The debt trap: a two-compartment train wreck

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Abstract

We aim to shed light on the debate among policy-makers trying to find prescriptions that will take troubled economies out of their debt trap. We do this with a highly stylized two-compartment dynamic model consisting of the stocks of money in Government and Society. The dynamics of the system are described by a simple four-parameter linear system of two differential equations. The solutions are investigated in closed form and provide precisely quantified "escape conditions" from the debt trap: receipts must be slightly larger than outlays and there must be sufficient annual inflows of funds into the system. The model fits the data for the U.S. between 1981 and 2012 with a coefficient of correlation of 0.996. The model is used to extrapolate the two stocks beyond 2012 with three escape scenarios which shed light on monetary flows needed to take the U.S. economy out of its debt trap.

Keywords: Compartmental model, debt, system of differential equations, dynamical system, fiscal policy.

JEL Classification: C51, C62, C63, E61, H63.

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1 Introduction

Until a few years ago, mainstream economists were convinced that they had finally understood the origins of big recessions like the one started in 1929. Above all they knew how to prevent them.[1] They use sophisticated instruments like Discrete Stochastic General Equilibrium (DSGE) models to inform macro-economic debates. These models have been criticized by some (Solow (2010); Kocherlakota (2010)) but are still used by Central Banks and the European Union among others (see Ratto et al. (2009) and Annicchiarico et al. (2013) for a recent application to Italy).

Inspired by these models Ben Bernanke expressed in 2004 his optimism that the *Great Moderation* (a reduction of the volatility of business cycles) would continue[2]. Today, in the midst of the worst and still ongoing economic crisis since 1929 such a belief sounds naive. The same Bernanke, at a conference organized at Princeton University in 2010 admitted that "economists failed to predict the nature, timing, or severity of the crisis". He added that "some observers have suggested the need for an overhaul of economics as a discipline, arguing that much of the research in macroeconomics and finance in recent decades has been of little value or even counter-productive" (Bernanke, 2010).

The reasons behind the failure of many beautiful and mathematically rigorous economic theories are the object of heated debates. Some suggested replacing the old models with better empirically-based ones. (Colander et al. (2008); Quiggin (2010) and others). Even before the crisis, Paul Krugman was sceptical of DSGE models. In his seminal paper Krugman (2000) suggested we not discard too easily old-fashioned *ad hoc* models like the Hicksian IS-LM model. These two proposals exemplify the criticism to the micro-foundation of the macro-economic models developed by many influential economists. Indeed, these macro models are essentially based on the aggregation of identical hyper-rational agents (the so-called representative agent hypothesis). Colander et al. (2008) point out that these models ignore heterogeneity among agents, who are at best bound-

[1] In 2003, one of the most important and influential economists, the Nobel laureate Robert Lucas, in his Presidential Address at the 115th meeting of the American Economic Association said that the "central problem of depression prevention has been solved".

[2] Meetings of the Eastern Economic Association, Washington, DC, 20 April 2004.

edly rational. As a result they suggest using agent-based models which capture individual heterogeneities more realistically.

Krugman argues that simple non-micro-founded models like IS-LM or AD-AS models have a role to play, next to "the bigger, more micro-founded models [which] have not lived up to their promises". More generally he adds that small *ad-hoc* models are useful and that "we need to keep them alive".

The model we propose in this paper follows Krugman's suggestion. Unlike DGSE models which rely on dozens of parameters ours is a highly stylized dynamical systems model that attempts to capture with only four parameters the flow of funds between the government and society.

We thus hope to make an objective and dispassionate contribution to the vital debate among economists trying to find policies that will get economies out of their debt trap. This debate rages both in Europe and in the U.S. Indeed, since the beginning of the current crisis the European Commission has urged Greece, Portugal, Italy, and Spain to increase taxes and reduce public expenditures. Many economists feel that debt-reduction is indeed the first priority and support this approach, both in Europe and the U.S. For example Reinhart and Rogoff (2010) warn against a debt above 90% of GDP - a finding put into question by Herndon et al. (2013) who have pointed out problems with the methodology used. Others, led by Krugman urge caution and warn that austerity can lead to doubts about government solvency (see Krugman (2012) and other articles on Krugman's blog). They also fear the long-term impact of reduced growth associated with austerity measures and advocate stimulus spending instead of austerity. One can see both sides: pro-austerity economists emphasize the need to stabilize public finances while pro-stimulus ones fear that austerity can be a vicious circle that keeps an economy permanently in its debt trap.

This trap is not unlike the Malthusian one faced by European economies for millennia up to the Industrial Revolution (Komlos and Artzrouni, 1990). Until then the Malthusian trap kept human populations at or below their subsistence level while a homeostatic mechanism insured that a period of crisis was always followed by a recovery. Similarly, the current debt trap keeps the government in debt - but sadly with no homeostasis in sight. For both traps the challenge is to describe the nature of the trap and the way out.

Our model consists of two compartments, Government and Society, with monetary flows between the compartments and the outside world (see Tramontana (2010) for a detailed discussion of the compartmental approach in economics and an application to macroeconomics). Many simplifying assumptions are made. The roles of inflation, GDP, financial markets and wealth distribution among others are ignored. The role of populations is ignored too - or rather it is implicitly assumed to grow with society's stock which we will see is monotonically increasing. The model is also time-invariant: parameters are fixed, although they can change at one point in time as will happen in our numerical illustration.

Despite its extreme simplifications, we will see that the model captures well the evolution of the stocks of money in the two compartments for the U.S. between 1981 and 2012. A simple mathematical analysis of the escape conditions is used to shed light on the conditions required for the U.S. to escape its debt trap.

In Section 2 we describe the model and provide simple closed-form expressions for the two stocks. Section 3 is devoted to a detailed analysis of the escape condition, i.e. the conditions on the four parameters under which the government's stock grows unhindered. Section 4 is devoted to an application to the U.S. federal debt. Three escape scenarios are given. In Section 5 we discuss the significance of the results in the context of the stimulus-austerity debate. We conclude in Section 6 with possible extensions.

