# Solving Optimal Timing Problems Elegantly

Todorova, Tamara

American University in Bulgaria

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Tamara Todorova[1]

Few textbooks in mathematical economics cover optimal timing problems. Those which cover them do it scantly or in a rather clumsy way, making it hard for students to understand and apply the concept of optimal time in new contexts. Discussing the plentiful illustrations of optimal timing problems, we present an elegant and simple method of solving them. Whether the present value function is exponential or logarithmic, a convenient way to solve it is to convert the base to the exponential number *e*, thus making it easy to differentiate the new objective function with respect to time *t*. This convenient method of base conversion allows to find a second-order derivative and to use the second-order condition as a proof of optimum.

***Keywords*** optimization of functions of one variable, continuous time, optimal timing, discounted present value, future value

***Jel codes*** A22, C02, C61, Q01, Q2, Q5

## 1. Introduction

Optimal timing problems represent an interesting set of illustrations related to economic dynamics and the role of the time factor in economic decision making. These are economic applications of univariate calculus where the argument is time and, given a specific objective function, usually the net present value of an asset or an economic resource, the optimal time or period of appropriating or harvesting the respective resource must be found which maximizes that value. These are problems which answer the question when is the best time, i.e., when is it best to pick up the resource, to harvest the crops or to sell the asset the value of which appreciates with time. Resources could generally be classified as appreciating or depreciating in value terms. Optimal timing problems study resources whose value appreciates with time in terms of rate of growth. When an economic resource loses value with the passage of time, that is, its discounted present value falls, optimal timing problems demonstrate its depreciation in the context of its rate of decay.

Various illustrations could be given of resources whose value grows as time goes by. Examples of this type answer questions such as when is the best time to pick olives, oranges, or peaches in the orchard, tomatoes in the garden, flowers in a flower plant or a greenhouse, grapes in the vineyard so that to produce wine. Except crops such examples ask when it is best to cut trees so that to maximize the value of lumber or to receive highest yields from selling it. Also from the realm of environmental and natural resource economics we may seek to find when to harvest fish or other from a common-pool or a common access resource. Optimal timing problems have

[1] Tamara Todorova is a full-time associate professor in the Department of Economics at the American University in Bulgaria (e-mail: ttodorova@aubg.bg).

relevance not only to mathematical but to environmental economics as well. Examples from mining and extraction depict resources whose value depreciates with time, for instance, extraction of oil from an oil well, mining from mines, etc. With extraction and mining the value of the resource will be declining with time as the resource gets depleted and it becomes harder to extract it.

Examples of appreciating assets beyond the scope of agricultural and environmental economics include artifacts, collection items and jewelry. Such examples ask when the best time is to sell a ruby, a Picasso picture, or a golden Rolex. In view of their diverse illustrations it is puzzling that most economic textbooks ignore optimal timing problems or present them mostly or only in the context of financial economics. The usual representation in economic literature is that of the principal and the interest where the discounted value of a financial asset (say a deposit) is sought to be maximized. The net present value can be traced using either a given simple or a compounded interest. This limited framework within which optimal timing is presented makes it uninteresting for undergraduate students who cannot always see the diverse and plentiful illustrations that optimal timing problems pose. The paper studies the essence, importance, validity and variety of optimal timing problems as they are covered in mathematical economics. Part 1 is a brief introduction. Part 2 elaborates on the diverse applications of optimal timing, its increasing importance and relevance to economic decision making. Part 3 covers some useful approaches to teaching and solving optimal timing problems elegantly including base conversion, derivatives of exponential and logarithmic functions, etc. The paper ends with conclusions.

