# A note on NIG-Levy process in asset price modeling: case of Estonian companies

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# A note on NIG-*Lévy* process in asset price modeling: case of Estonian companies

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**Abstract**

The purpose of this note is three folds. First, we review *Lévy* processes and analyse jumps. Second, we correct mistakes relating to terminology and analysis of results in Teneng [7]. Third, we extend results by showing returns of companies trading on Tallinn Stock Exchange between 01 January 2008 and 01 January 2012 cannot be modeled by NIG distribution; both in cases where closing prices can and cannot be modelled by NIG distribution. Thus, the NIG-*Lévy* process cannot be used to forecast the future prices of these assets.

**Keywords:** NIG, Levy process, Jumps, forecasting, goodness of fits.

**JEL Classification:** C68, C29
**AMS Classification:** 60G51, 65C20

## 1 Introduction

As an asset is traded at fair value, its varying price trace an interesting trajectory reflecting in a general way the asset's value and underlying stochastic properties or economic activities. Accurate asset price models are therefore important in describing and capturing these stochastic properties, constructing optimal portfolios, underwriting financial derivatives, maximizing investor profits and possibly predicting future prices. Of late, many processes have been suggested to replicate these price trajectories. Famous among is Brownian Motion, known to have serious modelling limitations like light tails, inability to effectively capture jumps, model stochastic volatility and a host of others.

*Lévy* process based models, of which Brownian Motion is a special case seek to eliminate shortcomings of Brownian Motion based models. They clearly can distingusih between large and small jumps, and do not necessarily have continuous paths. As well, two major classes of *Lévy* processes are of interest here: Jump diffusion and infinite activity *Lévy* processes. For jump diffusion cases, jumps are considered rare events and in any finite interval, there are only finitely many jumps. Opposite case applies to infinite activity models i.e. in any finite interval, there are infinitely many jumps. Infinite activity models are our focus, and particularly the NIG case which was implemented with daily closing data of some companies trading on Tallinn Stock Exchange in Teneng [7] and extended for their returns in this note.

The Black-Scholes model makes use of the exponential of Brownian Motion with drift. This ensures its probability distribution is lognormal. This case is especially interesting when considering log returns which is not our case in this note. We work simply with one period returns. One period returns have little difference with log returns but for long time scales. It must be pointed out that lognormal model has continuous paths but cannot model jumps, one of the reasons why *Lévy* processes are attractive here.

Section two deals with jumps analysis and a brief overview of notions related to *Lévy* Processes. Section three outlines model selection strategy, implements this for returns and handles corrections to Teneng [7].

## 2 Jumps

From Figure 1, it can be observed that there are jumps in prices. These jumps are random, and of different heights i.e. beyond any set equilibrium say $X_0 = 0.1$. Suppose we choose a random time $T$ and

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Figure 1: Price trajectory of Olympic Entertainment Grupp

the corresponding price $X_T$. Will the value at $t$, i .e. $X_t$ be below or above $X_T$? This same question can be represented with an indicator function as follows ($T$ a stopping time)
$X_t = \mathbf{1}_{\{t \geq T\}}$, $t > 0$ i.e. $X_t$ has a value 0 until the event $t \geq T$ when it takes the value 1 or this tells us when $X_t$ becomes equal or bigger than $X_T$. Another major problem here is determining the height of the jump or how far above $X_T$ is $X_t$? To answer this, we can characterize the value $X_t - X_T$ to be bigger than or smaller than one for bigger or smaller jumps respectively. Further, if we seize from considering a point $t$ to considering an interval $\{0, t\}$, $X_t$ will then characterize the number of jumps in this interval. This $X_t$ can now be represented easily by a Poisson distribution of parameter $\lambda t$. On the other hand, if we require that the jump sizes are no longer 1, then these jumps can be characterized by a probability distribution. We return to such cases later.

