An extension of the Sard-Smale Theorem to domains with an empty interior

Accinelli, Elvio and Covarrubias, Enrique

Universidad Autónoma de San Luis Potosí, Banco de México

22 May 2013

An extension of the Sard-Smale Theorem to domains with an empty interior

E. Accinelli* E. Covarrubias†

June 11, 2013

Abstract

A stumbling block in the modelling of competitive markets with commodity and price spaces of infinite dimensions, arises from having positive cones with an empty interior. This issue precludes the use of tools of differential analysis, ranging from the definition of a derivative, to the use of more sophisticated results needed to understand determinacy of equilibria and, more generally, the structure of the equilibrium set. To overcome these issues, this note extends the Preimage Theorem and the Sard-Smale Theorem to maps between spaces that may have an empty interior.

1 Introduction

Two closely related mathematical results, the Preimage Theorem and Sard's Theorem, are useful tools with many applications in economics, particularly within the theory of general economic equilibrium. The first of these, is stated as follows:

*Economics Faculty, Universidad Autónoma de San Luis Potosí, elvio.accinelli@eco.uaslp.mx

†Bank of Mexico, ecovarrubias@banxico.org.mx

Theorem. ***(Preimage Theorem)*** *If y is a regular value of the map $f : M \to N$ between differentiable manifolds M and N, then $f^{-1}(y)$ is a submanifold of M.*[1]

The Preimage Theorem is usually applied to show results of the following nature:

Consider the excess demand function $Z : \Omega \times S \to X$ of a pure exchange economy, where Ω is the set of parameters (e.g. initial endowments), S is the set of prices and X is the commodity space. Then, if 0 is a regular value of Z, we have that the equilibrium set, $Z^{-1}(0)$, is a manifold.

The set $\Gamma = Z^{-1}(0)$ is called the "equilibrium manifold" and the seminal paper of Balasko (1975) introduced this point of view of general equilibrium theory. The second result in this spirit is Sard's Theorem that states that almost all the values of a function are regular. Formally:

Theorem. ***(Sard's Theorem, 1942)*** *Let U be an open set of $\mathbb{R}^p$ and $f : U \to \mathbb{R}^q$ be a C^k map where $k > \max(p-q, 0)$. Then, the set of critical values in $\mathbb{R}^q$ has measure zero.*

Sard's Theorem also has many applications, usually to show results similar to this:

Consider the equilibrium manifold $\Gamma \subset \Omega \times S$ and the projection map $\pi : \Omega \times S \to \Omega$, restricted to Γ, given by $\pi(\omega, p) = \omega$. Then, the regular values are almost all of Ω. In other words, almost all equilibria are determinate.

Sard's Theorem turned out indeed to be the appropriate tool to study determinacy of equilibria since Debreu's (1970) seminal paper. These tools have been used in many other areas such as general equilibrium with incomplete financial markets where Chichilnisky and Heal (1996) have shown that

[1]Recall that for a map $f : M \to N$, a point x in M at which the derivative of f has rank less than n is called a *critical point* and its image a *critical value* of f. Other points y in N, that is, such that f has rank n at all points in $f^{-1}(y)$, are called *regular values* of f.

the equilibrium set is a manifold, while determinacy of equilibria was shown by Magill and Shafer (1990).

In spite of these general results and vast applications, many models of competitive markets have an infinite number of commodities which naturally lead to consumption and price spaces of infinite dimensions. At a first glance, it would seem appropriate to use Smale's generalisation of the Submanifold and Sard's Theorem to infinite dimensions, as follows:

Theorem. ***(Smale Theorem, 1965)*** *If $f : M \to V$ is a C^s Fredholm map between differentiable manifolds locally like Banach spaces with $s > \max(index f, 0)$, then*

1. *For almost all $y \in V$, $f^{-1}(y)$ is a submanifold of M;*
2. *The regular values of f are almost all of V.*

The statement and proof of Smale's Theorem is local, and requires for M and V to have a nonempty interior. However, there are many instances in economic modelling that require a domain with an empty interior. For example, many models of competitive markets use a consumption space in the positive cone of an ℓ_p or L_p space, for $1 \leq p \leq \infty$.[2] Unfortunately, the only spaces among L_p and ℓ_p space whose positive cone have a nonempty interior are L_∞ and ℓ_∞. To complicate things, prices are elements of the positive cone of the dual space of the commodity space.[3] Recall that the dual space of ℓ_p (L_p, resp.), $1 \leq p < \infty$, is the space ℓ_q (L_q, resp.) where