2 The model

2.1 Description

We will use a simple dynamical system to describe our two-sector economy consisting of a Government ("G") and Society ("S"). The model is compartmental in the sense that both the G and S sectors are characterized at every instant t by their total stocks of money $G(t)$ and $S(t)$ (Figure 1). The model is a purely financial one with monetary flows between the two compartments and the outside world. When a stock is positive it increases with interest rate ι (inflow). Society's stock will always be positive, but the government's stock can be negative ("debt"). In this case the interest paid $\iota G(t)$ is an outflow. The G sector spends money (public expenditures or "outlays") in the S sector and also draws from the S sector the revenues (tax revenues or "receipts") needed to provide vital services (infrastructure, civil servants, etc.). The S sector incorporates all forms of economic activity (manufacturing, services, etc.). For example in the U.S. the G compartment might be the federal government characterized by a stock called the public debt when it is negative. Society's stock might be the M2 or M3 money supply - although this supply must exclude funds held by banks.

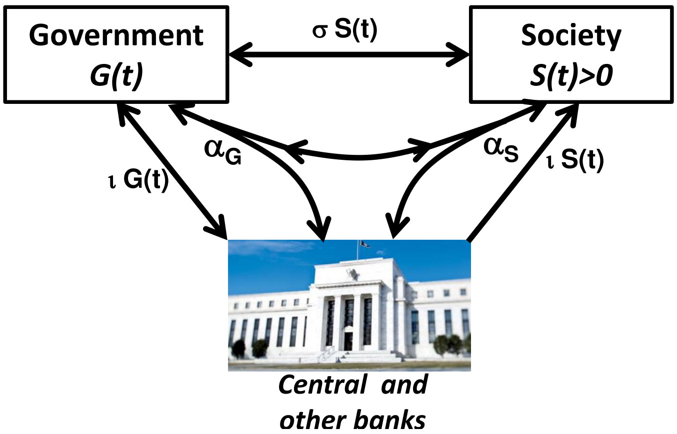

Figure 1: Government and Society compartments with a transfer $\sigma S(t)$ between the two. The direction of the flow is determined by the sign of σ. The annual flows α_G and α_S between the two compartments and the outside world are positive or negative. The interest rate ι is positive with interests $\iota S(t)$ flowing into S because society's stock is always positive. The government pays an interest $\iota G(t)$ when it is in debt ($G(t) < 0$) and receives $\iota G(t)$ when its stock is positive ($G(t) < 0$).

The model is in continuous time and we will describe the temporal dynamics of $(G(t), S(t))$ with a system of two linear differential equations. There are two components to each one of the derivatives dG/dt and dS/dt. The first one is $\iota G(t)$ and $\iota S(t)$, respectively, where $\iota > 0$ is an "intrinsic interest rate": the stocks increase per year by the quantities $\iota G(t)$ and $\iota S(t)$ that are received or paid out depending on the sign of $G(t)$ and $S(t)$. When $G(t)$ is negative then $\iota G(t)$ is negative and is an interest paid. With $S(t)$

being positive the S compartment earns an interest $\iota S(t)$ per unit of time. A simplification of the model is that the lending and borrowing are thus at the same unchanging rate ι. These flows link the G and S compartments to an outside world of banks and other financial institutions that lend, borrow and print money. (For this reason these financial institutions must not be included in the S sector).

The second components of the flows dG/dt and dS/dt are the affine functions $\alpha_G + \sigma S(t)$ and $\alpha_S - \sigma S(t)$ of $S(t)$. The system is, with explanations below:

$$\frac{dG}{dt} = \iota G + \alpha_G + \sigma S \tag{1}$$
$$\frac{dS}{dt} = \iota S + \alpha_S - \sigma S. \tag{2}$$

The two affine functions capture crudely but with some flexibility the transfers of money between the two compartments and the outside world. The quantity $\sigma S(t)$ is a net annual flow from S into G. This flow captures the net effect of receipts (e.g. taxes which are related to GDP) and of outlays (e.g. schools, civil servants which are related to the size of the population). These receipts and outlays are intrinsically proportional to the size of the economy/population crudely measured by society's stock $S(t)$. The coefficient of proportionality σ (or "transfer rate") is positive when receipts exceed outlays and negative otherwise (always assuming $S(t)$ remains positive).

The constant annual flows α_G and α_S are also of arbitrary sign. They can reflect exchanges of money with the outside world or with central banks (foreign trade or investments, money printing, etc.) They can also reflect a net balance of constant *annual* flows between the two compartments (unlike the flow $\sigma S(t)$ which is proportional to $S(t)$).

Fiscal and monetary policies are tuned through the signs and sizes of σ and the α's. Fiscal policies can change the sign and magnitude of σ. Monetary policies can determine the α's. For example the government (or the Central Bank) can on an annual basis print the same amount of money, buy bonds, or engage in various others forms of quantitative easing. For example the government may annually give itself 0.2 units of free money and stimulate the economy by spending 0.3 on public works in S. Its annual debt increases by

0.1 ($\alpha_G = -0.1$) while society's stock increases by 0.3 ($\alpha_S = 0.3$).

Alternatively the government may want to spend the same 0.3 on public works by borrowing the entire amount, not printing money. This increases the government debt by $\alpha_G = -0.3$ with the same result as above for S ($\alpha_S = 0.3$). In this case the sum $\alpha_G + \alpha_S$ is zero and Eqs. (1)-(2) show that the sum of derivatives $\frac{dG}{dt} + \frac{dS}{dt}$ is equal to $\iota(G+S)$. Because $G(0)+S(0)$ can be negative we conclude that when $\alpha_G + \alpha_S = 0$ then the absolute value of the sum $G(t)+S(t)$ of the stocks increases at the interest rate ι.