Denote $\Delta X_t = X_t - X_{t-}$ to be the jump at time $t$. Counting both large and small jumps respectively, we have

$$\Delta X_t = X_t - \sum_{t \geq s} \Delta X_s \mathbf{1}_{\{|X_s|>1\}} - \sum_{t \geq s} \Delta X_s \mathbf{1}_{\{|X_s| \leq 1\}}. \tag{1}$$

We can now introduce a random measure of jumps

$$\mu^x(\omega : dt, dx) = \sum_{s>0} \mathbf{1}_{\{\Delta X_s(\omega) \neq 0\}} \epsilon(s, \Delta X_s(\omega))(dt, dx)$$

i.e. for fixed $\omega$, $\mu^x$ places a point mass of size 1 on each pair $(s, \Delta X_s(\omega)) \in \Re_+ \times \Re$ if the process has a jump of size $\Delta X_s(\omega)$ at time $s$. This transforms equation 1 in to

$$\Delta X_t = X_t - \int_0^t \int_{\Re} X_s \mathbf{1}_{\{|X_s|>1\}} \mu^x(ds, dx) - \int_0^t \int_{\Re} X_s \mathbf{1}_{\{|X_s| \leq 1\}} \mu^x(ds, dx). \tag{2}$$

We have finitely many large jumps and can have infinitely many small jumps within the interval $\{0, t\}$. To make the whole count process meaningful, we reduce and average of these small jumps (predictable compensator) $\upsilon^x$ in such a way that $(\mu^x - \upsilon^x)$ is in general non-separable.

$$\Delta X_t = X_t - \int_0^t \int_{\Re} X_s \mathbf{1}_{\{|X_s|>1\}} \mu^x(ds, dx) - \int_0^t \int_{\Re} X_s \mathbf{1}_{\{|X_s| \leq 1\}} (\mu^x - \upsilon^x)(ds, dx). \tag{3}$$

Let us return to the case where the jump size is not one. We can assume jump size $Y_k$ has law $L(Y_k) = \mu$. Let us also assume $(N_t)_{\{t \geq 0\}}$ counts the number of jumps. Then $X_t$ can be represented by $X_t = \sum_{k=1}^{N_t} Y_k$, a compound poisson process with characteristic function $E[e^{iuX_t}] = e^{\{t\lambda \int_{\Re}(e^{iux}-1)\mu(dx)\}}$.

From a practical point of view, it is desirable to answer some questions before modelling asset prices. Questions like are there jumps? Are these frequent? Are Jumps large or small? Do upward movements differ from downward? And a host of others. Addressing these independetly and combining them to build the asset evolution process gives an extra edge. These questions can be answered effectively by *Lévy* process models, the subject of the next subsection.

## 2.1 General *Lévy* Process

*Lévy* processes are named after famous French mathematician Paul *Lévy* who helped bring together an understanding and characterization of processes with independent and stationary increments.

A *Lévy* process is a continuous time stochastic process $X = \{X_t : t \geq 0\}$ defined on the probability space $(\Omega, F, P)$ with the following basic properties:

1. $P\{X_0 = 0\} = 1$ i.e. starts at zero
2. $\forall s, t \geq 0$, $X_{s+t} - X_t \stackrel{d}{=} X_s$, stationary increments
3. $\forall s, t \geq 0$, $X_{s+t} - X_s$ is independent of $X_u : u \leq t$
4. $t \rightarrow X_t$ is *a.s.* right continuous with left limits (*Cadlag*).

The law of a *Lévy* Process is completely determined by its characteristic triplet. We will return to the characteristic triplet after discussing *Lévy*-Khintchine formula. Before delving into this formula, lets consider the issue of infinite divisibility introduced by Italian Mathematician Bruno De Fineti in 1929.

## 2.2 Infinite divisibility

The concept of infinite divisibility was introduced by De Fineti and he related it to *Lévy* processes, making the class of *Lévy* functions much wider.

