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Abstract:
In the Czech Republic each elementary or secondary school decides which textbook will be used for a given class and a given subject of education. As a supply of textbooks is wide, a selection of the most suitable textbook by a teacher is a typical case of multi-criteria decision making situation where an evaluation of different textbooks on selected criteria is rather ordinal in nature than cardinal: it is not possible to assign textbooks some numerical value with regard to criteria such as content, comprehensibility, adequacy to children's age and knowledge, etc. (with the exception of textbook's price), but textbooks can be ranked from the best to the worst by such criteria, and the best textbook can be found by a new and simple mathematical method developed for this purpose in this paper. The aim of the paper is to show how this multi-criteria decision making method with ordinal information can be used for the selection of the most appropriate textbook for elementary science education, because a right choice of a textbook plays an important role in children's education. And we shall not forget that decisions made today influence the world tomorrow, and the World of Tomorrow is also a World of Our (well-educated) Children.

Keywords: elementary education, multi-criteria decision making, ordinal information.

JEL: C61, D89.

1. Introduction

Multi-criteria decision making (MCDM) is an important part of everyday's life as well as it is succesfully applied in many areas of human action such as management, marketing, engineering, social science, politics, environmental protection, etc. With regard to character of information available about criteria and alternatives MCDM can be divided into MCDM with *cardinal* and *ordinal* information respectively.

In the former case criteria and alternatives are assigned numerical values (weights) with respect to a goal or criteria respectively, and the best alternative attains the highest value (when criteria are maximization ones). This is typical for the weighted sum approach (WSA), TOPSIS, analytic hierarchy/network process (AHP/ANP), etc., see e.g. Saaty (1980, 1996) or Ramík and Perzina (2008).

In the latter case only ordinal information is available, that is ordering (ranking) of n alternatives from the 1^{st} to the n^{th} place with regard to some criteria is known. This problem dates back to the late 18^{th} century and preferential elections context, see Borda (1781) or Condorcet (1785). In this type of MCDM ordinal information is usually transformed into cardinal information. An easy way to do so provides Borda-Kendall's method of marks (counts), see Borda (1781) or Kendall (1962): each alternative is assigned the number of points (marks) corresponding to its rank and an alternative with the lowest total sum (or average) of marks is the best. However, this and similar methods treat positions in a ranking as weights, but this is an ad-hoc assumption as real weights are not known. Cook and Kress (1991) and Cook et al. (1997) proposed more sophisticated method to generate (cardinal) weights of criteria by the data envelopment analysis. Also, several methods for group decision making with ordinal information were proposed, see e.g. Cook (2006) or Tavana et al. (2007), but none of these methods is suitable for multiple criteria.

However, at present there is no strictly ordinal MCDM method known to author though ordinal information might be sufficient to compare alternatives in many cases. This is somewhat surprising as ordinal way of comparison of objects is more general and also more natural than cardinal approach. We rank objects in accord with our preferences every time when we are selecting a car, a computer, a drink or a dinner in a restaurant, a holiday destination, a book to read, etc. We rank objects by criteria such as beauty, style, design, safety, taste, elegance, prestige, knowledge, etc., but it is not natural or even possible to evaluate objects with some numerical values by such criteria, e.g. on Saaty's 1-9 fundamental scale.

Hence, the aim of the article is to propose a simple method for MCDM with strictly ordinal information about criteria and alternatives. The use of the method is illustrated on a real-world problem: a selection of the most appropriate textbook for elementary science education. This is an important task because a right choice of a textbook plays an important role in children's education. And we shall not forget that decisions made today influence the world tomorrow, and the World of Tomorrow is also a World of Our (well-educated) Children.

