Interest rate paradox

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Acribia Management

20 June 2013

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Abstract

Maximization of result from operations with securities is not always the ultimate goal of participants. For example, result can be exchanged into different currencies. There can be different utility functions that transform result into some asset. Different risk-neutral probability densities could be derived from one set of option prices by participants using different utility functions. Integral of derived density function must be equal to one. There have to be no such utility function for which this condition is not met. Otherwise, derived function is not a probability density. This allows using of risk-free profitable arbitrage strategies. However it was shown that such utility function almost always exist. It is hard to use on nowadays markets. By this reason such opportunity was called "weak arbitrage".

Keywords: market efficiency, probability density, interest rate, arbitrage, efficiency conditions.

AMS subject classification: 91G20.
JEL classification codes: G10, G12

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1. Introduction

Market efficiency is to some extent crucial property of a market. It can be used for searching profitable, for example, arbitrage strategies. Such strategies drive markets to its efficient state (at least to equilibrium). If market is not efficient, especially if inefficiency is fundamental, then it is of high importance to understand how it has to change to be efficient and how this shift influences other markets (even not financial) and relations between different participants.

There are different methods and hypotheses, which are discussed and often argued. For example, empirical [7, 9, 11] and analytical findings [3, 12] have challenged the efficient market hypothesis. Some scientists use statistical arbitrage analysis and testing of historical data for understanding to what extent market if efficient. It was used by Bondarenko [1] and Hogan, Jarrow, Teo, and Warachka [4], and later improved in Jarrow, Teo, Tse, and Warachka [6].

This paper presents theoretical research, which can be used for further statistical testing. Probability density is observed from the point of view of different participants. For example, operations' result can be transformed in different currencies. In other words participant may have different utility functions that transform result from operation to some asset that is preferable to participant. This function was used for analyzing option prices. Option prices are equal for all participants, but utility functions may vary. Consequently, there can be different probability density functions implied in option prices for different utility functions and participants. For example, European and American participants should derive different probability density functions from EUR/USD options.

This approach potentially can be used for obtaining constraints on option prices and market efficiency conditions. For example, all derived functions have to be probability density functions.

2. Internal interest rate

Examine options on exchange rate between assets B and A. In first case our goal is to maximize amount of asset A. Then call option premium is

$$C(K) = c_a \cdot \int_K^\infty d_a(\Delta S) \cdot (\Delta S - K) d(\Delta S) \tag{1}$$

$c_a = e^{-r \cdot t}$ is discount coefficient. If premium is paid (in theory) at the moment of expiration then $c_a = 1$.

K is difference between strike price and initial price of B.

ΔS is price change of the underlying asset B.

$d_a(\Delta S)$ is risk-neutral probability density function.

Premium's first derivative is

$$\frac{d}{d(K)}C(K) = -c_a \cdot \int_K^\infty d_a(\Delta S)d(\Delta S) \tag{2}$$

First derivative is a price of a binary option.

Premium's second derivative is

$$\frac{d^2}{d(K)^2}C(K) = c_a \cdot d_a(K) \tag{3}$$

In second case our goal is to maximize amount of some asset C. When option expires, market participant transforms result of operation into asset C using utility function E(X). Premium in C should be

$$C'(K) = c_c \cdot \int_K^\infty d_c(\Delta S)\cdot(\Delta S - K)\cdot \mathrm{E(X)}d(\Delta S) \tag{4}$$

X – is some set of parameters. In most cases it includes ΔS . It does not contain K.

Option is priced in asset C. To compare its price with one obtained above it should be converted in asset's A units.

$$C(K) = \frac{C'(K)}{E(\mathrm{X}_0)} = c_c \cdot \int_K^\infty d_c(\Delta S)\cdot(\Delta S - K)\cdot \frac{\mathrm{E(X)}}{E(\mathrm{X}_0)} d(\Delta S) \tag{5}$$

$E(\mathrm{X}_0)$ is $E(\mathrm{X})$ at the moment of option writing.

