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Bloomberg LP

2013

Noname manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: The correct dates will be entered by the editor

Abstract We study the risk index of an additive gamble proposed in Aumann and Serrano (2008). We establish a generalized duality result for this index and use it to prove Yaari's (1969) alternative characterization of DARA utilities. A new characterization result for the risk index is obtained through essentially monotonic risk aversion utilities. We also extend the domain of gambles by introducing a price for gambles. We then develop a theory on the risk index for multiplicative gambles. Relative risk aversion functions for multiplicative gambles play the same role as absolute risk aversion functions for additive gambles.

Keywords Risk index · Attractiveness index · Duality · Additive gambles · Multiplicative gambles

JEL Classifications: C00; D80; D81

1 Introduction

Decision under risk has been at the core of modern economic theory. For some recent developments on this topic, see Cox et al (2013), Eguia (2013), Chen and Luo (2013). One particularly interesting development is Aumann and Serrano (2008), in which the authors proposed an index of riskiness that assigns to each gamble (risky asset) a single fixed number. The index is designed in isolation of a person's risk attitude, that is, it only depends on the gamble's

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own distributional attributes. Mathematically, the index is the reciprocal of the parameter for a constant absolute risk aversion (CARA) agent to be indifferent towards the gamble. The central ingredient in their development of the theory is the duality axiom, which states that if an agent accepts a gamble at a fixed wealth, then a uniformly less risk averse agent would accept any gamble with smaller risk index at that wealth. Aumann and Serrano (2008) also discuss the relations between their risk index and other proposed risk measures in the literature, such as different measures of dispersion (Markowitz 1952), the Sharpe ratio (Sharpe 1966), Value at Risk (see Pearson 2002), and the "coherent" risk proposed in Artzner et al. (1999). The risk index in Aumann and Serrano (2008) has some nice properties compared to other risk measures. It is measured in dollars, and is positive homogeneous of degree one. It is also subadditive. Another very nice property is that it respects first- and second-order stochastic dominance orders.

In this paper, we study the theoretical aspects of the risk index proposed in Aumann and Serrano (2008) in further detail. The main contributions can be divided into four groups, as we describe below.

First, we study the fundamental theoretical aspects of the risk index. We define a gamble's attractiveness index as the reciprocal of the risk index, as many results are more naturally stated in terms of the attractiveness index. We first give a necessary condition for an agent to accept or reject a gamble as well as a sufficient condition. Roughly speaking, for a risk averse agent to accept an additive gamble at a fixed wealth, there has to be at least one wealth level in the wealth range of taking the gamble such that the local absolute risk aversion is smaller than the attractiveness of the gamble. On the other hand, if we know that the attractiveness index of a gamble is larger than the local absolute risk aversions on the whole wealth range of taking the gamble, then the gamble is accepted by the agent. This result is useful in practice as it allows for the possibility of a quick decision on whether we should accept or reject a gamble without a detailed computation of the expected utility. By utilizing this result, we point out that the risk index actually allows for a more general duality result. In order for the duality result to hold between two fixed gambles, we only need to check the two agents' risk aversions on two intervals on the real line. This strengthened duality result is then used to derive an alternative characterization of utilities with nonincreasing or nondecreasing absolute risk aversions (DARA and IARA), first proposed in Yaari (1969) and rigorously proven in Dybvig and Lippman (1983). It states that an agent's absolute risk aversion function is nonincreasing if and only if gambles accepted at a given wealth level are also accepted at any higher wealth level. Furthermore, by considering agents with essentially monotonic absolute risk aversions, we give a characterization theorem for Aumann and Serrano's risk index in place of the duality axiom.

Second, we study in more detail sums of gambles. Aumann and Serrano (2008) obtain that for any two independent gambles, the risk index of their sum always lies between the risk indices of these two gambles. We study gambles that are not necessarily independent. In particular, we show that if the dependent parts of two gambles are similarly ordered, or more generally positively quadrant dependent, then the risk index of the sum of two gambles is always larger than the minimum of the risk indices of the two gambles. For negative dependence, the risk index of the sum is always smaller than the maximum of the two risk indices. The above results agree with our intuitions well. For example, the result for negative dependence agrees with our intuition of risk diversification. Thus this result can be considered another attractive property of Aumann and Serrano's risk index.

Third, we extend the domain of additive gambles to essentially any random payoff by introducing a price for each gamble. The price can come from a pricing functional or can be fixed exogenously. Each generalized gamble is associated with an additive gamble which is the generalized gamble net of its price. The additive gambles considered in Aumann and Serrano (2008) are then special zero price gambles. We define the risk index of the generalized gamble to be the risk index of its associated additive gamble. By considering the associated additive gambles, all previous results on additive gambles can be readily translated to the generalized gambles. An interesting result is that the risk index for a generalized gamble is always strictly increasing and strict convex with respect to the price.

The gambles considered in Aumann and Serrano (2008) are all additive gambles. Our final contribution is that we show the whole theory on the risk index for additive gambles can be translated into a theory on the risk index for multiplicative gambles. If an agent takes a multiplicative gamble, his final wealth will be the product of his current wealth and the random realization of the gamble. The exponentials of additive gambles are all multiplicative gambles, but not all multiplicative gambles are logarithms of additive gambles. Nonetheless, the theories on the risk indices of additive and multiplicative gambles are parallel to each other. For additive gambles, the risk index is the level of constant *absolute* risk aversion for an agent to be indifferent about taking or not taking the gamble. For multiplicative gambles, we define a risk index that is exactly the level of constant *relative* risk aversion for an agent to be indifferent towards the gamble. A similar duality result holds for multiplicative gambles. We then use this duality result to give an alternative characterization for utilities with nonincreasing and nondecreasing relative risk aversions (DRRA and IRRA). As far as we know, this alternative characterization for DRRA and IRRA has not appeared in the literature. Thus, it complements Yaari's (1969) alternative characterization for DARA and IARA utilities well. Finally, by considering agents with essentially monotonic relative risk aversions, we give another characterization for the risk index of multiplicative gambles, similar to what we do for additive gambles.

The paper is organized as follows. In Section 2, we study further properties of the risk index for additive gambles. We use a generalized duality result to to prove Yaari's (1969) alternative characterization of monotone absolute risk aversion utilities, and characterize the risk index in place of the duality axiom. We also extend the domain of gambles. Section 3 studies multiplicative gambles. Section 4 concludes.

2 Additive gambles

2.1 Duality on intervals

Same as in Aumann and Serrano (2008), throughout the paper, a *utility function* u is a twice continuously differentiable function on $\mathbb{R}$ which is strictly increasing and strictly concave. We write $\rho(w) = -u''(w)/u'(w)$ for the agent's absolute risk aversion function and $r(w) = -wu''(w)/u'(w)$ for the agent's relative risk aversion function as defined in Pratt (1964). A gamble g is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with real values such that $\mathbb{E}g$ is finite and positive and $\mathbb{P}(g < 0) > 0$. However, unlike Aumann and Serrano (2008), we do not assume that g necessarily takes finitely many values. Furthermore, we do not assume that g is bounded or g has a continuous density function. An agent i with wealth w_i makes decisions with regard to gambles according to his utility function u_i. He accepts a gamble g at wealth w_i if and only if $\mathbb{E}u_i(w_i + g) > u_i(w_i)$, is indifferent about g if and only if $\mathbb{E}u_i(w_i + g) = u_i(w_i)$, and rejects g if and only if $\mathbb{E}u_i(w_i + g) < u_i(w_i)$. Notice that here we followed the convention used in Aumann and Serrano (2008) where acceptance means strict preference and does not include indifference.

For any gamble g, Aumann and Serrano's index of risk $R(g)$ is defined to be the unique positive solution (if exists) of

$$\mathbb{E}e^{-g/R(g)} = 1. \tag{1}$$

In Aumann and Serrano (2008), g takes finitely many values, so the existence is always guaranteed. In our treatment, it could be that a positive solution of the above equation does not exist.[1] If this is the case, we set $R(g) = +\infty$.

It is obvious that $R(g)$ is positive homogeneous of degree 1: $R(tg) = tR(g)$ for any $t > 0$. The reciprocal of $R(g)$ is often of interest on its own. Thus, we will write $\alpha(g) \equiv 1/R(g)$ and call it the *attractiveness* of the gamble g. It is positive homogenous of degree -1. A graph of the function $f(\beta) \equiv \mathbb{E}e^{-\beta g} - 1$ is very helpful in what follows and is presented schematically in Figure 1. Notice that $f(\beta) < 0$ if and only if $0 < \beta < \alpha(g)$, $f(\beta) > 0$ if and only if $\beta < 0$ or $\beta > \alpha(g)$. In particular, if we know that $\beta > 0$, then $f(\beta) > 0$ if and only if $\beta > \alpha(g)$.