2. Formulation of the problem and the method for its solution

Setting of the problem *of multi-criteria decision making with ordinal information* considered in this article is as follows:

Let $C=\{C_1,...,C_k\}$ be the set of criteria and let $A=\{A_1,...,A_n\}$ be the set of alternatives. Let the ordering of all criteria according to their importance be as follows: $C_1 \succ C_2 ... \succ C_k$, so C_1 is the most important criterion and C_k is the least important criterion. Let all alternatives be ranked from the best to the worst by all criteria C_i (such rankings are nothing else than permutations of all alternatives). The goal of the problem is to rank the alternatives from the best to the worst with regard to all criteria. For better intelligibility alternatives will be denoted without lower indices as *A*, *B*, *C*, etc., thereinafter.

Definition 1. *For each pair of alternatives A and B there is a binary index-vector* $I_{(A,B)}$ *such that* $I_{(A,B)}=(i_1,...,i_k)$ *and* $i_j=1$ *if an alternative A is ranked better than B with regard to a criterion j, otherwise* $i_j=0$.

Example 1. Consider four criteria (from the most important C_1 to the least important C_4) and an alternative *A* is ranked better than *B* by criteria C_2 and C_4. Then $I_{(A,B)}=(0,1,0,1)$.

Remark 1. As a consequence of Definition 1 there exists also an index-vector $I_{(B,A)}$ such that $I_{(B,A)}=(1,...,1)-I_{(A,B)}$. In Example 1 we obtain: $I_{(B,A)}=(1,0,1,0)$. Both $I_{(A,B)}$ and $I_{(B,A)}$ are binary and are inverse to each other.

Index-vectors provide immediate information about alternatives' pair-wise comparisons by each and every criterion, and they can be used for determining overall dominance between two alternatives. If an alternative *A* dominates some other alternative *B*, it means that *A* is better (ranked higher) than *B* overall (by all criteria).

Definition 2. Let $H_{(A,B)} = (h_1, ..., h_k)$ *be a cumulative index-vector such that* $H_{(A,B)} = \left(i_1, \sum_{j=1}^{2} i_j, ..., \sum_{j=1}^{k} i_j \right)$. $H_{(B,A)}$ *is defined analogously from the index-vector* $I_{(B,A)}$, *and let* $H_{(B,A)} = \left(\widehat{h_1}, ..., \widehat{h_k} \right)$.

Example 2. In Example 1 we have $H_{(A,B)} = (0,1,1,2)$ and $H_{(B,A)} = (1,1,2,2)$.

Definition 3a *(a pair-wise dominance relation): An alternative A dominates an alternative B* ($A \succ B$) *iff :*

$$h_i \geq \widehat{h_i} \quad \forall i \in \{1, ..., k\}, \tag{1}$$

and at least one inequality in (1) is strict (and vice versa).

Example 3. From Example 1 we have $H_{(A,B)} = (0,1,1,2)$ and $H_{(B,A)} = (1,1,2,2)$. According to Definition 3a) an alternative *B* dominates an alternative *A*, because: $\widehat{h_i} \geq h_i \ \ \forall i \in \{1, ..., 4\}$, and from these four inequalities two inequalities are strict.

A pair-wise dominance relation from Definition 3a) can be equivalently reformulated without the use of mathematical notation either:

Definition 3b *(equivalent formulation of a dominance relation): An alternative A dominates an alternative B* ($A \succ B$) *iff for each criterion by which B is ranked better than A, there is a unique and more important criterion, by which A is ranked better than B.*

Consequence of Definition 3b). If an alternative *A* dominates an alternative *B* then *A* is ranked better than *B* by the same or higher number of criteria than vice versa. And also, an alternative *A* is ranked better than *B* by the most important criterion.

The logic behind Definition 3b) is simple: If one alternative is ranked better by more important criteria (and also by majority of criteria or at least by the same number of criteria) then it should be ranked better overall too.

Consider Example 1 once more: in this example an alternative *A* is ranked better than *B* by criteria C_2 and C_4, so an alternative *B* is ranked better than *A* by criteria C_1 and C_3. No matter what are the (unknown) weights of all criteria, *B* should be ranked better than *A* overall, because it is ranked better by more important criteria (C_1 is more important than C_2, and C_3 is more important than C_4) than *A*.