## 2. Validity and importance of optimal timing

Optimal timing problems are a suitable pedagogical tool to learning in an introductory course in mathematical economics or quantitative methods in the social sciences as part of program requirements. From a purely mathematical point of view, they are a very good setting for presenting optimization, one of the key aspects of mathematical economics, in particular, and of economics, in general. As they represent simple optimization of a function of one variable, they serve a good pedagogical purpose where the student can see how an economic function is maximized, what its best value is as well as that of the argument. Rather than using the standard and somewhat routine examples of profit maximization (in terms of output in the simple case of one variable or the output levels of two or more products in the multivariate case) and production function (where output is maximized given certain input conditions), optimal timing problems optimize a function in terms of time and study economic events dynamically. Thus they represent a simple illustration of economic dynamics without the need to go into dynamic programming or dynamic optimization at this very early stage of education and just using the tools of static optimization. Also mathematically, optimal timing problems are richer than the simple polynomial functions of profit (where total revenues and total costs are involved and where the highest degree is cubic) in that they demonstrate interesting exponential and logarithmic functions. Thus, it is normal to cover optimal timing problems in the chapter on exponential and logarithmic functions as types of inverse functions. The introduction of the exponential number $e$, having wide applications in economics, is a further advantage to the teaching of optimal timing. The professor could, for instance, relate the discussion to elasticity, its estimation and econometric analysis. While students learn a new perspective of optimization and best values, they do not have to go into the bogey of multivariate calculus and optimizing functions of two and more variables. The latter require knowledge of quadratic forms, first- and second-order differentials, Hessian matrices and determinants at best (in the unconstrained case) and bordered Hessian matrices and determinants at worst (in the constrained case).

From a general pedagogical and didactic point of view, this set of problems teaches students about natural and environmental resources, how they can be used optimally, how they should be stored, what waiting time is, how it relates to interest rate, etc.[2] Many dwell on examples from agriculture which expands student knowledge to the subject of agricultural economics. Meanwhile, the student understands that optimal time does not mean optimal quality and that the two are not quite identical. For instance, students learn that waiting time to grow grapes in order to produce wine for sale to consumers is shorter than that needed to produce and store wine for personal consumption. They are surprised to find out that bananas are picked unripe in order to be exported or sold later in the stores where optimal time accounts for their growth at the time of transportation and distribution. With the help of optimal timing they learn to distinguish firm profit from customer satisfaction where optimal time maximizes profit to the firm or entrepreneur, not consumer utility. If the entrepreneur were to produce for self-satisfaction, that is, being the consumer himself, and in the form of a household activity (or as part of the do-it-yourself sector) he would wait much longer to obtain the best taste of products.[3]

Optimal timing problems are rarely covered in mathematical economics textbooks and their coverage is usually superficial. One reason why they are neglected might be the disagreement among economists about the validity of optimal timing models. Generally, they are refuted by some economists who do not consider them realistic in that they fail to correctly describe the path or function by which an economic resource or asset grows. More specifically, Samuelson (1976), discussing the economics of forestry, observes that "standard managerial economics, and actual commercial practice, both tend to lead to an optimal cutting age of a forest that is much shorter than the 80 or even 100 years one often encounters in the forestry literature."[4] Exploring his tree-cutting example further, he maintains that many of the assumptions made in an optimal tree-cutting model are unrealistic. These, in his view, include: "(1) knowledge of future lumber prices at which all outputs can be freely sold, and future wages of all inputs; (2) knowledge of future interest rates at which the enterprise can both borrow and lend in indefinite amounts; … (3) knowledge of technical lumber yields that emerge at future dates once certain expenditure inputs are made (plantings, sprayings, thinnings, fellings, etc.) … (4) [the condition] that each kind of land suitable for forests can be bought and sold and rented in arm's length transactions between numerous competitors … so as to *earn the same maximum rent* obtainable at the postulated market rate of interest."[5] He observes that tomorrow's lumber prices, as well as future interest rates, are not knowable today, and that given the uncertainties of interest and profit

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[2] For optimal timing problems in relation to environmental policy see Pindyck, Robert S. "Optimal Timing Problems in Environmental Economics," *Journal of Economic Dynamics and Control*, 26 (2002), pp. 1677-1697.

[3] Industrial wine producers, for instance, are said to produce wine for domestic consumption by methods different from those applied in the market, for instance, picking grapes later than the general produce, picking the best grapes, using a different technology or different ingredients, storing the wine longer and keeping the best wine for home consumption.

[4] Samuelson, P. A. "The Economics of Forestry in an Evolving Society," *Economic Inquiry*, Vol. XIV, December, 1976, pp. 466-492, p. 467

[5] Italics in the original, p. 470

yields one cannot guarantee that the enterprise will at each date be able both to borrow and to lend in indefinite amounts at one interest rate.[6] Furthermore, the rotation period which maximizes the sustained yield of a forest is so long that at positive interest rates and market rent for land, it will bankrupt the enterprise, or alternatively, ignoring rent land and maximizing the internal rate of return will bring so short a rotation period that growing trees will be impossible. To these limitations optimal timing problems face one can add the difficulty of estimating the particular function by which an economic resource grows. The process of estimating requires robust knowledge of the specific field of environmental science, mining, arts or other plus further expertise in marketing the respective product. The future value function must be estimated promptly usually by empirically observing the growth of the asset or natural resource, which is a difficult endeavor. Thus, the growth of a specific tree species to be planted, logged and sold as timber or other, must be observed tightly so that to maximize the correct present value function and estimate the exact waiting time.

