Overnight Index Rate: Model, Calibration, and Simulation

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Working paper

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Abstract

In this study the extended Overnight Index Rate (OIR) model is presented. The fitting function for the probability distribution of the OIR daily returns is based on three different Gaussian distributions which provide modelling of the narrow central peak and the wide fat-tailed component. Calibration algorithm for the model is developed and investigated using the historical OIR data.

Keywords:
Overnight Index Rate, Fat tailed distribution, Calibration, Interest Rate Simulation

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1 Overnight Index Rate Model

Overnight indexed swap (OIS) rates are considered as the risk-free rate for valuation of collateralized portfolios (Hull and White, 2013). The development of Overnight Index Rate (OIR) models are very important. There are several publications on this topic, such as Poisson–Gaussian models (Das, 2002) for the Fed Funds rates, (Benito et al., 2006) for Eonia, and the OIR model based on short-term "memory" (auto-correlation) and its highly leptokurtical nature in (Yashkir and Yashkir, 2003). In the present study the daily changes of the OIR for a Monte Carlo scenario s are modelled as:

$$\begin{cases} r_1^{(s)} & = r_0 \\ x_{i+1}^{(s)} & = \sum\limits_{k=1}^{\min(i,m)} \beta_k \cdot \epsilon_{i-k+1}^{(s)}(\vec{q}) \\ x_i^{(s)} & = r_i^{(s)}/r_{i-1}^{(s)} - 1 \end{cases} \tag{1}$$

The OIR daily return x_{i+1} at a time point t_{i+1} is correlated to m previous daily returns. It is accounted for by the weighted sum of corresponding random driver $\epsilon(\vec{q})$. The probability distribution function $g(x, \vec{q})$ of the random number driver $\epsilon(\vec{q})$ is introduced as the linear combination of three normal distributions:

$$\begin{cases} g(x, \vec{q}) = w_1 G(x, \mu_1, \sigma_1) + w_2 G(x, \mu_2, \sigma_2) + w_3 G(x, \mu_3, \sigma_3) \\ G(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \end{cases} \tag{2}$$

where

$$\begin{cases} t_i = (i-1)\Delta t & \text{time points} \quad (i = 1, \cdots, n+1) \\ \vec{q} = (\sigma_1, \sigma_2, \sigma_3, w_1, w_2, \mu_1, \mu_2, \mu_3) & \text{parameters to be calibrated} \\ \sigma_{1,2,3} & \text{standard deviations of Gaussian functions } G(x, \mu, \sigma) \\ w_{1,2} & \text{weight coefficients} \\ w_3 = 1 - w_1 - w_2 & \\ \mu_{1,2,3} & \text{centering parameters} \end{cases} \tag{3}$$

The proposed distribution function (2) has enough flexibility to fit a typical historical distribution with a narrow central peak, and fat tails. The possible upward/downward rate drifts are reflected in non-zero values of $\vec{\mu}$. The auto-correlation of the daily returns of the OIR model (1) should satisfy the historical auto-correlations $\vec{\rho}$. Therefore, the auto-correlation factors $\vec{\beta}$ must satisfy the following equation

$$\sum_{k=1}^{m-p+1} \beta_k \beta_{k+p-1} = \rho_p \quad (p = 1, \cdots, m), \tag{4}$$

where $\vec{\rho}$ is the historical auto-correlation vector (the overline indicates averaging by i):

$$\rho_p = \frac{\overline{(x_i - \overline{x})(x_{i-p+1} - \overline{x})}}{\overline{(x_i - \overline{x})^2}} \tag{5}$$

2 OIR Model Calibration

2.1 Random driver parameters

Given the set of historical overnight rates[1] $\vec{r}^{(h)}$ for a chosen time period we calculate the historical density distribution $y^{(h)}(x)$ of the overnight returns $\vec{x}^{(h)}$. The calibration of the distribution $g(x, \vec{q})$ (2) is obtained by minimizing the objective function $H(\vec{q})$:

$$\begin{cases} \vec{q}^{(*)} & = \arg \min\limits_{\vec{q} \subset \mathbf{Q}} H(\vec{q}) \\ H(\vec{q}) & = \sum\limits_i (y^{(h)}(x_i) - g(x_i, \vec{q}))^2, \end{cases} \tag{6}$$

where $\mathbf{Q}$ is a user-defined argument hyper-box:

$$\mathbf{Q} = \begin{cases} \sigma_k^{(min)} < \sigma_k < \sigma_k^{(max)} & k = 1, 2, 3 \\ w_k^{(min)} < w_k < w_k^{(max)} & k = 1, 2 \\ \mu_{min} < \mu_k < \mu_{max} & k = 1, 2, 3 \end{cases} \tag{7}$$

2.2 Auto-correlation factors

The auto-correlations $\vec{\rho}^{(h)}$ are calculated using $\vec{x}^{(h)}$ in (5). Taking into account (4) the factors $\vec{\beta}$ are obtained by minimizing the objective function $V(\vec{\beta})$:

$$\begin{cases} \vec{\beta}^{(*)} & = \arg \min\limits_{\vec{\beta}} V(\vec{\beta}) \\ V(\vec{\beta}) & = \sum\limits_{p=1}^{m} \left(\sum\limits_{k=1}^{m-p+1} \beta_k \beta_{k+p-1} - \rho_p^{(h)} \right)^2. \end{cases} \tag{8}$$

[1] http://www.euribor-info.com/en/eonia

2.3 Implementation

The calibration of the OIR model and its use for the simulation are based on R-codes (R statistical package). The optimization procedures (6) and (8) are based on the R function **optim**() using the method "L-BFGS-B" that incorporates the box constraints[2]. The random driver function code for $\epsilon(\vec{q})$ (2) is as follows:

```
eps <- function(q){
  flag1 = flag2 = flag3 = 0
  y1    <- rnorm(1,q[6], q[1])
  y2    <- rnorm(1,q[7], q[2])
  y3    <- rnorm(1,q[8], q[3])
  flag1 <- rbinom(1,1,q[4])     # if 1 then y1
  if(flag1 == 0){               # if 0 then either y2 or y3
      flag2 <- rbinom(1,1,q[5]) # if 1 then y2
      if(flag2 == 0)            # if 0 then not y1, not y2, but y3
          flag3 = 1
  }
  y <- y1 * flag1 + y2 * flag2 + y3 * flag3
  return(y)
}
```

where the function **rnorm**() generates normally distributed random numbers.
The source codes[3] are:

$$\begin{cases} \text{OIR.v.3.5.R} & \text{(calibration)} \\ \text{OIRSim.v.2.R} & \text{(simulation)} \end{cases} \tag{9}$$

3 The Overnight Index Rate Model

3.1 Calibration

The three input data sets were used for the model calibration. The long time period data set (January 4, 1999 to July 11, 2012; Long Period **A**) covering 3464 time points and the short time period data set (July 11, 2011 to July 11, 2012; Short Period **B**) corresponding to 259 time points were chosen to investigate how calibration depends on a wide range or rate drifts (Long Period) compared to the relatively small rate drifts (Short Period). The other long time period data set (January 4, 1999 to December 31, 2004; Long Period **C**; 1534 time points) was chosen as the calibration base for OIR simulation for a long term (more than 5 years).

3.1.1 The Long Period A

The OIR model was calibrated based on Eonia rates (January 4, 1999 to July 11, 2012). The time dependence of Eonia rates and daily returns $\vec{x}^{(h)}$ are presented in Figure 1 and in Figure 2.