[2] Recall the following definitions. Let p be a real number $1 \leq p < \infty$. The space ℓ_p consists of all sequence of scalars $\{x_1, x_2, \dots\}$ for which $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$. The norm of an element $x = \{x_i\}$ in ℓ_p is defined as $\|x\|_p = (\sum_{i=1}^{\infty} \|x_i\|^p)^{1/p}$. The space ℓ_∞ consists of bounded sequences. The norm of an element $x = \{x_i\}$ in ℓ_∞ is defined as $\|x\|_\infty = \sup_i \|x_i\|$. The L_p spaces are defined analogously. For $p \geq 1$, the space $L[a, b]$ consists of those real-valued functions x on the interval $[a, b]$ for which $\|x(t)\|$ is Lebesgue integrable. The norm on this space is defined as $\|x\|_p = \left(\int_a^b \|x(t)\|^p\right)^{1/p}$. The space $L_\infty[a, b]$ consists of all Lebesgue measurable functions on $[a, b]$ which are bounded, except possible on a set of measure zero. The norm is defined by $\|x\|_\infty = \text{ess sup}\|x(t)\|$.

[3] If X is a normed linear vector space. The space of all bounded linear functionals on X is called the *normed dual* of X and is denoted by X^*. The norm of an element $f \in X^*$ is $\|f\| = \sup_{\|x\|=1} \|f(x)\|$.

$1/p + 1/q = 1$. In other words, even if the commodity space had a positive cone with a nonempty interior, the positive cone of the dual space -that is, the price space- will have an empty interior, and vice versa. The dual spaces of L_∞ and ℓ_∞ are subtler, but this problem still holds.[4] There are plenty of examples in different directions, but to name a few consider the following list:

- Duffie and Huang (1985) model financial markets through the space L_2;
- Bewley (1972) uses the space L_∞ to model infinite variations in any of the characteristics describing commodities. These Characteristics could be physical properties, location, the time of delivery, or the state of the world (in the probabilistic sense) at the time of delivery;
- The infinite horizon model which requires the set ℓ_∞ as modelled in Kehoe and Levine (1985) and Balasko (1997a,b,c);

The purpose of this note is precisely to extend Sard's and Smale's Theorems to maps between subsets of Banach manifolds which may have an empty interior. To this end, we will prove in the next sections the following result.

Theorem. ***(Main theorem)*** *Let $f : M \to V$ be a C^r star Fredholm map between star Banach manifolds, with $r > max(index\, f, 0)$. Suppose that M and V are connected and have a countable basis. Furthermore, suppose that f is locally proper and that it has at least one regular value. Then, the regular values of f are almost all of V.*

2 Analytical preliminaries

Let B be a Banach space, and let B_+ denote the positive cone of B which may have an empty interior. Notice that B_+ is a convex subset of B. The results of this paper can be generalized to any convex subset, not just the positive cone, but we restrict the analysis to this set because of an interest in economic applications.

[4] The dual space of $L\infty$ can be identified with bounded signed finitely additive measures that are absolutely continuous with respect to the measure. There are relatively consistent extensions of Zermelo-Franekel set theory in which the dual of ℓ_∞ is ℓ_1.

Definition 1. ***(α-admissible directions)*** *We say that $h \in B$ is an α-* admissible direction for $x \in B_+$ *if and only if there exist $\alpha > 0$ such that $x + \alpha\frac{h}{\|h\|} \in B_+$. The set of $\alpha-$admissible directions at x will be denoted by $\mathcal{A}_\alpha(x)$.*

Note that since B_+ is a convex subset of B then, if y and x are points in B_+, it must be that $h = (y - x)$ is α-admissible for x. To see this, consider $z = \alpha' y + (1 - \alpha')x$, $0 \leq \alpha' \leq 1$. Then $z \in B_+$ and $z = x + \alpha'(y - x)$, $\forall\, 0 \leq \alpha$. Let $\alpha = \frac{\alpha'}{\|h\|}$. Similarly, it follows that if h is α-admissible, it is also β-admissible for all $0 < \beta \leq \alpha$. Note then that for $\alpha \geq \beta > 0$, we have $\mathcal{A}_\alpha(x) \subseteq \mathcal{A}_\beta(x)$.