One key concept in the development of Aumann and Serrano's economic index of risk $R(g)$ is the following. Call agent i *uniformly no less risk-averse than* agent j (written as $i \trianglerighteq j$) if whenever i accepts a gamble at **some** wealth, then j accepts that gamble at **any** wealth. It is shown in Aumann and Serrano (2008) that

$$i \trianglerighteq j \quad \text{if and only if} \quad \inf_{w\in\mathbb{R}} \rho_i(w) \geq \sup_{w\in\mathbb{R}} \rho_j(w). \tag{2}$$

[1] An example would be a short position in a forward contract with final payoff $K - S_T$, where future asset price S_T follows a lognormal distribution. This is because the moment-generating function for the log-normal distribution does not exist.

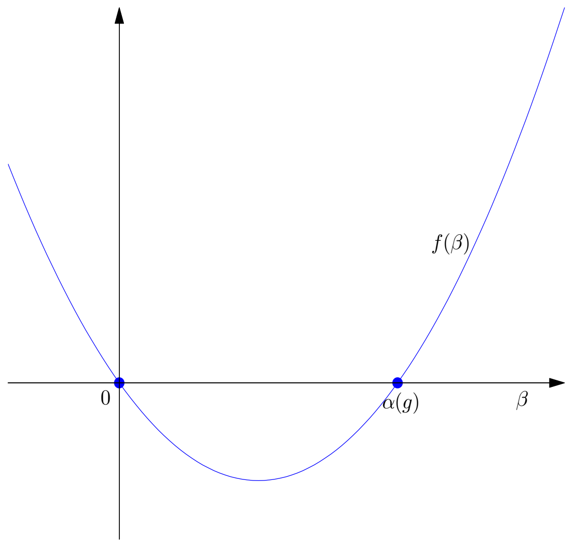

Fig. 1 The defining equation for an additive gamble's attractiveness $\alpha(g)$**.** The function $f(\beta) \equiv \mathbb{E}e^{-\beta g} - 1$ for a fixed gamble g is drawn. For gambles with finite and positive mean, $f(\beta)$ is convex in β and has two roots, one at 0 and the other at $\alpha(g) > 0$. Notice also that $f(\beta)$ is not monotonic. In particular, $f'(0) < 0$ and $f'(\alpha(g)) > 0$.

Besides positive homogeneity, Aumann and Serrano's risk index $R(g)$ also satisfies the following defining property. The intuition is that if the more risk-averse of two agents (in the sense of $\trianglerighteq$) accepts a gamble, then the less risk-averse agent should accept any less risky gambles (in the sense of $R(\cdot)$). Aumann and Serrano (2008) actually use a stricter relation "$\triangleright$" which is defined as follows: $i \triangleright j$ if and only if we have $i \trianglerighteq j$ but not $j \trianglerighteq i$.

Duality as in Aumann and Serrano (2008): Let g and h be two gambles such that $R(g) > R(h)$. If $i \triangleright j$ and i accepts gamble g at wealth w, then j accepts h at w.

One of our main purposes below is to give a generalization of the above duality property, which states that for the duality result to hold, it is not necessary to check the condition of uniform more risk aversion on the whole real line. We also show that it is not necessary to have the two strict inequalities in the conditions for duality. That is, it suffices to have weaker conditions $R(g) \geq R(h)$ and $i \trianglerighteq j$. For this, we need some notational preparations. The essential infimum $\operatorname{ess\,inf} h$ of a random variable h on $(\Omega, \mathcal{F}, \mathbb{P})$ is defined to be the largest possible real number L such that $\mathbb{P}(h \geq L) = 1$. When g

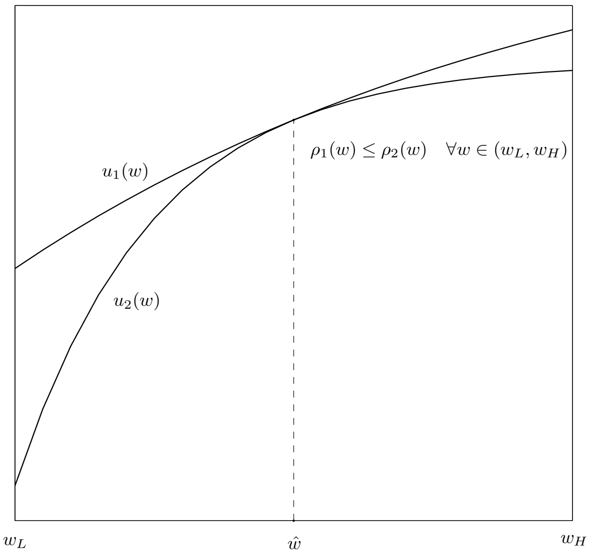

Fig. 2 Nested utilities. This figure depicts two utilities nested at $\widehat{w}$. Utility u_2 is assumed to be locally more risk-averse than u_1 in terms of absolute risk aversion for all wealth levels in (w_L, w_H). The values of u_1 and u_2 as well as their first-order derivatives are normalized to be equal to each other at $\widehat{w}$.

takes finitely many values with nonzero probabilities, this is exactly $\min g$. Similarly define $\operatorname{ess\,sup}$. Notice that since any gamble g satisfies $\mathbb{E}g > 0$ and $\mathbb{P}(g < 0) > 0$, we always have $\operatorname{ess\,inf} g < \operatorname{ess\,sup} g$. For any wealth level w and any gamble g, define $D(w, g)$ to be the closed interval $[w+\operatorname{ess\,inf} g, w+\operatorname{ess\,sup} g]$ if $\operatorname{ess\,inf} g$ and $\operatorname{ess\,sup} g$ are both finite, and with obvious modifications if either is infinite. This is the smallest closed interval of $\mathbb{R}$ containing essentially all the values of $w + g$. The reason we need to consider essential bounds rather than the bounds themselves is that values of g taken on null sets of $\mathbb{P}$ do not have any effect on computing the expected utility.

One fact that was used heavily in Aumann and Serrano (2008) and that will be used repeatedly in this paper is the following. Let $u_1(w)$ and $u_2(w)$ be two utility functions and suppose that for some $\widehat{w}$ we have $u_1(\widehat{w}) = u_2(\widehat{w})$ and $u_1'(\widehat{w}) = u_2'(\widehat{w})$. Suppose further that the continuous absolute risk aversion functions satisfy $\rho_1(w) \leq \rho_2(w)$ for all w in the interval (w_L, w_H) containing $\widehat{w}$. Then, we have $u_2'(w) \leq u_1'(w)$ for any $w \in (\widehat{w}, w_H)$, and $u_2'(w) \geq u_1'(w)$ for any $w \in (\widehat{w}_L, \widehat{w})$. Furthermore, $u_2(w) \leq u_1(w)$ for all $w \in (\widehat{w}_L, \widehat{w}_H)$. That is, the two utilities are "nested" at $\widehat{w}$. If $\rho_1(w) < \rho_2(w)$ for at least one $w \in (w_L, w_H)$, then there exists $w \in (w_L, w_H)$ such that the above inequalities become strict. Figure 2 depicts two nested utilities. A mechanic proof of the above claims is readily available if one notices that for any w and any utility function, we have

$$u'(w) = u'(\widehat{w})e^{-\int_{\widehat{w}}^{w} \rho(x)\mathrm{d}x}, \tag{3}$$

and

$$u(w) = u(\widehat{w}) + u'(\widehat{w}) \int_{\widehat{w}}^{w} e^{-\int_{\widehat{w}}^{y} \rho(x)\mathrm{d}x} \,\mathrm{d}y, \tag{4}$$

with the understanding that the integrals above are Riemann integrals so that we have $\int_a^b = -\int_b^a$ whenever $a > b$.

First we need the following proposition, which is related to statement (4.3.2) in Aumann and Serrano (2008). It gives a necessary condition for accepting or rejecting a gamble, as well as a sufficient condition. It roughly states that if a gamble g is accepted at wealth w_i, then its attractiveness has to overcome at least some values of the local risk-aversions in the range $D(w_i, g)$. On the other hand, if a gamble's attractiveness is larger than all the local risk-aversions in the range $D(w_i, g)$, then it is accepted.

Proposition 1 *We have the following statements with respect to acception and rejection of gambles:*

1. *If i accepts g at wealth w_i, then there exists $w \in D(w_i, g)$ such that $\alpha(g) > \rho_i(w)$. In particular, we must have $\alpha(g) > \inf_{w\in D(w_i,g)} \rho_i(w)$. On the other hand, if i rejects g at wealth w_i, then there exists $w \in D(w_i, g)$ such that $\alpha(g) < \rho_i(w)$. In particular, we must have $\alpha(g) < \sup_{w\in D(w_i,g)} \rho_i(w)$.*
2. *If $\alpha(g) \geq \rho_i(w)$ for any $w \in D(w_i, g)$ with strict inequality for at least one such w, then i accepts g at w_i. On the other hand, if $\alpha(g) \leq \rho_i(w)$ for any $w \in D(w_i, g)$ with strict inequality for at least one such w, then i rejects g at w_i.*

Proof: We will only prove the two first sentences in statements 1 and 2. The other claims can be proven similarly. Because utility functions are equivalent up to an affine transformation, without loss of generality, we may assume $w_i = 0$, $u_i(0) = 0$, and $u_i'(0) = 1$.