Generally speaking, an $\Re^d-$valued random variable $Y$ has infinite distribution if for each $n = 1, 2, \cdots$, there exist a sequence of independently identically distributed random variables $Y_1^n, Y_2^n, \cdots, Y_n^n$ such that $Y \stackrel{d}{=} Y_1^n + Y_2^n + \cdots + Y_n^n$. Alternatively, this relation can be expressed in terms of the characteristic exponent.

Suppose $Y$ has a characteristic exponent $\Psi(u) = -log(E[e^{iuY}])$. Then $Y$ is infinitely divisible if and only if for $n \geq 1$, there exist a characteristic exponent say $\Psi_n$ such that $\Psi(u) = n\Psi_n(u)$ for all $u \in \Re^d$. An extension of this property is achieved with the *Lévy*-Khintchine formula for *Lévy* processes.

## 2.3 *Lévy*-Khintchine formula

$Y$ is infinitely divisible if and only if there exist a triplet $(b, c, \mu)$ such that

$$\Psi(u) = iub - \frac{uc^2}{2} + \int_{\Re} (e^{iux} - 1 - iux\mathbf{1}_{\{|x| \leq 1\}}))\mu(dx) \tag{4}$$

where $b \in \Re$ and is a drift term, $c \in \Re^+$ a diffusion term and $\mu$ is a positive measure on $\Re/\{0\}$ such that $\int_{\Re/\{0\}} (1 \wedge |x|^2)\mu(dx) < \infty$. In other words,

$$\mu^x(\omega : dt, dx) = \sum_{s>0} \mathbf{1}_{\{\Delta X_s(\omega) \neq 0\}} \epsilon(s, \Delta X_s(\omega))(dt, dx)$$

.

The truncation function $\mathbf{1}_{\{|x| \leq 1\}}$ equals 1 when $|x| \leq 1$ and 0 otherwise. The purpose of this truncation function is to analyze jump properties around the singular point of zero. When $\int_{\Re/\{0\}} \mu(dx) = \lambda < \infty$, $\lambda$ the mean arrival rate of jumps, then the *Lévy* process is said to have finite activity. A finite activity jump process generates a finite number of jumps within any finite time interval. When $\lambda$ is infinite, the process is infinite or exhibit infinite activity i.e. it generates infinite number of jumps within any finite time interval. The truncation function $\mathbf{1}_{\{|x|<1\}}$ is needed only for infinite variation jumps.

Also, when $c \neq 0$, the sum of small jumps does not converge[1] i.e. $\int_{|x| \leq 1} |x| \, \mu(dx) = \infty$. This means almost all paths of the *Lévy* process will have infinite variation. On the other hand, when $c = 0$ and $\int_{|x| \leq 1} |x| \, \mu(dx) < \infty$, then almost all paths of the *Lévy* process have finite variation.

To see how the *Lévy*-Khintchine formula disintegrates in to Brownian motion and Poisson distribution terms, the Itô-*Lévy* decomposition theorem is presented below.

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[1] But the sum of jumps compensated by their means does converge(compensated Poisson process). This special behavior generates the necessity of the truncation function $\mathbf{1}_{\{|x| \leq 1\}}$

| Company | alpha($\alpha$) | beta($\beta$) | delta($\delta$) | mu($\mu$) | Skew | Kurtosis | KS p | KS D | $\chi^2 stat$ | $\chi^2 - p$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Arco Vara | 0.96 | 0.756 | 0.015 | -0.0017 | 31.47 | 994.45 | $<10^{-5}$ | 0.1433 | $<10^{-5}$ | 36365.44 |
| Baltika | 20.923 | 0.858 | 0.032 | -0.003 | -0.06 | 2.07 | 0.002 | 0.082 | $<10^{-5}$ | 14338.35 |
| Ekpress Grupp | 15.59 | 0.7032 | 0.024 | -0.002 | 0.253 | 2.308 | 0.0002 | 0.095 | $<10^{-5}$ | 23977.53 |
| Harju Elekter | 14.32 | 0.774 | 0.015 | -0.0008 | 0.247 | 3.098 | $<10^{-5}$ | 0.128 | $<10^{-5}$ | 38758.52 |