Definition 2. ***(Star-differentiable functions)*** *Let $u : B_+ \to \mathbb{R}$ be a real function defined on B_+. We say that u is* star-differentiable at $x \in B^+$ *if the Gâteaux derivative of u at x exists for all $h \in \mathcal{A}_\alpha$.*

Definition 2, in other words, states that u is star-differentiable at x if and only if there exists a map $L_x \in L(B_+, \mathbb{R})$ such that

$$\lim_{\alpha \to 0} \frac{u(x + \alpha h) - u(x)}{\alpha} = \frac{d}{d\alpha}|_{\alpha=0} u(x + \alpha h) = L_x h$$

for all $h \in \mathcal{A}_\alpha(x)$; that is, if $u(x + \alpha h) - u(x) = \alpha L_x h + o(\alpha h)$.

Definition 3. ***(Star-neighborhoods)*** *Let $x \in B_+$. We define a* star-neighborhood of x *by*

$$V_x^*(\alpha) = \left\{ y \in B_+ : y = x + \beta \frac{h}{\|h\|}, \forall\, h \in \mathcal{A}_\alpha(x),\ where\ 0 \leq \beta \leq \alpha \right\}.$$

We will say that O is a star-open subset of B_+ *if for each $x \in O$ there exist $V_x^*(\alpha) \subset O$.*

One can check that star neighborhoods form a base of a topology, called the *star-topology.*

Remark 1. *From now on to represent admissible directions we consider vectors h such that $\|h\| = 1$.*

Definition 4. ***(Star-charts)*** *Let Γ be a Hausdorff topological space. A* star-chart *on Γ is a pair (U^*, ϕ^*) where the set U^* is an open set in Γ and $\phi : V^* \to U^*$ is an homeomorphism from the star-neighborhood $V^* \subset B^+$ onto U^*. We call ϕ a parametrization. In such case, we say that the parametrization is of class C^k if the function ϕ is k times star-differentiable.*

Definition 5. ***(Star-manifold)*** *Let Γ be a Hausdorff topological space. We say that that Γ is a C^k* star-manifold *if for every $p \in \Gamma$, there exists an open star-neighborhood of B_+, denoted $V_a^*(\alpha)$, and a C^k parametrization, $\phi : V_a(\alpha) \to V_p$, where $V_p \subset \Gamma$ is an open neighborhood of p.*

To highlight the structure of Γ as a star manifold, we will use the notation Γ^*. We wish to remark in Definition 5 that, since we consider $\Gamma \subset B$ where B is a Banach space, then the open set in question can be considered to be $V_p^* = V_p \cap \Gamma$ where V_p is an open neighborhood of p in the topology of the norm.

Definition 6. ***(Star-atlas)*** *A C^k* star-atlas *is a collection of star charts $(V_{p_i} \cup M, \phi_i)$, $i \in I$, that satisfies the following properties:*

(i) *The collection $V_{p_i} \cup M$, $i \in I$, covers Γ.*

(ii) *Any two charts are compatible.*

[iii) *The map $\phi : V_{a_i}(\alpha) \to V_{p_i}$ is C^k star differentiable.*

(iv) *The set $\phi_i^{-1}(V_{p_i} \cup M)$ is a star-open subset of B_+.*

Definition 7. ***(Star-submanifolds)*** *Let Γ^* a C^k Banach manifold, $k \geq 0$. A subset S of Γ^* is called a* star submanifold of Γ^* *if and only if for each point $x \in S$ there exists an admissible chart in Γ^* such that*

(i) $\phi_i^{-1}(S \cap V_{p_i}) \subset V_{a_i}^*(\alpha)$.

(ii) The admissible directions $\mathcal{A}_a$ contain a closed subset $\mathcal{B}_a$ which splits $\mathcal{A}_a$.

(iii) The star chart image $\phi^{-1}(V_p \cap S)$ is an open star set $V^* = V_a^*(\alpha) \cap \mathcal{B}_a$.