Now assume that i accepts g at wealth 0, but $\alpha(g) \leq \rho_i(w)$ for all $w \in D(0, g)$. Let $u_{\alpha(g)}(w) = (1 - e^{-\alpha(g)w})/\alpha$ be a CARA utility function with constant absolute risk aversion $\alpha(g)$. Then, u_i and $u_{\alpha(g)}$ are nested at $w = 0$. In particular, $u_i(w) \leq u_{\alpha(g)}(w)$ for all $w \in D(0, g)$. However, this contradicts the assumption that i accepts g at $w_i = 0$ since then $\mathbb{E}u_i(g) \leq \mathbb{E}u_{\alpha(g)}(g) = 0 = u_i(0)$.

Now assume $\alpha(g) \geq \rho_i(w)$ for any $w \in D(w_i, g)$ with strict inequality for at least one such w. Notice by continuity of $\rho_i(w)$, if $\alpha(g) \geq \rho_i(w_*)$ for some w_*, there is a neighborhood V_{w_*} of w_*, such that $\alpha(g) \geq \rho_i(w)$ for any $w \in V_{w_*}$. Let $w_L = \sup\{w \in D(0, g) : w < 0, \alpha(g) > \rho_i(w)\}$, and $w_H = \inf\{w \in D(0, g) : w > 0, \alpha(g) > \rho_i(w)\}$. Then $[w_L, w_H]$ is a proper subset of $D(0, g)$. Examining equation (4) then shows that $u_i(w) = u_{\alpha(g)}(w)$ on $[w_L, w_H]$, but

$u_i(w) > u_{\alpha(g)}(w)$ for any $w \in D(0,g) \setminus [w_L, w_H]$. Now by the definition of $D(0,g)$, $\mathbb{P}[g \notin [w_H, w_H]] > 0$. Thus, $\mathbb{E}u_i(g) > \mathbb{E}u_{\alpha(g)}(g) = 0 = u_i(0)$, and i accepts g at $w_i = 0$. □

Proposition 2 (Duality strengthened) *Let g and h be two gambles defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $R(h) \leq R(g) < \infty$. Suppose*

$$\inf_{w \in D(w_i,g)} \rho_i(w) \geq \sup_{w \in D(w_j,h)} \rho_j(w). \tag{5}$$

Then if i accepts gamble g at wealth w_i, j accepts h at wealth w_j. Furthermore, if j rejects h at wealth w_j, then i rejects g at w_i.

Proof: We first prove the first statement in the conclusion. The two conditions in the Proposition allow us to chain inequalities. Since i accepts gamble g at wealth w_i, by statement 1 of Proposition 1,

$$\alpha(g) > \inf_{w \in D(w_i,g)} \rho_i(w). \tag{6}$$

Therefore, we have

$$\alpha(h) \geq \alpha(g) > \inf_{w \in D(w_i,g)} \rho_i(w) \geq \sup_{w \in D(w_j,h)} \rho_j(w). \tag{7}$$

By statement 2 of Proposition 1, j accepts h at wealth w_j. Notice that since the second inequality is strict, it is not necessary for the first and last inequalities to be strict.

The proof of the second statement is similar except that now we have

$$\inf_{w \in D(w_i,g)} \rho_i(w) \geq \sup_{w \in D(w_j,h)} \rho_j(w) > \alpha(h) \geq \alpha(g). \tag{8}$$

The strict inequality in the middle is due to statement 1 of Proposition 1. Statement 2 of Proposition 1 now tells us that i rejects g at w_i. □

Notice that $\rho_i(w) > \rho_j(w)$ says something about the *local* risk aversion behavior around wealth level w, while $i \trianglerighteq j$ concerns *global* risk aversion behavior on the whole real line (see Section IV.A of Aumann and Serrano (2008)). The condition in equation (5) can be thought of as something in between since we are considering *intervals* of the real line.

The following numerical example illustrates a typical situation where we can apply the duality on an interval, but not the duality on the whole real line. For simplicity, assume $w_i = w_j = 0$. Let $\rho_i(w) = 1/10$ for all $w \in \mathbb{R}$. Let $\rho_j(w) = \Phi(-w-4)/5$, where $\Phi(\cdot)$ is the cumulative normal distribution function. The two functions $\rho_i(w)$ and $\rho_j(w)$ only cross each other once at $w = -4$. Now let g be a gamble which pays off -4 and $+2$ with probabilities $1/4$ and $3/4$, and h a gamble which pays off -2 and $+3$ with equal probabilities, respectively. The attractiveness can be computed to be $\alpha(g) \doteq 0.13$ and $\alpha(h) \doteq 0.16$. Thus $R(g) > R(h)$. Since $\alpha(g) > \rho_i(w)$ for all $w \in \mathbb{R}$, i accepts g at any wealth, in particular, at $w_i = 0$. Thus, by Proposition 2, we know that j must accept h at $w_j = 0$. Notice that in this example, the interval dominance condition in equation (5) is satisfied, but we do not have the global dominance $i \trianglerighteq j$.

2.2 Alternative characterization of DARA and IARA utilities

A utility function is said to *belong to the DARA class* if and only if its absolute risk aversion function is nonincreasing. As a nontrivial corollary of Proposition 2, we have the following alternative characterization of DARA utilities, first proposed in Yaari (1969). This corollary was also rigorously proved in Dybvig and Lippman (1983) based on Pratt's well-known characterization that a utility function u is in DARA if and only if for any gamble g, the risk premium function $\pi_g(w)$ is nonincreasing, where $\pi_g(w)$ is defined to be the unique solution to $u(w+\mathbb{E}g-\pi_g(w)) = \mathbb{E}u(w+g)$. Our proof is an interesting alternative approach that makes use of Aumann and Serrano's risk index. In Dybvig and Lippman (1983), when $u \notin DARA$, the counterexample gamble g that is accepted at lower wealth but rejected at higher wealth is not constructed explicitly, but rather its existence is guaranteed by referring to one of Pratt's theorems. Based on the concept of attractiveness, we give a completely explicit construction for the counterexample g.

Proposition 3 *A utility function is in DARA if and only if any gambles accepted at a given wealth level will be accepted at all higher wealth levels. More generally, assuming $w_l < w_r$, a utility's absolute risk aversion function is nonincreasing on $[w_l, w_r]$ if and only if any gamble g accepted at a wealth level w_L will be accepted at any higher wealth level w_H so long as $D(w_L, g) \in [w_l, w_r]$ and $D(w_H, g) \in [w_l, w_r]$.*

Proof: Suppose u is in DARA with absolute risk aversion function $\rho(\cdot)$. Let i and j be two agents with the same utility function $u(w)$ but different wealth levels w_L and w_H, where $w_L < w_H$. Because utilities are equivalent up to an affine transformation, we can assume that i and j have the same initial wealth 0, but agent i has the utility $v_i(w) = (u(w_L + w) - u(w_L))/u'(w_L)$ and agent j has the utility $v_j(w) = (u(w_H + w) - u(w_H))/u'(w_H)$. Now $v_i(0) = v_j(0)$, $v_i'(0) = v_j'(0)$, and $\rho_i(w) = \rho(w + w_L) \geq \rho(w + w_H) = \rho_j(w)$ for any $w \in D(0, g)$. Thus, v_i and v_j are nested at $w = 0$. In particular, $v_i(w) \leq v_j(w)$ for any $w \in D(0, g)$. Therefore, if g is accepted by i, then it is also accepted by j, since $\mathbb{E}v_j(g) \geq \mathbb{E}v_i(g) > v_i(0) = 0 = v_j(0)$.

For the converse, let $u \notin DARA$. We need to show that there exists a gamble which is accepted by u at a lower wealth but rejected at a higher wealth. Since $u \notin DARA$, there exists two wealth levels w_L and w_H with $w_L < w_H$ such that $\rho(w_L) < \rho(w_H)$. By continuity of $\rho(\cdot)$, there exists $\epsilon > 0$, such that the sets $\{|w - w_L| < 2\epsilon\}$ and $\{|w - w_H| < 2\epsilon\}$ are disjoint. In particular,

$$\sup_{|w-w_L|<2\epsilon} \rho(w) < \inf_{|w-w_H|<2\epsilon} \rho(w). \tag{9}$$

Now let g be a probability premium gamble which pays off $+\epsilon$ with probability p and $-\epsilon$ with probability $1-p$, where $1/2 < p < 1$. It can be computed that $\alpha(g) = \log(p/(1-p))/\epsilon$. As p varies in the range $(1/2, 1)$, the attractiveness of g takes all values in $(0, +\infty)$. Fix p such that $\alpha(g)$ is sandwiched between the left and right sides of equation (9). Then,

$$\sup_{w \in D(w_L, g)} \rho(w) < \alpha(g) < \inf_{w \in D(w_H, g)} \rho(w). \tag{10}$$

By statement 2 of Proposition 1, g is accepted by u at wealth level w_L but rejected at w_H.

The proof for the more general interval case is almost exactly the same and thus omitted. □

It is clear that the above proposition could also be phrased in terms of rejections instead of acceptions. Instead of doing this, we give the following characterization for IARA utilities, which have nondecreasing absolute risk aversion functions. The characterization on intervals is probably more useful in this case as some economists have reservations for IARA utilities. Again, the proposition below could be phrased in terms of rejections.