The long-term consequences of these various strategies are uncertain. However things will become clearer once we obtain and analyze the solutions to the system of differential equations (1)-(2).

2.2 Solution

The linear system (1)-(2) is simple particularly as (2) is a differential equation in S alone. The solution is

$$G(t) = \left(G(0) + S(0) + \frac{\alpha_G + \alpha_S}{\iota}\right) \mathrm{e}^{\iota t} - \left(S(0) + \frac{\alpha_S}{\iota - \sigma}\right) \mathrm{e}^{(\iota - \sigma)t} + \frac{\sigma(\alpha_G + \alpha_S) - \iota\alpha_G}{\iota(\iota - \sigma)} \tag{3}$$

$$S(t) = \left(S(0) + \frac{\alpha_S}{\iota - \sigma}\right) \mathrm{e}^{(\iota - \sigma)t} - \frac{\alpha_S}{\iota - \sigma}. \tag{4}$$

We note that the sum of the two stocks is

$$G(t) + S(t) = \left(G(0) + S(0) + \frac{\alpha_G + \alpha_S}{\iota}\right) \mathrm{e}^{\iota t} - \frac{\alpha_G + \alpha_S}{\iota}. \tag{5}$$

This shows that up to the additive constant $-(\alpha_G + \alpha_S)/\iota$ the absolute value of the sum of the stocks will grow exponentially at rate ι. The sign of the sum, asymptotically, is that of the coefficient of $\mathrm{e}^{\iota t}$ in Eq. (5).

3 Escape analysis

3.1 Escape conditions

We will say that the economy escapes the debt trap if in the long-run both $G(t)$ and $S(t)$ remain positive. Equations (3)-(4) show that this can only happen with $G(t)$ and $S(t)$ asymptotically exponential with rates $\iota > 0$ and $\iota - \sigma > 0$ respectively. ($G(t) \sim \mathrm{e}^{\iota t}$ and $S(t) \sim \mathrm{e}^{(\iota-\sigma)t}$). Equation (4) shows that both $S(0) + \alpha_S/(\iota - \sigma)$ and $\iota - \sigma$ must be positive in order to have $S(t) \sim \mathrm{e}^{(\iota-\sigma)t}$. Equation (3) shows that $G(0) + S(0) + \dfrac{\alpha_G + \alpha_S}{\iota}$ must be positive and σ smaller than ι in order to have $G(t) \sim \mathrm{e}^{\iota t}$. In short the economy escapes the debt trap with $G(t) \sim \mathrm{e}^{\iota t}$ and $S(t) \sim \mathrm{e}^{(\iota-\sigma)t}$ if and only if the following three conditions are satisfied:

C1. $G(0) + S(0) + \dfrac{\alpha_G + \alpha_S}{\iota} > 0$.

C2. $0 < \sigma < \iota$.

C3. $S(0) > -\alpha_S/(\iota - \sigma)$.

Under conditions C1-C3 we have $S(t) \sim \mathrm{e}^{(\iota-\sigma)t}$ and $S(t)$ increases monotonically from time $t = 0$. On the other hand $G(t) \sim \mathrm{e}^{\iota t}$ either after an initial decrease ("eventual escape") or like $S(t)$ with a monotone increase from $t = 0$ ("immediate escape"). The escape is eventual (or immediate) if the initial value $dG(0)/dt$ of the derivative of $G(t)$ is negative (or positive). Equation (1) provides the expression for the derivative and shows that the condition

C4. $\iota G(0) + \alpha_G + \sigma S(0) > 0$

insures an immediate escape and also implies C1 when combined with C2-C3.

3.2 Escape locus in $(\alpha_G + \iota G(0), \sigma S(0))$ space

The model (and escape conditions) depend on four parameters $(\iota, \sigma, \alpha_G, \alpha_S)$ and on the initial values $G(0), S(0)$ of the stocks. We will formulate the escape conditions C1-C4 in the 2D space (λ, ζ) defined by

$$\lambda \stackrel{\text{def.}}{=} \alpha_G + \iota G(0),\ \zeta \stackrel{\text{def.}}{=} \sigma S(0). \tag{6}$$

In this space we define the vertical and horizontal lines

$$\lambda_0 \stackrel{\text{def.}}{=} -\alpha_S - \iota S(0),\ \zeta_0 \stackrel{\text{def.}}{=} \alpha_S^- + \iota S(0), \tag{7}$$

where x^- denotes $\min(x, 0)$ for any real x.[3]

The escape conditions C1-C4 translate into the four regions R_k (k=1,2,3,4) of the space (λ, ζ) corresponding to the different asymptotic behaviours of the stocks (see Figure 2 and its caption for details). Conditions C2 and C3 are equivalent to ζ between 0 and ζ_0 which must be positive (λ_0 is then negative). The condition C4 is equivalent to (λ, ζ) on the right of the diagonal $\zeta = -\lambda$. The condition C1 is equivalent to $\lambda > \lambda_0$.

The fact that $\zeta = \sigma S(0)$ must be between 0 and ζ_0 means that for an escape σ must be positive but not too large. If ζ and therefore σ are negative the excessive G to S transfer means either one of the two stocks goes to $-\infty$ while the other one grows. If ζ is larger than ζ_0 (i.e. $\sigma > \iota$ at least when $\alpha_S > 0$) then the government receipts are so high that $S(t) \to -\infty$. (However $S(t) \to -\infty$ is not an economically realistic scenario).

The interesting case arises when ζ is between 0 and ζ_0. Society's stock in this case always goes to $+\infty$. The behaviour of the government stock depends on $\lambda = \alpha_G + \iota G(0)$. If λ is too far on the left (Region R_1) this means the flow α_G into G or the initial $G(0)$ are too negative. If λ is in the middle region R_2 then an escape occurs after an initial period of decrease for $G(t)$. The closer λ is to the border value λ_0 between R_1 and R_2 the more protracted the period of decrease. An immediate escape takes place for (λ, ζ) in R_3; λ can still be negative but ζ must be larger than $-\lambda$. When λ becomes positive, then an immediate escape takes place for any ζ in $(0, \zeta_0)$.