Definition 8. ***(Tangent spaces)*** *Let M^* be a star manifold. The* tangent set *at $p \in M^*$ is the subset T_pM^* that can be described in the following way. Let $\phi : V_a^*(\alpha) \to V_p$ with $p = \phi(a)$. We write $T_pM^* = \phi'(a)(\mathcal{A}_\alpha(a))$. That is, $y \in T_pM^*$ if and only if $y = \phi'(a)h,\ h \in \mathcal{A}_\alpha(x)$.*

Definition 9. ***(Submersions)*** *Let $f : D(f) \to Y$ be a mapping between $B_+ = D(f)$ and the Banach space Y. Then, f is called a* submersion *at the point x if and only if*

1. *f is a C^1 mapping on a star-neighborhood $V^*(x)$ of x;*

2. *$f'(x) : B \to Y$ is surjective; and,*

3. *the null space $N(f'(x))$ splits B.*

We also say f is a submersion *on the set M if it is a submersion at each $x \in M$.*

Definition 10. ***(Regular points and regular values)*** *Let $f : D(f) \to Y$ be a mapping between $B_+ = D(f)$ and the Banach space Y. Then, the point $x \in D(f)$ is called a* regular point *of f if and only if f is a submersion at x. The point $y \in Y$ is called a* regular value *if and only if f^{-1} is empty or it consists solely of regular points. Otherwise y is called a* singular value, *i.e. $f^{-1}(y)$ contains at least one singular point.*

3 Results

Theorem 1. ***(The preimage theorem)*** *Let $f : M^* \to N$ a C^k mapping from a star manifold M^* to a Banach space N. If y is a regular value of f, then $S = f^{-1}(y)$ is a star-submanifold of M^**

Proof. It suffices to study the local problem. Let $V_a^*(\epsilon)$ be a star neighborhood of $a \in B^+$ and consider the C^k star differentiable map $\phi : B_+ \to M^*$ such that $\phi(a) = p$. Without loss of generality, let $f(p) = 0$. Let $h \in \mathcal{A}_\epsilon(a)$ and let V_p be a neighborhood of p. Then $V_p \cap M^* = \phi(V_a^*(\epsilon))$. Thus, $\phi(a + \alpha h) \in V_p \cap M^*$. From the local submersion theorem if f is a submersion, there exists a parametrization ϕ such that, $\phi(a) = p$, $\phi'(a) = I$. From Definition 5, for all $p' \in V_p \cap M^*$, there exists h and α such that $h \in \mathcal{A}_a(\alpha)$. Since *ker* $f'(p)$ splits B, there exists a projection $P : B \to N$. Let $P^\perp = I - P$ and $N^\perp = P^\perp$. Thus, we obtain that $B = N \oplus N^\perp$ and that $f'(p) : N^\perp \to Y$ is bijective. Denote its inverse by $A : Y \to N^\perp$. So let

$$a = P(p) + Af(p) \ \ and \ \ a + \alpha h = [P(p') + Af(p')],$$

$$\phi^{-1}(x) = Px + Af(x)$$

where $A = f'(p)^{-1}$.

Multiplying both sides of this equation we obtain: $f'(p)\phi(x) = f(x)$. So, from the local submersion theorem, given that $f(\phi(a)) = 0$, the equality

$$f(\phi(a + \alpha h)) = f'(\phi(a))(\alpha h) + y,$$

implies that, the solution of the equation $f(z) = y$ in a star-neighborhood V_p^* of p, corresponds to the solution of the equation $f'(\phi(a))h = 0$. Therefore, S is the set $h \in \mathcal{A}_a$ such that $f'(\phi(a))h = 0$ for some $h \in \mathcal{A}_\alpha(a)$. Hence, S is a star-submanifold of M^*.

□

Definition 11. ***(Star Fredholm maps)*** *A* star Fredholm operator *is a star continuous linear map $L : E_1 \to E_2$ such that*

(i) dim ker$L < \infty$;

(ii) range L is closed;

(iii) dim coker$L < \infty$.