[2](Byrd et al., 1995)

[3]The R-codes for OIR calibration and simulation were developed by Yashkir Consulting (www.yashkir.com)

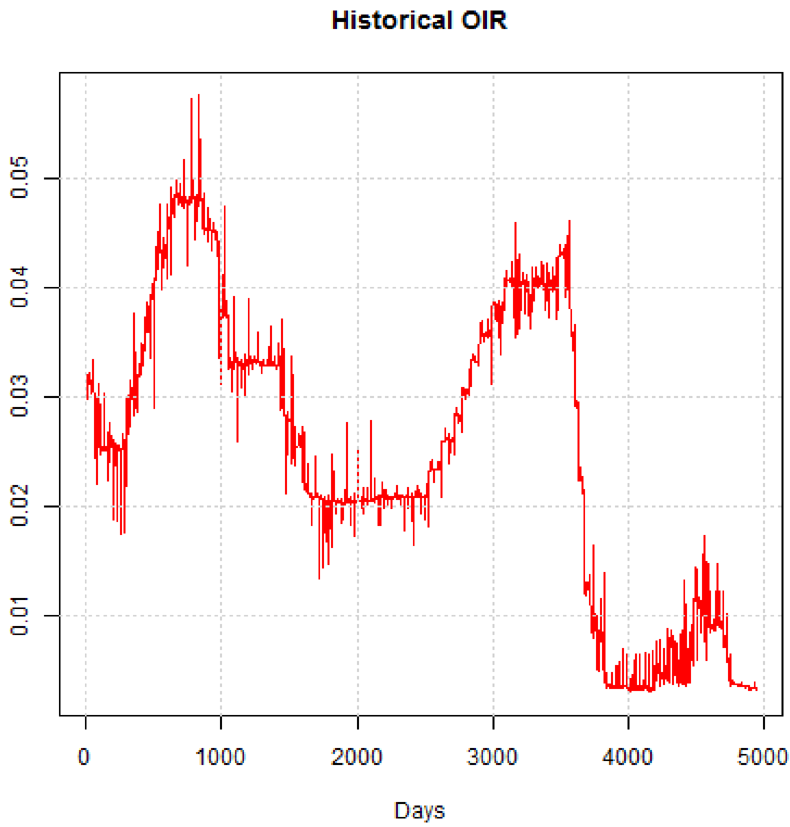

Figure 1: Eonia (Long Period **A**: January 4, 1999 to July 11, 2012)

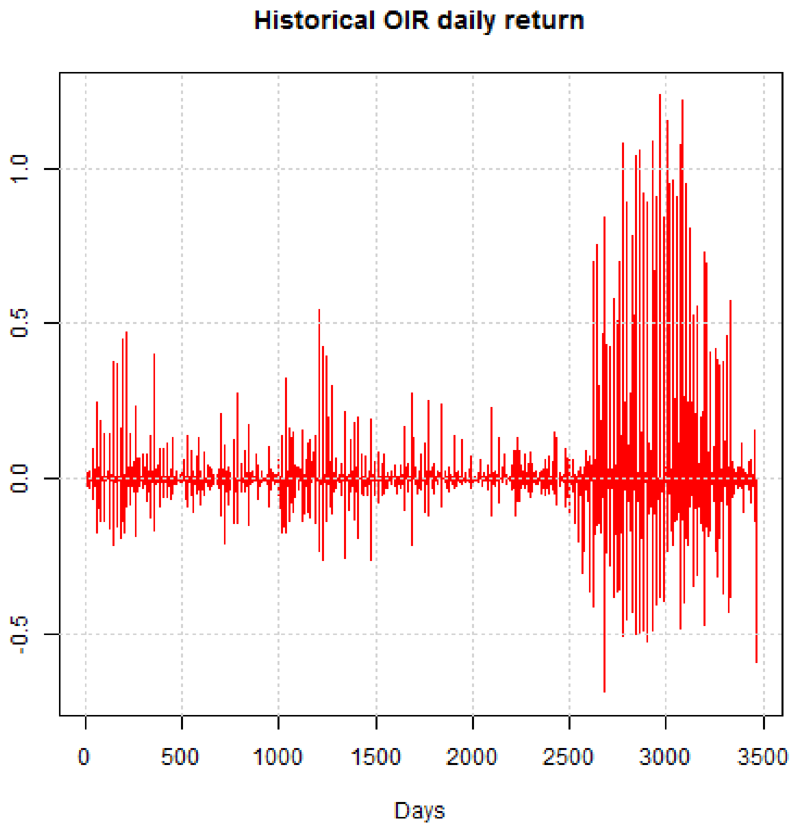

Figure 2: Eonia daily returns (Long Period **A**: January 4, 1999 to July 11, 2012)

The hyper-box for finding random driver parameters $\vec{q}$ was set as:

$$\mathbf{Q} = \begin{cases} 0.0001 < \sigma_1 < 0.01 \\ 0.0001 < \sigma_2 < 0.02 \\ 0.0001 < \sigma_3 < 0.95 \\ 0 < w_k < 0.5, & k = 1,2 \\ w_3 = 1 - w_1 - w_2 \\ 0 < \mu_k < 0.003, & k = 1,2,3 \end{cases} \tag{10}$$

The result of the optimization procedure (6) was reached after 8 iterations (from $H = 535.2$ to $\min H = 53.8$), and the resulting vector $\vec{q}$ is presented in Table 4 (Long Period **A**). The process of the convergence is shown in Figure 3.

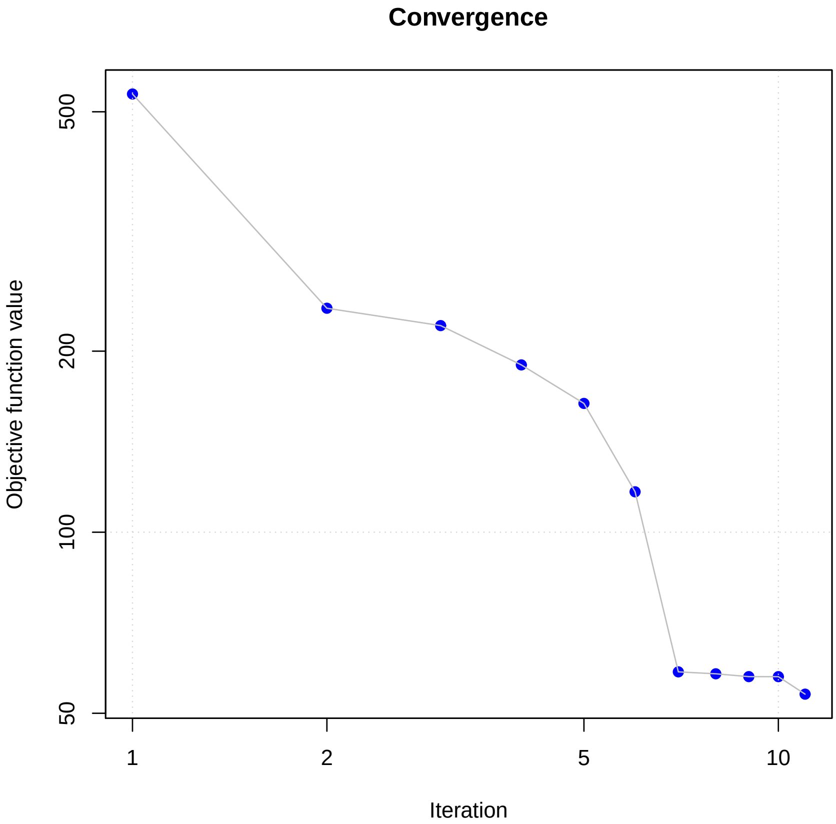

Figure 3: Convergence of the optimization procedure (6) (Long Period **A**)

The Long Period **A** calibration results demonstrate that the best-fit distribution has a narrow ($\sigma_1 = 0.38\%$) peak ($w_1 = 45\%$ weight), a wide band ($\sigma_2 = 2.0\%$ with the $w_2 = 45\%$ weight), and a fat-tail band ($\sigma_2 = 9.25\%$ with the $w_3 = 9.7\%$ weight). The optimal fit of the calibrated probability distribution function (2) to the historical density distribution $y^{(h)}$ is shown in Figure 4.

