Risk-Based Pricing of High Loan-To-Value Mortgage

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Abstract

High loan-to-value (LTV) mortgage are residential mortgage loans with LTV ratio greater or equal to 90%. Lenders are increasingly engaged in risk-based pricing. If properly quantified, the additional credit risk taken when originating high LTV mortgage can be compensated by higher interest rate charged to customers. High LTV mortgage is regulated to meet higher capital requirement and thus have higher funding cost. Current regulation raises regulatory capital requirement of banks on all high LTV mortgage holdings. However, it is not efficient to differentiate the risk between a high LTV first mortgage and a second lien mortgage with the same LTV.

In the paper, I show how LTV ratio affects credit risk in mortgage. A structured credit modeling approach is taken to quantify the credit risk of first mortgage and second mortgage. The total risk in a combination of first and second mortgage is shown to be equal to that of a first mortgage with the same aggregate LTV. Default risk is derived implicitly. Optionality of defaultable debt results in an upward sloping credit supply curve in terms of a function of interest rate with respect to LTV. Current regulation in high LTV mortgage creates a funding advantage in seperating a high LTV mortgage into a lower funding cost first mortgage and a higher cost second mortgage.

Keywords: mortgage lending, risk-based pricing, credit risk, regulatory capital

JEL Classification: G21, G14, D82

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1 Introduction

In the past few years, mortgage lending market has changed dramatically. New products have been developed to meet the needs of borrowers to achieve home ownership. It is estimated that there are more than 200 kinds of mortgage products in the market[1]. Many of the new products are tailored to the population that may not otherwise qualify for a traditional mortgage, either by lowering borrower's monthly payment or reducing down payment of a mortgage loan. It has not been enough time for most of the new mortgage products to go through a full credit cycle. As the housing market cools down, little is known about their likely performance in the near future. Certain sectors of the mortgage market, such as subprime, raise red flags to banks, regulators, and mortgage investors in general. It is imperative to study and understand the credit risk taken in these new and riskier products. I dedicate this paper to analyze high loan-to-value mortgage and its credit risk.

Loan-to-value ratio (LTV), is the ratio of outstanding mortgage loan balance over appraised property value. High LTV residential mortgage is defined by regulators as "any loan, line of credit, or combination of credits secured by liens on or interests in owner-occupied 1- to 4-family residential property that equals or exceeds 90 percent of the real estate's appraised value, unless the loan has appropriate credit support [2] ". High LTV mortgage can take two forms. The first form is a single senior lien mortgage with LTV ratio greater than 90%. The second form is a combination of a senior lien loan and a junior lien loan. The senior lien loan normally have LTV ratio below 80% and the combined senior and junior loan balance have close to 100% LTV ratio.

In traditional prime mortgage market, borrowers are required to provide at least 2 years of income and asset verification, make down payment with LTV ratio below 80%, and have high FICO scores. With unprecedented housing price appreciation, high LTV first mortgage opens doors to home buyers without sufficient savings to pay for the required 20% down payment. Existing home owners use second lien loan to cash out gain in home value.

Most high LTV first mortgage is issued in Alt-A market. Alt-A mortgage is a major non-prime sector. It refers to loans with non-standard features such as high LTV ratio and issued mostly to borrowers with good but less than perfect credit. Alt-A origination volume

[1] "Increasing Risks in Mortgage Lending", Supervisory Letter, 2006.

[2] "Interagency Guidance on High LTV Residential Real Estate Lending", Office of the Comptroller of the Currency, Board of Governors of the Federal Reserve System, Federal Deposit Insurance Corporation, and Office of Thrift Supervision, October 1999.

increased from 11.0 billion in 1996 to 206.1 billion in 2006.[3] 46% of all Alt-A mortgage origination have LTV ratio higher than 80% and 17% with LTV ratio more than 90%.[4]

Second lien loans can be originated simultaneously with a first mortgage or later on. It is estimated that in the first half of 2004, about 42% of home purchase mortgage loan involved with a simultaneous second lien mortgage.[5] More recent industry survey shows that simultaneous second mortgage accounts for 39% of Alt-A mortgage origination in 2006, up by 36% from 3% in 2002.[6] Simultaneous second mortgage normally have very high combined LTV ratio. About 80% of simultaneous second lien mortgage have 100% combined LTV ratio.[7] A popular choice is borrowing a 80% LTV first lien together with a 20% LTV second lien mortgage loan. It is popularly referred to as "80/20" mortgage.

From mortgage originators' perspective, as prime mortgage profit margin shrinks, issuing new product like high LTV mortgage are ways to stay profitable. Mortgage companies are in the business of managing risk. If risks are measured properly, lenders can price additional risk into higher lending rates and be profitable. If not, lenders will face tremendous risks that lead to a situation like what happened recently in sub-prime market. Regulators are concerned about risks in high LTV mortgage and issued Interagency Guidance in late 1999. It cited increased default risk and losses and limited default remedies as major credit risk concerns.

Currently, high LTV mortgage origination is regulated by policies set forth in the Interagency Guidelines for Real Estate Lending Policies (the Guidelines), jointly issued by four federal regulators. The regulation is based on capital reserve requirement, which limits banks's ability to leverage out on equity. It requires that lenders apply 100 percent of *total capital* to holdings of high LTV mortgage loans. *Total capital* refers to the aggregate of a company's equity, certain subordinated debt, and loss reserves. It is a broader definition of equity capital. As a comparison, the 1988 Basel Accord assigns regular mortgage portfolio holdings a risk weighting of 50%. That requires only 2% *tier one* (equity) capital. Banks originating and holding high LTV mortgage loans are thus facing significant capital constraints. Since debt normally demands a lower rate of return than equity, the capital requirement on high LTV loans increases the cost of funds to originate them.

Given the fact that high LTV mortgage has higher credit risk, it is a proper step for

[3] "Alt-A Credit Deterioration", UBS Mortgage Strategies, February 2007.
[4] "Fixed-Rate Alt-A MBS: Commonly Asked Questions Answered", Credit Suisse, October 2004.
[5] According to Calhoun (2005), SMR Research Corporation conducted a study with such findings.
[6] "Alt-A Credit Deterioration", UBS Mortgage Strategies, February 2007.
[7] "Silent Are Not Golden: Silent Seconds and Subprime Home Equity ABS," Credit Suisse, March 2005.

regulators to set up more stringent capital requirement. The policies established in the Guidelines, however, do not differentiate nuances in high LTV mortgage loans. Its definition of high LTV mortgage applies to first mortgage with LTV greater or equal to 90% but does not apply to a first mortgage with LTV lower than 90% having a second mortgage with combined LTV greater or equal to 90%. In the later case, only the second mortgage falls into high LTV category and requires higher capital requirement. This exception provides lenders with capital arbitrage opportunities to separate a high LTV loan into a package of a conforming first mortgage and a simultaneous second mortgage.

In the paper, I will show how LTV ratio affects credit risk of mortgage. A structured credit modeling approach is taken to quantify the credit risk of first mortgage and second mortgage. The total risk in a combination of first and second mortgage is shown to be equal to that of a first mortgage with the same aggregate LTV. Default risk is derived implicitly. Optionality of defaultable debt results in an upward sloping credit supply curve as a function of interest rate with respect to LTV. Funding advantage in separating a high LTV into a lower funding cost first mortgage and a higher cost second mortgage is shown to create new market equilibrium.