For asset B:

$$\mathrm{E(X)} = \frac{1}{S + \Delta S} \tag{6}$$

$$\mathrm{E(X_0)} = \frac{1}{S} \tag{7}$$

S is a price of asset B at the moment of option writing.

Derivatives of premium are

$$\frac{d}{d(K)}C(K) = -c_c \cdot \int_K^\infty d_c(\Delta S)\cdot \frac{\mathrm{E(X)}}{\mathrm{E(X_0)}} d(\Delta S) \tag{8}$$

$$\frac{d^2}{d(K)^2}C(K) = c_c \cdot d_c(K) \cdot \frac{E(\mathrm{X})}{E(\mathrm{X}_0)} \tag{9}$$

Option prices and derivatives have to be equal. Then

$$c_c \cdot d_c(K) \cdot \frac{E(\mathrm{X})}{E(\mathrm{X}_0)} = c_a \cdot d_a(K) \tag{10}$$

If price is expected to be changing then probability densities do not have to be equal to each other. In the case of asset B they are not equal.

Examine the case of asset B. Next equations have to be true.

$$\int_{-\infty}^{\infty} d_a(\Delta S) d(\Delta S) = 1 \tag{11}$$

$$\int_{-\infty}^{\infty} d_b(\Delta S) d(\Delta S) = 1 \tag{12}$$

They show simple thing: probability that price will be in interval $(-\infty; -\infty)$ have to be equal to one. Otherwise, such "scenario" is underpriced or overpriced and corresponding arbitrage opportunities arise. In this case it is possible, using combination of options, to gain asset A or C for free.

Transform equation (12) using equation (10):

$$\begin{aligned} \int_{-\infty}^{\infty} d_b(\Delta S) d(\Delta S) &= \int_{-\infty}^{\infty} \frac{c_a}{c_b} \cdot d_a(\Delta S_T) \cdot \frac{S + \Delta S}{S} d(\Delta S) = \\ &= \int_{-\infty}^{\infty} \frac{c_a}{c_b} \cdot d_a(\Delta S) d(\Delta S) + \int_{-\infty}^{\infty} \frac{c_a}{c_b} \cdot d_a(\Delta S) \cdot \frac{\Delta S}{S} d(\Delta S) = 1 \end{aligned} \tag{13}$$

Consequently,

$$\int_{-\infty}^{\infty} d_a(\Delta S) \cdot \Delta S d(\Delta S) = S \cdot \frac{c_b - c_a}{c_a} \tag{14}$$

Option prices reflect differences between interest rates of assets. This equation does not tell that interest rate parity have to be met on markets. There are evidences against [2, 8, 10] or for it [5]. However, this equation tells that interest rate parity have to be expected. If not then equation (11) or equation (12) is not true and there are arbitrage opportunities.

Equation (14) is one of efficiency conditions. For the general case analogous equations can be derived:

$$\frac{c_a}{c_c}\cdot\int_{-\infty}^{\infty} d_a(\Delta S)\cdot\frac{1}{E(\mathrm{X})}d(\Delta S)=\frac{1}{E(\mathrm{X}_0)} \quad (15)$$

$$\frac{c_c}{c_a}\cdot\int_{-\infty}^{\infty} d_c(\Delta S)\cdot E(\mathrm{X})d(\Delta S)=E(\mathrm{X}_0) \quad (16)$$

If there is such asset C and E(X) that makes equation (15) or equation (16) false then market is not efficient and there are arbitrage opportunities.

3. Options on interest rates

Asset at different moments of time (in fact, futures on asset with expiration at these moments) could be observed as different assets. There are time-varying prices between them. Also there can be corresponding financial instruments. Money market is the most common example. However, almost every asset has its internal interest rate.

Examine options on such assets. Let asset A be underlying asset at some moment T1 and B – at T2. Such assets have zero internal interest rate, i.e. $c_a = c_c = 1$, because their prices are determined only by internal interest rate of underlying asset A. Otherwise participant could buy one "dollar at some moment t" (futures with expiration time t) and gain at this moment more (or less) than one dollar.