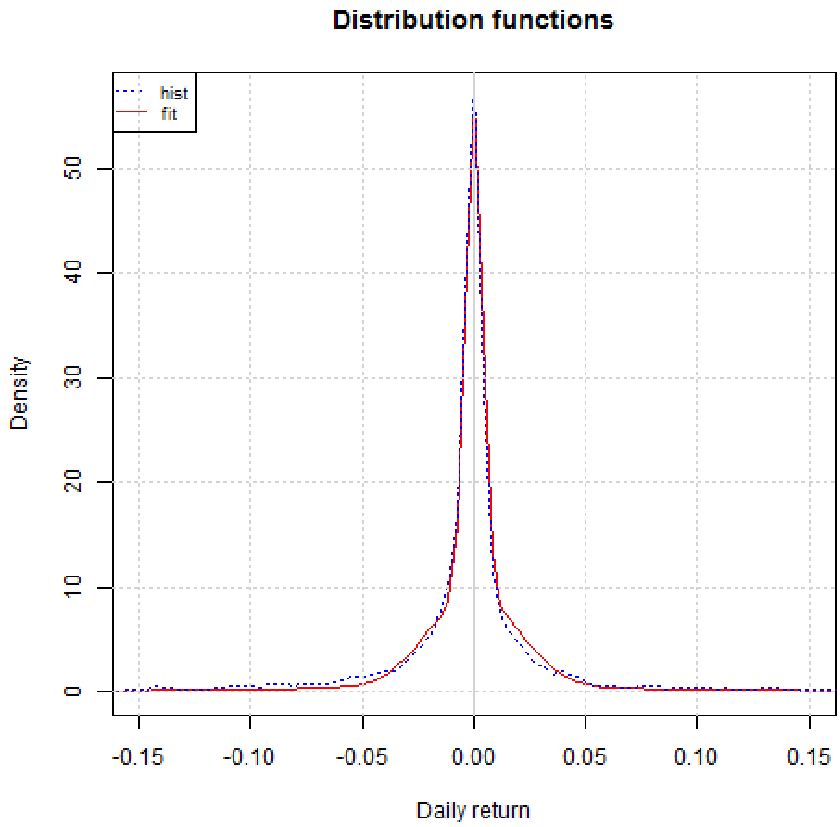

Figure 4: Fitting of the model daily return distribution to historical data (Long Period **A**)

Using the algorithm (8) we obtained the $\vec{\beta}$ values (presented in Table 1 and in Figure 5). Note that the one-day lag auto-correlation coefficient is negative which reflects the auto-compensation feature of the OIR time dynamics.

Table 1: The historical auto-correlation $\vec{\rho}$ and factors $\vec{\beta}$

Auto-correlation lag (days)	$\rho^{(h)}$	β
0	1	0.9656
1	-0.1986	-0.2333
2	-0.0541	-0.0760
3	-0.0420	-0.0594
4	-0.0564	-0.0615

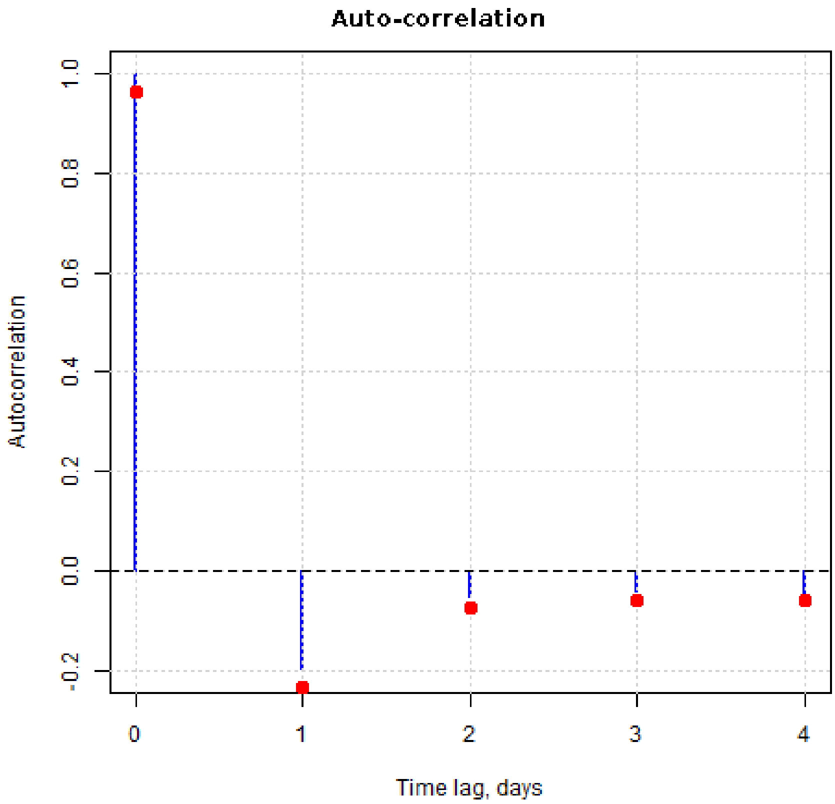

Figure 5: Historical auto-correlations $\vec{\rho}$ and factors $\vec{\beta}$

The efficiency of the calibration, and of the model itself, can be verified by the backtesting procedure. It is the simulation of the OIR using calibrated model and comparing simulation results with historical OIR series. We assume that the model performs well if the historical OIR time series lies between low and high confidence levels of simulated rates.

3.1.2 The "in-sample" backtesting of the Long Period A calibration

The "in-sample" backtesting was done using the OIR model (1) with calibration parameters presented in Table 4 (Long Period **A**) and in Table 1. The number of Monte Carlo scenarios was $N = 5000$. Results of the simulation are presented in Figure 6.

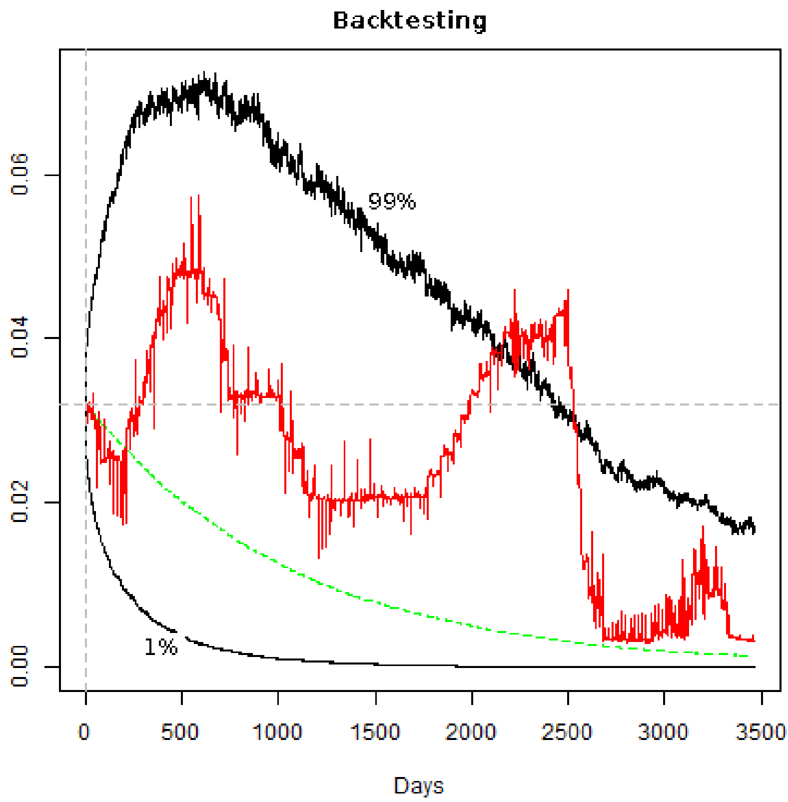

Figure 6: Backtesting (Long Period **A**): historical OIR (red), the upper/lower percentiles (99% and 1%) of simulated rates, and the OIR simulated average (dashed curve)

The historical OIR time series is mostly covered by low/high quantiles of simulated rates in spite of a very wide range of rate changes (the historical ratio of the highest rate to the lowest rate $= 5.75\%/0.131\% > 40$!).

3.1.3 The Short Period B

The OIR model was calibrated based on Eonia rates (July 11, 2011 to July 11, 2012). The time dependence of Eonia rates and daily return $x^{(\vec{h})}$ are presented in Figure 7 and in Figure 8.