2 The Model of Home Financing

In this section, we outline a basic credit model of property finance. To ensure availability of analytical solutions, I make the assumption that property value follows a Geometric Brownian Motion process. It follows the structured credit modeling methodology originated from Merton (1974). I analyze the payoff of home owner, senior lien debt, and junior lien debt holders. For simplicity, both senior and junior debts are assumed to be zero coupon bonds that are issued at discount and paid off by a lump sum payment at maturity.

In this simplified setting, credit risk is caused by volatility of property value and borrower's leverage in mortgage financing. Down payment and interest rates on senior and junior debts are the key determinants that impact each party's payoff and risk. Since the focus is credit risk, interest rate and prepayment risk are ignored. I assume there is no principal amortization and prepayment. In housing market, dynamic hedging of exposure to property value is not possible. Therefore, risk-neutral valuation and Black-Shole's option pricing formula cannot be applied. Expected payoffs in the future with probability measure in the real world are each party's objective for decision making.

Home buyers use their own financial assets and borrow money from bank to finance purchasing of a property. A property financing package is a combination (D, B_S, B_J, r_S, r_J), where

$$\begin{aligned} D &: \text{ Borrower's down payment} \\ B_S &: \text{ Senior debt amount borrowed} \\ B_J &: \text{ Junior debt amount borrowed} \\ r_S &: \text{ Senior debt interest rate} \\ r_J &: \text{ Junior debt interest rate} \end{aligned}$$

It is possible that the borrower finances purchase of the property without junior lien debt. In that case, $B_J = 0$. All interest rates are continuously compounded. The maturity date of debt is T. Debt is zero coupon. Principal and interest are due at debt maturity and there is no interim payment. Assume that borrower is not allowed to borrow more money than the initial property value $H(0)$. It is straight forward that,

$$H(0) = D + B_S + B_J. \tag{1}$$

Down payment is home owner's equity and is often confused with personal wealth. If comparing home purchasing with capital structure of a corporation, the home owner is at a position no different from that of equity holder of a corporation. Equity ownership is obtained when the home owner acquires the property. It will not change by how much equity the home owner has. Down payment determines leverage of the financing and the payoff structure. It also affects how much home owner's personal financial asset is at risk for the property investment. With debt financing, home owner can leverage out and afford a larger property than that solely using his own financial asset. When property value decreases, home owner can default and protect himself from further loss beyond down payment. However, when that happens, home equity will all be taken by debt holder. It is thus not favorable to have a disproportionately high home equity position. As we will discuss later, home owner's payoff is equivalent to a call option with debt payment as strike price. The call option represents home owner's equity value. Down payment is a cash outflow to pay for that call option. The moment the property is financed with debt, home owner no longer owns the down payment. Instead, he owns only that call option.

Down payment should be considered with opportunity cost in terms of returns from alternative investment. By putting money down as equity in property investment, home owner expects a return no worse than that from alternative investment opportunities. Otherwise, it is rational for home owner to reduce home equity and invest it in the higher earning investment. The liquidity cost for home owner can be thought of as the return on available alternative investment or the interest rate on other means of borrowing. We also have to recognize that in reality home owners normally view home more than a financial investment. In that sense, home equity is not exactly a decision driven by investment return. The emotional factor may reduce home owner's required rate of return on home equity.

Home buyer is liquidity constrained with an opportunity cost of η. With the above understanding about borrower's opportunity cost, η can take a low value when the borrower see home with high emotional value and a high value when the borrower is focused on investment return.

Banks' funding cost for senior debt is η_S and that for junior debt η_J. Funding costs are both above the risk-free rate r. For high LTV mortgage loans, the regulatory capital requirement, as mentioned in the introduction section, will make banks' funding cost on senior mortgage jump to a higher level after first lien LTV reaches 90%. If senior mortgage LTV is lower than 90% and CLTV is greater than 90%, senior debt funding cost will be at the lower level while junior debt funding cost will be at the higher level,

$$\eta_J > \eta_S > r.$$

The difference in senior and junior debt is in their priority to claim the underlying collateral. Whenever the borrower defaults, senior lien holder has claim to the property liquidation value before junior lien holder. Junior lien holder will not be able to recover anything until senior debt is fully repaid. Intuitively, junior lien holder bears higher risk than senior lien holder. This also justifies a higher funding cost for junior debt.

Property value and total outstanding liabilities are the only two factors that matter in borrower's decision to repay debt or default. At payment due date, if property value is less than the total outstanding balance of liabilities, whether borrower defaults on either one or both of the liabilities, lender will start foreclosure process. Property will be liquidated to repay debt. Lenders' losses are affected by priority in claim to collateral but not the order of default.

We introduce the blended lending rate, $\bar{r}$. Define the blended lending interest rate $\bar{r}$ as

$$\bar{r} = \frac{1}{T} ln(\frac{B_S e^{r_S T} + B_J e^{r_J T}}{B_S + B_J}) \tag{2}$$

Applying blended rate $\bar{r}$ to combined balance of senior and junior debt, the payment due at debt maturity will equal the total payment due on senior and junior debt. It will help us to understand payoff of the borrower and risk of junior debt holder in the next few sections.

LTV ratio is widely used in mortgage underwriting as a standard measure of borrower's leverage. The higher the LTV is, the higher the leverage of the borrower and the higher the risk of debt. We will use LTV to denote the senior debt LTV only. By assumption,

$$LTV(0) = \frac{B_S}{H(0)} \quad (3)$$

Similarly, Combined-Loan-To-Value or CLTV is the combined borrowed amount of senior and junior debt over property value.

$$CLTV(0) = \frac{B_S + B_J}{H(0)}.$$

Assume that property value $H(t)$ follows a Geometric Brownian Motion process with drift parameter $\mu - q$ and volatility parameter σ,

$$\frac{dH(t)}{H(t)} = (\mu - q)dt + \sigma dz(t), \quad (4)$$

where $dz(t)$ is a standard Brownian Motion process. The parameter q represents the rent equivalent of property's function as a shelter for property owner. The parameter μ is the average rate of housing price appreciation (HPA) after adjusting for rent equivalent income. For example, if national HPA is 5% and rent equivalent income is 4%, μ would be 9%. Property value is log Normally distributed,

$$lnH(t) \sim N(lnH(0) + (\mu - q - \frac{\sigma^2}{2})t, \sigma\sqrt{t}),$$

with expected value given by

$$E[H(t)] = H(0)e^{(\mu-q)t} \quad (5)$$

and variance by

$$Var[H(t)] = H(0)^2 e^{2(\mu-q)t}(e^{\sigma^2 t} - 1).$$

This assumption about property value is similar to that of a dividend paying stock in Black-Shole's option pricing model. In the world of Black-Shole's, financial derivatives can be replicated by dynamically trading the underlying assets and therefore risk-neutral pricing can be applied. In risk-neutral world, all assets grow at the risk-free rate and all risky payoffs can be discounted at risk-free rate to derive the present value. In Merton's seminal paper on corporate credit, similar assumption is made for firm value. Unfortunately, properties are illiquid assets and are not traded regularly. Without dynamic replication, risky asset pricing cannot be transformed to their equivalent in the risk-neutral world and has to be discounted with risk-adjusted rate of return. Before determining the risk-adjusted rate of return, financial derivatives cannot be priced properly.

