A Generalization of Gray and Whaley's Option

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Abstract

Options markets display interesting features. Most options are executed when they are near the money. However, the underlying asset price varies significantly during the life-time option. It is therefore difficult to predict the future option position.

In order to make options' markets more liquid, the paper proposes to replace all options into At-the-Money (ATM) ones by resetting the strike price X to the asset price at pre-specified time point t, before maturity time T. Strike price is locked in at the then underlying asset price S_t regardless whether it is above or below S_t.The reset condition is in exchange for deposit in the Clearing House. The idea is to provide a general valuation of reset option of Gray and Whaley (1999) in which reset condition does not depend on the relation between the strike price and the underlying asset price.

The contribution of this paper is double. First, it shows that our general model option, under specific conditions, can be generalized to the most common ones like for example Black-Scholes-Merton, forward-start and strike reset pricing formulae etc...

Second, in line with Haug and Haug (2001), we use the CRR binominal approach (Cox et al., 1979) and an estimation program of the cumulative bivariate normal distribution to provide closed-form solution for the pricing of the generalized European reset option.

Keywords: strike reset, at-the-money option, liquidity, reset option.

JEL Classification codes: G12, G13.

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1 Introduction

The current paper is related to the large body of work on the pricing of forward start options. It proposes a closed-form solution for a general forward start option and an analytical pricing formula that is an extension of CRR binomial tree. Under specific conditions, the *generalized reset* (hereafter GR) option converges to the most common ones like for example Black-Scholes-Merton (hereafter BSM), forward-start and strike reset pricing formulae etc...

The main idea is to lock in profit over the life-time of the option by resetting its strike price at a preagreed time point, regardless of the relation between strike and asset prices. Specifically, the strike price is automatically reset to the underlying asset price in exchange for deposit in clearing houses. This is supposed to enhance liquidity of option markets. In fact, all strike prices of out-of-the-money (OTM) and in-the-money (ITM) options should be reset, keeping in the market only at-the-money (ATM) options. The deposit can be the cost paid by the holders of OTM options to have more liquid options or the profit obtained by the holders of ITM options who want to lock it.

The GR option looks like a reset option in the sense it is a path-dependent option where the strike price can be reset based on a certain conditions/criteria. For example, the strike price of a call reset option can be reset downward if the underlying asset price falls below a predetermined value. Reset condition enables to protect investors amid declines (respectively increases) in asset price in reset call (respectively put) option. Reset option can be regarded as an insurance portfolio. In fact, reset option is like a standard option except that the strike price is reset to the minimum (respectively maximum) of the underlying asset price on reset dates for the call (respectively put) reset option. There are single-asset reset options, but reset options can involve two or more risky assets, in this case they are called rainbow options. Rainbow options have been applied to derivative products for many years.

Unlike reset European options, the reset condition in GR option does not depend on the underlying asset price. At the reset time point t, the strike price will be equal to the underlying price S_t whatever its value. This option is similar to a forward start options that come into existence at the reset time when the underlying asset price reaches a certain barrier and expire at maturity time. Under specific conditions, using the binomial approach of Cox et al. (1979) shows that the pricing formula of GR option converges to standard ones like for example BSM, forward-start, strike reset, lock in, ..

Despite the fact that there is an extensive literature on valuation problems for options, particularly options with a reset condition or a forward-start condition. Surprisingly, pricing options combining the two features is still an open problem because of the inherent path dependency coming from the difficulty of taking jointly into account the two features. Unlike standard options, forward start options start in a pre-specified date in the future. This date is based on a decision of some contractual terms. For instance, the strike price of forward start-options is determined in a pre-specified date in the future. They are also called *delayed* options. If the strike price is the only contractual term to be determined, the forward start options are called *delayed-strike* options. They can also be combined in a series to form a ratchet option (also called cliquet option) such that each forward start option starts with an at-the money (hereafter ATM) strike price when the previous one expires. The idea is to enable the investor to lock in profit over the life-time of the corresponding option. Ratchet options are commonly used in equity market.