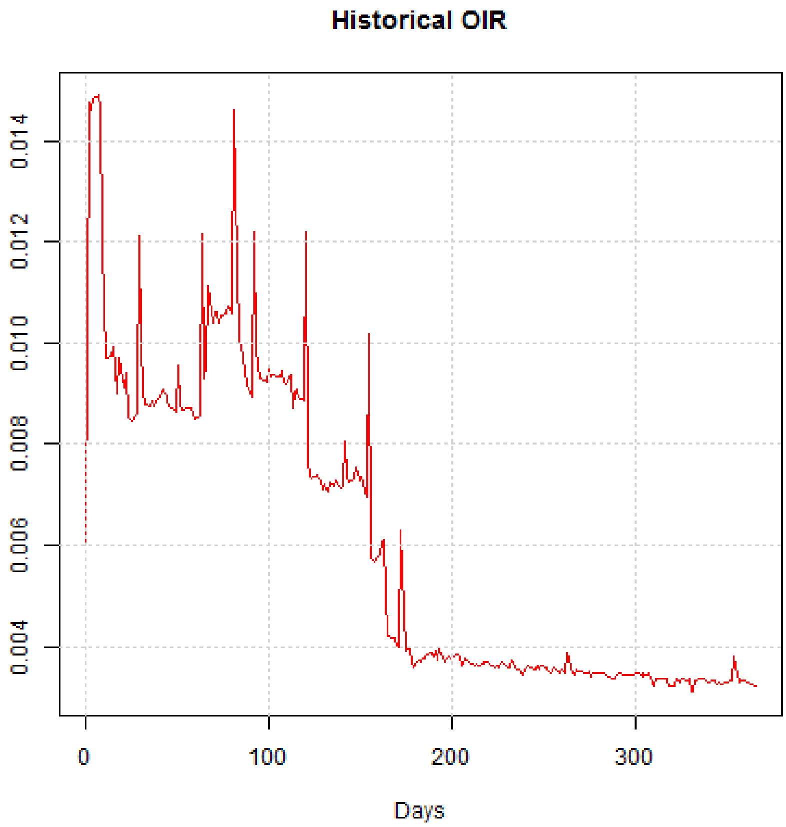

Figure 7: Eonia historical rates (Short Period **B**: July 11, 2011 to July 11, 2012)

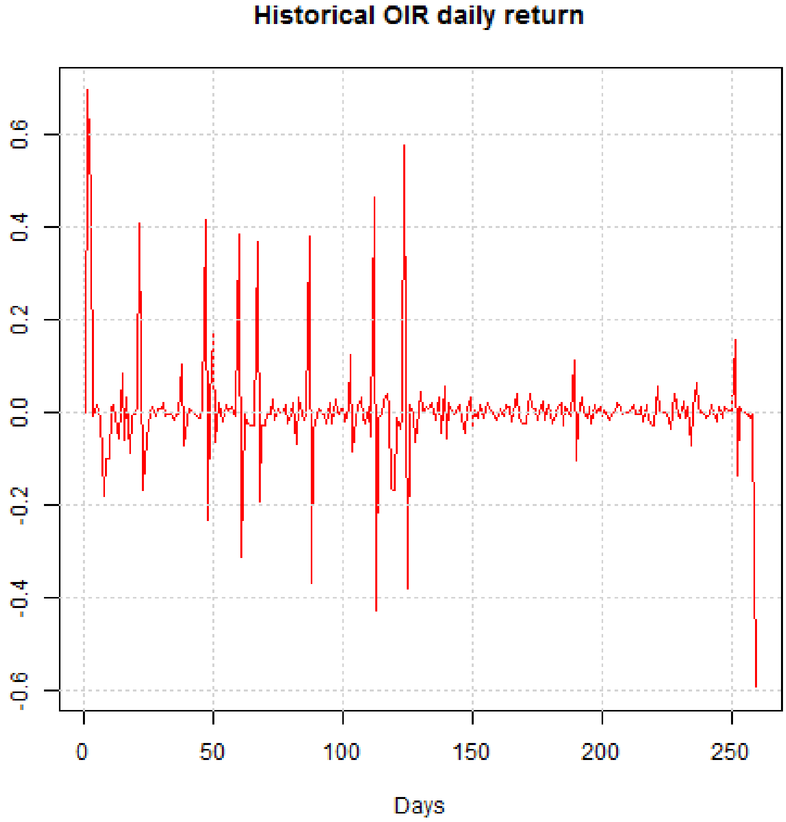

Figure 8: Eonia historical daily returns (Short Period **B**: July 11, 2011 to July 11, 2012)

The hyper-box for finding random driver parameters $\vec{q}$ was set as follows:

$$\mathbf{Q} = \begin{cases} 0.0001 < \sigma_1 < 0.1 \\ 0.0001 < \sigma_2 < 0.5 \\ 0.0001 < \sigma_3 < 0.95 \\ 0 < w_k < 0.4, & k = 1, 2 \\ w_3 = 1 - w_1 - w_2 \\ 0 < \mu_k < 0.0001, & k = 1, 2, 3 \end{cases} \tag{11}$$

The result of the optimization procedure (6) was reached after 24 iterations (from $H = 275.9$ to $\min H = 37.7$), and the resulting vector $\vec{q}$ is presented in Table 4 (Short Period **B**). The convergence process dynamics for (6) is shown in Figure 9.

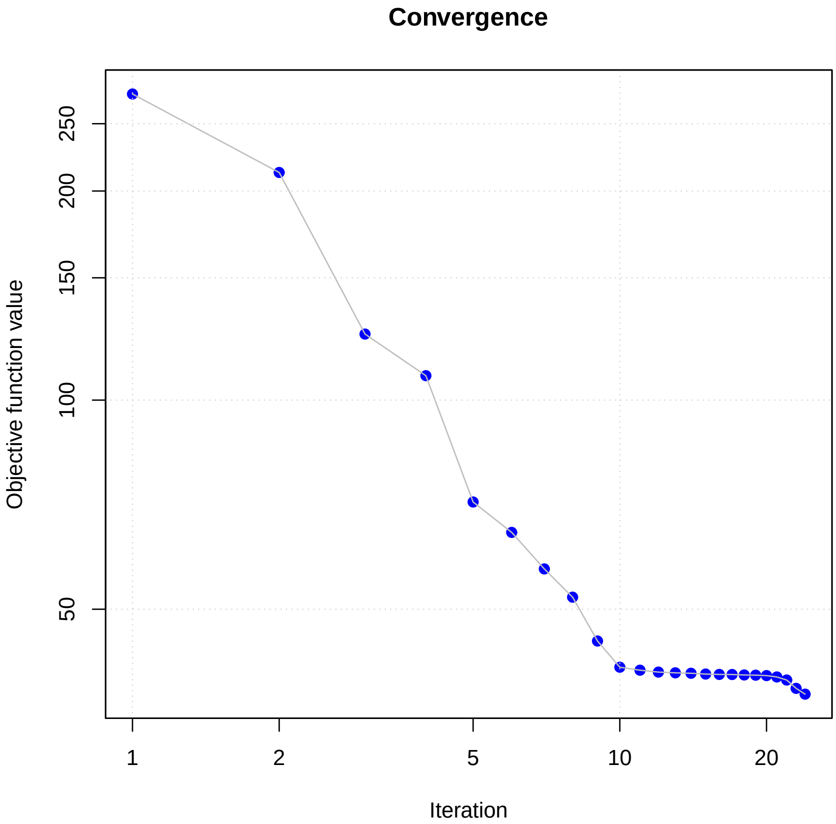

Figure 9: Convergence of the optimization procedure (6) (Short Period **B**)

The optimal fit of the calibrated probability distribution function (2) to the historical density distribution $y^{(h)}$ is shown in Figure 10.

