All-Pay Auctions with Polynomial Rewards

Olivier Bos and Martin Ranger

University Panthéon-Assas (Paris 2), University of Bonn

11. May 2013

All-Pay Auctions with Polynomial Rewards*

Olivier Bos†
University Panthéon-Assas (Paris 2)

Martin Ranger‡
University of Bonn

This version: May, 2013

Abstract

This paper examines a perfectly discriminating contest (all-pay auction) with two asymmetric players. We focus on unordered valuations. Valuations are endogenous (polynomial functions) and depend on the effort each player invests in the contest. The shape of the valuation function is common knowledge and differs between the contestants. Some key properties of R&D races, lobbying activity and sport contests are captured by this framework. After analyzing the unique mixed strategy equilibrium, we derive a closed form of the expected expenditure of both players. We characterize the expected expenditure by means of incomplete Beta functions.

Keywords: All-pay auctions, contests
JEL Classification: D44, D72

*A previous version of this paper circulated under the title "All-pay auctions with endogenous rewards". The first author thanks the University of Bonn and CORE (Université Catholique de Louvain) for their hospitality and acknowledges financial support from the European Commission (Marie-Curie Fellowship). We would like to thank Atsu Amegashie, Raouf Boucekkine, Subhasish Chowdhury, Gabrielle Demange, Aner Sela and Ron Siegel for their valuable comments, discussions and remarks. We also thank two anonymous referees for their valuable comments. All remaining errors are ours.

†Corresponding author. University Panthéon-Assas (Paris 2), LEMMA, 5/7 avenue Vavin, 75006 Paris, France. E-mail: olivier.bos@u-paris2.fr.

‡University of Bonn. E-mail: martin.ranger@uni-bonn.de.

1 Introduction

Although commonly assumed to be fixed, the size of the prize in a contest may in fact be endogenous and depend on the effort made by the contestants. In particular, a higher level of effort may lead to higher valuations.[1] In other words, the effort expended in a contest may increase both the probability of winning *and* the size of the prize. Moreover, contestants may differ with respect to the magnitude of this effect, and the same effort levels may lead to a different valuation of winning the contest (Kaplan, Luski, Sela, and Wettstein, 2002).

Such an environment is descriptive of several economic issues. In R&D races, for example, an increase in the amount of resources spent on developing new technologies may result in a shorter product pipeline and in the firm winning the race. At the same time, the additional resources may improve the quality of the final product and therefore its market value. Asymmetric market structures and differences in marketing, existing product variety or spill-over effects to related research projects are likely to lead to differences in the marginal value of R&D spending. Organizational differences in research departments or a different composition of inputs into the research process may likewise lead to different values of winning the race. In some sense, academic hiring efforts may follow a similar pattern. As long as universities attempt to attract faculty by offering productivity enhancing inducements, such as research funds, expanded seminar series or access to data sets, they are likely to increase the productivity of the potential new hire and at the same time the value of being able to hire the desired candidate.

In the classical example of a lobbying contest, the value of the legislation enacted or the project awarded may depend on the magnitude of the contribution to the political institution involved. Asymmetries may enter the contest through the pre-existing political connection of the lobbyist, so that an organization with conservative credentials would obtain a more favourable outcome with a conservative government than a more liberal lobbyist. Lastly, in professional sports, the effort invested by a team increases its expected score making a win more likely. In addition, conditional on having won the game, a higher score may raise the reputation of the team. For teams quoted on the stock market, such as several European soccer teams, this may translate into additional stock price gains.[2] Again, differences between teams may lead to asymmetries in this effect.

This paper examines the equilibrium of a contest with endogenous rewards and derives the aggregate expenditure of the contestants – which is the main contribution of the paper. Indeed, it is useful for the organizer of the contest to know the expected aggregate expenditures if he would like to maximize it and then to affect the design of the contest. It might be the case

[1] Higher effort levels can also lead to a lower value of winning a contest. In a war of attrition, for example, the effort spent to win is likely to make the winner more vulnerable in future conflicts and contestants prefer to win at lower effort levels.

[2] Indeed, players were also stockholders when soccer clubs issued the first shares.

in R&D races. We examine a contest modeled as an all-pay auction, that is a situation in which the contestant investing the largest effort will win with certainty. Regardless of whether they won or not, all contestants have to pay their effort cost. Information is assumed to be complete and valuations are asymmetric and endogenous (polynomial functions). For simplicity, the analysis is limited to two participants. The payoff from winning the contest, that is the valuation less the cost of effort, is assumed to decline strictly in effort despite valuations that increase in the invested effort. Similar to traditional all-pay auctions, the participants in the contest thus prefer to win at lower effort levels. This assumption contrasts with Amegashie (2001) who investigates situations where the returns to additional expenditures in all-pay auctions exceed the costs and players therefore could aim to win with higher efforts. Contrary to the case where payoffs strictly decrease he finds that a Nash equilibrium in pure strategies exists.

A second related paper is Kaplan, Luski, and Wettstein (2003), who investigate a model of innovation and R&D races with a structure similar to an all-pay auction. Information is complete, values and innovation cost are time dependent and firms compete in when to bring the innovation to the market. In consequence, the usual equivalence of staying out of the contest and exercising zero effort does not apply. In contrast to our paper, a closed form solution for expected aggregate expenditure is not given. Siegel (2009, 2010) also studies all-pay auctions with complete information and non-ordered contestants. The differences are in the levels of generality. Whereas Siegel's broad approach covers quite general model specifications – in particular he focuses on contests where all-pay auction is a sub-class of contests – our more specific set-up allows us to determine a closed form solution for expected equilibrium expenditures thanks to the incomplete Beta functions.[3] This result could be useful for applications of contests with – polynomial – endogenous rewards as in R&D races.

Three other recent papers with complete information are also related to our work. Interestingly, Chowdhury (2009) investigates the two contestants all-pay auction with winning payoffs as non-monotonic functions of their own efforts. He finds conditions under which a pure strategy Nash equilibrium exists and determines the unique mixed strategy Nash equilibrium. Furthermore, Siegel (2012) generalizes the results of Siegel (2009) for n contestants to a model with non-monotonic payoff functions. Sacco and Schumtzler (2008) characterize pure and mixed strategies Nash equilibria with contestants' valuation as a function of both own and rival efforts. In addition to their theoretical results they focus on an experimental analysis. Yet, as the other papers with complete information these do not provide a closed form solution for expected equilibrium expenditures.

A related paper with incomplete information is Kaplan, Luski, Sela, and Wettstein (2002) who investigate an all-pay auction where the rewards are additively or multiplicatively sepa-

[3] Siegel (2010) provides an algorithm to solve equilibrium expenditures in the general framework of Siegel (2009).

rable in the type of the players. This setting seems well-suited to R&D races, political contest or lobbying activities. Kaplan, Luski, Sela, and Wettstein (2002) solve for the equilibrium bid function and link the size of the reward as well as the costs of bidding to the expected sum of equilibrium bids. Cohen, Kaplan, and Sela (2009) study an all-pay auction with additively and multiplicatively separable rewards under incomplete information where the designer can set the shape of the reward function. In particular, they determine that the optimal additively separable reward is not necessarily positive. In our model, we investigate additively separable rewards with complete information.