The dominance relation from Definition 3 induces only quasi-order on a set of all alternatives. Hence, there might be pairs of alternatives which are incomparable, as in the following Example 4.

Example 4. Consider four criteria (from the most important C_1 to the least important C_4) and an alternative *A* is ranked better than *B* by criteria C_1 and C_4, so $I_{(A,B)} = (1,0,0,1)$ and $I_{(B,A)} = (0,1,1,0)$, $H_{(A,B)} = (1,1,1,2)$ and $H_{(B,A)} = (0,1,2,2)$. As can be seen from both cumulative index-vectors, there is no dominance between A and B, because $H_{(A,B)}$ has greater value on the first position, while $H_{(B,A)}$ has greater value on the third position. Hence, we cannot conclude, which alternative is ranked better overall. It is again logical, because for some weights of criteria A would be evaluated better (typically when C_1 acquires large weight), while for some other set of weights B would be evaluated better (typically when C_2 and C_3 acquire almost the same weight as C_1, and C_4 acquires a low weight).

The final overall ranking of all alternatives is every ranking consistent with all alternative pair-wise comparisons by Definition 3.

To summarize, the proposed model for MCDM with ordinal information proceeds in the following steps:

1. All criteria are ranked from the most important to the least important.
2. All alternatives as ranked from the best to the worst with regard to all criteria.
3. Index-vectors and cumulative index-vectors for all pairs of alternatives are established.
4. All pairs of alternatives are compared by a pair-wise dominance relation (1).
5. All alternatives are ranked into one or more final rankings.

From examples we will now turn to the question of a solution of the MCDM problem with ordinal information in general. There are three possible cases:

- The final overall ranking of all alternatives constitutes a complete order on a set of alternatives. Then the best alternative is unique and constitutes the best solution to a given problem.
- The final overall ranking of all alternatives is not complete, it is only a quasi-order (some alternatives cannot be compared and are regarded as ‘ties’), but the best alternative is unique and constitutes the best solution to a given problem. The fact that some other (and worse) ranked alternatives are incomparable is usually unimportant.
- The final overall ranking of all alternatives is not complete, it is only a quasi-order (some alternatives cannot be compared and are regarded as ‘ties’), and the best alternative is not unique. In such a case the best solution to a given problem cannot be found by the described method. All best alternatives can be considered equally good. A decision maker might try to repeat the method with only these best alternatives, or he/she can use some other method of MCDM, this time with cardinal information (such as AHP/ANP), which might be more suitable for a given problem.

3. The selection of the most appropriate elementary science textbook

In this Section six elementary science textbooks for the 6th grade pupils abbreviated by letters A, B, C, D, E and F (see Figure 1) are going to be evaluated by four criteria in order to find the most suitable textbook by the proposed MCDM method with ordinal information.

The criteria are:

- *Content (C)*: a textbook should include a required topic, for example energy and its changes, in a depth and an extent required by a given school (teacher). Because each elementary school in the Czech Republic has its own school educational programme, needs for a suitable textbook differ from school to school.

- *Comprehensibility and adequacy (CA)*: text, explanations, concepts, examples, etc. should be comprehensible and adequate to the children's age and prior knowledge.
- *Graphic design (GD)*: text and pictures should be clear and well arranged to foster understanding.
- *Price (P)*: price of a textbook should be as low as possible.

It is clear that the first three criteria cannot be cardinal as it doesn't make sense to assign a numerical value to content, comprehensibility, adequacy to children age and knowledge or graphic design. The only cardinal criterion is the price, but it can be made ordinal too.

In the first step all criteria were ranked from the most important (1.) to the least important (4.):

1. *C*	2. *CA*	3. *GD*	4. *P*

Somewhat unusually, the price is the least important criterion, because it has no sense to use the cheapest textbook if it is useless (it doesn't include required topic, or it is incomprehensible, etc.). But if there are two textbooks satisfying all more important criteria evenly, then the cheaper textbook is more suitable. Moreover, the price of different textbooks is very similar (around 6 euro).