Options expire at some t0<T1<T2. If our goal is to maximize return at some other point of time T1+T then

$$E(\mathrm{X}) = e^{-r_1(\Delta\mathrm{S},\mathrm{T})\cdot\mathrm{T}} \quad (17)$$

$$E(\mathrm{X}_0) = e^{-r_0(\mathrm{T})\cdot\mathrm{T}} \quad (18)$$

$r_1(\Delta\,\mathrm{S},\mathrm{T})$ is interest rate at which we transform just after expiration asset A at moment T1 to asset A at moment T1+T.

$r_0(\mathrm{T})$ is the same interest rate, but at the moment at option writing.

From equation (15) follows

$$\int_{-\infty}^{\infty} d_a(\Delta S)\cdot e^{r_1(\Delta S,T)\cdot\mathrm{T}}d(\Delta S) = e^{r_0(\mathrm{T})\cdot\mathrm{T}} \quad (19)$$

Equation (19) has to be true for every T.

Consequently,

$$\lim_{T\to\infty}(\mathrm{r}_1(\Delta \mathrm{S},\mathrm{T})) = \lim_{T\to\infty}\mathrm{r}_0(\mathrm{T}) \qquad (20)$$

Expectations of reflected in prices long-term interest rates have to be constant. Thus analysis of equations (15) and (16) can give constraints on interest rate dynamics.

4. Efficiency conditions test

Equations (15) and (16) are market efficiency conditions, because if they do not hold then risk-free arbitrage is possible. These strategies can be used by participants and drive market to efficient state. To prove that market is inefficient it is needed to find such asset and E(X) that make equations (15) and (16) false.

In this section proof of next thesis is proposed: if exchange rate between assets X and Y is equal to exchange rate between asset Y and Z during some time then such market is inefficient and risk-free profitable strategies are possible. X, Y and Z are variations of some asset A at different moments of time (futures on some asset A with different expiration times).

Such situation is possible on stock market, on money market and on every market where assets have non-zero internal interest rate.

It is possible to create a portfolio of paying dividends shares. Dividends can be partly spent on buying new shares to portfolio. If this decision is announced in advance then we expect price increase and according to equation (14) internal interest rate decrease. Also some of shares can be sold. In this case we expect price decrease and internal interest rate increase. Interest rate can change from 0 to infinity.

Let asset A be some currency, asset B be this portfolio, asset B1 be asset B at moment t1, asset B2 be asset B at moment t2, asset B3 be asset B at moment t3. Examine options on exchange rate between B3 and B2. Expiration time is t0<t1. Result from operation is in asset B2.

It is possible to manage price and internal interest rate in the next way:

$$S_b(\mathrm{t}1,\mathrm{t}2) = S_a(\mathrm{t}1,\mathrm{t}2) \qquad (20)$$

$$S_b(\mathrm{t}1,\mathrm{t}2) = S_b(\mathrm{t}2,\mathrm{t}3) \qquad (21)$$

$S_x(\mathrm{t}\,i,\mathrm{t}\,j)$ is exchange rate between asset X at the moment tj and asset X at the moment ti.

Just before t0 portfolio manager defines dividend policy and interest rates on (t1, t2) and (t2, t3).

First case. Our goal is to maximize B1. Then

$$E(\mathrm{X}) = S + \Delta S \tag{22}$$

$$E(X_0) = S \tag{23}$$

Assume that

$$\int_{-\infty}^{\infty} d_{b1}(\Delta S)d(\Delta S) = 1 \tag{24}$$

$$\int_{-\infty}^{\infty} d_{b2}(\Delta S)d(\Delta S) = 1 \tag{25}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} d_{b2}(\Delta S)d(\Delta S) = \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\frac{(\mathrm{S}+\Delta S)}{S}d(\Delta S) = \int_{-\infty}^{\infty} d_{b1}(\Delta S)d(\Delta S) + \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\frac{\Delta S}{S}d(\Delta S) = \\ 1 + \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\frac{\Delta S}{S}d(\Delta S) = 1\end{aligned} \tag{26}$$

Consequently,

$$\int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\Delta S d(\Delta S) = 0 \tag{27}$$