The next section introduces the model. While valuations are not ordered, they are regular in the sense that over some range of effort one player has higher valuations while above a certain threshold effort level the valuations of the other player are higher. The existence of a Nash equilibrium in mixed strategies and equilibrium effort levels are derived in Section 3. A closed form solution for the aggregate expected equilibrium expenditure is derived in Section 4 – this is indeed the main contribution of this paper.

2 The Model

Consider two players or group of players, $i = 1, 2$ who choose effort levels $x_i \in \mathbb{R}_+$ simultaneously and independently in a contest. The shape of their valuations is given by $V_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ which specifies the size of the prize as a function of their effort. The valuation of winning the contest V_i for each $i = 1, 2$ consists of a common element v and the additive polynomial function $\beta_i x_i^{\alpha_i}$ with $0 < \beta_i < 1$ and $\alpha_i \geq 1$. The shape of every player's valuation – the constant v and parameters α_i and β_i – are common knowledge at the beginning of the contest. Both players pay their effort cost and final payoffs for each player $i = 1, 2$ are given by

$$u_i(x_i, x_j) = \begin{cases} v + \beta_i x_i^{\alpha_i} - x_i & \text{if } x_i > x_j \\ \dfrac{v + \beta_i x_i^{\alpha_i}}{2} - x_i & \text{if } x_i = x_j \\ -x_i & \text{if } x_i < x_j \end{cases}$$

In an all-pay auction bidders have to pay their effort independently of the outcome of the contest and only the winner keeps a balance from his effort spent (unlikely to the first-price auction with complete information for example). The following assumption captures this element.

Assumption 1 (A1)**.** *The winning payoff is strictly decreasing with the effort spent.*

Assumption $A1$ makes the contestants prefer winning at lower effort levels. In other words, the net payoff from an additional unit of effort is negative and then $\beta_i \alpha_i x^{\alpha_i - 1} < 1$ for all $x \in [0, \tilde{x}]$ where $\tilde{x}$ is the maximum effort such as assumption $A1$ is satisfied. Remark that $\tilde{x}$ needs to be sufficiently small and is determined in the next Section. Although our set-up is more specific than Siegel (2009, 2010) – it is actually a special case of their model – it allows us to determine a closed form of the aggregate expected equilibrium expenditure (which is not possible in Siegel (2009, 2010)).[4] Moreover, the shape of the valuations can be derived because of an axiomatization.[5]

The case where $V_i(x_i) = v$ for all x_i corresponds to a pure common value setting. It is well-known that under these circumstances there is no Nash equilibrium in pure strategies. As Amegashie (2001) has shown, if utility is not monotonically decreasing in effort, a pure strategy Nash equilibrium exists. Although his framework is different, the link between increasing utility and equilibrium existence is likely to apply to our set-up as well. The reward components of the all-pay auction are depicted in Figure 1.

As valuations are dependent on the effort of the two contestants, they need not to be ordered. In other words, for two different effort levels, the ranking of the valuations could be reversed. If $\alpha_i > \alpha_j$, for example, $V_j(x) > V_i(x)$ for all $x < \left(\frac{\beta_j}{\beta_i}\right)^{\frac{1}{\alpha_i - \alpha_j}}$ and $V_j(x) \leq V_i(x)$ otherwise. Non-ordered valuations seem well-suited for the real-world applications that motivate our analysis. Indeed, with different marginal returns to effort in R&D races, lobbying or sports contests, the ranking of the valuations depends on effort levels and valuations are unlikely to be ordered.[6]

[4] Siegel (2009, 2010) do not specify the shape of the valuation and cost functions. Yet Siegel (2009, 2010) also use a similar assumption to assumption $A1$.

[5] Let us denote $e_i(.)$ a differentiable and continuous functions such as $V_i(.) = v + e_i(.)$. Given the three following assumptions

- $e_i(.)$ is a non-negative and increasing function for all $x_i > 0$ and $e_i(0) = 0$
- contestants prefer winning at lower effort levels (which is equivalent to our assumption $A1$)
- $e_i(.)$ is a homogeneous function

the shape of $e_i(.)$ is our polynomial function. Details and economic interpretations are available in a previous draft of this paper, Bos and Ranger (2009).

[6] The case of ordered valuations is considered in Appendix B.

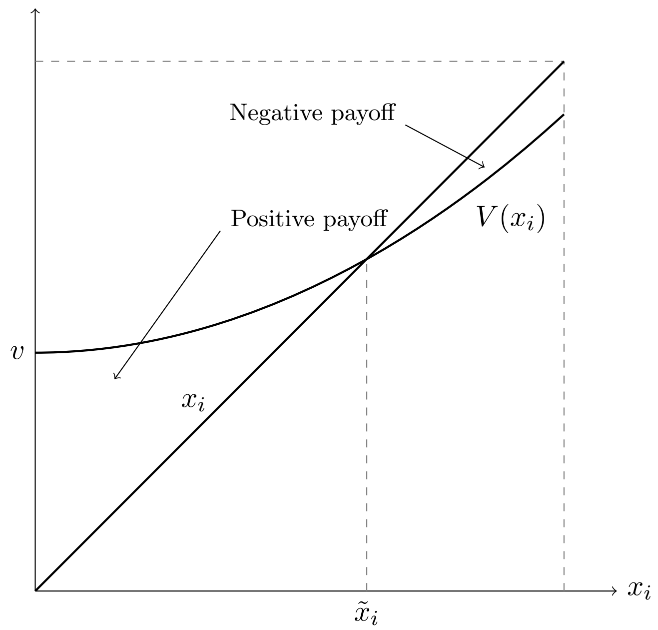

Figure 1: Payoff of Winning

3 Equilibrium Characterization

It is a well known result that all-pay auctions with constant heterogeneous valuations, that is valuations that are independent of the submitted bids, have a unique equilibrium in mixed strategies (Hillman and Riley (1989), Baye, Kovenock, and de Vries (1996) for linear costs, Che and Gale (1998), Che and Gale (2006), Kaplan and Wettstein (2006) and Vartiainen (2007) for non-linear cost functions). In recent papers, Siegel (2009, 2010, 2012) extends this result to non-ordered contestants in a general framework.

To simplify the notation we define the "weak" and the "strong" player and denote them by the subscripts w and s, respectively. The intuition is simple: since payoffs are falling in effort by assumption A1 there will be a level of effort after which the payoff obtained will be negative even if the contest is won. The "weak" player determines the maximum effort any player is willing to exercise in the contest. At this effort level its utility from winning the contest is zero; the "strong" player, in contrast, still obtains a positive payoff at the same effort level. Non-ordered valuations and ordered valuations are depicted in Figure 4 and Figure 5 in Appendix C.