Yet, things have changed since the time Samuelson wrote about the economics of forestry. With the development of science waiting time has consistently been reduced rendering optimal timing problems valid and more relevant to today's economic reality. Artificial selection, cultivation, genetic modification, and other methods of biotechnology have led to substantially reducing the growing time for biological resources. The time to harvest plants such as trees, flowers, vegetables, and fruits has been shortened dramatically with the help of fertilizers, phosphates, artificial lightening, etc., all of which stimulate species growth immensely but few of which existed at the time Samuelson described environmental management. Bio-cultivation has led to planting and breeding the most rapid and prosperous breeds and species with the sole purpose to reduce waiting time. This reduction is, in effect, a reduction of the opportunity cost of waiting since, in congruence with Samuelson's conclusion, the interest rate is the opportunity cost of money which the investor could alternatively put in the bank for an interest or other, instead of a difficult, strenuous and risky agricultural undertaking. Trees used to produce lumber which in the past have taken 40 years to grow, now take 20 years. By bringing down the opportunity cost of money the reduction in waiting time has made various environmental and natural resource projects much more attractive than in the past. Generally, science and biotechnology have somewhat changed the behavior of environmental managers and businessmen, from one of exploration and depletion to one of preservation and sustainability. Such is the example of fisheries where with the help of biotechnology fishermen have turned to growing fish in artificial or semi-artificial conditions rather than overuse common-pool resources, as the established cliché. Fishermen today are far less prone to overharvesting but, on the contrary, much more oriented to the optimal use, preservation and harvesting of a natural endowment. In the special

[6] Samuelson continues with an example of a consulting firm which applied dynamic programming to the tree-cutting problem where the "computer spun out of control and generated a negative… imaginary root for the equation." This according to Samuelson demonstrates that if economic modeling is applied and realistic profit rates are to be reached, it does not pay to keep a forest in existence at all (Samuelson, 1976, p. 467). Furthermore, he notes that the assumption of low interest rates as they apply to forestry is faulty where "the interest rate is the enemy of long-lived investment projects" (Samuelson, 1976, p. 473), and that lumber prices are wrongly presumed by analysts to be constant at relatively low levels. Samuelson's paper stresses some optimal timing mistakes made even by prominent economists such as Irving Fisher, Kenneth Boulding, H. Hotelling, John M. Keynes, to name a few. The paper also uncovers some of the divergences between economists, on the one hand, and foresters and environmentalists, on the other.

conditions of artificial fish farming, the use of artificial insemination, optimal temperature, feeding, etc., has increased yields from sustainable fishing immensely.

Time has also worked favorably for arts, increasing the popularity of artifacts, jewelry, antiques, etc. As people become more affluent, they get more sophisticated and as they get more sophisticated, they tend to appreciate art more. Thus, valuable or rare art objects become more desirable and a much preferred investment even by ordinary people. This is especially true at the time of an economic or financial crisis when material or financial assets of low value must be converted into objects that serve as a store of value. Mighty investors find themselves more willing to buy expensive ware and luxury items when market demand for their products has fallen by putting their money in an expensive jewel, golden watch or a famous artist's drawing, the value of which would appreciate with time. Waves of bank runs, consistent inflation that devalues fiat money and other shocks have taught investors that keeping a deposit for an interest in the bank is not always the best thing to do.

Generally, textbook authors do a poor job revealing the nature and applications of optimal timing problems. Interesting illustrations are missing and the numerical examples given are rather dull and elementary. Problems present the simplest case of a natural exponential function or expression for the future value of an economic resource where the base is the natural exponential number *e* and never go deeply into more sophisticated cases or situations of a general base or a logarithmic function (natural or general).