The index *of L is A* star Fredholm map *is a star continuous map between star manifolds such that at each point in the domain, its star Gateaux derivative is a star Fredholm operator. The index of a star Fredholm map is the index of its linearization.*

Definition 12. ***(Locally star proper maps)*** *A star Fredholm map $F : M \to V$ is said to be* locally star-proper *if for every $x \in M$ there is a star neighborhood U of x such that f restricted to U is proper.*

Theorem 2. ***(Main theorem)*** *Let $f : M \to V$ be a C^r star Fredholm map between star Banach manifolds, with $r > max(index\, f, 0)$. Suppose that M and V are connected and have a countable basis. Furthermore, suppose that f is locally proper and that it has at least one regular value. Then, the regular values of f are almost all of V.*

Proof. The proof follows closely [14]. The theorem is proved locally, since we assume M has a countable base and first category. Thus, let U be a star neighborhood of $x_0 \in M$. In this case, U is a subset of some Banach space E. Then, $A = Df(x_0) : E \to E'$ for some Banach space E'. Since A is a star Fredholm operator, we can write $x_0 = (p_0, q_0) \in E_1 \times kerA = E$. Thus, the Gateaux-star derivative $D_1 f(p, q) : E_1 \to E$ maps E_1 injectively onto a closed subspace of E for all (p, q) sufficiently close to (p_0, q_0). From the generalized implicit function theorem of [1], we know there is a star neighborhood $U_1 \times U_2 \subset E_1 \times kerA$ of (p_0, q_0) such that D_2 is compact and if $q \in U_2$, f restricted to $U_1 \times q$ is a homeomorphism onto its image. Since f is is locally proper by assumption, the critical points of f (which by assumption there is at least one) are closed.

□

4 Conclusions

This notes provided a generalised Sard Theorem and Preimage Theorem which are mathematical tools widely used in economic theory to study the structure of the equilibrium set. Through our approach, this tool can be used in situations where the spaces involved might have a positive cone with an empty interior.

References

[1] Accinelli, E. (2010) “A generalization of the implicit function theorem.” *Applied Mathematical Sciences* 4-26, 1289 - 1298.

[2] Balasko, Y. (1975) "Some results on uniqueness and on stability of equilibrium in general equilibrium theory." *Journal of Mathematical Economics* 2, 95-118.

[3] Balasko, Y. (1997a) "Pareto optima, welfare weights, and smooth equilibrium analysis." *Journal of Economic Dynamics and Control* 21, 473-503.

[4] Balasko, Y. (1997b) "Equilibrium analysis of the infinite horizon model with smooth discounted utility functions." *Journal of Economic Dynamics and Control* 21, 783-829.

[5] Balasko, Y. (1997c) "The natural projection approach to the infinite-horizon model." *Journal of Mathematical Economics* 27, 251-265.

[6] Bewley, T.F. (1972) "Existence of equilibria in economies with infinitely many commodities." *Journal of Economic Theory* 4, 514-540.

[7] Chichilnisky, G. and G. Heal (1996) "On the existence and the structure of the pseudo-equilibrium manifold," *Journal of Mathematical Econonomics* 26, 171-186.

[8] Debreu, G. (1970) "Economies with a finite set of equilibria." *Econometrica* 38, 387-392.

[9] Duffie, D. and C.F. Huang (1985) "Implementing Arrow-Debreu equilibria by continuous trading of a few long-lived securities." *Econometrica* 53, 1337-1356.

[10] Kehoe, T.J. and D.K. Levine (1985) "Comparative statics and perfect foresight in infinite horizon models." *Econometrica* 53, 433-453.

[11] Magill, M.J.P. and W.J. Shafer (1990) "Characterisation of generically complete real asset structure." *Journal of Mathematical Economics* 19, 167-194.

[12] Mas-Colell, A. and Zame, W.R. (1991) Equilibrium theory in infinite dimensional spaces. Handbook of Mathematical Economics IV, 1835 - 1898.

[13] Sard, A. (1942) "he measure of the critical values of differentiable maps." *Bulletin of the American Mathematical Society* 48, 883-890.

[14] Smale, S. (1965) "An infinite dimensional version of Sard's theorem." *American Journal of Mathematics* 87-4, 861-866.

[15] Zeidler, E. (1993) Non Linear Functional Analysis and its Applications. Springer Verlag.