Definition 1. *A player is called "weak" if he determines the maximum effort $\tilde{x}$ in the contest, that is $v + \beta_w \tilde{x}^{\alpha_w} - \tilde{x} = 0$ and $v + \beta_s \tilde{x}^{\alpha_s} - \tilde{x} > 0$. His opponent is called the "strong" player.*

Let us consider $\alpha_i > \alpha_j$. If $\tilde{x}$ is such as $v + \beta_j \tilde{x}^{\alpha_j} - \tilde{x} = 0$ and $\tilde{x} > \left(\frac{\beta_j}{\beta_i}\right)^{\frac{1}{\alpha_i - \alpha_j}}$ then the player i is the "strong" player. Otherwise, the player j is the "strong" player and the maximum effort is given by $\tilde{x}$ such as $v + \beta_i \tilde{x}^{\alpha_i} - \tilde{x} = 0$. Unlike in a standard all-pay auction, it is not enough for a particular player to have the higher valuation over an interval of x in order to be the "strong" player. Rather, the relative strength of a player is determined not only by the difference between the valuations at a particular x but also – implicitly – by the distance from the threshold $\left(\frac{\beta_j}{\beta_i}\right)^{\frac{1}{\alpha_i - \alpha_j}}$ which defines the order of the valuations on each sub-interval.

It follows from the implicit function theorem that the maximum effort is increasing in β_w, decreasing in α_w if $\tilde{x} < 1$ and increasing in α_w if $\tilde{x} > 1$,

$$\frac{d\tilde{x}}{d\beta_w} = \frac{\tilde{x}^{\alpha_w}}{1 - \beta_w \alpha_w \tilde{x}^{\alpha_w - 1}} > 0 \text{ and } \frac{d\tilde{x}}{d\alpha_w} = \frac{\beta_w \tilde{x}^{\alpha_w} \ln \tilde{x}}{1 - \beta_w \alpha_w \tilde{x}^{\alpha_w - 1}} \begin{Bmatrix} < 0 \text{ if } \tilde{x} < 1 \\ = 0 \text{ if } \tilde{x} = 1 \\ > 0 \text{ if } \tilde{x} > 1 \end{Bmatrix} \tag{1}$$

The signs follow from assumption A1.

Define the mixed strategies at the equilibrium by $F_i(\cdot) = \mathbb{P}(X \leq \cdot)$ for both players $i = s, w$. The following proposition determines the unique Nash equilibrium strategies for the two players and the corresponding equilibrium payoffs.

Proposition 1. *The unique Nash equilibrium is in mixed strategies as follows. Players choose their effort randomly according to the cumulative distributions functions*

$$F_s(x) = \frac{x}{v + \beta_w x^{\alpha_w}} \quad \textit{for all } x \in [0, \tilde{x}]$$

$$F_w(x) = \frac{v + \beta_s \tilde{x}^{\alpha_s} - \tilde{x} + x}{v + \beta_s x^{\alpha_s}} \quad \textit{for all } x \in [0, \tilde{x}]$$

And the expected equilibrium payoffs are

$$u_s^\star = \beta_s \tilde{x}^{\alpha_s} - \beta_w \tilde{x}^{\alpha_w}$$

$$u_w^\star = 0$$

Proof. Let us define $\tilde{v}_i(x) = v + \beta_i x^{\alpha_i} - x$ and $\tilde{c}_i(x) = x$. Then, the bidders' expected utility could be written as $F_i(x)\tilde{v}_i(x) - (1 - F_i(x))\tilde{c}_i(x)$. Observe that $\tilde{v}_i$ and $-\tilde{c}_i$ are continuous and strictly decreasing, $\tilde{v}_i(0) = v$ and $\lim_{x \to +\infty} \tilde{v}_i(x) < 0$ and $\tilde{v}_s^{-1}(0) > \tilde{v}_w^{-1}(0) = \tilde{x}$. Consequently all assumptions of Siegel (2010) are satisfied. In addition what he called the threshold T of the contest is in our case the maximum effort $\tilde{x}$ of the "weak" bidder. Thus, Theoreom 3 of Siegel (2010) can be applied and our result follows. ■

Observe that the equilibrium expected payoff of the "weak" player is independent of the parameters of the contestants' value functions. The expected equilibrium payoff of the "strong" depends on its own valuation and, via its equilibrium strategy, on the parameters of its opponent.

Corollary 1. *The expected equilibrium payoff of the "strong" player is (i) decreasing in β_w and (ii) decreasing in α_w if $\tilde{x} > 1$ and increasing otherwise.*

Proof. See Appendix A. ■

This asymmetry in the parameter effects is interesting for its implications. As the payoff of the "strong" player is given by $\beta_s \tilde{x}^{\alpha_s} - \beta_w \tilde{x}^{\alpha_w}$ the impact of the reward parameters β_w and α_w comes from two sources. On one hand we can identify a *parameter effect* from the endogenous valuation of the "weak" player independent of the effort level, in which a higher β_w (respectively a smaller α_w if $\tilde{x} < 1$ and a higher one if $\tilde{x} > 1$) reduces the payoff of the "strong" player independent of the effort levels. The *maximum effect*, on the other hand, works through the impact of the relative values of β_w and α_w on $\tilde{x}$. If the *maximum effect* and the *parameter effect* have contradictory signs, the latter dominates.

It is possible to compare the (standard) all-pay auction with exogenous valuations $V_i(x) = v_i$ and our setup where $V_i(x) = v + \beta_i x^{\alpha_i}$.[7] Even if rewards lead to either a higher or a lower valuation than in the standard case of all-pay auction, it is convenient to assume that the maximum effort is the same in the endogeneous and standard all-pay auctions. Indeed, the maximum effort could be decided *ex ante*, for example as a limit of the expenditure in an R&D race. In consequence, in the standard all-pay auction valuations are ordered and the "weak" contestant is the one with the lowest valuation. She determines her maximum effort equal to her valuation v_w such that $v_w = \tilde{x} = v + \beta_w \tilde{x}^{\alpha_w}$. Then, v_s, the valuation of the "strong" contestant is superior to $v + \beta_w \tilde{x}^{\alpha_w}$.

Corollary 2. *The expected equilibrium payoff of the "strong" player in the all-pay auction with polynomial rewards is lower than in a standard all-pay auction if and only if her valuation v_s is superior to $v + \beta_s \tilde{x}^{\alpha_s}$.*

This result comes from the comparison of the players' expected equilibrium payoff given by Proposition 1 and the contestants' expected equilibrium payoff given by Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996) in the standard all-pay auction which is $v_s - v_w$ for the "strong" player.