Table 1: Estimated NIG Parameters, Skews, Kurtoses, Kolmogorov-Smirnov(KS) and Chi-square (CS) test results for NIG-*Lévy* models; (returns)

### 2.4 Itô-*Lévy*-decomposition

For every $X$-$\Re^d$ value *Lévy* process with a *Lévy* measure $\mu$, the following properties hold:

1. The jump measure $\mu(dx)$ is a Poisson random measure on $\Re_+ \times \Re$ with intensity $\lambda = dt \times \mu$
2. The *Lévy* measure $\mu$ stisfies $\int_{\Re^d}(1 \wedge |x|^2)\mu(dx) < \infty$
3. There exist $b \in \Re^d$ and a d-dimensional Brownian motion $B_t$ such that $X_t = ut + B_t + N_t + M_t$ where $N_t = \int_{|x|>1, s\in[0,t]} x\mu^x(ds, dx)$ and

$$M_t = \int_{0\le|x|\le1, s\in[0,t]} x[\mu^x(ds, dx) - \nu^x(ds, dx)] = \int_{0\le|x|\le1, s\in[0,t]} x(\mu^x - \nu^x)(ds, dx) \quad (5)$$

.

It is easy to see that $M_t$ counts the small jumps and $N_t$ the big jumps within the finite interval $[0, t]$.

Considering characteristic exponents and the Itô-*Lévy*-decomposition , it clear that $\Psi(u) = \Psi^{(1)}(u) + \Psi^{(2)}(u) + \Psi^{(3)}(u)$ where

- $\Psi^{(1)}(u) = iub$ , linear or constant drift with parameter $b$
- $\Psi^{(2)}(u) = \frac{uc^2}{2}$, Brownian motion with coeficient $c$
- $\Psi^{(3)}(u) = \int_{\Re}(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\le1\}}))\mu(dx)$, compensated Poisson process.

Hence, Brownian motion and Poisson based models are just limiting cases of a general *Lévy* process model[1].For NIG, there is no Brownian Motion component or say $c = 0$.

## 3 Model selection and corrections to Teneng [7]

**Käärik and Umbleja proposed strategy for model selection**

1. choose a suitable class of distributions (using general or prior information about the specific data) ;
2. estimate the parameters (by finding maximum likelihoods);
3. estimate goodness of fit by 1) visual estimation, 2)classical goodness-of-fit tests (Kolmogorov-Smirnov, Chi-squared with equiprobable classes) and 3)probability or quantile-quantile plots.

In Teneng [7], the closing prices of assets trading on the Tallinn stock exchange between 01 January 2008 to 01 January 2012 were fitted with the normal inverse Gaussian (NIG) distribution[2]. Interpretation of results concluded Baltika and Ekpress Grupp were suitable candidates for NIG-*Lévy* asset model. Unfortunately, there were mistakes in terminology and analysis. For the closing prices to be described by NIG-*Lévy* process, the returns should be NIG distributed. This is not the case in this article (see Table 1 and Figure 3, applying Käärik and Umbleja proposed strategy for model selection) and further research also concludes this is not the case; meaning future prices of these assets cannot be forecasted with NIG-*Lévy* process. Second correction deals with the definition of general *Lévy* process; the independence criteria (*pg*.2). It is suppose to read $\forall s, t \ge 0, X_{s+t} - X_t$ **is independent of** $X_u$, $u \le t$**, i.e. independent increments.** Third correction is related to analysis of data (*Pg*.4). From Table 2, we can clearly see that Ekpress Grupp has a very small Kolmogorov-Smirnov (KS) test p-value (0.012). This means we need to reject this model as it says data does not come from theoretical probability distribution; in our case NIG distribution. We have included an updated version of graphs to display correctly the goodness of fits (Figure 2).