In this model, however, we assume the risk-adjusted return as exogenous variables.

Banks' funding rates are the risk-adjusted rate of returns required of senior and junior mortgage debt. Regulators normally set up economic capital reserve requirements that banks have to satisfy. Capital reserve requirement limits banks' ability to borrow funding at cheaper financing rate and constrains banks' leverage. I am interested in knowing the interaction among market participants and the market equilibrium lending rates as a consequence of bank regulation. Exogenous funding rate can be viewed as an instrument that central bank can utilize to control retail banks's leverage. This assumption will not alter the qualitative outcomes of the model. It would be of interest from regulator's perspective to determine regulator capital requirement that truly reflects the riskiness of debt lending and direct financial resources efficiently. This framework can act as a benchmark for empirical study to value the efficiency of regulatory policies.

With the underlying asset following Geometric Brownian motion process, expected value of option payoff has closed form solution. Since $H(t)$ is log normally distributed, it can be proved that

$$E[Max(H(T) - X, 0)] = E[H(T)]N(\delta_1) - XN(\delta_2) \tag{6}$$

and that

$$E[Max(X - H(T), 0)] = XN(-\delta_2) - E[H(T)]N(-\delta_1). \tag{7}$$

where

$$\delta_1 = \frac{ln[E(H(T)/X)] + \sigma^2/2}{\sigma\sqrt(T)} \tag{8}$$

and

$$\delta_2 = \delta_1 - \sigma\sqrt(T). \tag{9}$$

These two relationships will be used to analyze each party's payoffs through out this paper. The associate probability measure is the real world probability measure on which the Geometric Brownian motion process in Equation (4) is defined.

3 Home Buyer

At debt maturity, if property value is greater than or equal to total liabilities due, borrower will repay debt and obtain the gain in property value. Otherwise, he will default on debt. Borrower's payoff at time T is

$$C(T) = Max(H(T) - B_S e^{r_S T} - B_J e^{r_J T}, 0).$$

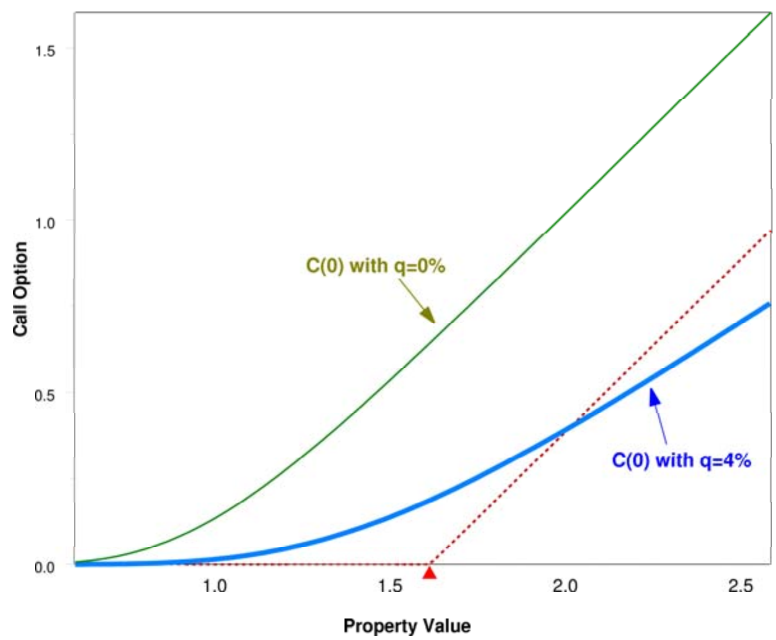

Figure 1: Borrower's Expected Call Option Value and Property Value

It is equivalent to the payoff of a call option on property value with a strike price of $B_S e^{r_S T} + B_J e^{r_J T}$. Call option entitles the borrower to upside gain without downside pain.

A call option is said to be in-the-money if the underlying asset value is greater than the option strike price. The more the call option is in the money, the more it is worth. If the underlying asset value is less than the option strike price, the call option value is still positive due to volatility or possibility of property value increase at maturity. Figure (1) illustrates borrower's call option value with respect to property value. I use Black-Shole's option pricing formula for illustrative purpose only. Parameters are close to reality. They

H_0	σ	$\bar{r}$	r	μ	q	T	η_S	η_J
1	5%	7%	4.5%	9.5%	4%	10	5.5%	6%

Table 1: Parameters Used in Figure 1

are shown in Table 1. Except otherwise specified, I use the same set of parameters in other figures or examples. In Figure (1), another case with $q = 0$ is demonstrated as well. The dotted line shows terminal payoff of the option. It starts at zero value. After crossing the strike price, it coincides with the 45 degree line above x-Axis. The thin brown line shows the option value when $q = 0$. It is always above the terminal payoff. Due to long maturity, it is quite high above the terminal payoff line. The thick blue line shows the option value when $q = 4\%$. With positive rent equivalent cash flow, the option value is lower as if property is less valuable.

Since CLTV provides a normalized measure of leverage irrespective of property value, it will be helpful to see how CLTV affects expected call option value. Let's express the call option strike price by CLTV, blended lending rate, and property value,

$$B_S e^{r_S T} + B_J e^{r_J T} = CLTV(0)H(0)e^{\bar{r}T}.$$

At time 0, expected value of borrower's terminal payoff can be determined using Equation (6). Denote it as $E[C(T)]$,

$$\begin{aligned} E[C(T)] &= H(0)e^{(\mu-q)T}N(d_1) - (B_S e^{r_S T} + B_J e^{r_J T})e^{-rT}N(d_2), \\ &= H(0)[e^{(\mu-q)T}N(d_1) - CLTV(0)e^{\bar{r}T}N(d_2)]. \end{aligned} \tag{10}$$

where

$$\begin{aligned} d_1 &= \frac{ln(H(0)/(B_S e^{r_S T} + B_J e^{r_J T})) + (\mu - q + \sigma^2/2)T}{\sigma\sqrt{T}}, \\ &= \frac{-lnCLTV(0) + ((\mu - q) - \bar{r} + \sigma^2/2)T}{\sigma\sqrt{T}} \end{aligned} \tag{11}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{12}$$

$N(.)$ is the cumulative Normal distribution function.

As CLTV goes up, the call option value is lower. I will discuss later that higher CLTV will lead to higher credit risk and debt holders will charge a higher risk premium. Higher interest rates will increase the strike price and decrease the option value. For the moment, I neglect this complication and assume constant interest rates. Formally, the relationship between CLTV and borrower's expected call option value is stated in Proposition 1. Proof is shown in the Appendix.