4 Aggregate Expenditures

We now derive an explicit expression for the expected equilibrium expenditure of both contestants by means of the incomplete Beta functions. We present a very short overview of the family of incomplete Beta functions which are a useful tool for the computations of the

[7] The analysis does not change if the exogenous part of the valuations v_i is not common to both contestants and $V_i(x) = v_i + \beta_i x^{\alpha_i}$.

expected revenues. The incomplete Beta functions belongs to the general class of hypergeometric functions and are studied in detail Spanier and Oldham (1987) Chapter 58 (see also Temme (1996) for a more recent textbook).[8]

Definition 2. *The incomplete Beta function, $B(\nu, \mu, x)$, with $0 \leq x \leq 1, \mu \in \mathbb{R}, \nu > 0$, is given by the Euler integral representation:*

$$B(\nu, \mu, x) = \int_0^x \frac{t^{\nu-1}}{(1-t)^{1-\mu}} dt$$

Theorem 1. *The incomplete Beta function, $B(\nu, \mu, x)$, with $0 \leq x < 1, \mu \in \mathbb{R}, \nu > 0$, is given by the expansion series:*

$$B(\nu, \mu, x) = \frac{(1-x)^{\mu}}{\nu} \sum_{j=0}^{\infty} \frac{(\mu+\nu)_j}{(1+\nu)_j} x^{j+\nu}$$

where $(y)_n$ denotes the Pochhammer symbol such that

$$(y)_n = \frac{\Gamma(j+y)}{\Gamma(y)},$$

where $\Gamma(.)$ is the special function Gamma.

Proposition 2. *The aggregate expected equilibrium expenditures are given by*

$$\mathbb{E}R_s = \begin{cases} -\dfrac{v}{\beta_w}\left[1 + \dfrac{1}{\beta_w}\ln(1-\beta_w)\right] & \text{if } \alpha_w = 1 \\ \tilde{x} - \dfrac{1}{\alpha_w v \phi^{\frac{2}{\alpha_w}}} B\left(\dfrac{2}{\alpha_w}, 1 - \dfrac{2}{\alpha_w}, \dfrac{\phi \tilde{x}^{\alpha_w}}{1+\phi\tilde{x}^{\alpha_w}}\right) & \text{if } \alpha_w > 1 \end{cases}$$

$$\mathbb{E}R_w = \begin{cases} \dfrac{1-\beta_s}{\beta_s}\left[\ln(1+\varphi\tilde{x})\left(\tilde{x} + \dfrac{v}{\beta_s}\right) - \tilde{x}\right] & \text{if } \alpha_s = 1 \\ \tilde{x} - \dfrac{1}{\alpha_s v \varphi^{\frac{2}{\alpha_s}}} B\left(\dfrac{2}{\alpha_s}, 1 - \dfrac{2}{\alpha_s}, \dfrac{\varphi\tilde{x}^{\alpha_s}}{1+\varphi\tilde{x}^{\alpha_s}}\right) - \dfrac{\beta_s\tilde{x}^{\alpha_s} - \beta_w\tilde{x}^{\alpha_w}}{\alpha_s v^2 \varphi^{\frac{1}{\alpha_s}}} B\left(\dfrac{1}{\alpha_s}, 1 - \dfrac{1}{\alpha_s}, \dfrac{\varphi\tilde{x}^{\alpha_s}}{1+\varphi\tilde{x}^{\alpha_s}}\right) & \text{if } \alpha_s > 1 \end{cases}$$

with $\tilde{x} = v + \beta_w \tilde{x}^{\alpha_w}$, $\varphi = \frac{\beta_s}{v}$ and $\phi = \frac{\beta_w}{v}$.

Proof. See Appendix A. ■

The Euler integral form of the incomplete Beta functions is used in the proof to identify it. An explicit form of the expected revenues are given in Appendix A by the means of the expansion series of the incomplete Beta functions.

[8] The incomplete Beta function converges to the standard Beta Function when $x = 1$.

Assuming a linear relationship between effort and the valuations, $\alpha_s = \alpha_w = 1$, one can simplify the expression for $\mathbb{E}R_w$ to

$$\mathbb{E}R_w = \frac{1-\beta_s}{1-\beta_w}\frac{v}{\beta_s}\left[-1 + \left(\frac{1-\beta_w+\beta_s}{\beta_s}\right)\ln\left(\frac{\beta_s}{1-\beta_w}+1\right)\right]$$

The relationship between the expected equilibrium expenditure for both players and the values for β_w and β_s can be shown graphically (Figure 2 and Figure 3 with $v = 2$).

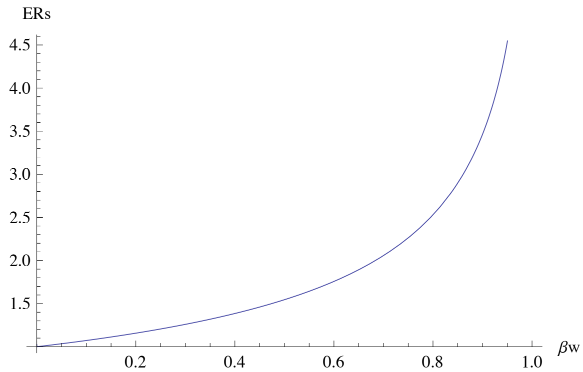

Figure 2: $\mathbb{E}R_s$ for $\alpha_s = \alpha_w = 1$

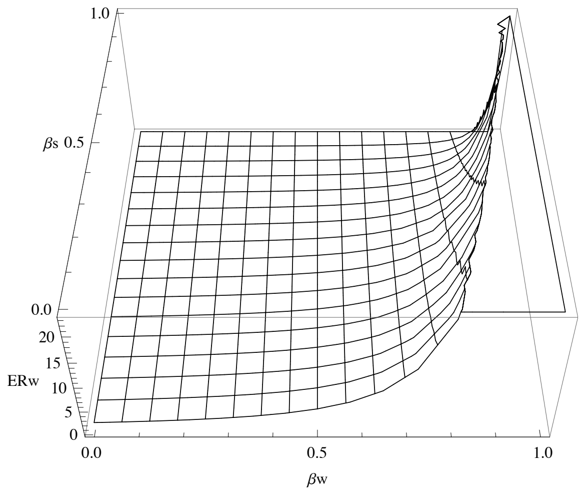

Figure 3: $\mathbb{E}R_w$ for $\alpha_s = \alpha_w = 1$

The features of the individual expenditures that we observe in graphs 2 and 3 can be extended to all values of α_s and α_w. It is thus possible to compare the standard all-pay auction with exogenous valuations $V_i(x) = v_i$ with our setup where $V_i(x) = v + \beta_i x^{\alpha_i}$. Qualitatively, β_i and α_i (when $\tilde{x} > 1$) play the same role as v_i in the standard framework and α_i (when $\tilde{x} \leq 1$) the inverse one and the results should comparatively similar with respect to individual expected equilibrium expenditure. The following corollary confirms this intuition for all value of $\alpha_s, \alpha_w, \beta_s, \beta_w$ and v.

Corollary 3. *The individual expected equilibrium expenditures*

(i) of the "strong" player are increasing in β_w and decreasing in α_w if $\tilde{x} \leq 1$.

(ii) of the "strong" player are independent of β_s and α_s.

(iii) of the "weak" player are decreasing in β_s, increasing in α_s if $\tilde{x} \leq 1$.

(iv) of the "weak" player are increasing in β_w, decreasing in α_w if $\tilde{x} \leq 1$ and increasing in α_w if $\tilde{x} > 1$.