[1] *Lévy* processes with a.s. increasing paths are called subordinators.

[2] See Teneng [7] for exposition on NIG distribution.

| Company | alpha($\alpha$) | beta($\beta$) | delta($\delta$) | mu($\mu$) | Skew | Kurtosis | $\chi^2 stat$ | $\chi^2 - p$ | KS d | KS p |
|---|---|---|---|---|---|---|---|---|---|---|
| Arco Vara | 468.9 | 468.86 | 0.03 | 0.02 | 0.38 | -1.53 | 2251.60 | $<10^{-5}$ | 0.23 | $<10^{-5}$ |
| Baltika | 7.06 | 6.62 | 0.22 | 0.52 | 1.67 | -1.53 | 1771.12 | $<10^{-5}$ | 0.06 | 0.06 |
| Ekpress | 2.68 | 2.15 | 0.49 | 0.85 | 1.70 | 2.53 | 1194.24 | $<10^{-5}$ | 0.07 | 0.012 |
| Harju | 3.20 | -2.07 | 0.72 | 2.95 | -0.82 | -0.05 | 1345.87 | $<10^{-5}$ | 0.09 | 0.0003 |

Table 2: Estimated NIG Parameters, Skews, Kurtoses, Kolmogorov-Smirnov(KS) and Chi-square (CS) test results for NIG distribution models (Daily closing prices)