Proposition 4 *A utility function is in IARA if and only if any gambles accepted at a given wealth level will be accepted at all lower wealth levels. More generally, assuming $w_l < w_r$, a utility's absolute risk aversion function is nondecreasing on $[w_l, w_r]$ if and only if any gamble g accepted at a wealth level w_H will be accepted at any lower wealth level w_L so long as $D(w_L, g) \in [w_l, w_r]$ and $D(w_H, g) \in [w_l, w_r]$.*

Proof: Mimic the proof for Proposition 3 by switching the roles of w_L and w_H. □

2.3 Attractiveness and essentially monotonic absolute risk aversion

Monotone risk aversion has long been a much-studied research topic in economic theory. For some recent development, see Chateauneuf et all (2005) and Nielsen (2005).

We now slightly generalize IARA and DARA utilities. We say that a utility has *essentially nondecreasing absolute risk aversion* if there exists $\widehat{w}$ such that $\sup_{w<\widehat{w}} \rho(w) \leq \inf_{w>\widehat{w}} \rho(w)$, and $\rho(w)$ is nondecreasing on $(\widehat{w}, +\infty)$. We say that a utility has *essentially nonincreasing absolute risk aversion* if there exists $\widehat{w}$ such that $\inf_{w<\widehat{w}} \rho(w) \geq \sup_{w>\widehat{w}} \rho(w)$, and $\rho(w)$ is nonincreasing on $(-\infty, \widehat{w})$. We say that a utility has *essentially monotonic absolute risk aversion* if it has either essentially nondecreasing absolute risk aversion or essentially nonincreasing absolute risk aversion. Essentially monotonic absolute risk aversion utilities include IARA and DARA as special cases. The generalization is that in essentially monotonic absolute risk aversion utilities, we do not require that the absolute risk aversion function is monotonic on the half real line where its values are small.

For a CARA utility agent, if he accepts a gamble at any wealth, then he also accepts any gamble with higher attractiveness at any wealth. The following proposition generalizes this result to all utilities with essentially monotone risk aversions. It can be considered as another characterization theorem for Aumann and Serrano's risk index in place of the duality axiom.

Proposition 5 *If an agent with essentially monotonic absolute risk aversion accepts a bounded additive gamble g at any wealth, then $\alpha(g) \geq \sup_{w\in\mathbb{R}} \rho(w)$ and he also accepts any gamble h with $\alpha(h) \geq \alpha(g)$ at any wealth. Conversely, if any essentially monotonic risk averse agent who accepts a gamble g at any wealth also accepts h at any wealth, then $\alpha(h) \geq \alpha(g)$.*

Proof: We first prove the first statement. Let w_m and w'_n be two sequences on $\mathbb{R}$ such that $\lim_{m\to\infty} w_m = +\infty$ and $\lim_{n\to\infty} w'_n = -\infty$. The reason we take two sequences is that this allows us to consider the essentially nondecreasing absolute risk aversion and essentially nonincreasing absolute risk aversion cases together. If ρ is essentially nonincreasing, actually we only need to take the sequence w'_n, and vice versa. For any m and n, since the agent accepts g at w_m and w'_n, by Proposition 1, there exist $\widetilde{w}_m \in D(w_m, g)$ and $\widetilde{w}'_n \in D(w'_n, g)$ such that $\alpha(g) > \rho(\widetilde{w}_m)$ and $\alpha(g) > \rho(\widetilde{w}'_n)$. Since g is bounded, we must have $\lim_{m\to\infty} \widetilde{w}_m = +\infty$ and $\lim_{n\to\infty} \widetilde{w}'_n = -\infty$. Thus $\alpha(g) \geq \sup_{w\in\mathbb{R}} \rho(w)$ by the essential monotonicity of the risk aversion function. The fact that g is accepted means that we either have $\alpha(g) > \sup_{w\in\mathbb{R}} \rho(w)$, or $\alpha(g) = \sup_{w\in\mathbb{R}} \rho(w)$ with $\alpha(g) > \rho(\widehat{w})$ for some $\widehat{w} \in \mathbb{R}$. In either cases, by Proposition 1, the agent accepts h at any wealth. For the converse, suppose $\alpha(h) < \alpha(g)$. Then a CARA utility agent with parameter $(\alpha(h) + \alpha(g))/2$ accepts g at any wealth, but rejects h at any wealth. Contradiction. □

We have required that g is bounded in the first statement of the above proposition. When g is unbounded, the above proof does not work since $D(w_m, g)$ and $D(w'_n, g)$ could be the whole real line. Thus, we are not guaranteed that $\lim_{m\to\infty} \widetilde{w}_m = +\infty$ or $\lim_{n\to\infty} \widetilde{w}'_n = -\infty$. It is unclear whether the statement is still true if g. A closer examination of the proof above shows that we actually only need g to be bounded from above if ρ is essentially nonincreasing, and bounded from below if ρ is essentially nondecreasing.

The condition that the agent has essentially monotonic absolute risk aversion function is very important. Without this condition, when an agent accepts a bounded gamble g at any wealth, we do not necessarily have that $\alpha(g) \geq \sup_{w\in\mathbb{R}} \rho(w)$. It is also not necessarily true that the agent will accept a more attractive gamble at any wealth. The following counterexample shows these points. Let the utility of the agent be $u(w) = \log(1 + w)$ if $w \geq 0$ and $u(w) = w - w^2/2$ if $w < 0$. One readily checks that this is a well-defined utility function with absolute risk aversion function $\rho(w) = 1/(1 + |w|)$. Notice that $\sup_{w\in\mathbb{R}} \rho(w) = 1$ and $\rho(w)$ is not essentially monotonic. Now let g be an additive gamble which pays $+1$ with probability $p = 7/10$ and -1 with probability $3/10$. It can be easily shown either graphically or analytically that g is accepted at any wealth. However, the attractiveness of g is $\alpha(g) = \log(p/(1-p)) \doteq 0.847$, and thus we do not have $\alpha(g) \geq \sup_{w\in\mathbb{R}} \rho(w)$. Now let h be another probability premium gamble that pays off $1/2$ with probability q and $-1/2$ with probability $1-q$, where $q = 0.605$. Then $\alpha(h) \doteq 0.853 > \alpha(g)$. However, h is rejected at wealth 0 since $\mathbb{E}u(h) \doteq -0.0016 < 0$.

Proposition 5 also gives rise to the following open question. For what class of utility functions do we have the property that if the agent accepts a bounded gamble g at any wealth, then he accepts any gamble h at any wealth so long as $\alpha(h) \geq \alpha(g)$? Proposition 5 shows that utilities with essentially monotonic risk aversions belong to this class. Are there other utility functions having this property? Our intuition strongly suggests no for the last question, but so far we have not been able to prove it or disprove it.

2.4 Sums of additive gambles

Considering sums of gambles are useful in practice. For example, an investor's portfolio might consist of different positions, each considered a different gamble. It might be useful to be able to get some quick idea of the riskiness of the whole portfolio given the riskiness of the components and their dependence structure. From a financial engineering point of view, many new financial products can be thought of as the result of adding gambles (such as sector index funds) or splitting a gamble into many others (such as collateralized mortgage obligations).

In the following, we will always assume that $g+h$ is a well-defined gamble for two gambles g and h. Aumann and Serrano (2008) show that the riskiness (and thus the attractiveness) of $g+h$ always lies between those of g and h. In addition, even without independence, we still have subadditivity: $R(g+h) \leq R(g) + R(h)$. In this section, we examine sums of additive gambles more closely. In particular, we will generalize (5.8.1) of Aumann and Serrano to situations where we do not necessarily have independence. It turns out that in line with our intuition, if two gambles g and h are positively dependent (in senses presented rigorously in Propositions 7 and 8), then the risk index of the gamble $g+h$ cannot be smaller than the minimum of the risk indices of g and h. On the other hand, the risk index of the gamble $g+h$ cannot be larger than the maximum of the risk indices of g and h if we have negative dependence.

The following proposition is a straight-forward generalization of the results in Aumann and Serrano (2008) to arbitrary number of additive gambles.