At the same time

$$\begin{aligned}\int_{-\infty}^{\infty} d_{b2}(\Delta S)\cdot\Delta S d(\Delta S) = \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\frac{(S+\Delta S)}{S}\cdot\Delta S d(\Delta S) = \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\Delta S d(\Delta S) + \\ + \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\frac{\Delta S^2}{S}d(\Delta S) = \int_{-\infty}^{\infty} d_{b1}(\Delta S)\cdot\frac{\Delta S^2}{S}d(\Delta S)\end{aligned} \tag{28}$$

All multipliers are above zero. Consequently,

$$\int_{-\infty}^{\infty} d_{b2}(\Delta S)\cdot\Delta S d(\Delta S) \neq 0 \tag{29}$$

Second case. Our goal is to maximize B3. Then

$$E(\mathrm{X}) = \frac{1}{S + \Delta S} \tag{30}$$

$$E(X_0) = \frac{1}{S} \tag{31}$$

Assume that

$$\int_{-\infty}^{\infty} d_{b2}(\Delta S) d(\Delta S) = 1 \tag{32}$$

$$\int_{-\infty}^{\infty} d_{b3}(\Delta S) d(\Delta S) = 1 \tag{33}$$

Then

$$\int_{-\infty}^{\infty} d_{b3}(\Delta S) d(\Delta S) = \int_{-\infty}^{\infty} d_{b2}(\Delta S) \cdot \frac{(\mathrm{S} + \Delta S)}{S} d(\Delta S) = \int_{-\infty}^{\infty} d_{b2}(\Delta S) d(\Delta S) + \int_{-\infty}^{\infty} d_{b2}(\Delta S) \cdot \frac{\Delta S}{S} d(\Delta S) = \\ 1 + \int_{-\infty}^{\infty} d_{b2}(\Delta S) \cdot \frac{\Delta S}{S} d(\Delta S) = 1 \tag{34}$$

Consequently,

$$\int_{-\infty}^{\infty} d_{b2}(\Delta S) \cdot \Delta S d(\Delta S) = 0 \tag{35}$$

However, in both cases one combination of options was used and $d_{b2}(\Delta S)$ is the same.

Consequently, at least one of three "probability densities" $d_{b1}(\Delta S)$, $d_{b2}(\Delta S)$ or $d_{b3}(\Delta S)$ is not a probability density. Independently from real expectations it allows using risk-free profitable strategies that result in obtaining for free asset B1, B2 or B3 – futures on asset B with different moments of expiration. Prices of B, B1, B2 and B3 are always positive.

Mentioned above strategies have to drive market to its efficient state. What this state has to be? There are two conditions:

1. Market is perfectly developed, i.e. there are no transaction costs and it is highly liquid. Underdevelopment was not considered in research.
2. There are assets which internal interest rates are expected to change.

First could be a problem, because difference between probability densities should be very small. By this reason found inefficiency is proposed to be called “weak arbitrage”. However, while market is developing transaction costs are decreasing and liquidity is increasing.

Second assumption is too fundamental for market economy. So, the paradox arises. For efficiency, markets and whole economy have to fundamentally differ from the existing. Moreover, it should tend to this state, because of risk-free profitable strategies existence.

5. Conclusion

It was shown that participants with different interests imply different probability density functions in option prices. When new participants with new interests arise they influence prices, which in-turn influence other probability functions. New interests refer to new asset, in which participant transform result from operations on markets. Interconnections between probability functions can be used to determine constraints on option prices and market efficiency conditions. Using this method it was found that interest rate parity has to be expected in prices and long-term interest rates have to be constant. In other case there are risk-free profitable strategies that drive market to the efficient state.

As a generalization of interest rate parity two equations were obtained. There have to be no such interests (assets) that make found efficiency conditions false. It was found that such asset almost always exists. To eliminate this opportunity serious fundamental shifts in economy have to take place.

Found inefficiency allows using risk-free profitable strategies. However, they are based on tiny, but fundamental, deviations. By this reason market underdevelopment could be a problem. But if market is developed enough or some participants find corresponding opportunities then markets should shift to new efficient state, which greatly differs from the current one. Participant’s benefits for making such shift should be extremely significant.

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