3.3 Escape conditions

In the numerical application below we will fit the model to U.S. data on total debt and the M2 money stock. We will obtain baseline (fitted) values of the parameters that not surprisingly place the country in a debt trap. The situation is of course not sustainable

[3]The sum $\lambda_0 + \zeta_0$ is non-positive because it is equal to 0 when $\alpha_S \leq 0$ and to $-\alpha_S$ otherwise.

in the long run. We are thus interested in exploring which perturbation of the baseline parameters might bring the system into an escape region. For example one could find the set of parameter values in the escape region that is closest to the baseline values. A simpler prescription is possible when the baseline parameter values put the system in R_1 of Figure 2. With a superscript "b" referring to baseline values one can then increase the sum $\alpha_G^b + \alpha_S^b$ in a way that brings the point (λ^b, ζ^b) into R_2. Indeed increasing α_G^b moves (λ^b, ζ^b) horizontally to the right toward and beyond the vertical line at λ_0. Increasing α_S^b moves that vertical line to the left and (λ^b, ζ^b) can find itself in R_2. We call this a "$\sum_\alpha$-increase prescription" in Figure 2.

Other possibilities include "zero-sum prescriptions" that only change the flow of money between the two compartments without any appeal to external funds. There are two ways of achieving this. One is to modify σ which only changes the flow $\sigma S(t)$ between the two compartments ("σ-zero-sum prescription"). Figure 2 shows that for a sufficiently large increase in σ such a prescription can move to R_2 or R_3 a (λ^b, ζ^b) in R_5 for which λ^b is larger than λ_0 but still negative.

Alternatively, we can shift constant amounts annually from one compartment to the other. We do this by adding a quantity δ to the baseline α_G^b and subtracting the same δ from the baseline α_S^b. We obtain a zero-sum perturbation of the flows which we labelled as a "α-zero-sum prescription" in Figure 2. The modified constant flows are of the form

$$\alpha_G' = \alpha_G^b + \delta,\ \alpha_S' = \alpha_S^b - \delta \tag{8}$$

where δ can be either positive or negative. The parameter ζ^b remains unchanged while λ is increased by δ to become

$$\lambda' = \lambda^b + \delta. \tag{9}$$

The vertical and horizontal lines at λ_0 and ζ_0 become

$$\lambda_0' = -\alpha_S^b + \delta - \iota S(0),\ \zeta_0' = (\alpha_S^b - \delta)^- + \iota S(0). \tag{10}$$

These equations show that any positive or negative δ moves a (λ^b, ζ^b) in R_1 horizontally while the border between R_1 and R_2 (i.e. the vertical line at $\lambda_0^{'}$) also moves in the same direction at the same rate. This means that even with a positive δ an α-zero-sum prescription cannot move (λ^b, ζ^b) from R_1 to R_2. Starting from R_1 an increase in σ is of no help either as it would just move $(\lambda^b, \zeta^b) = (\lambda^b, \sigma^b S(0))$ vertically upward and into R_4.

We recall that a baseline (λ^b, ζ^b) in R_2 means an escape after a initial period of decline for $G(t)$. We ask whether an α-zero-sum prescription with a positive δ could move a (λ^b, ζ^b) from R_2 into R_3 thus insuring a quicker escape. The perturbed $\lambda^{'}$ moves to the right while the vertical border value $\lambda_0^{'}$ also moves to the right at the same rate. Therefore the perturbed $(\lambda^{'}, \zeta^b)$ cannot fall back into R_1. However as we let δ grow there is now a race between the rightward shift of $(\lambda^{'}, \zeta^b)$ that we are trying to bring to the (unchanging) diagonal and the downward move of the perturbed horizontal line at $\zeta_0^{'}$. When $\alpha_S^b < 0$ then the escape region R_2 is reduced to a triangular area in which (λ^b, ζ^b) in R_2 is closer to the diagonal then to the baseline $\zeta_0^b = \iota^b S(0)$. The perturbed $(\lambda^{'} = \lambda^b + \delta, \zeta^b)$ wins the race because it reaches the diagonal for

$$\delta = -\lambda^b - \zeta^b > 0 \tag{11}$$

before $\zeta_0^{'}$ reaches ζ^b.

A positive α_S^b means the existence of a rectangular area in R_2, as in Figure 2. In this case $(\lambda^{'}, \zeta^b)$ can also be brought into R_3 because $\zeta_0^{'}$ starts decreasing only when δ reaches α_S^b.