Proposition 1: CLTV and Borrower's Expected Call Option Value

Holding interest rates constant, the higher the CLTV, the lower the borrower's expected call option value. Mathematically,

$$\frac{\partial E[C(T)]}{\partial CLTV(0)} = -H(0) \cdot e^{\bar{r}T} \cdot N(d_2) < 0$$

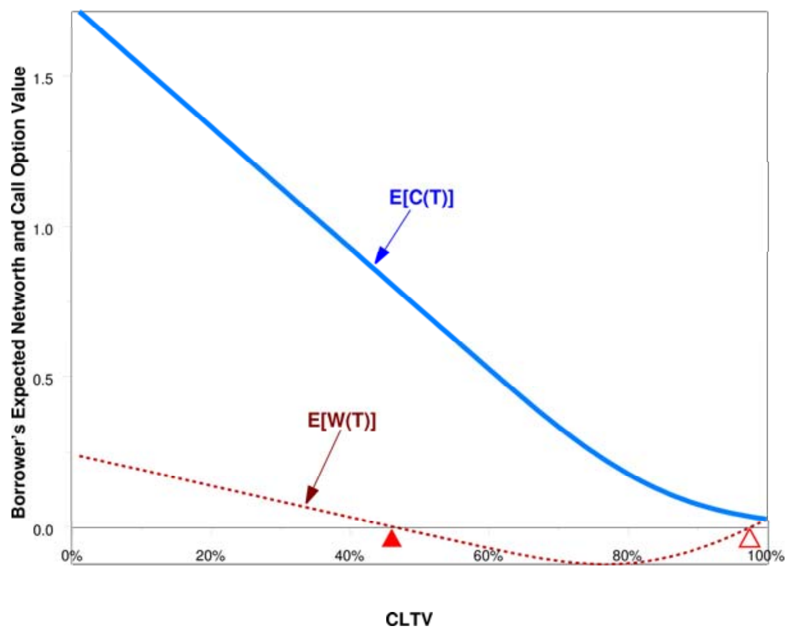

Figure 2: Borrower's Call Option Value and CLTV

where d_2 is defined as in Equation (12). Moreover, the rate of decreasing in the borrower's call option value decreases as CLTV increases,

$$\frac{\partial^2 E[C(T)]}{\partial CLTV(0)^2} = \frac{H(0) \cdot e^{\bar{r}T} \cdot N'(d_2)}{\sigma\sqrt{T} \cdot CLTV(0)} > 0.$$

The expected call option value as a function of CLTV is plotted in Figure 2. It is the solid blue line. The expected option value is at its maximum of 1.71 when CLTV is at the lowest, 0%. Expected option value decreases as CLTV increases and is always positive. It approaches 0 as CLTV increases towards 100%.

It is easy to see that the higher the lending rates, the higher the call option strike. Higher lending rates will decrease expected call option value and lower borrower's net worth. The impact of lending rate to home buyer's payoff is stated in Proposition 2. It can be proved by applying the "very important relationship" defined in the Appendix in the proof of Proposition 1.

Proposition 2: Blended Interest Rate and Borrower's Expected Call Option Value

Borrower's expected call option value is a decreasing function of blended lending rate $\bar{r}$.

$$\frac{\partial E[C(T)]}{\partial \bar{r}} = -T \cdot H(0) \cdot CLTV(0) \cdot e^{\bar{r}T} N(d_2) < 0 \tag{13}$$

Now let's turn the attention to borrower's net worth, the ultimate objective to consider when purchasing a property. Borrower's expected net worth at acquiring the property, $E(W(T)]$, is the expected call option value minus down payment with interest accrued at η.

$$E[W(T)] = E[C(T)] - De^{\eta T} \tag{14}$$

Although the call option is more valuable with lower CLTV, lower CLTV also corresponds to higher down payment. If the borrower makes down payment more than the value of the call option, his initial net worth will be negative. Borrower's net worth at time 0 as a function of CLTV is shown in Figure 2. In the graph, the borrower's net worth starts at 0.33 when down payment is zero. It gradually drops to the minimum as CLTV increases. After passing certain point, the net worth increases toward 0 as CLTV gets closer to 100%. Borrower's net worth minimum is reached when CLTV is around 39%. The minimum point is marked with a red triangle in the graph. When CLTV exceeds 99%, borrower's net worth starts to be positive. It is not shown very clearly in the graph due to scale. If we use a lower rent equivalent income parameter q, borrower's net worth will become positive with a lower CLTV. From borrower's perspective, it is beneficial to borrow at CLTV as close to 100% as possible.

I state the impact of CLTV on borrower's net worth $W(0)$ rigorously below. Proof is omitted. If utilizes the fact that if $\eta \geq \bar{r}$, then

$$e^{(\eta - \bar{r})T} \geq 1 > N(d_2). \tag{15}$$

Proposition 3: CLTV and Borrower's Expected Net Worth

Borrower's expected net worth $E[W(T)]$ is affected by CLTV. The rate of change is given by,

$$\frac{\partial E[W(T)]}{\partial CLTV(0)} = H(0) \cdot [e^{\eta T} - e^{\bar{r}T} N(d_2)]. \tag{16}$$

If borrower's liquidity constraint is higher than blended mortgage rate, $\eta \geq \bar{r}$, the expected net worth is increasing with respect to CLTV.

The impact of blended interest rate on borrower's net worth is the same as that on the call option value. We have the following proposition.

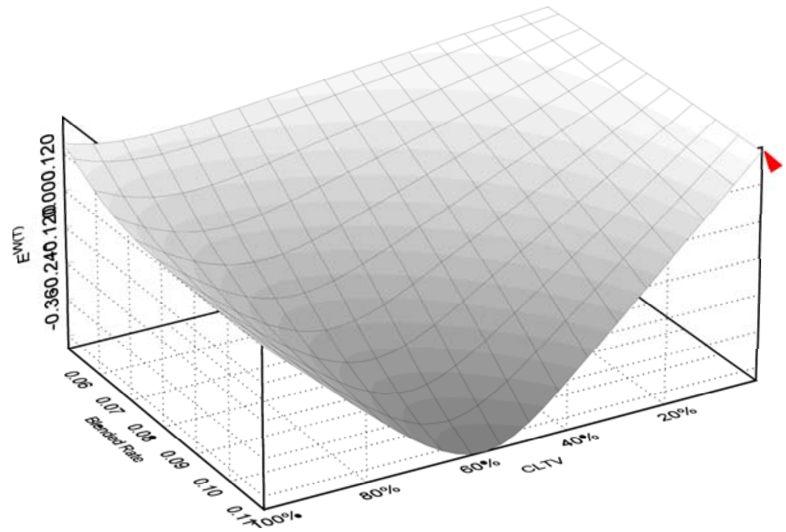

Figure 3: Borrower's Expected Net Worth Surface - CLTV and Blended Interest Rate

Proposition 4: Blended Debt Interest Rate and Borrower's Expected Net Worth
Borrower's expected net worth $E[W(T)]$ is a decreasing function of blended debt interest rate $\bar{r}$,

$$\frac{\partial E[W(T)]}{\partial \bar{r}} = -H(0) \cdot CLTV(0) \cdot T \cdot e^{\bar{r}T} \cdot N(d_2) < 0. \tag{17}$$

Borrower's expected net worth at $t = 0$ as a function of CLTV and blended interest rate is plotted in Figure (3). The maximum is achieved at the front right hand side corner as identified by a red Triangle. The senior debt interest rate has to be equal to risk-free rate, to get there. It is therefore not obtainable. Borrower's expected net worth is much more sensitive to CLTV than to blended interest rate. In this example, for a loan with 100% CLTV, borrower's expected net worth will decrease by 0.003 if the blended lending rate changes from 7% to 10%. Holding blended lending rate the same at 7%, borrower's expected net worth will decrease by 0.010 if CLTV drops from 100% to 99%.