Proposition 6 *Let g_i where $i = 1, \cdots, N$ be N additive gambles.*

1. *(Subadditivity) Let $\lambda_i > 0$ for $i = 1, \cdots, N$, then*
$$R\Big(\sum_{i=1}^{N} \lambda_i g_i\Big) \leq \sum_{i=1}^{N} \lambda_i R(g_i); \tag{11}$$
2. *If all gambles are independent, the riskiness of $\sum_{i=1}^{N} g_i$ lies between the minimum riskiness and the maximum riskiness. That is,*
$$\min_i R(g_i) \leq R\Big(\sum_{i=1}^{N} g_i\Big) \leq \max_i R(g_i); \tag{12}$$

Proof: Statement 1 in the special two gambles case has been proven using the convexity of the exponential function in Aumann and Serrano (2008) by Sergiu Hart. The general statement follows from induction and the homogeneity of the risk index. Below we give another proof based on generalized Hölder's inequality (see, for example, Finner 1992, or Kuptsov 2001). For any $k = 1, \cdots, N$, let $p_k = \sum_{i=1}^N \lambda_i R(g_i)/(\lambda_k R(g_k)) > 1$. Then $\sum_{k=1}^N 1/p_k = 1$. We have

$$\begin{aligned} \mathbb{E} \exp\Big(-\frac{\sum_{k=1}^N \lambda_k g_k}{\sum_{i=1}^N \lambda_i R(g_i)}\Big) &= \mathbb{E} \prod_{k=1}^N \exp\Big(-\frac{\lambda_k g_k}{\sum_{i=1}^N \lambda_i R(g_i)}\Big) \\ &= \Big\|\prod_{k=1}^N \exp\Big(-\frac{\lambda_k g_k}{\sum_{i=1}^N \lambda_i R(g_i)}\Big)\Big\|_1 \\ &\le \prod_{k=1}^N \Big\|\exp\Big(-\frac{\lambda_k g_k}{\sum_{i=1}^N \lambda_i R(g_i)}\Big)\Big\|_{p_k} \\ &= \prod_{k=1}^N \Big(\mathbb{E} e^{-g_k/R(g_k)}\Big)^{1/p_k} = 1. \end{aligned} \tag{13}$$

This proves the subadditivity. In particular, the equality obtains if and only if all the g_i's are multiples of each other. Statement 2 follows from (5.8.1) in Aumann and Serrano (2008) and induction. □

The independence assumption in the second statement is quite strong for actual applications. For example, the profit/loss of a call option (viewed as a gamble) is positively correlated with that of a digital call option, and negatively correlated with that of a put option. The following proposition gives some partial results when we do not have independence. Notice that equation (12) in the independence case follows immediately from the more general proposition below.

Proposition 7 *We have the following statements for sums of additive gambles:*

1. *Suppose there exists a random variable Z such that g_1 and g_2 are both nonincreasing functions (or both nondecreasing) in Z, then $R(g_1 + g_2) \ge \min(R(g_1), R(g_2))$. More generally, suppose there exist $N + 1$ independent random variables $\widetilde{g}_i$ $(i = 1, \cdots, N)$ and Z, such that $g_i - \widetilde{g}_i$ are all nonincreasing functions (or all nondecreasing) in Z, then $R(\sum_{i=1}^N g_i) \ge \min_i R(g_i)$.*
2. *Suppose there exists a random variable Z such that g_1 is nonincreasing in Z and g_2 is nondecreasing in Z (or vice versa), then $R(g_1 + g_2) \le \max(R(g_1), R(g_2))$. More generally, suppose there exists three independent random variables $\widetilde{g}_1$, $\widetilde{g}_2$ and Z, such that $g_1 - \widetilde{g}_1$ is a nonincreasing function in Z while $g_2 - \widetilde{g}_2$ is nondecreasing in Z (or vice versa), then $R(g_1 + g_2) \le \max(R(g_1), R(g_2))$.*

Proof: The main ingredient for the proof is Čebyšev's algebraic inequality (see Mitrinović, Pečarić, and Fink 1993, or Theorem 236 in Hardy, Littlewood and Pólya 1934), which was used by Merton in his development of portfolio selection theory (p. 25, Merton 1990). It states that if f_1 and f_2 are two random variables both nonincreasing (or nondecreasing) functions in Z, then $\mathrm{cov}(f_1, f_2) \geq 0$, and $\mathrm{cov}(f_1, f_2) \leq 0$ if one is nonincreasing and the other nondecreasing.

For statement 1, we prove the more general conclusion. Let $\beta > 0$, then by independence,

$$\mathbb{E}e^{-\beta\sum_{i=1}^N g_i} = \mathbb{E}\prod_{i=1}^N e^{-\beta\widetilde{g}_i}e^{-\beta(g_i-\widetilde{g}_i)} = \prod_{i=1}^N \mathbb{E}e^{-\beta\widetilde{g}_i}\cdot\mathbb{E}\prod_{k=1}^N e^{-\beta(g_k-\widetilde{g}_k)}. \tag{14}$$

The product of two positive nonincreasing functions is still nonincreasing. The same is true for nondecreasing functions. Thus, by repeated use of Čebyšev's algebraic inequality, we have

$$\begin{aligned}\mathbb{E}\prod_{k=1}^N e^{-\beta(g_k-\widetilde{g}_k)} &\geq \mathbb{E}e^{-\beta(g_N-\widetilde{g}_N)}\cdot\mathbb{E}\prod_{k=1}^{N-1} e^{-\beta(g_k-\widetilde{g}_k)} \\ &\geq \cdots \geq \prod_{k=1}^N \mathbb{E}e^{-\beta(g_k-\widetilde{g}_k)}. \end{aligned}\tag{15}$$

Putting the above two equations together, we have by independence again

$$\mathbb{E}e^{-\beta\sum_{i=1}^N g_i} \geq \prod_{i=1}^N \mathbb{E}e^{-\beta\widetilde{g}_i}\cdot\prod_{k=1}^N \mathbb{E}e^{-\beta(g_k-\widetilde{g}_k)} = \prod_{i=1}^N \mathbb{E}e^{-\beta g_i}. \tag{16}$$

Now let $\beta = \max_i \alpha(g_i)$, then $\mathbb{E}e^{-\beta\sum_{i=1}^N g_i} \geq 1$ since $\mathbb{E}e^{-\beta g_i} \geq 1$ for all $i = 1, \cdots, N$. Thus,

$$\beta \geq \alpha\Big(\sum_{i=1}^N g_i\Big), \tag{17}$$

or equivalently, $R(\sum_{i=1}^N g_i) \geq \min_i R(g_i)$.

The proof for statement 2 is similar and thus omitted. Notice we only consider two gambles in statement 2, as it does not have a natural generalization to N gambles. □

An interesting application of the above proposition is the following. Let g_1 and g_2 be multivariate normally distributed gambles with positive means μ_1 and μ_2, variances σ_1^2 and σ_2^2 and correlation coefficient ϱ. We already know that when $\varrho = 0$, $\min(R(g_1), R(g_2)) \leq R(g_1 + g_2) \leq \max(R(g_1), R(g_2))$ by Proposition 6. Proposition 7 above allows us to draw conclusions when $\varrho \neq 0$. Through a Gram-Schmidt orthogonalization, we see that when $\varrho > 0$, we have $\min(R(g_1), R(g_2)) \leq R(g_1 + g_2) \leq R(g_1) + R(g_2)$. When $\varrho < 0$, we have $R(g_1 + g_2) \leq \max(R(g_1), R(g_2))$. With some elementary but interesting algebra, these statements can be verified explicitly since we have $R(g_i) = \sigma_i^2/\mu_i$ for $i = 1, 2$, and

$$R(g_1 + g_2) = \frac{\sigma_1^2 + \sigma_2^2 + 2\varrho\sigma_1\sigma_2}{\mu_1 + \mu_2}. \tag{18}$$

When dealing with the sum of two gambles, Proposition 7 can be further generalized. Two random variables X and Y are said to be *positively quadrant dependent* (Lehmann 1966) if for any x and y we have

$$\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\ \mathbb{P}(Y \leq y). \tag{19}$$

We say X and Y are negatively quadrant dependent if the above equation reverses sign. Intuitively, X and Y are positively quadrant dependent if the probability that they are simultaneously small (or simultaneously large) is at least as great as it would be were they independent. In Proposition 7, the dependent parts of g_1 and g_2, namely $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$, are assumed to be concordant (or in another terminology, similarly ordered). The positive quadrant dependence is a weaker notion than concordance. Concordance implies positive quadrant dependence but the reverse is not true. The proposition below shows that we can replace the concordance with positive quadrant dependence.

Proposition 8 *We have the following statements for sums of additive gambles:*

1. *Suppose g_1 and g_2 are positively quadrant dependent, then $R(g_1 + g_2) \geq \min(R(g_1), R(g_2))$. More generally, suppose there exist independent random variables $\widetilde{g}_1$, $\widetilde{g}_2$, such that $\widetilde{g}_1$ and $\widetilde{g}_2$ are both independent with $g_1 - \widetilde{g}_1 + g_2 - \widetilde{g}_2$. If $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$ are positively quadrant dependent, then $R(g_1 + g_2) \geq \min(R(g_1), R(g_2))$.*
2. *Suppose g_1 and g_2 are negatively quadrant dependent, then $R(g_1 + g_2) \leq \max(R(g_1), R(g_2))$. More generally, suppose there exist independent random variables $\widetilde{g}_1$, $\widetilde{g}_2$, such that $\widetilde{g}_1$ and $\widetilde{g}_2$ are both independent with $g_1 - \widetilde{g}_1 + g_2 - \widetilde{g}_2$. If $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$ are negatively quadrant dependent, then $R(g_1 + g_2) \leq \max(R(g_1), R(g_2))$.*

Proof: A very useful characterization for positive quadrant dependence is as follows. Two random variables X and Y are positively quadrant dependent if and only if $\mathrm{cov}(s(X), t(Y)) \geq 0$ for all nondecreasing functions of s and t such that the integrals in the covariance are well-defined. Notice that $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$ are positively quadrant dependent, then so are $e^{-\beta(g_1 - \widetilde{g}_1)}$ and $e^{-\beta(g_2 - \widetilde{g}_2)}$. The proof is almost exactly the same as that of Proposition 7 for $N = 2$. Instead of relying on Čebyšev's algebraic inequality, we use the characterization result for positive quadrant dependence. □

As one example of applying the above proposition, let S_1, S_2 and S_3 be three independent random variables standing for three future financial quantities. Let $g_1 \equiv \max(S_1 + S_2 - K_1, 0) - p_1$ be the profit or loss of a spread option with strike price K_1 and price p_1. Similarly for $g_2 \equiv \max(S_1+S_3-K_2, 0)-p_2$. Assume that the strike prices and option prices are such that g_1 and g_2 are gambles. Then by Example 1.(iv) in Lehmann (1966), g and h are positively quadrant dependent. Proposition 8 then tells us that $R(g_1+g_2) \geq \min(R(g_1), R(g_2))$.

