}\right). \tag{214}$$

Similarly, we have

$$T_0' = \frac{1}{\kappa'\theta'} \log\left(\frac{1+z'}{z'}\frac{z_0'}{1+z_0'}\right) = \frac{1}{\kappa'\theta'} \log\left(\frac{R'(1+z')}{1+R'z'}\right), \tag{215}$$

with

$$z_0' \equiv \frac{V-\theta'}{\theta'}, \tag{216}$$

$$z' \equiv \frac{V-\theta'}{\theta'} e^{-\kappa'(B-\xi)}, \tag{217}$$

$$R' \equiv \frac{z_0'}{z'} = e^{\kappa'(B-\xi)}. \tag{218}$$

6.4.2 Solving T to Second Order in η in the 3/2 Model

The derivatives we need here are

$$T_{0,V} = \frac{z-z_0}{\kappa\theta^2 z_0(1+z_0)(1+z)}, \tag{219}$$

$$T_{0,VV} = \frac{(z_0-z)(z+z_0+2zz_0)}{\kappa\theta^3 z_0^2(1+z_0)^2(1+z)^2}. \tag{220}$$

The quantity $H(\xi, V; \kappa, \theta, r)$ satisfies the following PDE

$$VH_\xi + \kappa V(\theta - V)H_V + \frac{1}{2}V^3\left[T_{0,VV} - r(T_{0,V})^2\right] = 0. \tag{221}$$

Using $V = \theta(1+z_0)$, we can write the source term in terms of z and z_0:

$$\frac{1}{2}V^3\left[T_{0,VV} - r(T_{0,V})^2\right] = h(z, z_0). \tag{222}$$

The solution H is then given by the integration

$$H = \int_z^{z_0} \frac{1}{\kappa\theta u(1+u)} h(z,u)\ \mathrm{d}u. \tag{223}$$

Though tedious, the algebra is simple, and we omit the details here. In the main text, we give H in terms of $R = e^{\kappa(B-\xi)}$ and V which is closer to our original variables ξ and V.

6.4.3 Solving Σ^2 to First Order in η in the 3/2 Model

Write

$$\Sigma^2 = B - \xi + 2\eta\rho(r-\delta)G(\xi, V). \tag{224}$$

The PDE we need to solve for $G = G(\xi, V)$ is:

$$VG_\xi + \kappa V(\theta - V)G_V + V^2 T_{0,V} = 0. \tag{225}$$

Writing $V^2 T_{0,V}$ as a function of z and z_0, this equation can be solved using the same technique above, and we omit the details here.

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Table 1: **Accuracy of Timer Call Price Approximation in the Heston Model**

Parameters used here are the same ones as in Liang, Lemmens and Tempere (2011): $V_0 = 0.087$, $\kappa = 2$, $\theta = 0.09$, $\eta = 0.375$, $B = 0.087$, $\xi = 0$, $r = 0.015$, $S_0 = 100$, and $\delta = 0$. Analytical results are taken from Liang, Lemmens and Tempere (2011). In Panel A, we give the timer option prices assuming that $r = 0$ and $\eta = 0$, where we have exact pricing formulas. In Panel B, we compare the analytic results with the approximation in this paper. `Approx1` is price obtained if we solve T, T' and Σ all to first order in η. In `Approx1-2`, Σ is solved exactly to first order in η, while T and T' are solved exactly to second order in η. All errors reported are percentage errors with respect to the analytic results.

Panel A: Naive Approximations with $r = 0$ or $\eta = 0$

K	ρ	Analytic	$r = 0$	Error	$\eta = 0$	Error
90	-0.5	17.8095	16.8371	-5.46%	17.6148	-1.09%
	0	17.7249	16.8371	-5.01%	17.6148	-0.62%
	0.5	17.6263	16.8371	-4.48%	17.6148	-0.07%
100	-0.5	12.5789	11.7263	-6.78%	12.3837	-1.55%
	0	12.4772	11.7263	-6.02%	12.3837	-0.75%
	0.5	12.3691	11.7263	-5.20%	12.3837	0.12%
110	-0.5	8.6515	7.9445	-8.17%	8.4697	-2.10%
	0	8.5449	7.9445	-7.03%	8.4697	-0.88%
	0.5	8.4393	7.9445	-5.86%	8.4697	0.36%