Based on the previous two propositions on borrower's expected net worth, I make the following statement about borrower's debt financing objective.

Proposition 5: Borrower's Debt Financing Objective

Borrower's objective function is,

$$\begin{aligned} MAX_{CLTV} \quad & E[W(T)] \\ s.t. \quad & \bar{r}(CLTV). \end{aligned} \tag{18}$$

where $\bar{r}(CLTV)$ *is lender's credit supply function. Borrower's marginal rate of substitution between blended interest rate and CLTV to maintain the same expected net worth is,*

$$\begin{aligned} MSR^{B}_{\bar{r},CLTV} &= -\frac{\partial E[W(T)]}{\partial CLTV(0)} / \frac{\partial E[W(T)]}{\partial \bar{r}} \\ &= \frac{1}{T \cdot CLTV(0)} \cdot \frac{e^{\eta T} - e^{\bar{r}T} N(d_2)}{e^{rT} N(d_2)}. \end{aligned} \tag{19}$$

If the borrower has a liquidity constraint higher than the blended lending rate, it is beneficial to borrow with as high CLTV as possible. It is true for the same reason as shown in Equation (15).

4 Senior Debt Holder

As we discussed in the previous section, at debt maturity, if property is worth less than liabilities, borrower will default on debt. Senior debt holder's risk lies in the possibility that the property liquidation value may not fully cover outstanding balance of senior debt when default occurs. Senior lien holder's payoff at maturity, $SL(T)$, is the minimum of senior lien notional amount and property value,

$$\begin{aligned} SL(T) &= Min(H(T), B_S e^{r_S T}) \\ &= B_S e^{r_S T} - Max(B_S e^{r_S T} - H(T), 0) \end{aligned}$$

The last term in the second line of the equation is terminal payoff of a put option with strike price of $B_S e^{r_S T}$. Senior lien holder essentially takes a long position in a risk-free bond and a short position in a put option on property value with a strike price at senior debt balance at maturity. Denote the put option value at time t as $P_S(t)$. We have

$$SL(t) = B_S e^{r_S T - r(T-t)} - P_S(t) \tag{20}$$

By Equation (7), at $t = 0$, the expected put option terminal value is,

$$E[P_S(T)] = H(0)[LTV(0)e^{r_S T} N(-d_4) - e^{(\mu-q)T} N(-d_3)] \tag{21}$$

where

$$\begin{aligned} d_3 &= \frac{lnH(0)/B_S + (\mu - r_S - q + \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{-lnLTV(0) + (\mu - r_S - q + \sigma^2/2)T}{\sigma\sqrt{T}} \end{aligned} \quad (22)$$

and

$$d_4 = d_3 - \sigma\sqrt{T} \quad (23)$$

Notice that junior debt balance is not involved in Equation (21). We are ready to show the first result on senior debt holder's risk.

Proposition 6: The Impact of Junior Debt on Senior Debt

Senior debt holder's risk is determined by LTV of senior debt and existence of junior debt is irrelevant.

Calhoun (2005) argues that simultaneous second lien loans expose senior debt holder to greater credit risk. Proposition 6 dismisses that statement.

The expected put option and senior debt value is plotted in Figure (4). I assumed $r_S = 6.5\%$. Expected put option value increases as LTV increases and reaches its maximum of 0.50 at 100% LTV.

If we take a step further, the impact of LTV on expected put option value can be quantified as,

$$\frac{\partial E[P_S(T)]}{\partial LTV(0)} = H(0)e^{r_S T}N(-d_4) > 0 \quad (24)$$

and

$$\frac{\partial^2 E[P_S(T)]}{\partial (LTV(0))^2} = H(0)e^{r_S T}N'(-d_4)\frac{\sqrt{T}}{\sigma} > 0 \quad (25)$$

Senior debt interest rate r_S affects expected senior debt value only through the embedded put option. LTV has additional influence through debt balance. Senior debt interest rate r_S has negative impact on senior debt value. For the same balance borrowed, higher interest rate makes borrower more likely to default.

Senior debt holder's expected profit is the expected senior debt value minus funding cost,

$$E[\Pi_S(T)] = H(0) \cdot LTV(0) \cdot (e^{r_S T} - e^{\eta_S T}) - E[P_S(T)].$$

I state LTV and senior debt interest rate's impact on expected senior debt value in the following two propositions without proof.

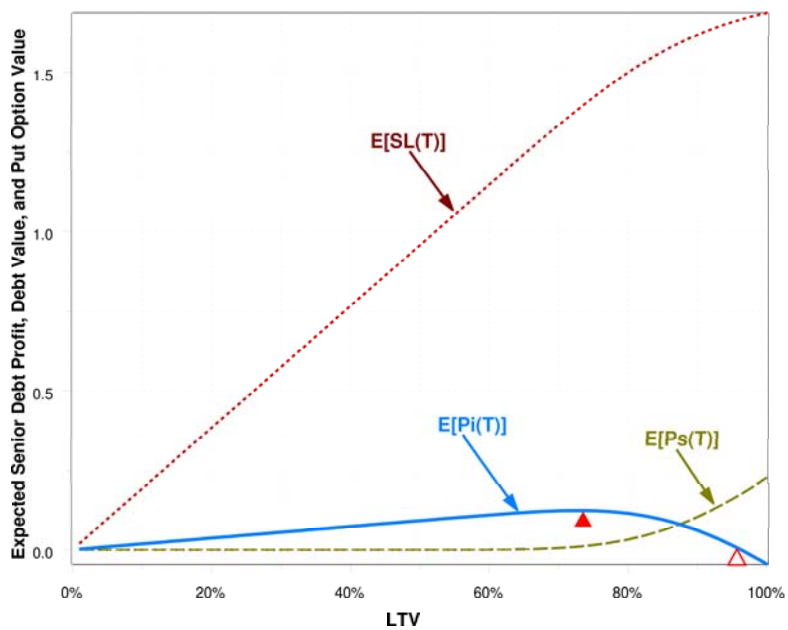

Figure 4: Expected Senior Debt and Put Option Value

Proposition 7: Senior Debt Interest Rate and Expected Profit

The impact of senior debt interest rate on expected senior debt profit is,

$$\frac{\partial E[\Pi_S(T)]}{\partial r_S} = H(0) \cdot LTV(0) \cdot T \cdot e^{r_S T}(1 - N(-d_4)) > 0.$$

Proposition 8: LTV and Expected Senior Debt Profit

The impact of LTV on senior debt expected profit is,

$$\frac{\partial E[\Pi_S(T)]}{\partial LTV(0)} = H(0)[e^{r_S T}(1 - N(-d_4)) - e^{\eta_S T}]. \tag{26}$$

Define the breaking point, $L\hat{T}V(r_S)$, such that

$$e^{r_S T}(1 - N(-d_4)) = e^{\eta_S T}.$$

Senior debt expected profit increases with LTV when LTV is below $L\hat{T}V(r_S)$ and decreases when LTV is above $L\hat{T}V(r_S)$.