Ian Jacque (2006) in his "Mathematics for Economics and Business," a basic-level text, discusses optimal time in the context of investment decisions and present value. He introduces interest compounding, relating thus present value to future value in the discrete case. Pemberton and Rau (2001) briefly introduce the formula for continuous compounding and discounting and discuss time in economics. However, they only define rate of growth in relation to rate of change and stop there leaving the reader without any practical illustrations of those two. Sydsaeter and Hammond (2006) also discuss present and future value showing the interesting case of implicit differentiation. They demonstrate optimal timing in the general case where they use the implicit-function rule to show the negative effect interest rate has on optimal time. However, more detailed, numerical examples showing different situations and diverse future value functions are missing and the only illustration given refers to harvesting trees again. The tree-cutting example seems to be an ongoing theme of most textbooks. Students, therefore, cannot fully comprehend the variety and width of optimal timing problems applied to diverse situations. Baldani et al. (2001) emphasize intertemporal consumption showing the relationship between present and future consumption. They also demonstrate net present value of an income stream and solve a constrained optimization of consumption. However, there is no relevance to natural resource economics, sustainable harvesting or growth of an initial endowment. Similar is the discussion in Silberberg (2000) who as well presents intertemporal choice in the context of utility maximization. Then a more thorough coverage follows discussing Fisherian investment in the context of trees again and giving interest rate as their optimal rate of growth. In fact, Silberberg (2000) moves into the Fisher equation of the real and nominal interest rate and, finally, into stocks and flows. Nicholson (1992) whose intermediate-level microeconomics text is also more mathematically oriented follows Silberberg's pattern. After discussing discounted present value from the point of view of financial economics, where he presents the perpetual case of a long-lived machine or of a perpetual asset or bond, he shows the continuous case of a sum (integral) of present values. In addition to the rental rate of a machine he covers the "tree-cutting" example again obtaining the optimal harvesting age for trees and analyzing the effect of interest rate on it. This is followed by an interesting discussion into human and physical capital as well as the standard case of future versus present consumption in the discrete case and considering two periods only. The analysis continues with optimal control theory, stocks and flows, similar to that in Silberberg (2000).

Of the higher-level texts, Miller (1979) presents compounding and discounting in the discrete and continuous case with no reference to harvesting or natural resource economics but sticking solely to financial economics and the example of principal and interest. An advanced text such as de la Fuente (2000) does not make reference to optimal timing problems. Simon and Blume (1994), a standard textbook used in undergraduate mathematical economics courses, also refers to the present value of an investment decision discussing the discounted flow of revenues and annuities. The authors demonstrate the example in the case of both discrete and continuous discounting where it does not become clear how one obtains from the other, given the natural exponential number *e*. This leads to the section on optimal holding time where they introduce the rate of growth by the first-order condition. The example given is rather basic, using the future value of a real estate which grows exponentially and taking the specific form $F(t) = 10{,}000e^{\sqrt{t}}$. The example solves the optimal time for selling the estate as 69.44 years, given that the interest rate will remain at 6% in the foreseeable future. A second-order condition is missing. The text continues with the differentiation of other functions and terms, all scattered around. Some interesting examples given in the exercises section include the sales of a rare book, wine and a parcel of land. However, the examples solved numerically cover only the simple case of a natural exponential number and never delve into more sophisticated situations.

Few of the textbooks present the derivatives of exponential and logarithmic functions, both general and natural, where the two types could actually be related and the derivatives of a general exponential or logarithmic function could be obtained by converting those conveniently into natural exponential or natural logarithmic function.[7] Of the standard textbooks Chiang (1984) alone presents the derivatives of exponential and logarithmic functions, rate of growth of a simple function, rate of growth of composite functions, etc. He gives the interesting example of wine storage (with storage cost ignored) and the usual one of timber cutting where his examples end. Both examples are awkwardly solved, which makes it hard for students to follow the second-order condition and in the second example it is even missing.