Proof. See Appendix A. ■

We are not able to compute the sign of the derivatives of the role of the parameters α_s on the "weak" player's and α_w on the "strong" player's expected equilibrium expenditure when $\tilde{x} > 1$. Yet, we did not find any example which could contradict the intuition given above.

These results (Corollary 3) may have implications for the designer of a contest. If the designer is interested in eliciting the largest amount of effort, in cases where the effort accrues directly to him, and if he can manipulate the contest technology of both players, Corollary 3 indicates a method to achieve this goal. In addition, the contestants themselves have an incentive to influence the parameters in the valuation functions. A "weak" firm in an R&D race, for example, would prefer a larger value for β_w and a smaller value for α_w if $\tilde{x} \leq 1$ (respectively a higher value if $\tilde{x} > 1$) in order to decrease the expected equilibrium payoff of its competitor (see Corollary 1). If the contestant can chose the parameters α_i and β_i at some cost before the beginning of the contest, the game can be extended to include the pre-contest selection of the contest technology.

As in Section 3 for Corollary 2, we compare our setup where $V_i(x) = v + \beta_i x^{\alpha_i}$ with the standard all-pay auction where the valuation v_w of the "weak player" is such that $v_w = \tilde{x}$.

Corollary 4. *The expected equilibrium expenditures of the "strong" player in the all-pay auction with polynomial rewards is lower than in the standard all-pay auction.*

Proof. As $v_w > v + \beta_w x^{\alpha_w}$ for all effort inferior to $\tilde{x}$, the "strong" player's mixed strategy in the contest with rewards is stochastically dominated by the one in the standard contest. Then, the result follows from $\mathbb{E}R_s = \tilde{x} - \int_0^{\tilde{x}} F_s(x)dx$. ■

The effect on the expected equilibrium expenditures of the "weak" contestant is not clear. Indeed, as the payoff of the "strong bidder" decreases if $v_s > v + \beta_s \tilde{x}^{\alpha_s}$, the effect of the reward on the mixed strategy of the "weak" contestant is ambiguous.

5 Conclusion

In this paper we examine a perfectly discriminating contest (all-pay auction) with two asymmetric players and polynomial valuations in a complete information environment. Similar to real-world situations, we postulate that the value of winning depends on the effort levels invested. In particular, we assume that higher effort levels lead to higher prizes but that this increase is smaller than the cost of effort. The contestants thus prefer to win at lower effort levels. We believe that this set-up captures the nature of many contests such as R&D races, lobbying games or sports events.

As the valuation functions are not symmetric, we can define the "strong" contestant as the one having the higher effort limit. The effort limit is level of effort after which the payoff obtained will be negative even if the contest is won. The "weak" contestant, analogously, has the lower effort limit. Within these limits, valuations need not be ordered, however, as (due to the asymmetry) both players may have the higher valuation at different levels of effort. This notwithstanding, we show that the equilibrium strategies and expected payoffs depend on the strength of the player.

In equilibrium, the expected equilibrium payoff of the "strong" player is positive and depends on the parameters of both players' valuation function. In particular, it is decreasing in the steepness of the "weak" player's valuations. We are able to characterize the expected expenditure thanks to the incomplete Beta functions. This result could be useful for applications of contests with polynomial rewards as in R&D races. For example, the organizer of the contest could provide incentives to contestants such as they would select a contest technology (their parameters α_i and β_i) which would increase the expected aggregate expenditures.

Appendix A: Proofs

Proof of Corollary 1. (i)

$$\frac{\partial u_s^\star}{\partial \beta_w} = \underbrace{\frac{d\tilde{x}}{d\beta_w}(\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - \beta_w\alpha_w\tilde{x}^{\alpha_w-1})}_{\text{maximum effect}} - \underbrace{\tilde{x}^{\alpha_w}}_{\text{parameter effect}} \tag{2}$$

$$= \frac{d\tilde{x}}{d\beta_w}\left(\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - 1\right) \tag{3}$$

$$< 0 \tag{4}$$

To arrive at equation (3) from (2) we apply the implicit function theorem to $\tilde{x} = v + \beta_w\tilde{x}^{\alpha_w}$ in $\tilde{x}$ and β_w such that $\frac{d\tilde{x}}{d\beta_w}\left(1 - \beta_w\alpha_w\tilde{x}^{\alpha_w-1}\right) = \tilde{x}^{\alpha_w}$. The result then follows from assumption A1 and (1).

(ii)

$$\frac{\partial u_s^\star}{\partial \alpha_w} = \underbrace{\frac{d\tilde{x}}{d\alpha_w}\left(\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - \beta_w\alpha_w\tilde{x}^{\alpha_w-1}\right)}_{\text{maximum effect}} - \underbrace{\beta_w\tilde{x}^{\alpha_w}\ln\tilde{x}}_{\text{parameter effect}} \tag{5}$$

$$= \frac{d\tilde{x}}{d\alpha_w}\left(\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - 1\right) + \tilde{x}^{\alpha_w}\left(\frac{d\tilde{x}}{d\alpha_w}\frac{d\beta_w}{d\tilde{x}} - \beta_w\ln\tilde{x}\right) \tag{6}$$

$$= \tilde{x}^{\alpha_w}\left(\frac{d\tilde{x}}{d\alpha_w}\frac{d\beta_w}{d\tilde{x}} - \beta_w\ln\tilde{x}\right) \tag{7}$$

$$= \frac{d\tilde{x}}{d\alpha_w}\left(\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - 1\right) \tag{8}$$

Using equations (2) and (3), $\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - \beta_w\alpha_w\tilde{x}^{\alpha_w-1} = (\beta_s\alpha_s\tilde{x}^{\alpha_s-1} - 1) + \tilde{x}^{\alpha_w}\frac{d\beta_w}{d\tilde{x}}$. Then

(6) follows. We get (7) as $\frac{d\tilde{x}}{d\alpha_w}\frac{d\beta_w}{d\tilde{x}} - \beta_w \ln \tilde{x} = 0$ from (1). In (8), the term between brackets is negative from assumption A1 and the sign of $\frac{d\tilde{x}}{d\alpha_w}$ is given by (1).

■

Proof of the Proposition 2. The expected equilibrium expenditure of effort is given by $\mathbb{E}R_i = \tilde{x} - \int_0^{\tilde{x}} F_i(x)dx$ for $i = w, s$.

1. Computation of $\mathbb{E}R_w$.

Let us denote $\varphi = \frac{\beta_s}{v}$. If $\alpha_s = 1$, it follows that

$$
\begin{aligned}
\int_0^{\tilde{x}} F_w(x)dx &= (v + (\beta_s - 1)\tilde{x}) \int_0^{\tilde{x}} \frac{dx}{v + \beta_s x} + \int_0^{\tilde{x}} \frac{x}{v + \beta_s x} dx \\
&= \frac{v + (\beta_s - 1)\tilde{x}}{v} \int_0^{\tilde{x}} \frac{dx}{1 + \varphi x} + \frac{1}{\beta_s} \int_0^{\tilde{x}} 1 - \frac{1}{1 + \varphi x} dx \\
&= -\frac{1 - \beta_s}{\beta_s} \left(\frac{v}{\beta_s} + \tilde{x} \right) \ln(1 + \varphi \tilde{x}) + \frac{\tilde{x}}{\beta_s}
\end{aligned}
$$

Therefore,

$$
\mathbb{E}R_w = \frac{1 - \beta_s}{\beta_s} \left[\ln(1 + \varphi \tilde{x}) \left(\tilde{x} + \frac{v}{\beta_s} \right) - \tilde{x} \right]
$$

The derivative in $\tilde{x}$ and the boundedness condition guarantee that $\mathbb{E}R_w$ is positive.