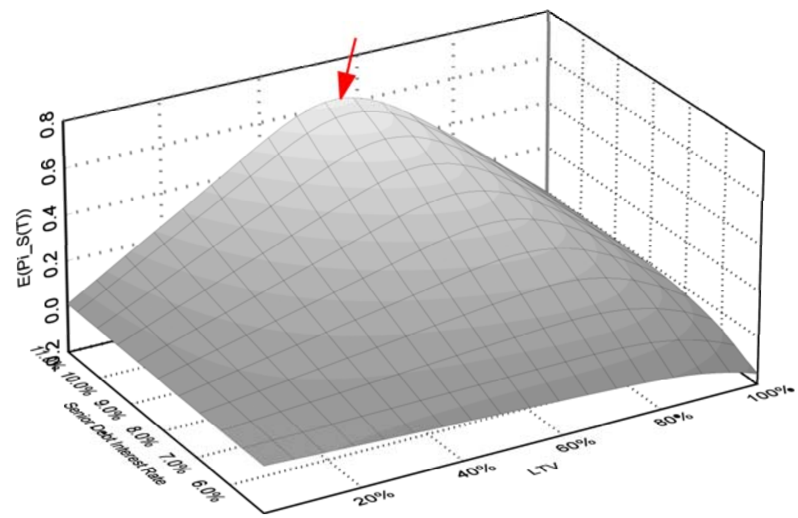

Figure 5: Senior Debt Expected Profit Surface - LTV and Senior Debt Interest Rate

Expected senior debt profit as function of LTV is depicted in Figure 4 by the dotted red line. With a short position in an embedded put option, expected senior debt profit initially increases with LTV. It starts decreasing after LTV passes 72%. Its peak value of 0.123 is marked by a red triangle in the graph.

Proposition 9: Senior Debt Holder's Lending Decision

To senior debt holder, the marginal rate of substitution between senior debt interest rate and LTV is

$$\begin{aligned} MSR^{SL}_{r_S,LTV(0)} &= -\frac{\partial E[\Pi_S(T)]}{\partial LTV(0)} / \frac{\partial E[\Pi_S(T)]}{\partial r_S} \\ &= \frac{e^{\eta_S T} - e^{r_S T}(1 - N(-d_4))}{T \cdot LTV(0) \cdot e^{r_S T}(1 - N(-d_4))}. \end{aligned} \tag{27}$$

I end this section by showing the graph of expected senior debt profit surface as function of LTV and senior debt interest rate r_S. In Figure 5, the surface is tilted higher towards the direction of higher senior debt interest rate. It is humped at the breaking point of LTV for every senior debt interest rate above funding rate. The lower the senior debt interest rate is, the higher the LTV breaking point. If senior debt interest rate is 10%, the LTV breaking point is 60%. It is marked by a red arrow.

5 Junior Debt Holder

Junior lien holder will encounter a loss of the excess of total liabilities and the property value up to junior lien notional amount if the property is worth less than total liabilities of both senior and junior debt at maturity. Junior debt holder's payoff at maturity, $JL(T)$, is the minimum of junior liability balance and property value after senior debt claim,

$$JL(T) \quad = Min(Max(H(T) - B_S e^{r_S T}, 0), B_J e^{r_J T})).$$

The terminal payoff of junior debt can be re-written as,

$$JL(T) = B_J e^{r_J T} - Max(B_J e^{r_J T} - Max(H(T) - B_S e^{r_S T}, 0), 0). \tag{28}$$

It is equivalent to the payoff of a risk-free bond minus the value of a compound option. The compound option is a put option on a call with the put strike price at $B_J e^{r_J T}$ and the call strike price at $B_S e^{r_S T}$. It is more involved to price a compound option. Since I assume that senior and junior debt have the same maturity, instead, I utilize the relationship that property value equals the total of senior debt, junior debt, and home owner's call option as stated below,

$$JL(t) = H(t) - C(t) - SL(t). \tag{29}$$

I make the first statement about junior debt in Proposition 10. It is true by observing that both senior debt and borrower's call option enters into junior debt's value in Equation (29).

Proposition 10: The Impact of Senior Debt on Junior Debt
Junior debt holder's risk is determined by LTV of senior debt and CLTV.

I will not specify junior debt's risk sensitivities to LTV and CLTV. It is straight forward from results in the previous sections. Junior debt holder's expected profit is,

$$E[\Pi_J(T)] = E[H(T)] - E[C(T)] - E[SL(T)] - B_J \cdot e^{\eta_J T}.$$

The impact of r_J, LTV and $CLTV$ on junior debt expected terminal profit is stated as follows.

Proposition 11: The Impact of r_J, LTV, and $CLTV$ on Junior Debt Holder's Expected Profit

Junior debt holder's profit is affected by its interest rate r_J, LTV of senior debt, and $CLTV$.

$$\frac{\partial E[\Pi_J(T)]}{\partial r_J} = H(0) \cdot e^{r_J T} \cdot (CLTV - LTV) \cdot N(d_2) > 0, \tag{30}$$

$$\frac{\partial E[\Pi_J(T)]}{\partial LTV(0)} = -\frac{\partial E[SL(T)]}{\partial LTV(0)}$$
$$= H(0) \cdot e^{r_S T} \cdot (N(-d_4) - 1) < 0, \tag{31}$$

$$\frac{\partial E[\Pi_J(T)]}{\partial CLTV(0)} = H(0)[e^{r_J T}(CLTV - LTV)N(d_2)] > 0 \tag{32}$$

Junior debt holder's expected profit increases with r_J and decreases with LTV. CLTV has positive impact on junior debt holder's profit.

6 Market Equilibrium

To derive the market equilibrium, I assume the lending market is an oligopoly and multiple lenders engage in Bertrend competition. Consequently, interest rates will be at a level that lenders make zero economic profit. Mortgage market in reality is well represented by this market structure. There are more than 20 major originators in each product sector of mortgage lending business, each having no more than 10% of the market share.

The senior debt supply schedule is a function of senior debt interest rate with respect to LTV. The senior debt interest rate and LTV satisfy,

$$E[\Pi_S(T)] = 0 \tag{33}$$

In Figure 6, the senior debt supply schedule without high LTV regulation is shown by the dashed green line. Senior debt rate starts close to funding cost at 5.5% when LTV is 65%. As LTV rises, senior debt originator will charge a higher interest rate to compensate for higher credit risk. The interest rate is flat for LTV below 80%. It is only 7 bps higher when LTV increases from 65% to 80%. The curve is still flat going from 80% LTV to 90%. Interest rate is at 5.86% when LTV is 90%. It rises dramatically after LTV exceeds 95%. When LTV approaches 100%, the interest will be extraordinarily high that prohibits lending.

Current regulation results in a higher funding cost for senior debt after LTV reaches 90%. If we assume that funding cost will increase by 50 bps (or 0.5%) to 6% for loans with high LTV loans, the senior debt supply schedule shifts to the dotted purple line after LTV passes 90%.