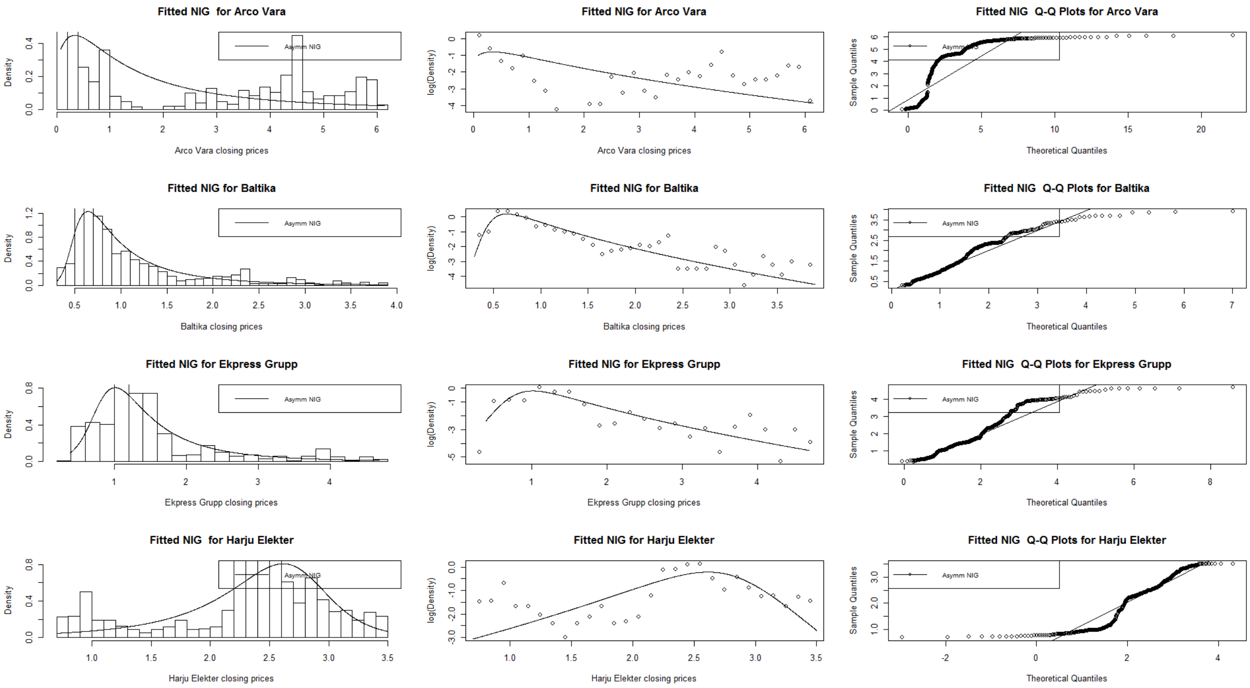


Figure 2: Fitted NIG density, log densities and Q-Q plots for Baltika, Arco Vara, Harju Elekter and Ekpress Grupp (Daily closing prices)

# 4 Conclusion

It has been demonstrated in this paper that jumps in asset prices can be captured more effectively with *Lévy* processes. It has also been made clear that Brownian motion and Poisson type models are limiting cases of a general *Lévy* process model. Specific kinds of *Lévy* processes, properties like time change dynamics, martingale measures, discussion of pricing with *Lévy* models have been left out of this work. Further, results of Teneng [7] have been extended to conclude that the closing prices of Baltika (company trading on the Tallinn Stock Exchange between 01 January 2008 and 01 January 2012) can be modeled with normal inverse Gaussian distribution, but its future prices cannot be forecasted with NIG-*Lévy* process. This is generalised for all the assets considered that none of their future prices can be forecasted with NIG-*Lévy* process.

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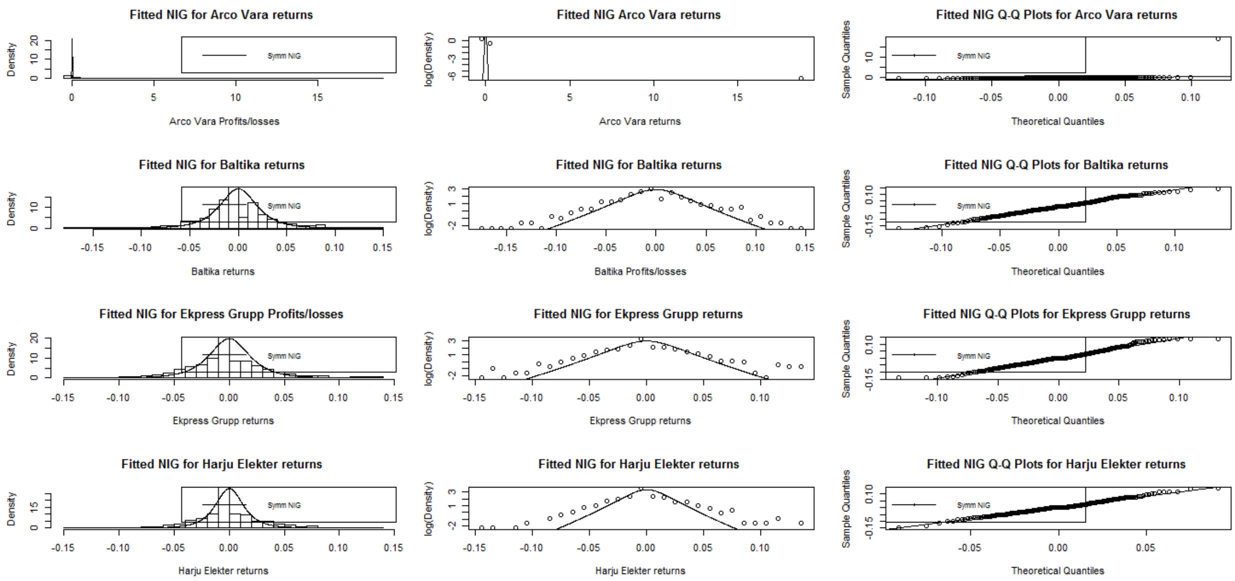


Figure 3: Fitted NIG density, log densities and Q-Q plots for Baltika, Arco Vara, Harju Elekter and Ekpress Grupp (returns)

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