Next, all textbooks were ranked by an expert (an experienced teacher) with regard to all criteria, see Table 1.

Table 1. Rankings of textbooks with regard to all criteria.

Criterion	1.	2.	3.	4.	5.	6.
C	A	D	C	E	F	B
CA	C	A	B	F	E	D
GD	E	B	A	D	C	F
P	D	A	C	B	F	E

In the following step index-vectors and cumulative index-vectors for each pair of alternatives were evaluated:

- Pair A-B: $I_{(A,B)} = (1,1,0,1)$, $I_{(B,A)} = (0,0,1,0)$, $H_{(A,B)} = (1,2,2,3)$, $H_{(B,A)} = (0,0,1,1)$, and according to a condition (1) from Definition 3 an alternative *A* dominates an alternative *B*: $A \succ B$.
- Pair A-C: $I_{(A,C)} = (1,0,1,1)$, $I_{(C,A)} = (0,1,0,0)$, $H_{(A,C)} = (1,1,2,3)$, $H_{(C,A)} = (0,1,1,1)$: $A \succ C$.
- Pair A-D: $I_{(A,D)} = (1,1,1,0)$, $I_{(D,A)} = (0,0,0,1)$, $H_{(A,D)} = (1,2,3,3)$, $H_{(D,A)} = (0,0,0,1)$: $A \succ D$.
- Pair$A$-E: $I_{(A,E)} = (1,1,0,1)$, $I_{(E,A)} = (0,0,1,0)$, $H_{(A,E)} = (1,2,2,3)$, $H_{(E,A)} = (0,0,1,1)$, $A \succ E$.
- ...
- Pair B-C: $I_{(B,C)} = (0,0,1,0)$, $I_{(C,B)} = (1,1,0,1)$, , $H_{(B,C)} = (0,0,1,1)$ $H_{(C,B)} = (1,2,2,3)$: $C \succ B$
- ...

After all 15 pairs were compared; the following 12 dominance relations were obtained:

$$A \succ B,\ A \succ C, A \succ D, A \succ E, A \succ F,\ C \succ B,$$

$$C \succ E\,,\ C \succ F\,,\ D \succ C\,,\ D \succ F\,,\ E \succ B\,,\ E \succ F\,.$$

The final overall ranking of all alternatives have to be consistent with these 12 relations. Three remaining pairs were incomparable, namely: *B*-*D*, *B*-*F* and *D*-*E*.

The final overall rankings are: (A, D, C, E, F, B) and (A, D, C, E, B, F). Hence, the best (most suitable) textbook is *A*.

As it would be interesting to compare this result with some other method, by Borda-Kendall's method of marks one obtains the final ranking (A, C, D, B, E, F), see Table 2. The main difference between both methods is in ranking of alternatives *C*, *D* and *B*.

Table 2. Rankings of textbooks with regard to Borda-Kendall's method of marks.

Overall ranking	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**
Alternative	A	C	D	B	E	F
Sum of points	8	12	13	15	16	20