If $\alpha_s > 1$

$$
\int_0^{\tilde{x}} F_w(x)dx = (\beta_s \tilde{x}^{\alpha_s} - \beta_w \tilde{x}^{\alpha_w}) \int_0^{\tilde{x}} \frac{dx}{v + \beta_s x^{\alpha_s}} + \int_0^{\tilde{x}} \frac{x}{v + \beta_s x^{\alpha_s}} dx
$$

Moreover,

$$\int_0^{\tilde{x}} \frac{dx}{v+\beta_s x^{\alpha_s}} = \frac{1}{v}\int_0^{\tilde{x}} \frac{dx}{1+\varphi x^{\alpha_s}}$$

$$= \frac{1}{\alpha_s v \varphi^{\frac{1}{\alpha_s}}} \int_0^{\varphi \tilde{x}^{\alpha_s}} \frac{y^{\frac{1}{\alpha_s}-1}}{1+y} dy \tag{9}$$

$$= \frac{1}{\alpha_s v \varphi^{\frac{1}{\alpha_s}}} \int_0^{\frac{\varphi \tilde{x}^{\alpha_s}}{1+\varphi \tilde{x}^{\alpha_s}}} \frac{t^{\frac{1}{\alpha_s}-1}}{(1-t)^{\frac{1}{\alpha_s}}} dt \tag{10}$$

$$= \frac{1}{\alpha_s v \varphi^{\frac{1}{\alpha_s}}} B\left(\frac{1}{\alpha_s}, 1-\frac{1}{\alpha_s}, \frac{\varphi \tilde{x}^{\alpha_s}}{1+\varphi \tilde{x}^{\alpha_s}}\right) \tag{11}$$

$$= \frac{\tilde{x}}{v}\Gamma\left(\frac{1}{\alpha_s}\right)\left(\frac{1}{1+\varphi \tilde{x}^{\alpha_s}}\right)^{1-\frac{1}{\alpha_s}} \sum_{j=0}^{\infty} \frac{(\varphi \tilde{x}^{\alpha_s})^j}{(1+\alpha_s j)\Gamma(\frac{1}{\alpha_s}+j)} \tag{12}$$

To obtain equations (9) and (10), we define $y = \varphi x^{\alpha_s}$ and $t = \frac{y}{1+y}$. As $\frac{\varphi \tilde{x}^{\alpha_s}}{1+\varphi \tilde{x}^{\alpha_s}} \in (0,1)$ and $\frac{1}{\alpha_s} > 0$, equation (10) is the Euler integral representation of an incomplete Beta function and equation (12) comes from his expansion series given by Theorem 1.

Moreover,

$$\int_0^{\tilde{x}} \frac{x}{v+\beta_s x^{\alpha_s}} dx = \frac{1}{\alpha_s v \varphi^{\frac{2}{\alpha_s}}} \int_0^{\frac{\varphi \tilde{x}^{\alpha_s}}{1+\varphi \tilde{x}^{\alpha_s}}} \frac{t^{\frac{2}{\alpha_s}-1}}{(1-t)^{\frac{2}{\alpha_s}}} dt \tag{13}$$

$$= \frac{1}{\alpha_s v \varphi^{\frac{2}{\alpha_s}}} B\left(\frac{2}{\alpha_s}, 1-\frac{2}{\alpha_s}, \frac{\varphi \tilde{x}^{\alpha_s}}{1+\varphi \tilde{x}^{\alpha_s}}\right) \tag{14}$$

As before, we find (13) after change in variables as for equation (9) and (10).

Therefore,

$$\begin{aligned}\mathbb{E}R_w = \tilde{x} - \frac{\tilde{x}}{v}\Gamma\left(\frac{2}{\alpha_s}\right)\left(\frac{1}{1+\varphi \tilde{x}^{\alpha_s}}\right)^{1-\frac{2}{\alpha_s}} \sum_{j=0}^{\infty} \frac{(\varphi \tilde{x}^{\alpha_s})^j}{(2+\alpha_s j)\Gamma(\frac{2}{\alpha_s}+j)} \\ - \tilde{x}\frac{\beta_s \tilde{x}^{\alpha_s} - \beta_w \tilde{x}^{\alpha_w}}{v}\Gamma\left(\frac{1}{\alpha_s}\right)\left(\frac{1}{1+\varphi \tilde{x}^{\alpha_s}}\right)^{1-\frac{1}{\alpha_s}} \sum_{j=0}^{\infty} \frac{(\varphi \tilde{x}^{\alpha_s})^j}{(1+\alpha_s j)\Gamma(\frac{1}{\alpha_s}+j)}\end{aligned}$$

2. Computation of $\mathbb{E}R_s$.

Let us denote $\phi = \frac{\beta_w}{v}$. If $\alpha_w = 1$

$$\begin{aligned}\mathbb{E}R_s &= -\frac{1}{\beta_w}[(1-\beta_w)\tilde{x} + \frac{1}{\phi\beta_w}\ln(1+\phi\tilde{x})] \\ &= -\frac{v}{\beta_w}\left(1+\frac{1}{\beta_w}\ln(1-\beta_w)\right)\end{aligned}$$

which is positive as $\beta_w < 1$.

If $\alpha_w > 1$ the calculation is the same as for $\mathbb{E}R_w$, thus

$$\mathbb{E}R_s = \tilde{x}\left(1 - \frac{1}{v}\Gamma\left(\frac{2}{\alpha_w}\right)\left(\frac{1}{1+\phi\tilde{x}^{\alpha_w}}\right)^{1-\frac{2}{\alpha_w}}\sum_{j=0}^{\infty}\frac{(\phi\tilde{x}^{\alpha_w})^j}{(2+\alpha_w j)\Gamma(\frac{2}{\alpha_w}+j)}\right)$$

■

Proof of the Corollary 3. We recall that the expected equilibrium expenditure is given by $\mathbb{E}R_i = \tilde{x} - \int_0^{\tilde{x}} F_i(x)dx$ for $i = w, s$.

(i)

$$\begin{aligned}\frac{\partial \mathbb{E}R_s}{\partial \beta_w} &= \frac{d\tilde{x}}{d\beta_w}\underbrace{(1-F_s(\tilde{x}))}_{=0} + \int_0^{\tilde{x}}\frac{x^{\alpha_w+1}}{(v+\beta_w x^{\alpha_w})^2}dx\\ &> 0\end{aligned}$$

$$\frac{\partial \mathbb{E}R_s}{\partial \alpha_w} = \frac{d\tilde{x}}{d\alpha_w}\underbrace{(1-F_s(\tilde{x}))}_{=0} + \int_0^{\tilde{x}}\frac{x^{\alpha_w+1}\beta_w \ln x}{(v+\beta_w x^{\alpha_w})^2}dx$$

which is negative if $\tilde{x} \leq 1$.