## 3. Solving optimal timing problems elegantly

Before throwing the student into sophisticated examples it is good to plainly cover base conversion and the rules of differentiating exponential and logarithmic functions. After discussing those generally we introduce base conversion which allows converting any general base $b$ into the number $e$, convenient to use in economic and mathematical analysis, or any general number *a*, that is,

[7] For reference see Todorova, T. *"Problems Book to Accompany Mathematics for Economists,"* Wiley-Blackwell, Hoboken, New Jersey, 2010

$b = e^{\ln b}$ $\qquad\qquad\qquad$ $b = a^{\log_a b}$

The method of base conversion to the natural exponential number $e$ is key to the teaching of optimal timing where it is much easier to work with $e$ as the base when it comes to finding first-order and second-order derivatives of objective functions. It is good to remind students that when the base is 10 we speak of a common log, when it is $e$ we speak of a natural log. The formula for base conversion is as important in deriving the derivatives of exponential and logarithmic functions. Practical experience shows that students taking introductory mathematical economics have no difficulty dealing with logarithms, logarithmic and exponential functions, but find their differentiation particularly hard. A student-friendly approach is to mention that students already know that the derivative of the function $e^t$ with respect to time $t$ is the function itself, that is,

$$\frac{de^t}{dt} = e^t$$

The logarithmic equivalent of this natural exponential function is the simple logarithmic function $\ln t$ whose derivative with respect to $t$ is

$$\frac{d\ln t}{dt} = \frac{1}{t}$$

These simple illustrations can be expanded to the more general cases where instead of the variable $t$ we have some function of it $f(t)$ with the following derivatives:

$$\frac{de^{f(t)}}{dt} = f'(t)e^{f(t)} \qquad\qquad \frac{d\ln f(t)}{dt} = \frac{f'(t)}{f(t)}$$

We can easily see that the first two simple derivatives are special cases of the last two when $f(t) = t$. A more interesting case is when the base is a number different from $e$, for example, some general base $b$, stressed only in Chiang (1984) both in the theoretical and problem section. Base conversion comes in handy now, where the derivative of the function $b^t$ with respect to $t$ can be found as

$$\frac{db^t}{dt} = \frac{de^{t\ln b}}{dt} = e^{t\ln b}\ln b = b^t \ln b \qquad\qquad \frac{d\log_b t}{dt} = \frac{d(\ln t/\ln b)}{dt} = \frac{1}{t\ln b}$$

In the most expansive case of a function $f(t)$ we have respectively

$$\frac{db^{f(t)}}{dt} = \frac{de^{f(t)\ln b}}{dt} = f'(t)b^{f(t)}\ln b \qquad\qquad \frac{d\log_b f(t)}{dt} = \frac{d(\ln f(t)/\ln b)}{dt} = \frac{f'(t)}{f(t)\ln b}$$

We can refer to the usual timber cutting problem where for a given planted forest the value of timber is an increasing function of time $V = 3^{\sqrt{t}}$ but note that the base is chosen to be general, rather than the easy exponential case. At an interest (discount) rate of $r$ the present value of timber is $A(t) = Ve^{-rt} = 3^{\sqrt{t}}e^{-rt}$ which serves as the objective function to be maximized. Chiang (1984) is the only author presenting this interesting case of a general base different from $e$, but followed by a very awkward solution using the rate of growth. Chiang takes the log of the net present value function, differentiates it with respect to time to present it in the form of a rate of growth and finally expresses the first derivative $A'(t)$ in terms of the present value function $A(t)$, where at the optimum the first derivative should be zero. A more elegant way to solve the example is converting the base to $e$ so that to account for the discounting factor. Rather than, for instance, applying the product rule, which is clumsy and hardly convenient to use in the second-order condition, we proceed as,

$$A(t) = 3^{\sqrt{t}} e^{-rt} = e^{\sqrt{t}\ln 3} e^{-rt} = e^{\sqrt{t}\ln 3 - rt}$$

At this point it is easy to apply the first-order condition of maximization by simply differentiating an exponential function of the type $e^{f(t)}$, as follows

$$A'(t) = \frac{dA}{dt} = e^{\sqrt{t}\ln 3 - rt}\left(\frac{\ln 3}{2\sqrt{t}} - r\right) = A(t)\left(\frac{\ln 3}{2\sqrt{t}} - r\right) = 0$$

Since present value is positive, i.e., $A(t) \neq 0$, the first derivative $A'(t)$ can be zero only when $r = \frac{\ln 3}{2\sqrt{t}}$, that is, for $t^* = \frac{\ln^2 3}{4r^2}$ giving the optimal number of years after which the timber should be cut. At an interest rate of 10%, for example, the optimal number of years is approximately 30 years, or

$$t^* = \left(\frac{1.098}{0.2}\right)^2 \approx 30 \text{ years}$$

To prove that present value is, indeed, maximized we resort to the second-order condition

$$A''(t) = \underbrace{A'(t)}_{0}\left(\frac{\ln 3}{2\sqrt{t}} - r\right) + A(t)\left(-\frac{\ln 3}{4t\sqrt{t}}\right) = A(t)\left(-\frac{\ln 3}{4t\sqrt{t}}\right) < 0$$