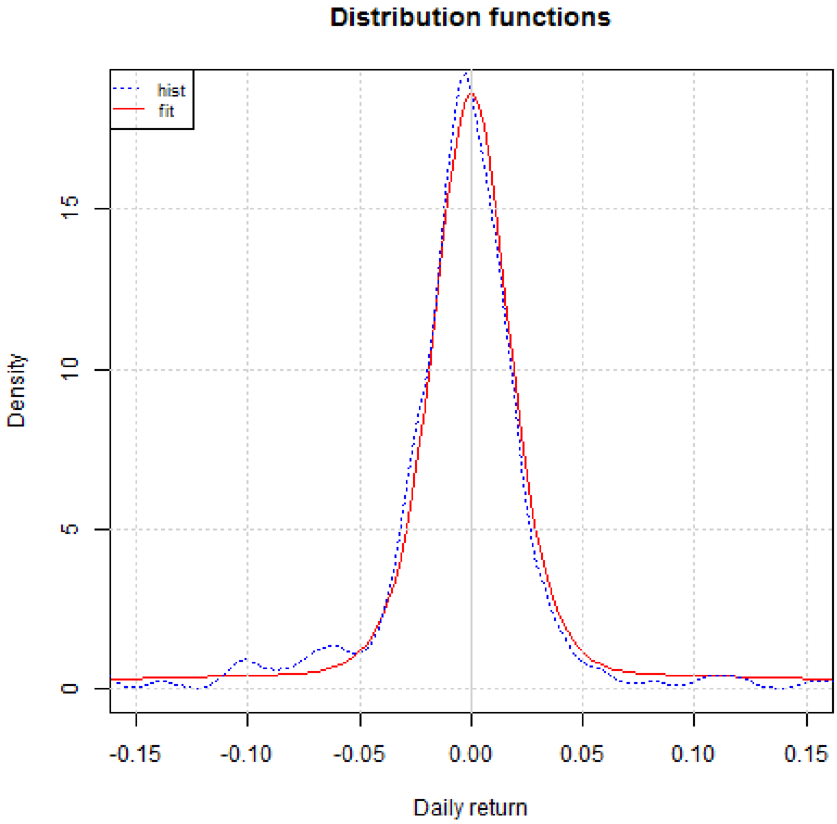

Figure 10: Fitting of the model returns distribution to the historical data (Short Period **B**)

Using the algorithm (8) we obtained $\vec{\beta}$ values (presented in Table 2 and in Figure 11). The main auto-correlation feature of the OIR daily log return (the negative value of ρ for the one-day lag) is present.

Table 2: Historical auto-correlations $\vec{\rho}$ and factors $\vec{\beta}$

Auto-correlation lag (days)	$\rho^{(h)}$	β
0	1	0.9750
1	-0.1941	-0.2050
2	-0.0227	-0.0212
3	0.0294	0.0142
4	-0.0618	-0.0716

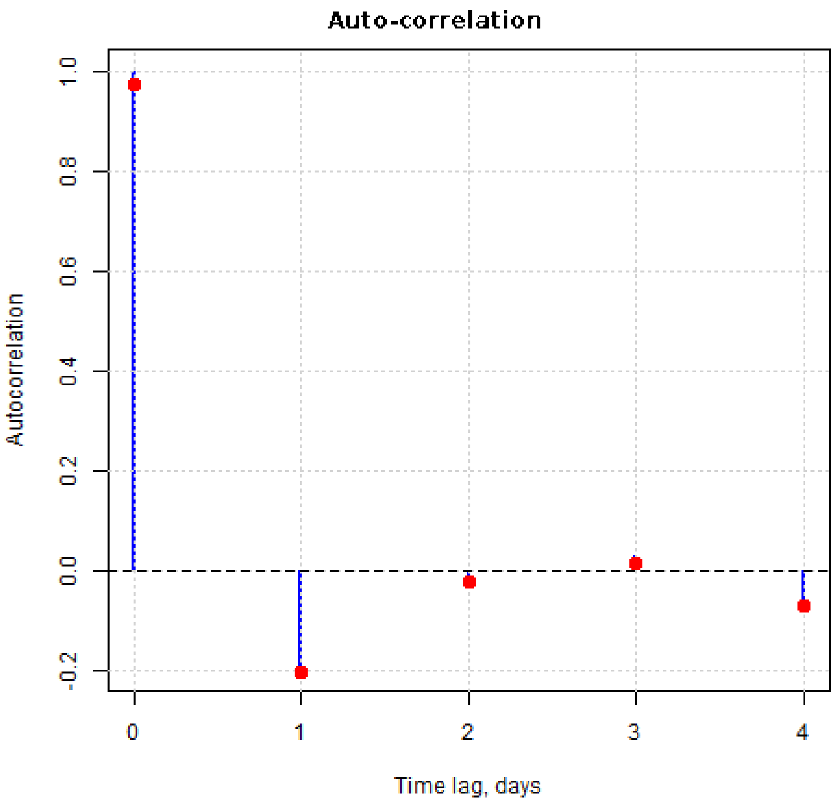

Figure 11: Historical auto-correlations $\vec{\rho}$ and factors $\vec{\beta}$ (Short Period **B**)

Note that in case of the Short Period the one-day negative auto-correlation is dominant.

3.1.4 The "in-sample" backtesting of the Short Period B calibration

The "in-sample" backtesting was done using the OIR model (1) with calibration parameters presented in Table 4 (Short Period **B**) and in Table 2. The number of Monte Carlo scenarios was $N = 10000$. Results of the simulation are presented in Figure 12.

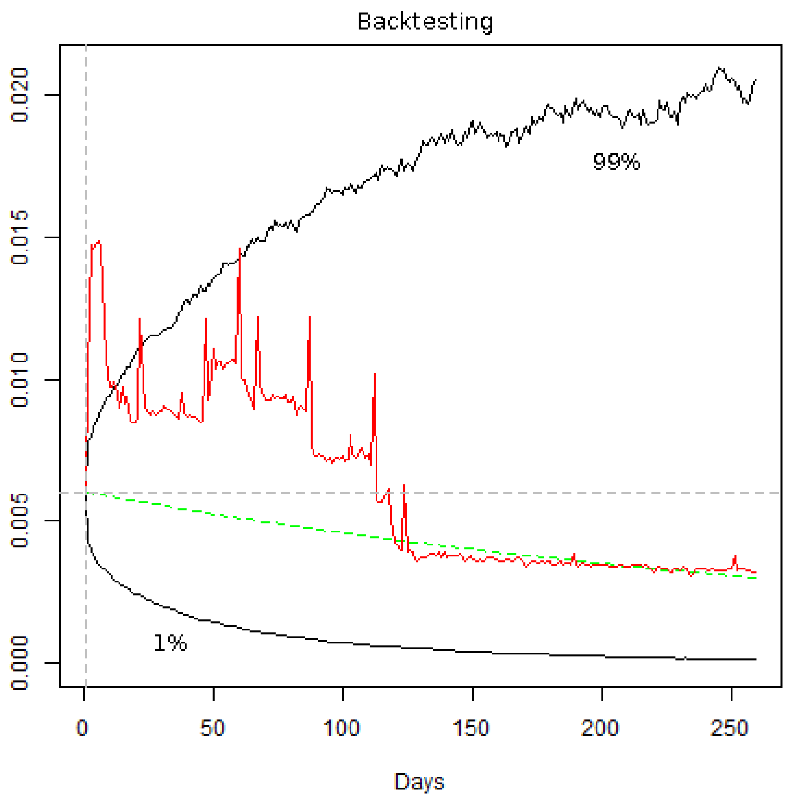

Figure 12: Backtesting (Short Period **B**): historical OIR (red), the upper/lower percentiles (99% and 1%) of simulated rates, and simulated average (dashed curve)

The historical OIR time series is mostly covered by low/high quantiles of simulated rates and (sic!) the simulated rate average tends to the historical rate trend.

3.1.5 The Long Period C

The OIR model was calibrated based on Eonia rates (January 4, 1999 to December 31, 2004). The time dependence of Eonia rates and daily returns $\vec{x}^{(h)}$ are presented in Figures 13 and in Figure 14.