2.5 Extending the Domain of Gambles

The original definition of additive gambles in Aumann and Serrano (2008) considers gambles with a discrete distribution, positive mean, and a positive probability of loss. We have extended the domain to allow for continuous or mixed distributions and infinite support. However, we still only consider positive probability of loss. On p. 821 of Aumann and Serrano (2008), the authors suggest one simple way of extending the domain to gambles where only positive gains are realized by setting their risk index to 0. This is a very sensible choice if we focus on a single gamble. However, it does not allow us to compare the riskiness across different gambles. Consider the following two gambles of coin tossing for example. Gamble g gives 1 dollar if head and 3 dollars if tail. Gamble h gives 2 dollars if head and 4 dollars if tail. The simple extension will say both gambles have zero risk index, but is silent on which one is riskier. A look at the definition of the risk index in equation (1) shows that there is no nontrivial solution for $R(g)$ if g only has positive gains. Apparently, some modification is needed in order to extend the concept of risk index to gambles with positive gains only.

We define a (generalized) gamble to be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A normal reaction to the two gambles above is that if they are offered at the same price, then gamble h should be more attractive. Therefore, the approach we take is to associate the gamble and risk index with an upfront price P. That is, we assume that there is a price $P \in \mathbb{R}$ associated with each gamble g, such that $g-P$ is an additive gamble in the sense in earlier subsections, and we define the risk index $R(g,P)$ of the gamble g with price P to be the unique positive solution (if exists) of

$$\mathbb{E}e^{-(g-P)/R(g,P)} = 1. \tag{20}$$

Although we associate P with g, P does not have to be internally determined from g. It could be given exogenously. Of course, one interesting application which is very relevant in practice is that P is determined from g by risk-neutral pricing or utility indifference pricing. Notice that we also allow for gambles with losses only and negative prices (an upfront compensation). The only requirement we impose is that $g-P$ is an additive gamble such that $\mathbb{E}g > P$ and $\mathbb{P}(g < P) > 0$. That is, on average there is a positive gain so that there is some incentive to take the gamble, but there is also a positive probability of losing money.

Apparently, the additive gambles we have considered previously are special cases of the gambles here with a zero price. By focusing on the shifted additive gamble $g - P$, we can parallel translate the previous results to the generalized gambles. For example, the duality in Aumann and Serrano (2008) would generalize to the following.

Duality for generalized gambles: Let $R(g, P_g) > R(h, P_h)$. If $i \triangleright j$ and i accepts gamble g with price P_g at wealth w, then j accepts h with price P_h at wealth w.

These generalizations, while somewhat trivial, extend all previous results for additive gambles to generalized gambles which are more relevant in real-life applications. The introduction of a price P opens up a nontrivial new direction to study the behavior of the risk index $R(g, P)$. As a first cut, we list some of the properties of $R(g, P)$ in the following proposition. The first statement shows that $R(g, P)$ is strictly increasing and convex in P.[2] The last two statements show that $R(g, P)$ is homogeneous of degree one and subadditive in g if the pricing functional is linear.

Proposition 9 *For any generalized gamble g, we define $D(g) = \{P \in \mathbb{R} : R(g, P)$ exists$\}$. We assume that g is well-behaved so that $D(g)$ is not empty.*

1. *The set $D(g)$ is connected. On the set $D(g)$, $R(g, P)$ is strictly increasing and strictly convex in P.*
2. *Let $P \in D(g)$. Then $\lambda P \in D(\lambda g)$ and we have $R(\lambda g, \lambda P) = \lambda R(g, P)$.*
3. *Let $P_g \in D(g)$ and $P_h \in D(h)$. Then $P_g + P_h \in D(g + h)$. In addition, we have*

$$R(g + h, P_g + P_h) \leq R(g, P_g) + R(h, P_h). \tag{21}$$

If g and h are independent, then $R(g + h, P_g + P_h)$ lies between $R(g, P_g)$ and $R(h, P_h)$.

Proof: The second and third statements follow easily from considering the shifted additive gambles $g - P_g$ and $h - P_h$, and using the properties of the additive gambles. Therefore, we focus on the first statement. To show that $D(g)$ is connected, let $P_1 \in D(g)$ and $P_2 \in D(g)$ with $P_1 < P_2$. For any $P \in (P_1, P_2)$, we have

$$\begin{aligned} \mathbb{E}\, e^{-(g-P)/R(g,P_1)} &= \mathbb{E}\, e^{-(g-P_1)/R(g,P_1)+(P-P_1)/R(g,P_1)} \\ &= e^{(P-P_1)/R(g,P_1)} > 1. \end{aligned} \tag{22}$$

Similarly we have $\mathbb{E}\, e^{-(g-P)/R(g,P_2)} < 1$. Since the function $f(z) \equiv \mathbb{E}\, e^{-(g-P)/z}$ is continuous in z, there exists z_0 between $R(g, P_1)$ and $R(g, P_2)$ such that $f(z_0) = 1$. Therefore, $P \in D(g)$ and $D(g)$ is connected. Notice that the equation above also shows that $R(g, P)$ is strictly increasing in P. To show the

[2] In an earlier version of this paper, we prove the convexity of $R(g, P)$ where g is the gamble in St. Petersburg's paradox by using the implicit function theorem coupled with Cauchy-Schwartz inequality. The current proof is much simpler. This idea of using the convexity of the exponential function was inspired by the proof of (5.8.2) in Aumann and Serrano (2008) which is due to Sergiu Hart.

convexity of $R(g,P)$, let $\overline{P} = (P_1 + P_2)/2$ and $\overline{R} = (R_1 + R_2)/2$, where we write $R_1 = R(g,P_1)$ and $R_2 = R(g,P_2)$ for notational ease. We need to show that $\overline{R} > R(\overline{P})$. This is equivalent to showing that $\mathbb{E}\, e^{-(g-\overline{P})/\overline{R}} < 1$ (see Figure 1). It is easy to check that for $\lambda = R_1/(R_1 + R_2)$, we have

$$(g - \overline{P})/\overline{R} = \lambda(g - P_1)/R_1 + (1 - \lambda)(g - P_2)/R_2. \tag{23}$$

The fact that $\mathbb{E}\, e^{-(g-\overline{P})/\overline{R}} < 1$ now follows the convexity of e^{-z} and Jensen's inequality. □

We make two remarks. First, a sufficient and necessary condition for the non-emptiness of $D(g)$ is that there exists $\lambda > 0$ such that $\mathbb{E}e^{-\lambda g} < \infty$. Second, in incomplete markets with transaction costs, different borrowing and lending rates, or non-hedgable default risk, the pricing functional P_g is often subadditive in g. That is, $P_{g+h} \le P_g + P_h$. The subadditivity of $R(g,P_g)$ in g in the above proposition still holds when P_g is subadditive since

$$R(g + h, P_{g+h}) \le R(g + h, P_g + P_h) \le R(g, P_g) + R(h, P_h). \tag{24}$$

Here in the first inequality we have used the monotonicity of $R(g,P)$ and the subadditivity of P_g, and in the second inequality we have used the subadditivity of $R(g,P_g)$ when the pricing functional is linear.

3 Multiplicative gambles

3.1 Multiplicative gambles and CRRA utilities

So far we have considered additive gambles, that is, an agent with wealth w will have final wealth $w+g$ if he takes an additive gamble g. In this subsection, we will take a look at multiplicative gambles. We will assume that wealth is always positive and utilities are defined on the positive real line. Operationally, an agent with wealth w accepting a multiplicative gamble ϕ will have random final wealth $w\phi$. Mathematically, we define a multiplicative gamble ϕ to be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with strictly positive values such that

$$1 < \mathbb{E}\phi < +\infty \quad \text{and} \quad \mathbb{P}[\phi < 1] > 0. \tag{25}$$

We also require that $\mathbb{E}\log\phi > -\infty$. The condition $\mathbb{E}\phi > 1$ requires that the gamble ϕ is actuarially attractive. An agent accepts a multiplicative gamble ϕ if and only if $\mathbb{E}u(w\phi) > u(w)$. Similarly for indifference and rejection. We will always use lowercase Greek letters to denote multiplicative gambles to differentiate from additive gambles.