The first term in the second derivative is 0 because at the maximum the first derivative is $A'(t) = 0$, while the second term has a negative sign. Thus, we prove that the second derivative is negative and $t^* \approx 30$ years is an optimal solution. The method is smooth and can be applied to various situations and optimal-timing decisions. A good illustration is a speculator in precious stones who has purchased a ruby that is increasing in value according to the function $V(t) = V_o e^{\sqrt{t}}$, where $t$ is time measured in years and $V_o$ is some initial value of the ruby. This time the base is the simple exponential number but the example is interesting in that students are surprised to find out that the initial value $V_o$ of the ruby (at the initial moment or at $t = 0$) has no effect on optimal time $t$. At a discount rate of 8% per year, for instance, we find

$$A(t) = V_o e^{\sqrt{t}} e^{-rt} = V_o e^{\sqrt{t} - 0.08t} \qquad A'(t) = V_o e^{\sqrt{t} - 0.08t}\left(\frac{1}{2\sqrt{t}} - 0.08\right) = 0 \qquad t^* = \frac{1}{0.16^2} \approx 39,$$

that is, the speculator should sell the ruby in approximately 39 years to maximize its present value. Students see that, since optimal time $t$ depends solely on the interest rate, it is unaffected by the initial value $V_o$. The second-order condition proves maximum, i.e.,

$$A''(t) = A'\left(\frac{1}{2\sqrt{t}} - 0.08\right) + V_o e^{\sqrt{t}-0.08t}\left(-\frac{1}{4t^{3/2}}\right) = 0 + A(t)\left(-\frac{1}{4t^{3/2}}\right) < 0$$

The example could be modified by giving a specific number for the initial value or slightly changing the function in the exponent. For instance, an investment which is valued approximately by the function $f(t) = 50{,}000e^{0.2\sqrt{t}}$ where the student has to find the relative change in the value of the investment, that is, its rate of growth, in, say, 10 years? A simple rate-of-growth formula gives

$$\frac{f'(t)}{f(t)} = \frac{0.2(50{,}000)e^{0.2\sqrt{t}}}{2\sqrt{t}(50{,}000)e^{0.2\sqrt{t}}} = \frac{0.1}{\sqrt{t}} = \frac{0.1}{\sqrt{10}} = 0.0316 \text{ or approximately } 3.16\%$$

Or the example of a coin and stamp dealer who calculates that his collection will appreciate after $t$ years by the formula $V(t) = \$1{,}000e^{\sqrt{t/4}}$ in dollar terms. An annual discount rate of 8% results in an optimal time $t^*$ of nearly 10 years when the dealer should sell the collection in order to maximize the return. An interesting pedagogical example is a general present value function $A(t) = V(t)e^{-rt}$, where the future value is not exactly known but taken generally. Again, using our simple technique of base conversion we can demonstrate to students that at the optimal time $t^*$, future value grows exactly at the rate of interest $r$, that is, $\dot{V}(t) = r$, where

$$A(t) = V(t)e^{-rt} = e^{\ln V(t) - rt} \qquad A'(t) = A(t)\left[\frac{d\ln V(t)}{dt} - r\right] = A(t)\left[\dot{V}(t) - r\right] = 0$$

Furthermore, using the second-order condition and the implicit function rule, we can check that even in the general case optimal time $t^*$ depends negatively on the interest rate $r$.

$$A''(t) = \underbrace{A'(t)\left[\dot{V}(t) - r\right]}_{0} + A(t)\left[\frac{V''V - (V')^2}{V^2}\right] = \underbrace{A(t)}_{+}\left[\frac{V''V - (V')^2}{V^2}\right] < 0$$

$$F(t^*, r) = \frac{V'(t^*)}{V(t^*)} - r = 0$$

$$\frac{dt^*}{dr} = -\frac{F_r}{F_{t^*}} = \frac{1}{\frac{V''V - (V')^2}{V^2}} = \frac{V^2}{V''V - (V')^2} < 0 \text{ (negative by SOC)}$$