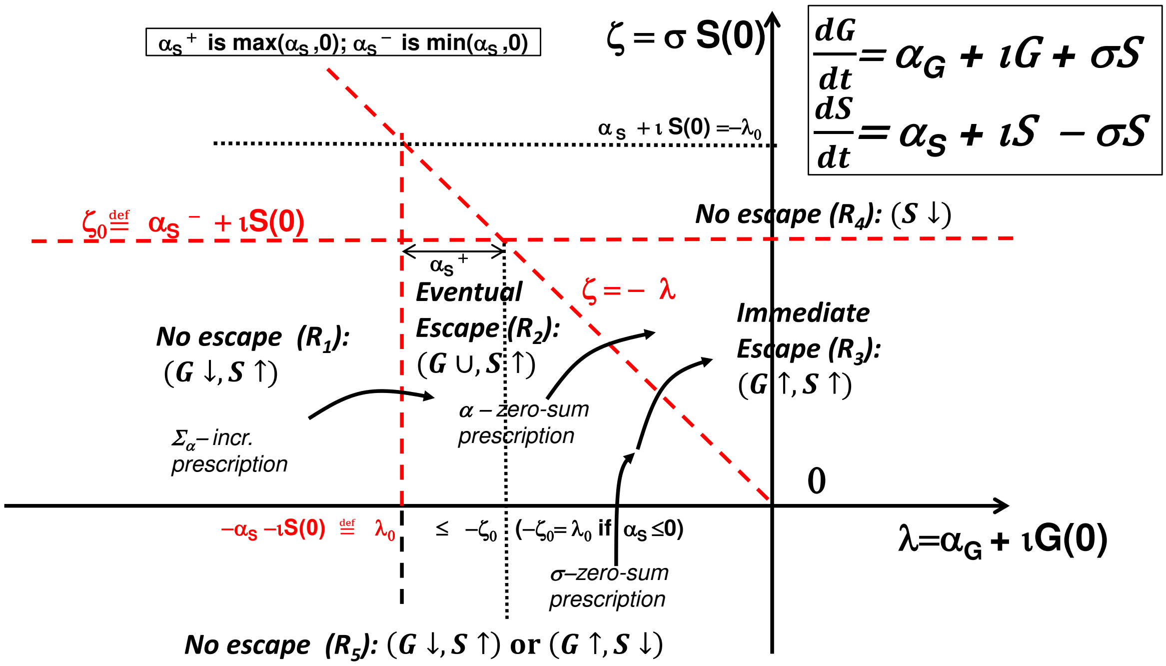

Figure 2: Locus of escape conditions in the (λ, ζ) space. The system can escape the debt trap only if (λ, ζ) is between the horizontal lines $\zeta = 0$ and $\zeta = \zeta_0$. The three regions R_1, R_2 and R_3 are between these two lines. To be in R_1 a point must also be on the left of the vertical line at $\lambda = \lambda_0$. To be in R_2 it must be below the diagonal $\zeta = -\lambda$ and to the right of the vertical line at $\lambda = \lambda_0$. To be in R_3 it must be above that diagonal - while always remaining between the two horizontal lines. If $\lambda = \alpha_G + \iota G(0)$ is too negative (in region R_1) there is no escape. For the intermediate region R_2 the escape will occur eventually, after an initial decrease in $G(t)$. For λ large enough (R_3) there is an immediate escape. (A variable name G or S followed by an up arrow $\uparrow$ (or down arrow $\downarrow$) means the variable goes to plus (or minus) infinity exponentially fast; "G $\cup$" means $G(t)$ first decreases before growing exponentially.)

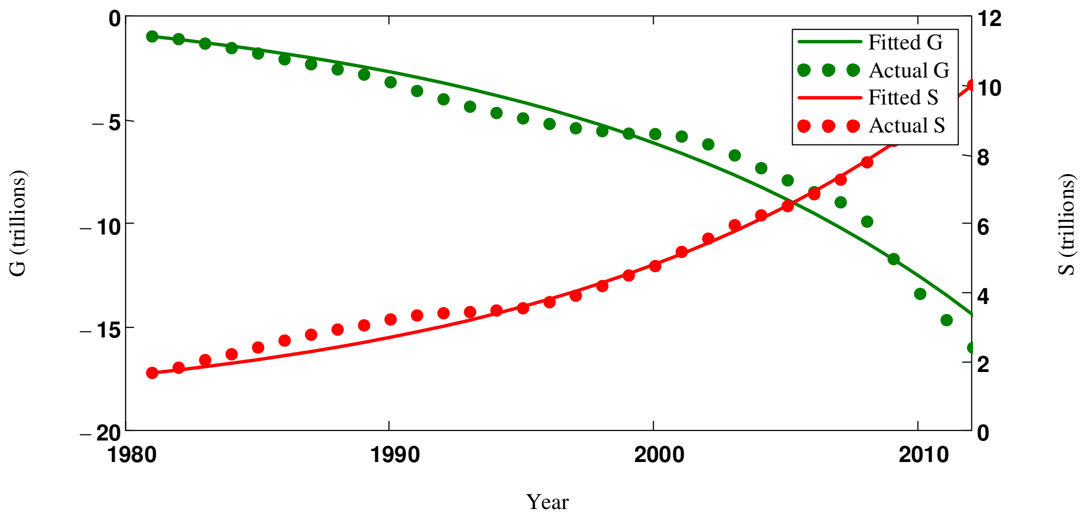

Figure 3: Fitted and actual trajectories of public debt G (left axis) and M2 money supply S (right axis) in the U.S., 1981-2012, trillions of USD (coefficient of correlation $r = 0.996$).

4 Application

4.1 Model fitting to U.S. data

We illustrate the model with data on the U.S. federal budget ($G(t)$) which is readily available. We recognize that this is gross simplification, mostly because we thus ignore the role of states which provide many services. For this reason we consider that the exercise is for illustrative purposes only. Society's stock is an even bigger challenge. Indeed, the money supplies M1-M3 come to mind but we are unsure which one is the most relevant; M1 or M3 may be too narrowly or broadly defined. In the U.S. no data on the M3 money supply is available after 1986. For these reasons we chose M2.

If society's real stock is quite different from M2 and/or immeasurable this complicates the use of the model, although all is not necessarily lost. Indeed suppose a "real" stock S were an affine but unknown function of the M2 stock: $S = \rho_0 + \rho_1 M2$. Substituting this expression for S in Eqs. (1)-(2) yields a model in $(G(t), M2(t))$ that is structurally the same as before with $M2(t)$ as a proxy stock for society. Indeed σ becomes $\rho_1\sigma$; α_G

becomes $\alpha_G + \rho_0\sigma$ and α_S becomes $\alpha_S - \rho_0\sigma$. The unknown ρ's have been folded into the model's unknown parameters.