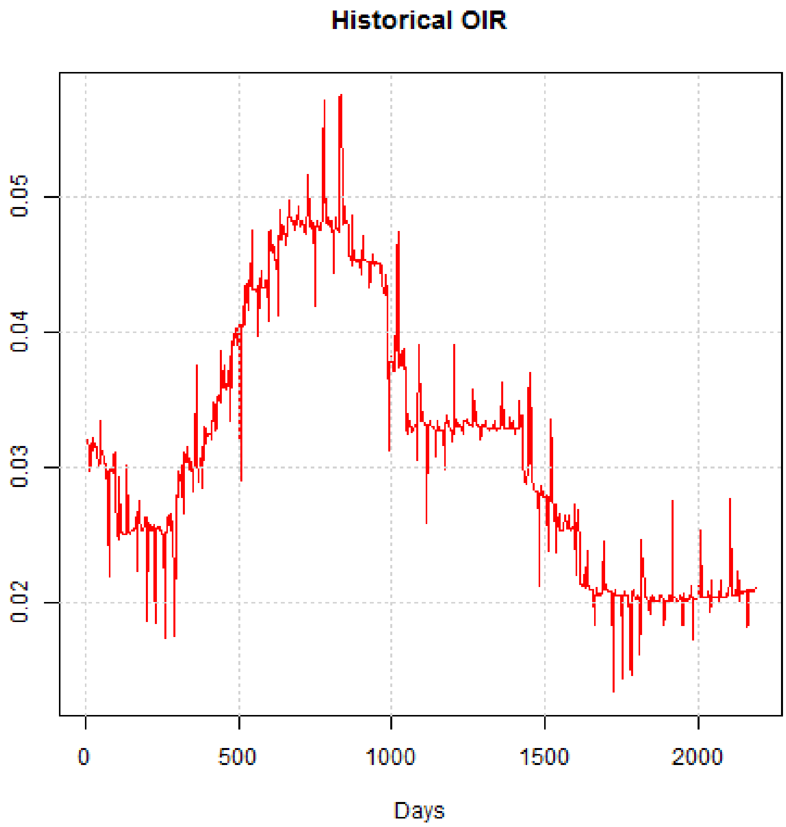

Figure 13: Eonia (Long Period **C**: January 4, 1999 to December 31, 2004)

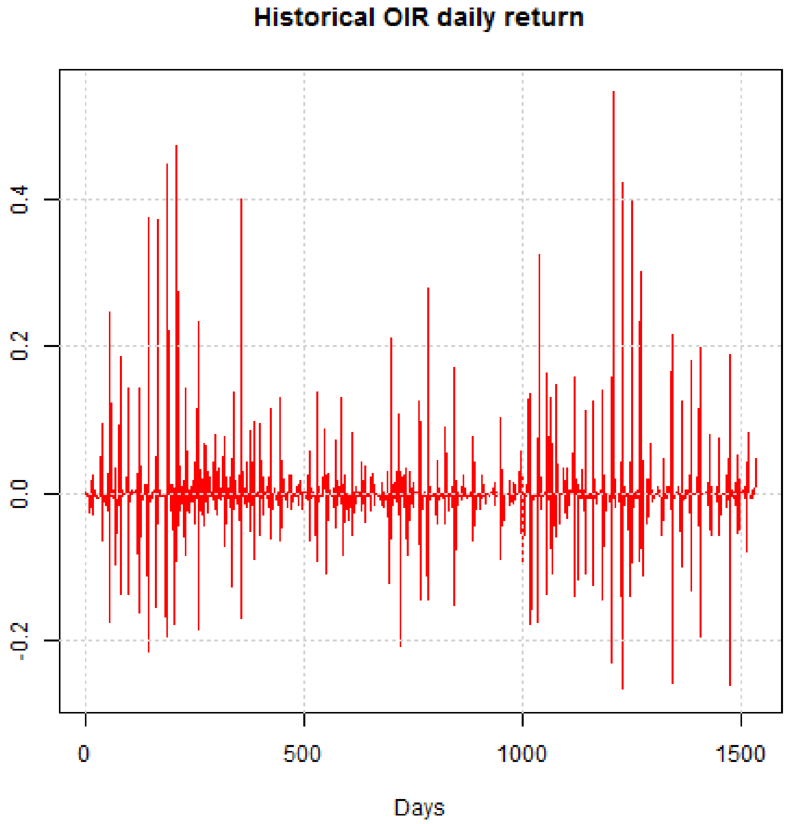

Figure 14: Eonia daily returns (Long Period **C**: January 4, 1999 to December 31, 2004)

The hyper-box for finding random driver parameters $\vec{q}$ was set as:

$$\mathbf{Q} = \begin{cases} 0.0001 < \sigma_1 < 0.01 & \\ 0.0001 < \sigma_2 < 0.02 & \\ 0.0001 < \sigma_3 < 0.95 & \\ 0 < w_k < 0.4, & k = 1,2 \\ w_3 = 1 - w_1 - w_2 & \\ 0.00001 < \mu_k < 0.01, & k = 1,2,3 \end{cases} \tag{12}$$

The result of the optimization procedure (6) was reached after 5 iterations (from $H = 4706$ to $\min H = 1490$), and the resulting vector $\vec{q}$ is presented in Table 4 (Long Period **C**). The process of convergence is shown in Figure 15.

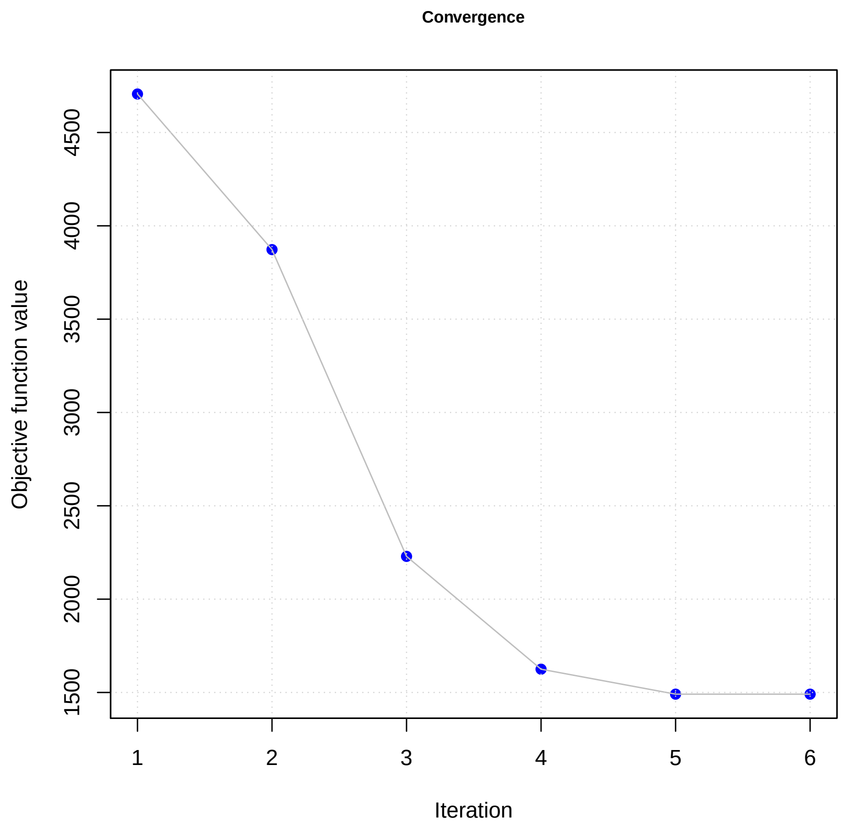

Figure 15: Convergence of the optimization procedure (6) (Long Period **C**)

The Long Period **C** calibration results (16) demonstrate that the calibrated probability distribution function (2) fits well the historical density distribution $y^{(h)}$.

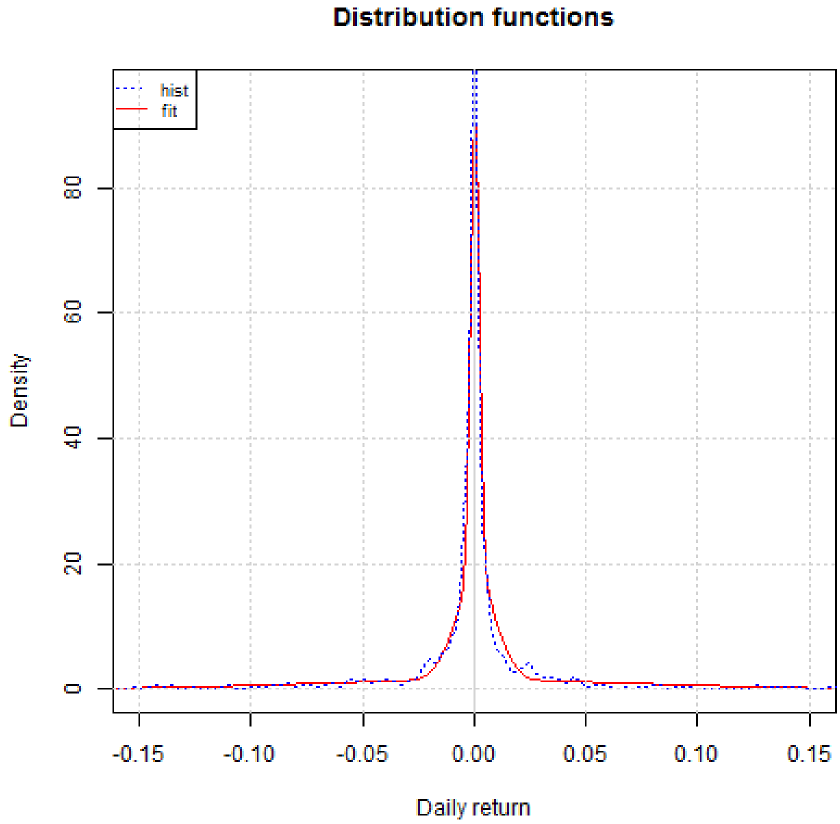

Figure 16: Fitting of the model daily return distribution to historical data (Long Period **C**)