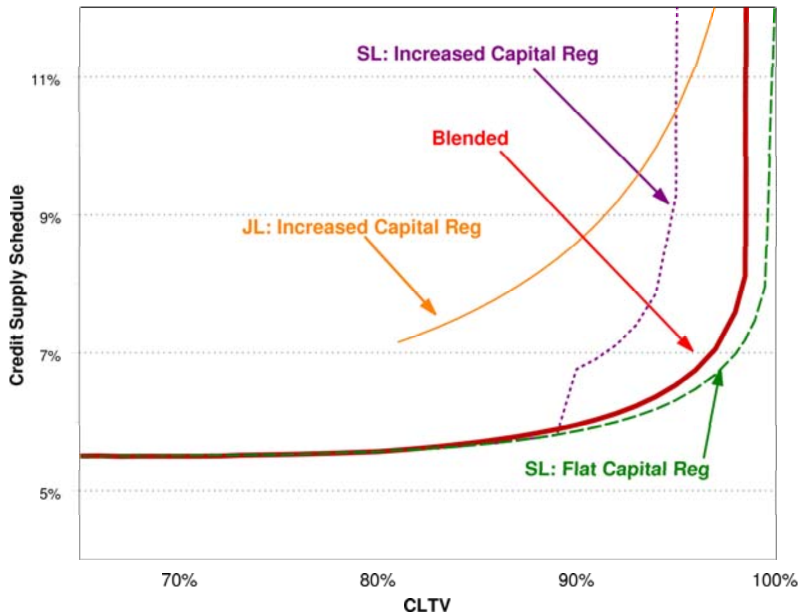

Figure 6: Lenders' Credit Supply Schedule

The Junior debt supply function will solve the zero expected profit condition for junior debt holder,

$$E[\Pi_J(T)] = 0. \tag{34}$$

In Figure 6, it is demonstrated by the orange solid thin line. I assumed that the senior debt in front of the junior debt has 80% LTV. Since junior debt takes a subordinated position for senior debt, junior debt interest rate is much higher than the senior debt at the same LTV. At 80% LTV, the junior debt interest rate is at 7.15% for 81% CLTV, which is 1.57% higher than the senior debt interest rate at the same LTV. It rises as CLTV increases and is always above senior debt rate. Compared with senior debt, the rate of increase is gradual after CLTV exceeds 95%.

The blended lending rate is depicted by the red thick solid line. It is above the unregulated senior lending rate but much lower than the regulated senior lending rate.

Borrowers with different liquidity constraint will have different demand schedule. Borrowers will be willing to borrow only at an interest rate lower than or equal to what makes his expected net worth to be zero. Mathematically, the demand schedule is $\bar{r}(LTV) < \bar{r}^*$

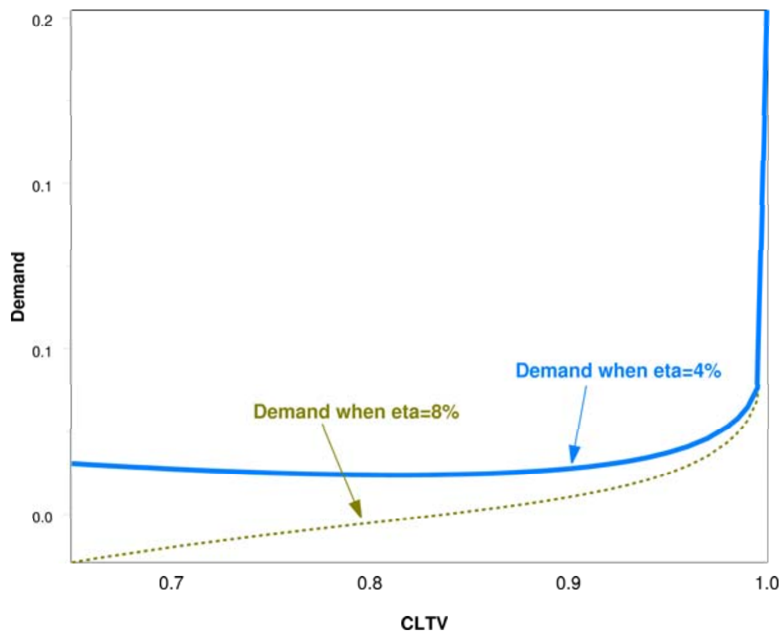

Figure 7: Borrower's Credit Demand Schedule

and $\bar{r}^*$ is such that,

$$E[W(T)] = 0. \tag{35}$$

As shown in Proposition 3 in Equation (16), borrowers with liquidity constraint higher than blended lending rate will have a demand schedule that is an increasing function of CLTV. For those with lower liquidity constraint, the demand schedule will reach bottom at certain point and pick up again. It is obvious from the definition of borrower's expected net worth as in Equation (14) that all else equal borrower with lower liquidity constraint will have higher expected net worth.

Borrower's demand schedule is shown in Figure 7. The solid blue line shows the demand of a borrower with liquidity constraint $\eta = 8\%$ and the dotted brown line is that of a borrower with $\eta = 4\%$.

The market equilibrium is achieved when borrower can borrow at an interest rate at or below his demand schedule and lenders can lend at an interest rate at or above their supply schedule.

Market equilibrium is shown in Figure 8. The demand schedule shown is for $\eta = 4\%$. Demand is the shaded blue area. Borrowers are screened on their liquidity constraint. Only

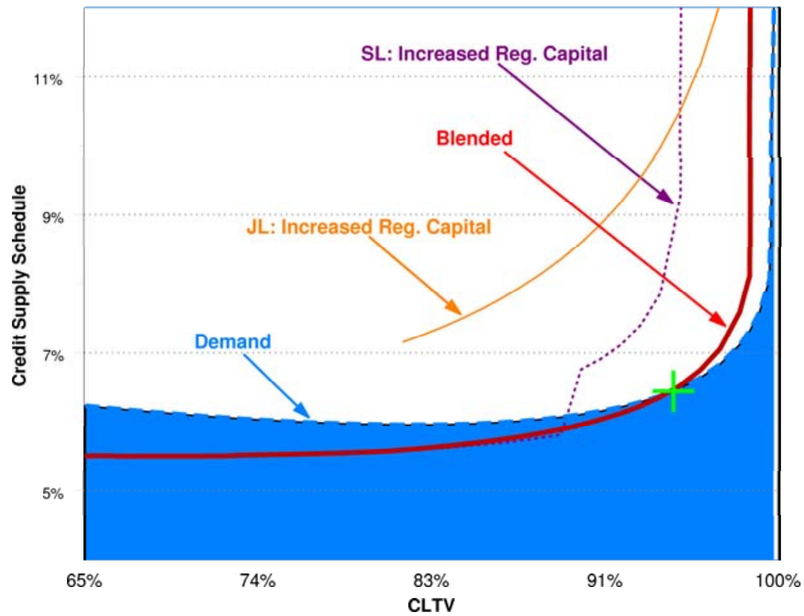

Figure 8: Market Equilibrium

those with liquidity constraint low enough to have demand above blended supply schedule will be offered credit. Borrower will search for the best combination of senior and junior debt to lowest blended lending rate. Senior lien holder will supply credit until the blended lending rate is lower than senior debt rate. The market equilibrium is achieved on a series of combination of CLTV and interest rate along the blended supply schedule. For a borrower with liquidity constraint $\eta = 4\%$, he will borrow along the blended supply schedule depicted by a thick solid red line until it intersects with his demand schedule, marked by a green cross.