(ii) The mixed strategies and the maximum effort are independent of β_s and α_s. Hence the result.

(iii) $\frac{\partial \mathbb{E}R_w}{\partial \beta_s} = -\int_0^{\tilde{x}}\frac{v(\tilde{x}^{\alpha_s} - x^{\alpha_s}) + x^{\alpha_s}(\tilde{x}-x)}{(v+\beta_s x^{\alpha_s})^2}dx < 0$. Moreover, if $\tilde{x} \leq 1$,

$$\begin{aligned}\frac{\partial \mathbb{E}R_w}{\partial \alpha_s} &= -\beta_s\int_0^{\tilde{x}}\frac{v\tilde{x}^{\alpha_s}\ln\tilde{x} - x^{\alpha_s}\ln x(v - \tilde{x} + x)}{(v+\beta_s x^{\alpha_s})^2}dx\\ &\geq \beta_s\int_0^{\tilde{x}}\frac{x^{\alpha_s}(\tilde{x}-x)\ln x}{(v+\beta_s x^{\alpha_s})^2}dx\\ &\geq 0\end{aligned}$$

(iv) Using assumption A1 and equation (1), it follows that $\frac{\partial \mathbb{E}R_w}{\partial \beta_w} = \frac{d\tilde{x}}{d\beta_w}\underbrace{(1-F_s(\tilde{x}))}_{=0} - \int_0^{\tilde{x}}\frac{d\tilde{x}}{d\beta_w}\frac{\alpha_s\beta_s\tilde{x}^{\alpha_s-1}-1}{v+\beta_s x^{\alpha_s}}dx > 0$ and $\frac{\partial \mathbb{E}R_w}{\partial \alpha_w} = \frac{d\tilde{x}}{d\alpha_w}\underbrace{(1-F_s(\tilde{x}))}_{=0} - \int_0^{\tilde{x}}\frac{d\tilde{x}}{d\alpha_w}\frac{\alpha_s\beta_s\tilde{x}^{\alpha_s-1}-1}{v+\beta_s x^{\alpha_s}}dx$ which is negative if $\tilde{x} \leq 1$ and non-negative if $\tilde{x} > 1$.

■

Appendix B: Ordered Valuations

An alternative way to analyze the problem would be to consider ordered valuations such that $V_i(x) > V_j(x)$ over the relevant range of x. Due to the form of the valuation functions, two separate cases have to be examined: $\alpha_i > \alpha_j$ for $x > \left(\frac{\beta_j}{\beta_i}\right)^{\frac{1}{\alpha_i-\alpha_j}}$ and $\alpha_i < \alpha_j$ for $x \leq \left(\frac{\beta_j}{\beta_i}\right)^{\frac{1}{\alpha_i-\alpha_j}}$. In the following, let us denote the threshold $\left(\frac{\beta_j}{\beta_i}\right)^{\frac{1}{\alpha_i-\alpha_j}}$ by $x^\star$.

Case (i): $\alpha_i > \alpha_j$, Positive Minima

If the maximum effort $\tilde{x}$ is superior to the effort level $x^\star$ the mixed equilibrium strategies for both players can be computed. Otherwise, there is no positive density in equilibrium. The main difference to the standard all-pay auction, in a sense, is that the non-participation level of effort and the minimum effort level have to be distinguished. In particular, the players do not participate below the threshold level $x^\star$. In the following, only the results that do not follow straightforwardly from Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996) will be given.

Lemma 1. *If the minimum effort $x^\star$ is strictly positive and inferior to $\tilde{x}$ then the two players' strategies have an atom such that*

$$F_i(x^\star) = \frac{x^\star}{V_j(x^\star)} \text{ and } F_j(0) = \frac{V_j(x^\star)}{V_i(x^\star)}F_i(x^\star) + \frac{V_i(\tilde{x}) - V_j(\tilde{x})}{V_i(x^\star)}$$

Proof. Since the strategy spaces are the same, and expected utilities are constant at the equilibrium we obtain $V_i(\tilde{x}) - \tilde{x} = F_j(x)V_i(x) - x$ and $V_j(\tilde{x}) - \tilde{x} = F_i(x)V_j(x) - x$ for all $x \in [x^\star, \tilde{x}]$. As $V_j(\tilde{x}) - \tilde{x} = 0$, the two last equations lead to the result. ∎

In this case, player i is "strong" and player j "weak" in the sense defined above with probablitiy one for all x. Thus, with the exception of the common mass point at the lower end of the distribution and the length of the strategy space, the mixed equilibrium strategies should be the same as in the case with non-ordered valuations and i as the strong player. In others words, for all $x \in [x^\star, \tilde{x}]$

$$F_i(x) = \frac{x}{v + \beta_j x^{\alpha_j}} \text{ and } F_j(x) = \frac{v + \beta_i \tilde{x}^{\alpha_i} - \tilde{x} + x}{v + \beta_i x^{\alpha_i}}.$$

Even if the distributions are the same, the expected revenue will differ as the strategy spaces are different. We do not provide the closed form solution here, but the computation is straightforward and similar to the one for non-ordered valuations.

Case (ii): $\alpha_i < \alpha_j$, Caps

Here, two cases have to be distinguished. In both, player i is "strong" and player j is "weak" for all x. If the maximum effort $\tilde{x}$ is inferior to $x^\star$ the situation is as same as that of ordered-valuations with polynomial rewards and the results of Proposition 1 apply. Alternatively, the agents face a cap in their bids that they could not exceed such as $\tilde{x} > x^\star$. This last case was studied by Che and Gale (1998) with exogenous valuations. As in their paper, we consider two cases. When $x^\star \leq \frac{\tilde{x}}{2}$, there is a pure strategy Nash equilibrium where the effort of the players is $x^\star$. Otherwise, mixed strategies have to be computed. It can be shown that the players have a nonzero density on $(0, x']$ and a zero density on $(x', x^\star)$ with a mass point at $x^\star$.[9] Then, with similar technical arguments than Che and Gale (1998) we find that for all $x \in [0, x']$ $F_i(x) = \frac{x}{V_j(x)}$ and $F_j(x) = \frac{x}{V_i(x)} + \frac{V_i(x') - V_j(x')}{V_i(x)}$ and for all $x \in [x', x^\star[$ $F_i(x) = \frac{x'}{V_j(x')}$ and $F_j(x) = \frac{x'}{V_i(x')} + \frac{V_i(x') - V_j(x')}{V_i(x')}$. To sum up, if $x^\star \in \left(\frac{\tilde{x}}{2}, \tilde{x}\right)$

$$F_i(x) = \begin{cases} \dfrac{x}{v + \beta_j x^{\alpha_j}} & \text{for all } x \in [0, x'] \\ \dfrac{x'}{v + \beta_j x'^{\alpha_j}} & \text{for all } x \in [x', x^\star[\\ 1 & \text{for } x = x^\star \end{cases} \quad \text{and}$$

$$F_j(x) = \begin{cases} \dfrac{x}{v + \beta_i x^{\alpha_i}} + \dfrac{\beta_i x'^{\alpha_i} - \beta_j x'^{\alpha_j}}{v + \beta_i x^{\alpha_i}} & \text{for all } x \in [0, x'] \\ \dfrac{x'}{v + \beta_i x'^{\alpha_i}} + \dfrac{\beta_i x'^{\alpha_i} - \beta_j x'^{\alpha_j}}{v + \beta_i x'^{\alpha_i}} & \text{for all } x \in [x', x^\star[\\ 1 & \text{for } x = x^\star \end{cases}$$

Appendix C: Figures

In the following figures, i is the "strong" player and j the 'weak" player.