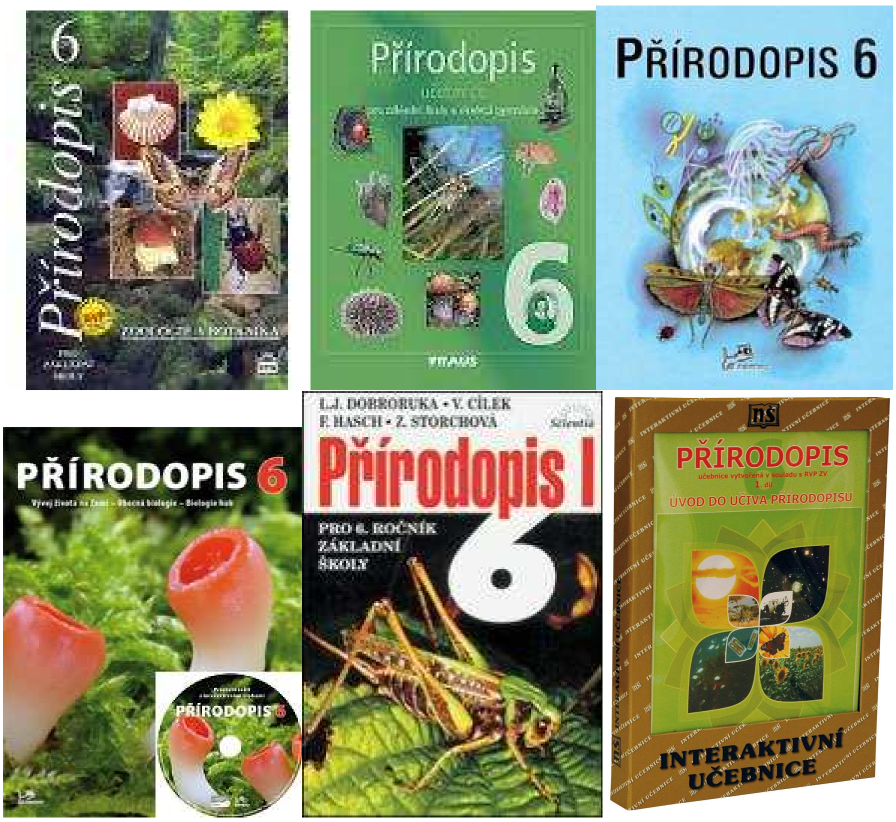

Figure 1. Elementary science textbooks from *A* to *F* for the 6th grade pupils. 'Přírodopis' means 'natural history'. This subject is taught at elementary level before science subjects such as physics, biology or chemistry.

4. Extensions

The method described in previous sections can be extended to include ties or uncertainty (intensity of preference) between alternatives:

- *Ties*: When a decision maker is not sure, whether an alternative A is better than an alternative B or vice versa, than he/she can assign both alternatives 0.5 points. In such a case index-vectors are not binary any more, as they can include values 0.5 as well. But the method (the model) doesn't change. From index-vectors cumulative index-vectors are derived, and the dominance relation (1) is used in the same manner as before to acquire the final ranking of alternatives.

- *Uncertainty*: The approach above can be further generalized to include uncertainty or intensity of preference. If $p_{j,k} \in [0,1]$ is the intensity of preference of an alternative J over an alternative K from a set of alternatives A, $p_{k,j} \in [0,1]$ vice versa, and $p_{k,j} + p_{j,k} = 1$, then the relation $p: A \times A \to [0,1]$ is called the fuzzy preference relation. Index-vectors now include values from the interval $[0,1]$, and the method can be used as before again, see Mazurek (2011, 2012).

Example 5. Consider five criteria (from the most important C_1 to the least important C_5) and two alternatives: A and B. An alternative A is ranked better than B by criteria C_1 and C_2, the same as B by a criterion C_3, and worse than B by remaining criteria. Then:

$$I_{(A,B)} = (1,1,0.5,0,0), I_{(B,A)} = (0,0,0.5,1,1)$$

$$H_{(A,B)} = (1,2,2.5,2.5,2.5), H_{(B,A)} = (0,0,0.5,1.5,2.5).$$

According to the dominance relation (1) an alternative A dominates B.

5. Conclusions

The aim of the article was to demonstrate new method for multi-criteria decision making with ordinal information about criteria and alternatives on the selection of the most appropriate textbook for elementary science education, which is a common teachers' problem in the Czech Republic. The proposed method is computationally simple and intuitive, so a teacher can follow its steps without problems. Also, the method can be used in many other areas of multi-criteria decision making where the problem is rather ordinal than cardinal in nature. To facilitate decision making in problems with the larger number of alternatives, the software support tool in the form of Microsoft Excel Add-in would follow soon. Further research may focus on problems with incomplete rankings or problems with dependent criteria.

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