Panel B: Singular Perturbation Method

K	ρ	Analytic	Approx1	Error	Approx1-2	Error
90	-0.5	17.8095	17.7033	-0.60%	17.8167	0.04%
	0	17.7249	17.6148	-0.62%	17.7287	0.02%
	0.5	17.6263	17.5258	-0.57%	17.6400	0.08%
100	-0.5	12.5789	12.4847	-0.75%	12.5815	0.02%
	0	12.4772	12.3837	-0.75%	12.4806	0.03%
	0.5	12.3691	12.2817	-0.71%	12.3788	0.08%
110	-0.5	8.6515	8.5719	-0.92%	8.6500	-0.02%
	0	8.5449	8.4697	-0.88%	8.5476	0.03%
	0.5	8.4393	8.3666	-0.86%	8.4444	0.06%

Table 2: **Accuracy of Timer Call Price Approximation in the 3/2 Model**

Parameters used here are the same ones as in Liang, Lemmens and Tempere (2011): $V_0 = 0.295^2$, $\kappa = 22.84$, $\theta = 0.4669^2$, $\eta = 8.56$, $B = V_0$, $\xi = 0$, $r = 0.015$, $S_0 = 100$, and $\delta = 0$. Analytical results are taken from Liang, Lemmens and Tempere (2011). In Panel A, we give the timer option prices assuming that $r = 0$ and $\eta = 0$, where we have exact pricing formulas. In Panel B, we compare the analytic results with the approximation in this paper. `Approx1` is price obtained if we solve T, T' and Σ all to first order in η. In `Approx1-2`, Σ is solved exactly to first order in η, while T and T' are solved exactly to second order in η. All errors reported are percentage errors with respect to the analytic results.

Panel A: Naive Approximations with $r = 0$ or $\eta = 0$

K	ρ	`Analytic`	$r=0$	`Error`	$\eta=0$	`Error`
90	−0.5	17.8064	16.8371	−5.44%	17.2853	−2.93%
	0	17.7046	16.8371	−4.90%	17.2853	−2.37%
	0.5	17.5839	16.8371	−4.25%	17.2853	−1.70%
100	−0.5	12.5780	11.7263	−6.77%	12.1045	−3.76%
	0	12.4619	11.7263	−5.90%	12.1045	−2.87%
	0.5	12.3300	11.7263	−4.90%	12.1045	−1.83%
110	−0.5	8.6518	7.9445	−8.18%	8.2461	−4.69%
	0	8.5339	7.9445	−6.91%	8.2461	−3.37%
	0.5	8.4026	7.9445	−5.45%	8.2461	−1.86%

Panel B: Singular Perturbation Method

K	ρ	`Analytic`	`Approx1`	`Error`	`Approx1-2`	`Error`
90	−0.5	17.8064	17.3662	−2.47%	17.7653	−0.23%
	0	17.7046	17.2853	−2.37%	17.6856	−0.11%
	0.5	17.5839	17.2038	−2.16%	17.6053	0.12%
100	−0.5	12.5780	12.1962	−3.04%	12.5356	−0.34%
	0	12.4619	12.1045	−2.87%	12.4443	−0.14%
	0.5	12.3300	12.0120	−2.58%	12.3522	0.18%
110	−0.5	8.6518	8.3383	−3.62%	8.6113	−0.47%
	0	8.5339	8.2461	−3.37%	8.5188	−0.18%
	0.5	8.4026	8.1532	−2.97%	8.4255	0.27%

Table 3: **Sensitivity Analysis of Timer Call Prices in Heston Model**

Base case parameters are the same ones as in Liang, Lemmens and Tempere (2011): $V_0 = 0.087$, $\kappa = 2$, $\theta = 0.09$, $\eta = 0.375$, $B = 0.087$, $\xi = 0$, $r = 0.015$, $S_0 = 100$, and $q = 0$. Here for the base case, we use $K = 110$ and $\rho = -0.5$. In each sensitivity analysis, one parameter is increased or decreased by 10% from its base value. We look at the effect of the parameter change on the following quantities: effective discounting e^{-rT}, effective total variance Σ, deterministic variance budget exceeding time T_0, risk-neutral expected time to exercise T^E, the timer call price C, as well as the delta Δ.

parameter		e^{-rT}	Σ	T_0	T^E	C	Δ
base case		0.9833	0.0885	0.9810	1.1228	8.6500	0.4542
κ	+10%	0.9836	0.0885	0.9801	1.1041	8.6349	0.4538
	−10%	0.9830	0.0886	0.9820	1.1451	8.6674	0.4548
θ	+10%	0.9844	0.0884	0.9300	1.0514	8.5992	0.4527
	−10%	0.9820	0.0887	1.0416	1.2100	8.7125	0.4562
η	+10%	0.9829	0.0887	0.9810	1.1526	8.6766	0.4550
	−10%	0.9837	0.0884	0.9810	1.0959	8.6249	0.4536
ρ	+10%	0.9833	0.0887	0.9810	1.1228	8.6601	0.4544
	−10%	0.9833	0.0884	0.9810	1.1228	8.6398	0.4541
V_0	+10%	0.9839	0.0883	0.9398	1.0815	8.6144	0.4532
	−10%	0.9827	0.0884	1.0233	1.1648	8.6653	0.4550
B	+10%	0.9817	0.0973	1.0781	1.2281	9.2786	0.4639
	−10%	0.9849	0.0795	0.8838	1.0160	7.9729	0.4431
r	+10%	0.9816	0.0885	0.9810	1.1227	8.7109	0.4565
	−10%	0.9850	0.0882	0.9810	1.1229	8.5688	0.4517