We used data from 1981, the earliest year available for the readily downloadable data on M2 from the Federal Reserve.[4] The last data point was for 2012 ($n = 32$). We fixed the initial values of the stocks $G(0)$ and $S(0)$ at their 1981 values. We then attempted to estimate the four parameters α_G, α_S, ι, and σ by minimizing the sum of the 64 squared deviations consisting of the 32 deviations between observed and fitted government stocks plus the 32 deviations between observed and fitted stocks for society. Because the model is non-linear we used standard numerical methods to find the minimum. However the algorithm converged to a different solution for each initial value of the four parameters - a most undesirable situation. The model is not strictly speaking over-parametrized: there are no two different sets of parameters that yield mathematically the same trajectories of Eqs. (3) - (4). However this "near-over-parametrization" arose because different sets of parameters produced fitted trajectories that were extremely close to one other during the period 1981-2012 - even though they diverged later on.

Because we have information on interest rates we decided to address this problem by fixing ι. We recall however that in our model this interest rate is assumed to be equal for government borrowing and for investors in society and also unchanging through time - two gross simplification. Still, to assess what this interest might be we looked at the series of "Interest Expense on National Debt". Dividing these amounts by the national debt provided perhaps naive estimates of the interest rate paid by the government. This interest rate declined from roughly 7 to 4 % during the 32 years between 1981 and 2012. We thus chose to estimate ι as the average value during that period which is $\widehat{\iota} = 5.875\%$.

With ι now fixed we define the vector θ of the three remaining parameters

$$\theta = (\alpha_G, \alpha_S, \sigma). \tag{12}$$

[4]Federal Reserve Economic Data; https://research.stlouisfed.org/fred2/.

Parameters	Value
Primary parameters and r	
$\widehat{\alpha_G}$	-0.07447
$\widehat{\alpha_S}$	-0.02359
$\widehat{\sigma}$	-0.00561
Coeff. corr. r	0.996
Derived parameters	
$\widehat{\zeta_0} = \widehat{\alpha_S}^- + \widehat{\iota}S(0)$	0.007503
$\widehat{\lambda_0} = -\widehat{\alpha_S} - \widehat{\iota}S(0)$	-0.007503
$\widehat{\zeta} = \widehat{\sigma}S(0)$	-0.00942
$\widehat{\lambda} = \widehat{\alpha_G} + \widehat{\iota}G(0)$	-0.13267

Table 1: Primary parameter estimates, coefficient of correlation and derived parameters in fitting of U.S. data (1981-2012).

These parameters were estimated by minimizing over θ the sum of squared deviations

$$SS_1(\theta) = \sum_k (G(k)^{actual} - G(k,\theta))^2 + \sum_k (S(k)^{actual} - S(k,\theta))^2 \tag{13}$$

where $G(k,\theta)$ and $S(k,\theta)$ are for the year k the modelled values given in Eqs. (3)-(4).

The initial values of the stocks are the debt and M2 money supply on 1/1/1981: $G(0) = -0.991; S(0) = 1.679$ (in trillions of USD). The search procedure converged nicely to the same estimated value $\widehat{\theta} = (\widehat{\alpha_G}, \widehat{\alpha_S}, \widehat{\sigma})$ for any reasonable initial values of the parameters. These three "primary parameter" estimates as well as the derived ones are given in Table 1.[5]

The solutions in Eqs. (3)-(4) are now

$$G(t) = -1.31213\,\mathrm{e}^{0.06437t} - 0.98105\,\mathrm{e}^{0.05875t} + 1.3026 \tag{14}$$

$$S(t) = 1.31213\,\mathrm{e}^{0.06437t} + 0.36657 \tag{15}$$

with $t = 0$ corresponding to Jan 1, 1981 (Figure 3). With a coefficient of correlation of 0.996 the model is able to describe with four parameters (one of which was fixed) the

[5] Ours is not an econometric model and we do not attempt to derive confidence interval for the parameters - which we estimate just for illustrative purposes.

dynamics of the U.S. debt and of the M2 stock of money over 32 years.

There are several ways in which an economy can be caught in a debt trap. The value of ζ_0 which defines the horizontal border between $R_1 - R_3$ and R_4 could be negative; λ_0 could be positive. Even if ζ_0 and λ_0 are of the right sign, λ could be smaller than λ_0, thus putting the system in the trap region R_1. Here the U.S. is in a debt trap because the estimated σ is negative: modelled outlays are larger than receipts - as is the case with actual values.

4.2 Conditions toward an escape from the U.S. debt trap

In order to investigate scenarios (i.e. parameter values) under which the U.S. may escape its current debt trap we will run the model from Jan. 1, 2012 with initial values of the two stocks equal to the actual ones on that date: $G(0) = -15.984$ and $S(0) = 10.003$ (these are relatively close to the "endpoint" modelled values obtained from Eqs.(14)-(15) with $t = 31$ i.e., -14.410 and 10.017). Our goal is to use the insights gained above to find the minimum modifications of the estimated parameter values for which the economy can escape the debt trap.

With a falling trend for interest rates during 1981-2012 we will run the model from 2012 using as a baseline interest ι^b the rate defined like above as the ratio of interest payment to debt on Dec 31, 2011 (last reliable value available): $\iota^b = 454/15223 = 0.02982$.

Because the system cannot escape with a negative σ we set $\sigma^b = 0$. This means $\zeta^b = \sigma S(0) = 0$. We take as baseline α's the values obtained above:

$$\alpha_G^b = \widehat{\alpha_G} = -0.07447,\ \alpha_S^b = \widehat{\alpha_S} = -0.02359. \tag{16}$$

We then have

$$\lambda_0^b = -\alpha_S^b - \iota^b S(0) = -0.27473, \tag{17}$$

$$\lambda^b = \alpha_G^b + \iota^b G(0) = -0.55117 \tag{18}$$

Parameter	Parameter values for three scenarios		
	Flat G	Immediate escape	Eventual escape
α_G	$\alpha_G^{'} = 0.4767$	$\alpha_G^{'} = 0.4767$	$0.7\alpha_G^{'} = 0.33369$
α_S	-0.02359	-0.02359	-0.02359
σ	0	0.01	0.01
ι	0.02982	0.02982	0.02982

Table 2: Primary parameters chosen for three projection scenarios of S and G stock in the U.S., beyond 2012.

and $\zeta_0^b = -\lambda_0^b = 0.27473$ because $\alpha_S < 0$.