The fitting result (Figure 16) shows that the three-component fitting function 2 provides a good replication of the historical distribution.
Using the algorithm (8) we obtained the $\vec{\beta}$ values (presented in Table 3 and in Figure 17). The one-day lag auto-correlation coefficient is negative which reflects the auto-compensation feature of time dependence of OIR.

Table 3: Historical auto-correlation $\vec{\rho}$ and factors $\vec{\beta}$

Auto-correlation lag (days)	$\rho^{(h)}$	β
0	1	0.9445
1	-0.1720	-0.2520
2	-0.1542	-0.1925
3	-0.0501	-0.0697
4	-0.0331	-0.0422

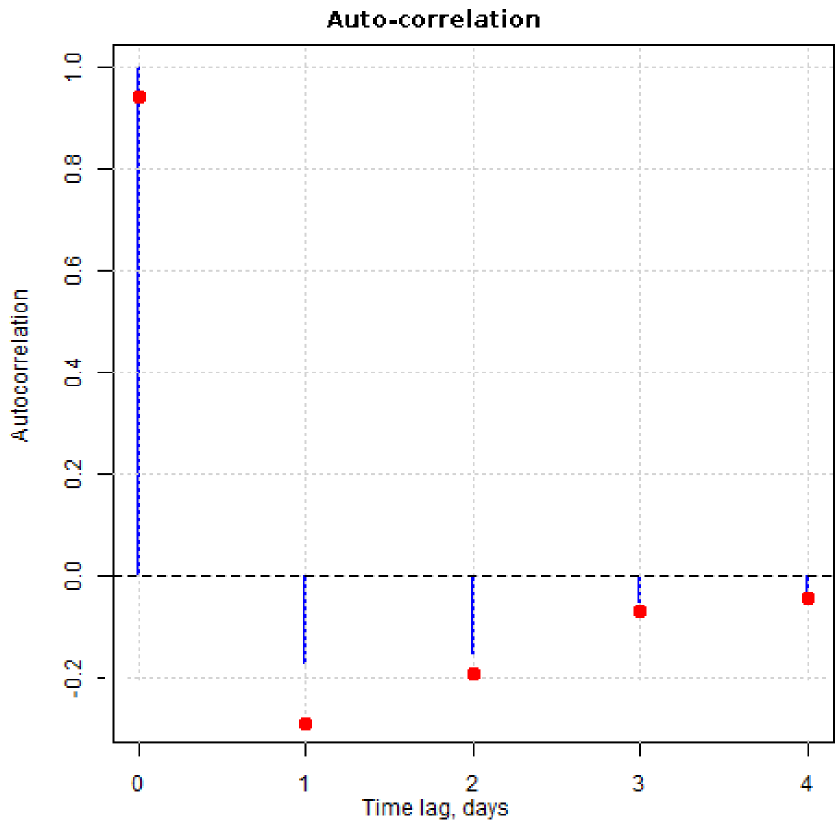

Figure 17: Historical auto-correlations $\vec{\rho}$ and factors $\vec{\beta}$

The efficiency of the calibration (and quality of the model itself) was verified by the backtesting procedure in the next subsection.

3.1.6 The "in-sample" backtesting of the Long Period C calibration

The "in-sample" backtesting was done using the OIR model (1) with calibration parameters presented in Table 4 (Long Period **C**) and in Table 3. The number of Monte Carlo scenarios was $N = 5000$. Results of the simulation are presented in Figure 18.

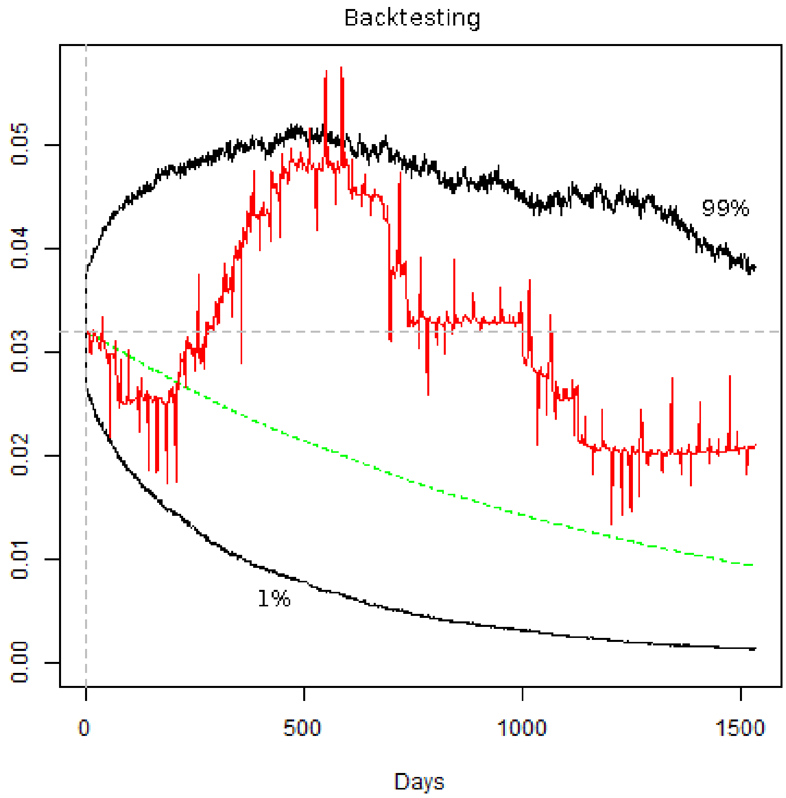

Figure 18: Backtesting (Long Period **C**): historical OIR (red), the upper/lower percentiles (99% and 1%) of simulated rates, and the simulated OIR average (dashed curve)

The historical OIR time series is mostly covered by low/high quantiles of simulated rates (in spite of very strong upward/downward rate drift periods and long periods with relatively stable rates.

3.2 Summary of OIR calibration tests

The results of the OIR calibration based on different data sets are summerized in Table 4.

Table 4: OIR Model Calibration results

Case	Time period From	Time period To	σ_1 w_1 μ_1	σ_2 w_2 μ_2	σ_3 w_3 μ_3	β_1	β_2	β_3
A	4-Jan-1999	11-Jul-2012	0.0038 0.4516 0	0.0200 0.4515 0	0.0925 0.0969 0.0003	0.9656	-0.2333	-0.0760
B	11-Jul-2011	11-Jul-2012	0.0230 0.4000 0	0.0142 0.3968 0	0.1585 0.2032 0	0.9750	-0.2050	-0.0212
C	4-Jan-1999	31-Dec-2004	0.0092 0.3680 0.0007	0.0019 0.3680 0	0.0762 0.2640 0.0008	0.9445	-0.2520	-0.1925

3.3 OIR simulation using the "out-of-sample" calibration

3.3.1 The short term OIR simulation

The "out-of-sample" OIR simulation for a short term (July 11, 2012 to June 5, 2013; 230 days) was done:

- using the Long Period **A** calibration (Table 4, case A and Table 1). Results of the simulation are presented in Figure 19.
- using the Short Period **B** calibration (Table 4, case B and Table 2). Results of the simulation are presented in Figure 20.