Recall that for any well-defined additive gamble g, we require $\mathbb{E}g > 0$. By Jensen's inequality, $\log\mathbb{E}e^g \ge \mathbb{E}g > 0$. Thus, $\phi \equiv e^g$ satisfies $\mathbb{E}\phi > 1$ and is a well-defined multiplicative gamble. However, not all multiplicative gambles come from this way. That is, there are multiplicative gambles whose logarithms are not additive gambles. In the jargon of functional analysis, if let $\mathcal{G}_a$ and $\mathcal{G}_m$

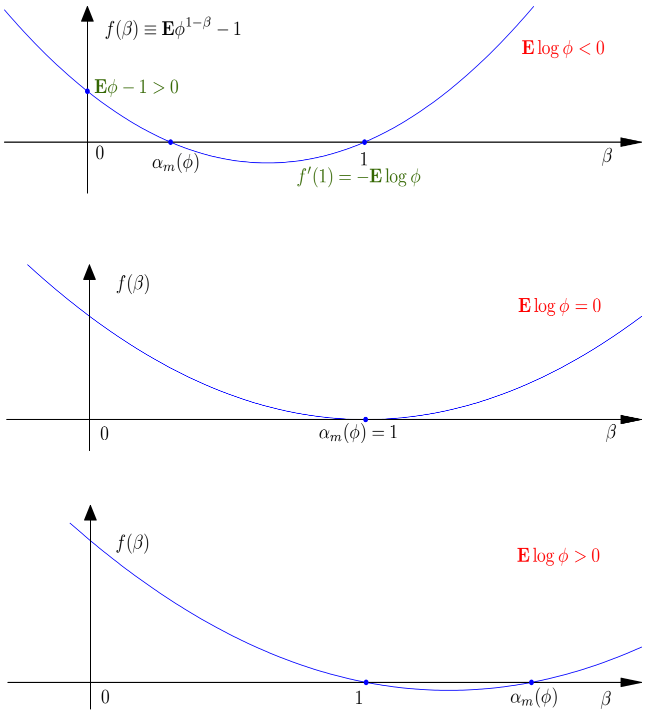

Fig. 3 The defining equation for a multiplicative gamble's attractiveness $\alpha_m(\phi)$. The function $f(\beta) \equiv \mathbb{E}\phi^{1-\beta} - 1$ for a fixed multiplicative gamble ϕ is drawn. For gambles with $1 < \mathbb{E}\phi < +\infty$, $f(\beta)$ is convex in β and has at most two roots. The sign of $\mathbb{E}\log\phi$ is always the same as that of $\alpha_m(\phi) - 1$.

be the sets of additive and multiplicative gambles, respectively, and let the mapping $\Upsilon : \mathcal{G}_a \to \mathcal{G}_m$ be $\Upsilon(g) = e^g$, then Υ is injective, but not surjective.

For any multiplicative gamble ϕ, if there is a solution $\beta \neq 1$ to the following equation

$$\mathbb{E}\phi^{1-\beta} = 1, \tag{26}$$

we define the (multiplicative) attractiveness index $\alpha_m(\phi) = \beta$. If $\beta = 1$ is the only solution to the above equation, we define $\alpha_m(\phi) = 1$. The risk index $R_m(\phi)$ of a multiplicative gamble is defined to be $R_m(\phi) \equiv 1/\alpha_m(\phi)$. It turns out that for any multiplicative gamble, the attractiveness index is always well-defined. Figure 3 depicts schematically the attractiveness index for a multiplicative gamble. Unlike additive gambles, it is useful to consider three different cases in accordance to the signs of $\mathbb{E}\log\phi$, as we do in the following proposition.

Proposition 10 *For any multiplicative gamble ϕ, its attractiveness index $\alpha_m(\phi)$ is always well-defined. Furthermore, we have $0 < \alpha_m(\phi) < 1$ if and only if $\mathbb{E}\log\phi < 0$, $\alpha_m(\phi) = 1$ if and only if $\mathbb{E}\log\phi = 0$, and $1 < \alpha_m(\phi) < +\infty$ if and only if $\mathbb{E}\log\phi > 0$. In the last case where $\mathbb{E}\log\phi > 0$, $\log\phi$ is a well-defined additive gamble, and we have $\alpha_m(\phi) = 1 + \alpha(\log\phi)$.*

Proof: For any $\beta \in \mathbb{R}$, define $f(\beta) \equiv E\phi^{1-\beta} - 1$. Notice that $f(0) = \mathbb{E}\phi - 1 > 0$, $f(1) = 0$, and $\lim_{\beta\to+\infty} f(\beta) = +\infty$. Furthermore, $f''(\beta) = \mathbb{E}\phi^{1-\beta}(\log\phi)^2 > 0$ so ϕ is globally strictly convex. The derivative of $f(\beta)$ at $\beta = 1$ is given by $f'(1) = -\mathbb{E}\log\phi$. It is easy to see that the sign of $f'(1)$ determines the sign of $\alpha_m(\phi) - 1$, as Figure 3 shows. Finally, when $\mathbb{E}\log\phi > 0$, it is easy to see that $\alpha_m(\phi) = 1 + \alpha(\log\phi)$ since $f(\beta) = \mathbb{E}e^{-(\beta-1)\log\phi}$. □

A CRRA utility is a utility with constant relative risk aversion function. The relative risk aversion function $r(w)$ of a utility u is defined to be $r(w) = -wu''(w)/u'(w)$. A CRRA utility with parameter $\gamma > 0$ is given by

$$u_\gamma(w) = \frac{w^{1-\gamma} - 1}{1 - \gamma} \tag{27}$$

if $\gamma \neq 1$, and $u_\gamma(w) = \log w$ if $\gamma = 1$. Notice that for $u_\gamma(w)$, $r(w) \equiv \gamma$ is a constant. Aumann and Serrano (2008) give $R(g)$ an operational meaning by showing that if we let $w_\gamma(g)$ denote the cutoff wealth of a CRRA agent with parameter γ at which he is indifferent towards an additive gamble g, then $\lim_{\gamma\to+\infty} w_\gamma(g)/\gamma = R(g)$. In this subsection, we explore the relations between multiplicative gambles and CRRA utilities. Later we will give a duality result for multiplicative gambles. This in turn allows us to give an alternative characterization for utilities with nonincreasing or nondecreasing relative risk aversion functions.

We have the following proposition with respect to multiplicative gambles and CRRA utilities. It states that just as $\alpha(g)$ is the critical absolute risk aversion parameter for a CARA agent to accept an additive gamble g, $\alpha_m(\phi)$ is the critical relative risk aversion parameter for a CRRA agent to accept a multiplicative gamble ϕ.

Proposition 11 *A CRRA agent with parameter $\gamma > 0$ accepts a multiplicative gamble ϕ at any wealth $w > 0$ if and only if $\gamma < \alpha_m(\phi)$.*

Proof: We first prove the "only if" part. If $\gamma = 1$, a CRRA agent (having a log utility) accepts ϕ at any wealth only if $\mathbb{E}\log\phi > 0$. By Proposition 10 or Figure 3, this happens only if $\alpha_m(\phi) > 1 = \gamma$. If $0 < \gamma < 1$, he accepts ϕ at any wealth only if $\mathbb{E}\phi^{1-\gamma} > 1$. There are three possible cases to consider: $\mathbb{E}\log\phi > 0$, $\mathbb{E}\log\phi = 0$, and $\mathbb{E}\log\phi < 0$. By Proposition 10 or Figure 3 again, in all three cases, this happens only if $\gamma < \alpha_m(\phi)$. Finally, if $\gamma > 1$, he accepts ϕ at any wealth only if $\mathbb{E}\phi^{1-\gamma} < 1$. Again this happens only if $\gamma < \alpha_m(\phi)$.

The proof for the "if" part is similar and thus omitted. □

3.2 Duality for multiplicative gambles

Aumann and Serrano's theory on additive gambles can be translated to multiplicative gambles without much modification, as the following propositions show. For any multiplicative gamble ϕ, we define its essential range $D_m(w,\phi)$ to be the closed interval $[w\operatorname{ess\,inf}\phi, w\operatorname{ess\,sup}\phi]$.