The fact that $\lambda^b < \lambda_0^b$ means the system is in Region R_1 on the border with R_5 (because $\zeta^b = \sigma^b S(0) = 0$.) Therefore the system is still in a debt trap. Bringing the system out of the trap with a minimum change in parameters requires a λ^b at least equal to the border value $\lambda_0^b = -0.27473$. As already discussed this can mean a very slow escape preceded by a protracted decline of $G(t)$. To attain an immediate escape we need to move $(\lambda^b, \zeta^b = 0)$ to $(0, 0)$, the closest border point between R_2 and R_3. Equation (18) shows that a zero λ^b means that α_G^b must become

$$\alpha_G^{'} = -\iota^b G(0) = 0.4767. \tag{19}$$

This last equation combined with $\sigma = 0$ means that $G(t)$ of Eq. (3) is constant. We will call this scenario "Flat G" meaning an unchanging government stock. With $\alpha_G = \alpha_G^{'}$ there will be an immediate escape if σ becomes infinitesimally larger than 0 since ζ^b becomes positive and $(\lambda^b = 0, \zeta^b > 0)$ is then inside R_3 ("Immediate escape" scenario, below with $\sigma = 0.01$). If, for illustrative purposes, $\alpha_G^{'}$ of Eq. (19) is reduced to 70% of its value (i.e. to $0.70\alpha_G^{'} = 0.33369$) then $(\lambda^b, \zeta^b > 0)$ falls back into R_2 and we have an "Eventual escape" scenario. The parameter values corresponding to these three projection scenarios are brought together in Table 2. Extrapolated values of the stocks are plotted in Figure 4.

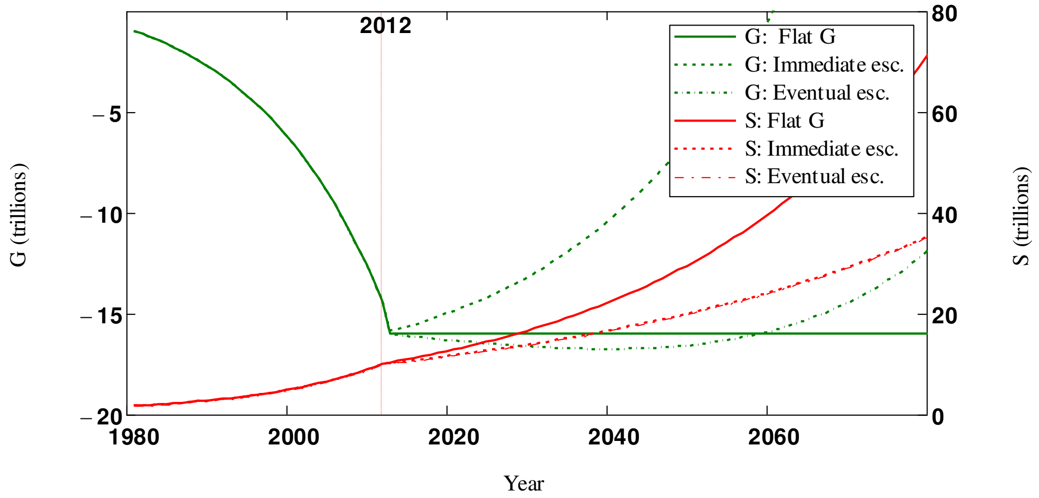

Figure 4: Fitted (until 2012) and extrapolated G stock (left axis) and S stock (right axis), U.S., 1981-2012, trillions of USD. There are three extrapolation scenarios. "Flat G" corresponds to the minimum $\alpha_G = 0.47670$ government annual inflow *above which* an immediate escape will occur (with $\sigma = 0$). The "Immediate escape" scenario corresponds to the same α_G but with $\sigma = 0.01$. (The system is in Region R_3 of Figure 2). The "Eventual escape" scenario corresponds to an α_G reduced to 70% of its initial value ($\alpha_G = 0.3337$) which puts the system back in Region R_2. The extrapolated stock S is the same for the third and second scenario because S does not depend on α_G.

In summary, the model suggests that the U.S. could start pulling itself out of its trap with a balance between outlays and receipts ($\sigma = 0$) combined with an infusion of funds at least equal to 0.47670 trillion USD a year. The timing of events can be read off Figure 4. A transfer rate σ of just 0.01 means an immediate escape with a stock G reaching 0 after 50 years around 2060. With an annual infusion of 0.33369 instead of 0.47670, the U.S. government's stock continues to sink until approximately 2040 and then recovers ever so slowly.

4.3 Discussion of escape conditions

4.3.1 The U.S.

Escape conditions can go beyond parameter values and affect initial values. Indeed, mirroring the way random population crises kept populations of the past in their Malthusian trap, a sufficiently large one-off positive deterministic shock on the initial values $G(0)$ and $S(0)$ can take the government's budget out of its trap. For example the effect of the Troubled Asset Relief Program (TARP) in the U.S. could have been captured, however crudely, by our model. We would have stopped the model in 2008 and restarted it after increasing initial values of the stocks by a total 0.475 trillion USD, the amount injected in the economy under TARP. A one-off injection of 0.475 was probably not enough to take the U.S. economy out of its debt trap - in reality or in our model. The model can then be used to assess, at least in broad terms, the size and G/S allocation of such a one-off stimulus that would be sufficient to bring about the escape.

The 0.475 amount is coincidentally very close to the prescribed minimum *annual* injection of funds $\alpha_G^{'} = 0.47670$ deemed sufficient to halt the drop in G if $\sigma = 0$ and to insure an immediate escape if $\sigma > 0$. Despite the purely illustrative nature of the example we can draw some very tentative conclusions: balanced books and a half-trillion USD annual inflow into government coffers could take the U.S. out of its debt trap.