7 Expected Default Rate and Losses

Default occurs when property value drops below balance of liabilities at debt maturity. Expected default rate is quantified in Proposition 12.

Proposition 12: Expected Default Rate

Expected default rate on mortgage debt is $N(-d_2)$.

Proof

By put-call parity, borrower's call option is equivalent to holding the property, longing a put option on it, and borrowing debt with balance of the strike price.

$$C(0) = H(0) + P(0) - H(0) \cdot CLTV$$

where the payoff of the put at maturity is

$$P(T) = Max(H(0) \cdot CLTV \cdot e^{\bar{r}T} - H(T), 0).$$

Expected default rate is therefore the probability that the put option gets exercised at maturity with real world probability measure. By Equation (7), it is $N(-d_2)$.

Q.E.D.

It would be of interest to note that expected default rate is increasing in both CLTV and $\bar{r}$.

$$\frac{\partial N(-d_2)}{\partial CLTV} = \frac{N'(-d_2)}{\sigma\sqrt{T}CLTV} > 0$$

and

$$\frac{\partial N(-d_2)}{\partial \bar{r}} = \frac{\sqrt{T}N'(-d_2)}{\sigma} > 0.$$

Dubitsky and Guo (2005) shows that the 80/20 first lien piggy back loan is more likely to become delinquent than 80% true LTV loans. The higher delinquency rate on senior debt is consistent with our model. If we use the same set of parameters as in the example of sub-section (3). The loan package of 80% LTV and 100% CLTV will have a delinquency rate of 7.7%. The loan package with only 80% senior debt LTV and no junior debt will have a delinquency rate of 1.7%. The delinquency rate is much higher on the senior debt with junior debt subordinate to it even though LTV is the same.

The expected loss on senior debt is, however, the same irrespective of the existence of junior debt. It is the value of the put option $P_S(0)$ as defined in Equation (21).

The expected loss on junior debt depends on both senior debt LTV and CLTV. It is the value of the compound option as defined in Equation (28). It can be expressed explicitly using compound option pricing as discussed in Geske (1979).

8 High LTV Mortgage Regulation

Current regulation requires banks to reserve capital according to LTV on holdings of residential mortgage loans[8]. It limits banks' high LTV holdings not to exceed *total capital* and

[8] "Interagency Guidance on High LTV Residential Real Estate Lending", Office of the Comptroller of he Currency, Board of Governors of the Federal Reserve System, Federal Deposit Insurance Corporation, Office of the Thrift Supervision, October 8, 1999.

hence banks' ability to leverage on high LTV mortgage holdings. The interpretation of the policy as stated in the Guidance applies the limit on holdings of high LTV senior mortgage and high CLTV junior mortgage. However, it does not apply to senior mortgage originated below the 90% LTV threshold in a high CLTV senior and junior mortgage combination if the bank does not originate the senior mortgage or sells it to secondary market. For example, if a bank originates senior mortgage debt with LTV of 100%, it will have to apply higher funding cost. If it originates a 80/20 mortgage package and keeps only the junior mortgage, the regulation will not apply to the 80% LTV senior mortgage.

If risk of mortgage is measured by expected loss as a percentage of outstanding balance, junior mortgage will be much riskier. As I demonstrated in Section 5, the expected loss of junior mortgage is much higher than that of the senior mortgage with the same CLTV since junior mortgage takes a subordinated position. Current regulation will reduce origination of high LTV senior mortgage. However, it will induce banks to engage in capital arbitrage to increase origination volume in high CLTV junior mortgage, which has even higher credit risk.

If there is no regulation on high LTV loans, market force alone will not fail to price the credit risk properly. As I showed in Section 6, credit supply schedule will be based on LTV and CLTV. Market equilibrium will be achieved on the unregulated credit supply curve. The need for regulation on high LTV mortgage is not well justified.

9 Conclusion

In the paper, I take structured credit modeling approach to quantify the credit risk of first mortgage and second mortgage. LTV as a measure of leverage is the most important indicator of credit risk. I derived default probabilities and expected losses on mortgage debt. Optionality of defaultable debt results in an upward sloping credit supply curve as a function of interest rate with respect to LTV. Market force alone is shown to be sufficient enough to match supply with demand and still account for credit risk.

Current regulation in high LTV mortgage creates a funding advantage in separating a high LTV mortgage into a lower funding cost first mortgage and a higher cost second mortgage. It explains the increased origination volume in second mortgage. The credit risk in high LTV mortgage however may not be reduced as the regulation intended given the fact that second mortgage has more concentrated risk due to subordination.

In the simplified model, there is no principal amortization or prepayment. Therefore it does not account for interim default at amortized principal and interest due date. It could underestimate default risk in the sense that there is less chance for default to occur. It could also overestimate default in the sense that principal amortization reduces option strike price. In a model with interim payments, the effect of other credit characteristics of mortgage loan such as income verification and credit score on credit risk will kick in. The linkage between junior mortgage and senior mortgage can be different due to the fact that junior mortgage increases borrower's burden of payments and the probability of default. We will be able to model adverse selection from the lenders' perspective to screen applicants with higher risk of interim default using income verification and credit scoring. Interim default can be introduced to the current simplified model by an additional independent hazard process. It will be the next step for future research.

It will also be interesting to test empirically if high LTV mortgage interest rate in current market is appropriate to compensate for the credit risk. That will be helpful for regulators to test efficiency of regulation and for mortgage originators to measure pricing of products with significant credit risk.

Appendix

Proof of Proposition 1:

Take partial derivative of $C(0)$ with respect to $CLTV(0)$ in Equation (10) yields,

$$\frac{\partial C(0)}{\partial CLTV(0)} = H(0)[e^{-qT}N'(d_1)\frac{\partial d_1}{\partial CLTV(0)} - e^{(\bar{r}-r)T}N(d_2) - CLTV(0)e^{(\bar{r}-r)T}N'(d_2)\frac{\partial d_2}{\partial CLTV(0)}]$$

It can be easily shown that

$$\frac{\partial d_1}{\partial CLTV(0)} = -\frac{1}{\sigma\sqrt{T}CLTV(0)} < 0$$

and

$$\frac{\partial d_2}{\partial CLTV(0)} = \frac{\partial d_1}{\partial CLTV(0)}.$$

Combining terms and applying Normal distribution density function, we can get

$$\frac{\partial C(0)}{\partial CLTV(0)} = -\frac{H(0)}{\sigma\sqrt{2\pi T}CLTV(0)}[e^{-qT-d_1^2/2} - CLTV(0)e^{-d_2^2/2+(\bar{r}-r)T}] - e^{(\bar{r}-r)T}N(d_2)$$

Applying the relationship between d_1 and d_2, it can be shown that the part inside the rectangular bracket equals 0. It is a very useful relationship to prove other propositions. I state it formally and refer to it as "very important relationship" in other sections,

$$e^{-qT}N'(d_1) - e^{(\bar{r}-r)T}N(d_2) - CLTV(0)e^{(\bar{r}-r)T}N'(d_2) = 0. \quad (36)$$

We are then ready to see that

$$\frac{\partial C(0)}{\partial CLTV(0)} = -H(0)e^{(\bar{r}-r)T}N(d_2) < 0.$$

The second order derivative can be taken from the above equation and shown to be positive.

Q.E.D.

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