The example of a large wood products company which has planted hybrid trees and has determined that the value $V(t)$ of this timber (in millions of dollars) is increasing over time according to the exponential function $V(t) = 16^{\lg(8t+16)}$ involves a common logarithm in the exponent. At an interest rate $r$ of 6% per year, the company should cut the timber for maximum profit in approximately 18 years since

$$A(t) = 16^{\lg(8t+16)}e^{-0.06t} = e^{2\ln 4 \lg(8t+16) - 0.06t}$$

$$A'(t) = e^{\cdots}\left[\frac{2\ln 4(8)}{\ln 10.(8t+16)} - 0.06\right] = e^{\cdots}\left(\frac{2\lg 4}{t+2} - 0.06\right) = 0$$

$$\frac{2\lg 4}{t+2} - 0.06 = 0$$
$$t^* \approx 18 \text{ years}$$

The example could be transformed into a general log such as $V(t) = 8^{\log_2 \sqrt{6+2t}}$ where at an interest rate $r$ of 10% per year, a diamond should be sold in 12 years which will maximize profit to his owner since

$$V(t) = 8^{\log_2 \sqrt{6+2t}} = 2^{\frac{3}{2}\log_2 (6+2t)} = (6+2t)^{\frac{3}{2}}$$

$$A(t) = (6+2t)^{\frac{3}{2}} e^{-0.1t} = e^{\frac{3}{2}\ln(6+2t) - 0.1t} \qquad A'(t) = e^{\cdots}\left[\frac{3}{(6+2t)} - 0.1\right] = 0, \text{ and } \quad t^* = 12 \text{ years}$$

The logarithmic form of the exponent could as well be transformed into a parametric problem where students could be asked to establish the effect of certain parameters on optimal time. An example could be a large orange producer in Greece the value $V(t)$ of whose produce (in millions of dollars) is increasing over time according to the exponential function $V(t) = a^{\ln \sqrt{bt+c}}$, where $a,b,c > 0$, $a > e$ and $t$ is time for oranges to ripe. Assuming that the discount rate is $r$, students may be asked to find optimal time $t$ for the producer to pick oranges such that the present value of the harvest is maximized and to see how $t$ is related to the three coefficients $a,b,c$ and the interest rate $r$. A similar task is for them to find the optimal time to pick bananas for a large banana producer in Ecuador who has determined that the value $V(t)$ of the produce (in millions of dollars) is given by $V(t) = a^{\ln \sqrt{t-bc}}$ again specifying the parameters, i.e., ($a,b,c > 0$ and $a > e$). The example of an olive producer in Spain for whom the value $V(t)$ of the olive produce is estimated to be $V(t) = a^{\log_b \sqrt{ct+d}}$ ($a > b > 1$ and $c,d > 0$), is taking a general, rather than a natural, log in the exponent. Examples could as well include paintings and museums as the famous Prado museum in Madrid which owns a painting by El Greco. Experts in the field have evaluated that its value $V(t)$ is increasing over time according to the exponential function $V(t) = 2^{\sqrt{2t+5}}$. At an annual interest rate constant at 8 percent, Prado computes that it is optimal to sell El Greco's painting in about 35 years. Or the famous Thyssen museum which owns a Picasso painting whose value grows by the function $V(t) = 2^{\sqrt{2t+8}}$ and which at an annual interest rate of 10 percent should be sold in 20 years.

**Conclusion**

These are only a few of the ample illustrations of optimal timing problems which could be covered in a course in mathematical economics. Examples are easy for students to comprehend while expanding their knowledge of mathematics or operations with logarithms, exponents, logarithmic and exponential functions. Through the method of base conversion students learn to swiftly differentiate a net present value function and its derivative so that to use first- and second-order condition. Optimal timing allows covering essential economic models and applications using the instruments of univariate calculus. Yet, the student feels like a true discoverer and an economist having found an optimal value within a simple framework and using simple techniques. Students see real-life illustrations of economic modeling which stops being abstract and obscure. Our purpose as teachers is to make math less intimidating and the subject of mathematical economics more lovable for students. Innovative and easy methods of familiarizing students with some mathematical tools as well as expanding economic applications of mathematics to new settings such as optimal decisions in time is one way to attract student attention to the subject of mathematical economics.

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