Proposition 12 *We have the following statements with respect to acception and rejection of multiplicative gambles:*

1. *If i accepts a multiplicative gamble ϕ at wealth w_i, then there exists $w \in D_m(w_i,\phi)$ such that $\alpha_m(\phi) > r_i(w)$. In particular, we must have $\alpha_m(\phi) > \inf_{w\in D_m(w_i,\phi)} r_i(w)$. On the other hand, if i rejects ϕ at wealth w_i, then there exists $w \in D_m(w_i,\phi)$ such that $\alpha_m(\phi) < r_i(w)$. In particular, we must have $\alpha_m(\phi) < \sup_{w\in D_m(w_i,\phi)} r_i(w)$.*
2. *If $\alpha_m(\phi) \geq r_i(w)$ for any $w \in D_m(w_i,\phi)$ with strict inequality for at least one such w, then i accepts a multiplicative gamble ϕ at w_i. On the other hand, if $\alpha_m(\phi) \leq r_i(w)$ for any $w \in D_m(w_i,\phi)$ with strict inequality for at least one such w, then i rejects ϕ at w_i.*

Proof: The proof is very similar to that of Proposition 1. Thus we only give proof for the first sentence in statement 1. Suppose $r_i(w) \geq \alpha_m(\phi)$ for all $w \in D_m(w_i,\phi)$. Let $u_{\alpha_m(\phi)}(w)$ be the CRRA utility with parameter $\alpha_m(\phi)$. By Proposition 11, $\mathbb{E}u_{\alpha_m(\phi)}(w_i\phi) = u_{\alpha_m(\phi)}(w_i)$. Let

$$\widetilde{u}(w) \equiv \frac{u'_{\alpha_m(\phi)}(w_i)}{u'(w_i)}\big(u(w) - u(w_i)\big) + u_{\alpha_m(\phi)}(w_i). \tag{28}$$

Then $\widetilde{u}(w)$ is equivalent to $u(w)$ since it is a positive affine transformation of $u(w)$. Furthermore, $\widetilde{u}(w_i) = u_{\alpha_m(\phi)}(w_i)$, and $\widetilde{u}'(w_i) = u'_{\alpha_m(\phi)}(w_i)$. For any $w \in D_m(w_i,\phi)$, $\rho_i(w) \geq \rho_{\alpha_m(\phi)}(w)$ where $\rho_{\alpha_m(\phi)}(w)$ is the absolute risk aversion function of $u_{\alpha_m(\phi)}(w)$. Thus, $\widetilde{u}$ and $u_{\alpha_m(\phi)}$ are "nested" at w_i: $\widetilde{u}(w) \leq u_{\alpha_m(\phi)}(w)$ for any $w \in D_m(w_i,\phi)$ except that we have equality at $w = w_i$. Therefore, $\mathbb{E}\widetilde{u}(w_i\phi) \leq \mathbb{E}u_{\alpha_m(\phi)}(w_i\phi) = u_{\alpha_m(\phi)}(w_i) = \widetilde{u}(w_i)$. This contradicts the assumption that i accepts ϕ at wealth w_i. □

The above proposition immediately gives the following duality result for multiplicative gambles, which is completely parallel to Proposition 2.

Proposition 13 (Duality for multiplicative gambles) *Let ϕ and ψ be two multiplicative gambles defined on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $R_m(\psi) \leq R_m(\phi) < \infty$. Suppose*

$$\inf_{w\in D_m(w_i,\phi)} r_i(w) \geq \sup_{w\in D_m(w_j,\psi)} r_j(w). \tag{29}$$

Then if i accepts gamble ϕ at wealth w_i, j accepts ψ at wealth w_j. Furthermore, if j rejects ψ at wealth j, then i rejects ϕ at w_i.

Proof: The proof is very similar to that of Proposition 2 so we omit the details. For example, the first conclusion can be seen from the following chain of inequalities:

$$\alpha_m(\psi) \geq \alpha_m(\phi) > \inf_{w \in D_m(w_i,\phi)} r_i(w) \geq \sup_{w \in D_m(w_j,\psi)} r_j(w), \tag{30}$$

where the strict inequality in the middle is by Proposition 12. □

3.3 Alternative characterization of DRRA and IRRA utilities

A utility defined on the positive real line is said to be in the DRRA class if its relative risk aversion function is nonincreasing. It is said to be in the IRRA class if its relative risk aversion function is nondecreasing. Proposition 13 above allows us to derive an alternative characterization of DRRA and IRRA utilities using multiplicative gambles, exactly parallel to Propositions 3 and 4 for DARA and IARA utilities using additive gambles.

Proposition 14 *A utility function defined on $\mathbb{R}^+$ is in DRRA if and only if any multiplicative gambles accepted at a given wealth level will be accepted at all higher wealth levels. More generally, assuming $w_l < w_r$, the relative risk aversion function of a utility is nonincreasing on $[w_l, w_r]$ if and only if any multiplicative gamble ϕ accepted at a wealth level w_L will be accepted at any higher wealth level w_H so long as $D_m(w_L, \phi) \in [w_l, w_r]$ and $D_m(w_H, \phi) \in [w_l, w_r]$.*

Proof: If u is in DRRA, then Proposition 13 tells us that any multiplicative gamble accepted by u at a lower wealth will be accepted at any higher wealth.

For the converse, suppose u is not in DRRA so that there exist two wealth levels w_L and w_H with $w_L < w_H$ and $r(w_L) < r(w_H)$. We need to show that there exists a multiplicative gamble which is accepted at w_L but rejected at w_H. By continuity of $r(w)$, there exists $\epsilon > 0$, such that

$$\sup_{|\log(w/w_L)|<2\epsilon} r(w) < \inf_{|\log(w/w_H)|<2\epsilon} r(w). \tag{31}$$

Now let ϕ be a multiplicative gamble taking two values: $\phi = e^\epsilon$ with probability p, and $\phi = e^{-\epsilon}$ with probability $1-p$, where $p > 1/(1+e^\epsilon)$. The condition $p > 1/(1+e^\epsilon)$ is to guarantee that $\mathbb{E}\phi > 1$. It can be computed that $\alpha_m(\phi) = 1 + \log(p/(1-p))/\epsilon$. As p varies in the range $(1/(1+e^\epsilon), 1)$, $\alpha_m(\phi)$ takes all values in $(0, \infty)$. Pick p such that $\alpha_m(\phi)$ is sandwiched between the two sides of equation (31). Then, ϕ is accepted at wealth w_L but rejected at wealth w_H. □

Proposition 15 *A utility function defined on $\mathbb{R}^+$ is in IRRA if and only if any gambles accepted at a given wealth level will be accepted at all lower wealth levels. More generally, assuming $w_l < w_r$, the relative risk aversion function* *of a utility is nondecreasing on* $[w_l, w_r]$ *if and only if any multiplicative gamble* ϕ *accepted at a wealth level* w_H *will be accepted at any lower wealth level* w_L *so long as* $D_m(w_L, \phi) \in [w_l, w_r]$ *and* $D_m(w_H, \phi) \in [w_l, w_r]$.

Proof: Mimic the proof for Proposition 14 by switching the roles of w_L and w_H. □

3.4 Attractiveness and essentially monotonic relative risk aversion

We now slightly generalize IRRA and DRRA utilities. We say that a utility has *essentially nondecreasing relative risk aversion* if there exists $\widehat{w}$ such that $\sup_{w<\widehat{w}} r(w) \leq \inf_{w>\widehat{w}} r(w)$, and $r(w)$ is nondecreasing on $(\widehat{w}, +\infty)$. We say that a utility has *essentially nonincreasing relative risk aversion* if there exists $\widehat{w}$ such that $\inf_{w<\widehat{w}} r(w) \geq \sup_{w>\widehat{w}} r(w)$, and $r(w)$ is nonincreasing on $(-\infty, \widehat{w})$. We say that a utility has *essentially monotonic relative risk aversion* if it has either essentially nondecreasing relative risk aversion or essentially nonincreasing relative risk aversion. Essentially monotonic relative risk aversion utilities include IRRA and DRRA as special cases. Similar to the case for additive gambles, we have the following proposition.

Proposition 16 *If an agent with essentially monotonic relative risk aversion accepts a bounded multiplicative gamble* ϕ *at any wealth, then* $\alpha_m(\psi) \geq \sup_{w\in\mathbb{R}^+} r(w)$ *and he also accepts any multiplicative gamble* ψ *with* $\alpha_m(\psi) \geq \alpha_m(\phi)$. *Conversely, if any essentially monotonic relative risk averse agent who accepts a multiplicative gamble* ϕ *at any wealth also accepts multiplicative gamble* ψ *at any wealth, then* $\alpha_m(\psi) \geq \alpha_m(\phi)$.

Proof: The proof is very similar to that of Proposition 5 for additive gambles. Thus, we omit the details here. The main ingredient needed is Proposition 12. □

4 Conclusion

We study the risk index proposed in Aumann and Serrano (2008) in more detail. First, we use a strengthened duality result to derive an alternative characterization of utilities with nonincreasing or nondecreasing absolute risk aversions. Furthermore, by considering agents with essentially monotonic absolute risk aversions, we give a characterization theorem for Aumann and Serrano's risk index in place of the duality axiom. Second, we study in more detail sums of gambles that are not necessarily independent. Third, we extend the concept of risk index to essentially any random payoff by introducing a price. An interesting result is that the generalized risk index is always strictly increasing and strict convex with respect to the price. Finally, we translate the theory on the risk index for additive gambles to multiplicative gambles. Relative risk aversion functions for multiplicative gambles play the same role as absolute risk aversion functions for additive gambles.

Acknowledgements The author thanks the anonymous referee for the constructive comments that have improved the paper, especially on extending the domain of additive gambles.

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