4.3.2 Japan

In early 2013 the Bank of Japan announced an aggressive stimulus package in the form of government bond purchases worth 50 trillion yens per year. This stimulus would typically be expressed in our model through a 50 trillion yen increase in the sum $\alpha_G + \alpha_G$. The analyst is left with the challenging task of deciding how to allocate this amount between the two compartments. No doubt there is an optimal such allocation. One issue is the knock-on effects of such prescriptions, for example on interest rates (our ι) and on inflation which seems to be a concern in Japan. Although beyond the scope of the present paper these questions raise the possibility of feedback mechanisms between parameters.

5 Discussion

Despite its extreme simplicity our model shows that an escape from a debt trap rests on fairly subtle conditions concerning monetary flows between the government and society.

In summary, two conditions are required for an escape from the debt trap. First there must be a non-negative flow from society into government that is proportional to society's stock without being too large: the transfer rate σ can be small and must be less than the interest rate ι. Second, recurrent constant annual expenditures are vital for an escape. Intriguingly, it is worth noting that regardless of the size of these annual flows, the asymptotic growth rate of the G and S stocks are always ι and $\iota - \sigma$: large *annual* expenditures do not cause inflation in the long run. We believe that these conditions may in a general, qualitative sense, apply within more complex models - and perhaps even to the real world.

Zero-sum perturbations of the sum of α's which divert funds from one sector to the other cannot alone take a system out of its debt trap. However in the case of an "eventual escape" the model shows than an annual diversion of funds from S to G brings forward the time at which the government's stock bottoms out.

We note that an exponential growth of the government's stock is a mathematical construct that does not correspond to reality. Once the economy has escaped the debt trap it is possible for the government to plan the next election and start spending more in Society - perhaps even making σ at least temporarily negative. The government's long-term aim then is to keep its budget at or near an equilibrium.

We emphasize that ours are not *policy prescriptions* but merely *monetary flow prescriptions*, which is not the same thing. For example we resist the temptation to translate an increasing σ into an increased tax rate - because things may not be that simple and one does not necessarily follow from the other. Complex feedback mechanisms can lead to counter-intuitive results which can however be fed into the model. For example a policy-maker may decide that a particular tax *decrease* will increase economic activity, growth, and consumer spending - thus enhancing revenues in ways that more than offset the original tax decrease. He will formulate this policy by increasing σ, not decreasing it. In short we suggest "directional flows" of funds leading to an escape, but do not take a position on how to achieve the result - thus eschewing the austerity-stimulus debate.

Things could be different with an improved version of the model in which the S sector has sub-compartments such as households, firms, etc. With a richer model and set of parameters it may be possible to equate flow prescriptions to austerity/stimulus prescriptions more plausibly than if we are limited to the single parameter σ.

6 Conclusion

The model performed well when applied to illustrative data on the U.S. and initial work suggests that the same may hold for other countries (France in particular). There are several ways however in which the model can be extended. We could add sub-compartments to one or both of the existing ones. We could endogenize relationships between the parameters. We could include inflation in the model beyond the parameter ι which already captures the intrinsic increase in the money stock. We could replace $\sigma S(t)$ with a nonlinear function of $S(t)$. However we then risk losing the benefit of closed form solutions, which are few and far between when it comes to solving systems of differential equations. Still, despite its extreme simplicity our model plausibly describes complex macro-economic realities even if it does not provide definitive answers to the austerity/stimulus debate. Our model may however point in the direction of sensible solutions to the difficult and burning problem of preventing the train wreck facing those economies unable to escape their debt trap.

References

Annicchiarico, B., Di Dio, F., Felici, F. 2013. Structural reforms and the potential effects on the Italian economy, Journal of Policy Modeling, 35, 88-109.

Bernanke, B. 2010. Implications of the Financial Crisis for Economics, Conference Co-sponsored by the Center for Economic Policy Studies and the Bendheim Center for Finance, Princeton University, Princeton, New Jersey, 24 Septmber 2010.

Colander, D., Howitt, P., Kirman, A., Leijonhufvud, A. and Mehrling, P. 2008. Beyond DSGE Models: Towards an Empirically Based Macroeconomics. American Economic Review 98, 236-240.

Herndon, T., Ash, M., and Pollin, R. 2013. Does High Public Debt Consistently Stifle Economic Growth ? A Critique of Reinhart and Rogoff. Working Paper No 322. Political Economy Research Institute, University of Massachsetts, Amherts.

Kocherlakota, N. 2010. Modern Macroeconomic Models as Tools for Economic Policies (2009 Annual Report). Banking and Policy Issues Magazines, The Federal Reserve Bank of Minneapolis.

Komlos, J. Artzrouni M. 1990. Mathematical investigation of the escape from the Malthusian trap, Mathematical Population Studies, 2(4), 269-287.

Krugman, P. 2000. How complicated does the model have to be? Oxford Review of Economic Policy, 16(4), 33-42.

Krugman, P. 2012. Blunder of Blunders. New York Times Blog, 22 March.

Quiggin, J. 2010. Zombie Economics: How Dead Ideas Still Walk Among Us. Princeton University Press, Princeton, USA.

Ratto, M., Roeger, W. and in't Veld, J. (2009) QUEST III: An estimated open economy DSGE model of the Euro area with fiscal and monetary policy. Economic Modelling, 26(1), 222-233.

Reinhart, C.M. and Rogoff, K.S. 2010. Growth in the Time of Debt. American Economic Review 100, 573-578.

Solow, R. 2010. Building a Science of Economics for the Real World. House Committee on Science and Technology. Subcommittee on Investigation and Oversight. July 20, 2010.

http://web.archive.org/web/20110204034313/http://democrats.science.house.gov/Media/file/Commdocs/hearings/2010/Oversight/20july/Solow_Testimony.pdf

Tramontana, F. 2010. Economics as a compartmental system: a simple macroeconomic example, International Review of Economics, 57(4), 347-360.