[9] See the Lemma 3 of Che and Gale (1998).

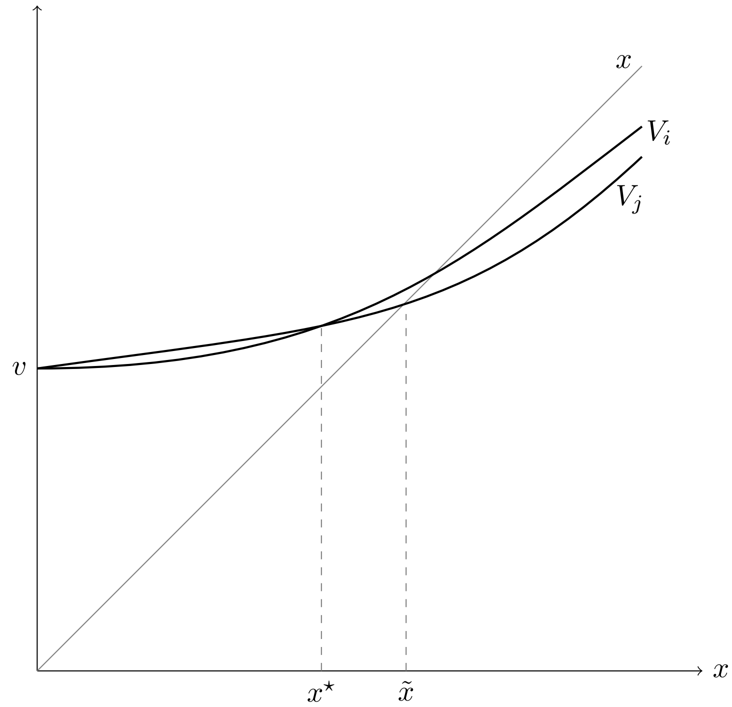

Figure 4: Unordered values

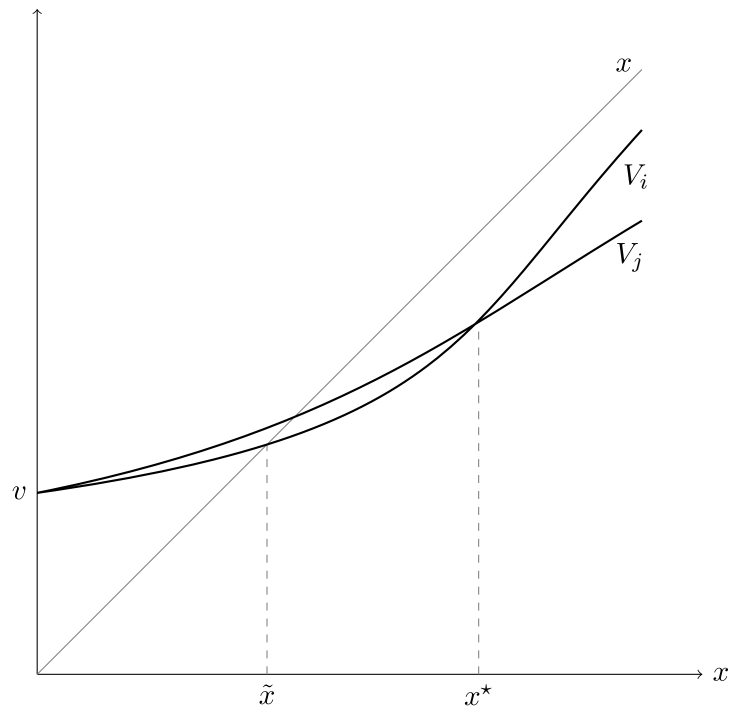

Figure 5: Ordered Values

References

AMEGASHIE, J. (2001): "An All-pay auction with a pure-strategy equilibrium," *Economics Letters*, 70, 79–82.

BAYE, M., D. KOVENOCK, AND C. DE VRIES (1996): "The All-pay Auction with complete information," *Economic Theory*, 8(2), 291–305.

BOS, O., AND M. RANGER (2009): "All-pay auctions with endogenous rewards," Working Paper Series in Economics, University of Cologne.

CHE, Y.-K., AND I. GALE (1998): "Caps on political Lobbying," *American Economic Review*, 88, 643–651.

——— (2006): "Caps on political Lobbying: Reply," *American Economic Review*, 96, 1355–1360.

CHOWDHURY, S. (2009): "The All-pay Auction with Non-monotonic Payoff," Purdue University Working paper.

COHEN, C., T. KAPLAN, AND A. SELA (2009): "Optimal rewards in contests," *Rand Journal of Economics*, forthcoming.

HILLMAN, A., AND J. RILEY (1989): "Politically contestable rents and transfers," *Economics and Policy*, 1, 17–39.

KAPLAN, T., I. LUSKI, A. SELA, AND D. WETTSTEIN (2002): "All-pay auctions with variable rewards," *Journal of Industrial Economics*, 4, 417–430.

KAPLAN, T., I. LUSKI, AND D. WETTSTEIN (2003): "Innovative Activity and Sunk Cost," *International Journal of Industrial Organisation*, 21, 1111–1133.

KAPLAN, T., AND D. WETTSTEIN (2006): "Caps on Political Lobbying: Comment," *American Economic Review*, 96(1351-1354).

SACCO, D., AND A. SCHUMTZLER (2008): "All-Pay Auctions with Negative Prize Externalities: Theory and Experimental Evidence," University of Zurich, Socioeconomic Institute, Working Paper No. 0806.

SIEGEL, R. (2009): "All-Pay Contests," *Econometrica*, 77, 71–92.

——— (2010): "Asymmetric Contests with Conditional Investments," *American Economic Review*, 100, 2230–2260.

——— (2012): "Contests with Productive Effort," mimeo.

Spanier, J., and K. Oldham (1987): *An Atlas of Functions.* Hemisphere Pubishing Corporation, Springer-Verlag.

Temme, N. (1996): *Special Functions, An Introduction to the Classical Functions of Mathematical Physics.* Wiley.

Vartiainen, H. (2007): "Comparing of all-pay auctions under complete information," *Review of Economic Design*, forthcoming.