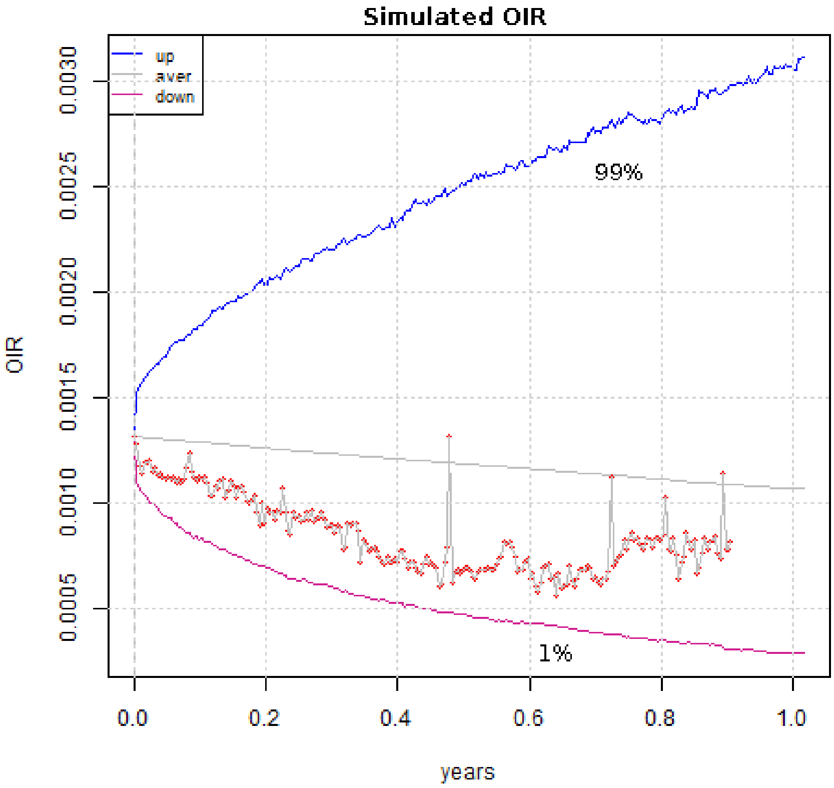

Figure 19: The short term OIR simulation using the Long Period **A** calibration: the 99% and 1% quantiles, the simulated OIR average, and historical rates (dots)

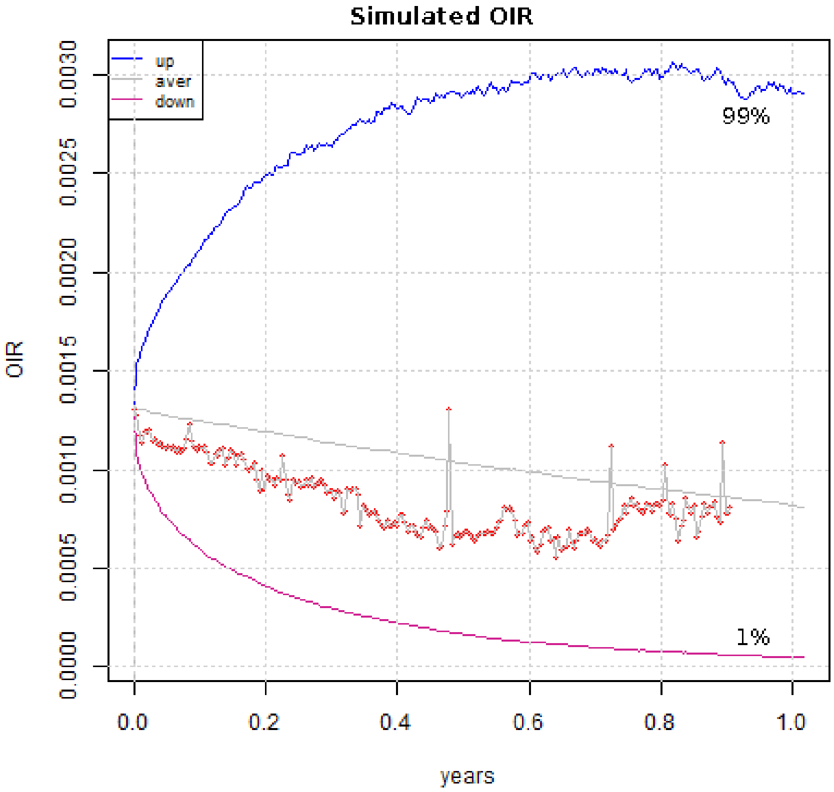

Figure 20: The short term OIR simulation using the Long Period **B** calibration: the 99% and 1% quantiles, the simulated OIR average, and historical rates (dots)

Simulated OIR are presented in Figure 19 and in Figure 20 by upper/lower percentiles (99% / 1%) and by the average of simulated rates. Historical rates (not used for calibration) are plotted as dots. In both cases historical rates do not deviated far from the simulated averages. In both cases (Figures 19 and 20) the historical "out-of-sample" rates lie within the quantile envelope (99% - 1%).

3.3.2 The long term OIR Simulation

The "out-of-sample" OIR simulation for a long term (December 31, 2004 to December 30, 2011; 1796 days) was done using Long Period **C** calibration (Table 4 and Table 3). Results of the simulation are presented in Figure 21.

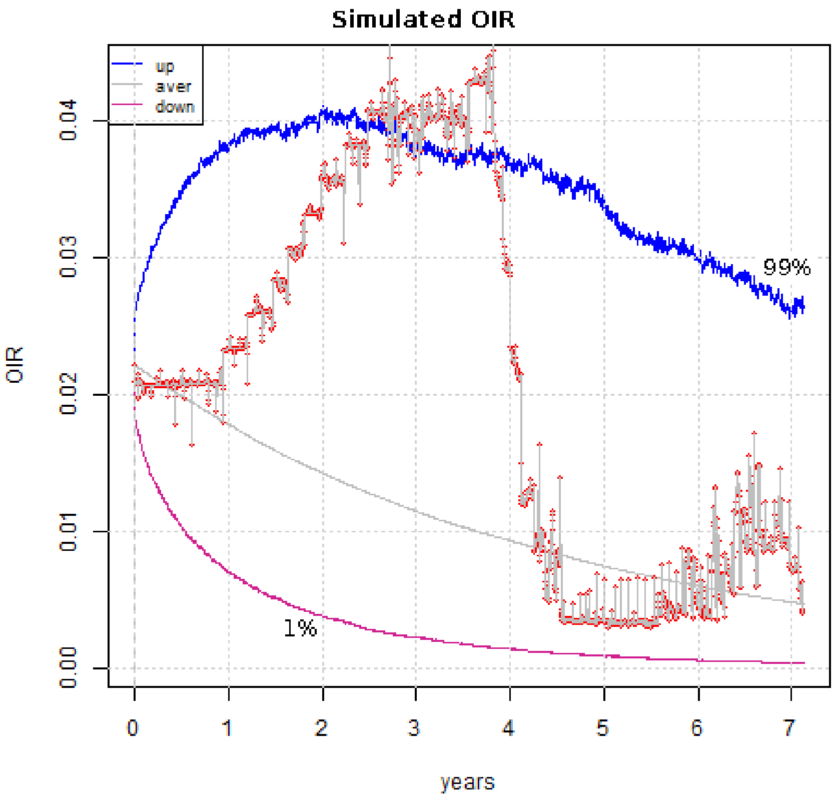

Figure 21: The long term OIR simulation using the Long Period **C** calibration

In spite of the strong upward/downward drifts of the rate during certain periods of time, the envelope of upper/lower quantiles covers most of historical rate changes. The simulated OIR average and the historical rates have similar time tendencies.

4 Summary

The extended Overnight Index Rate model was developed and validated. The model is based on auto-correlated daily log returns with the special stochastic driver (represented by the mix of three different Gaussian processes). The density distribution of this stochastic driver provides flexible modelling of the narrow central peak, the medium width component, and the wide fat-tailed band. The calibration algorithm was developed, tested and validated using both "in-sample" and "out-of-sample" OIR simulations. The model is well suited for the OIR simulation in both quiet and stressed market conditions. It can be used both for OIR forward estimation and for pricing of OIR-based derivatives (such as Overnight Interest rate Swaps).

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