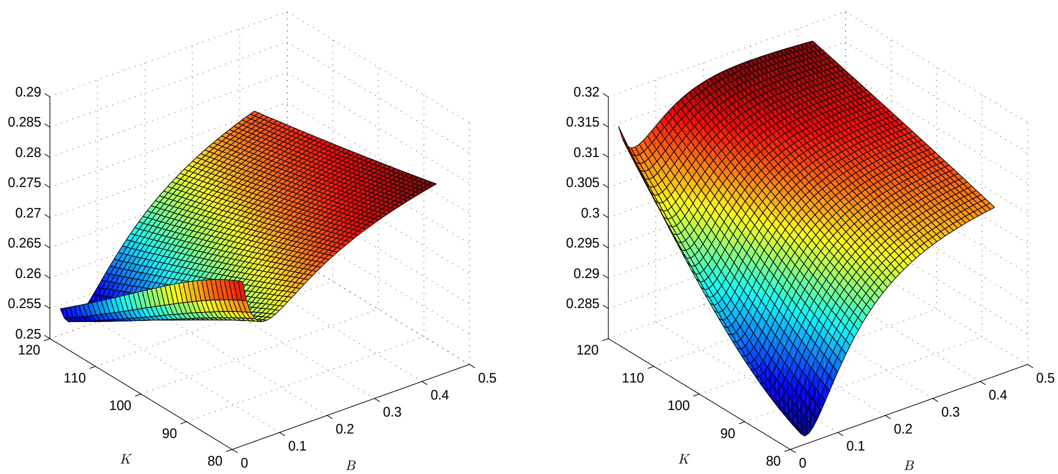

Figure 1: **Black-Scholes implied volatility surface for timer options under the Heston model.** The x and y axes are the variance budget B and strike level K. The left and right subplots are for $\rho = -0.5$ and $\rho = 0.5$, separately. The other parameters used are the same as those in Table 1. The precise definition of Black-Scholes implied volatility for timer options is given in equation (147).

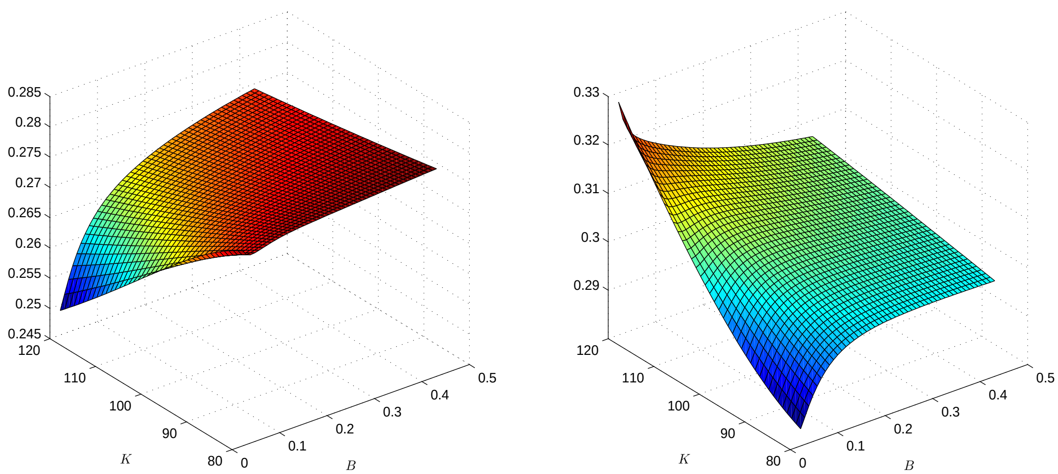

Figure 2: **Black-Scholes implied volatility surface for timer options under the $3/2$ model.** The x and y axes are the variance budget B and strike level K. The left and right subplots are for $\rho = -0.5$ and $\rho = 0.5$, separately. The other parameters used are the same as those in Table 2. The precise definition of Black-Scholes implied volatility for timer options is given in equation (147).