Not surprisingly, our paper is linked to several studies on pricing these options. For instance, Rubinstein (1991) provides a pricing formula of standard forward-start option for which the strike price is set at a future time point such that the option becomes ATM at that time point. Guo and Hung (2008) generalize the Rubinstein formula under specific conditions.

Many papers propose pricing formulae for reset and barrier options. Our paper is related to the following studies: Gray and Whaley (1999) are the first to investigate the pricing formula for put reset option while Haug and Haug (2001) provide a closed-form solution and an analytical pricing formula for European call reset option. Cheng and Zhang (2000) study a reset option with multiple reset dates in which the strike price is reset only if the option is OTM at the reset dates. Liao and Wang (2003) provide a closed-form pricing formula with stepped reset of the strike price on pre-specified reset dates.

There are also several studies on the valuation of rainbow options. Stultz (1982) uses the solution of partial differential equations to derive the pricing formula for rainbow option on the maximum or minimum of two assets. The general case of rainbow put option with more than two assets was considered by Johnson (1987) based on a previous study of Margrabe (1978). Kargin (2005) proposes a numerical pricing method based on sophisticated calculus. All these studies are designed for path independent rainbow options.

In a more recent work, Chen and Wang (2008) focus on path dependent rainbow options and study the impact of the forward start feature on rainbow options. They propose a general martingale pricing method to value forward-start rainbow option and derive analytical pricing formula that is applicable to general settings and covers Johnson (1987), Gray and Whaley (1999) and Black and Scholes (1973).

The contribution of this paper is double.

First, we generalize the reset option so that the percentage of near the money options increases which improves the liquidity of options' market. According to Rubinstein (1991) and Gray and Whaley (1999), a closed-form solution for the pricing of the generalized European reset option is derived. Under specific conditions, the general model converges to BSM, forward-start and strike reset pricing formulae.

Second, in line with Haug and Haug (2001) and using the binomial tree of CRR[1], we propose an analytical pricing formula of the generalized option model.

The rest of this paper is organized as follows. In Section 2, we present the generalized Euopean reset option and provide a closed-form solution for its pricing. We derive an analytical pricing formula based on the binomial tree of CRR approach and compare our closed-form solution and, Gray and whaley, BSM and Rubinstein formulae with our binomial method using 5000 time steps in Section 3. Section 4 concludes the paper.

2 Reset versus non reset options: when option becomes ATM?

Before defining and valuing the GR call option, we present a brief reminder of the main options discussed in this paper to which our model can converge. For the sake of simplicity, we focus on the particular case of call options but provide closed-form solutions for put options.[2]

Consider a standard European call option with maturity T and the exercise price X. The underlying asset price at date $t = 0$ is denoted S_0 and its volatility per year is σ. We will assume r the risk free-interest rate and d the dividend yield, such that $r \geq d$. The underlying asset price at maturity is denoted S_T. Let $C\left(S_0\ ,\ X\ ,\ 0,\ T\right)$ denotes the call option price at time 0.

[1] For the estimation of the cumulative bivariate normal distribution, we rely on an estimation program available at globalderivatives.com.

[2] Further details about put options are available upon request.

2.1 Black-Scholes-Merton call option

According to Black, Scholes and Merton (1973), the pricing formula of a standard call option is written:

$$\begin{aligned} C_{BSM}\left(S_0\ ,\ X\ ,\ 0,\ T\right) &= E\left(S_T - X\right)P\left(S_T \geq X\right) \\ &= S_0 e^{-dT} N\left(d_{1,T}\right) - Xe^{-rT} N\left(d_{2,T}\right) \end{aligned} \tag{1}$$

where

$$d_{1,T} = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - d + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_{2,T} = d_{1,T} - \sigma\sqrt{T}$$

and $N\left(a\right)$ is a univariate cumulative normal distribution function with upper integral limit a.

Let $C_{BSM}\left(S_t,\ X,\ t,\ T\right)$ denotes the call option price at date t. Then, we can write

$$C_{BSM}\left(S_0\ ,\ X\ ,\ 0,\ T\right) = E\left[C_{BSM}\left(S_t\ ,\ X\ ,\ t,\ T\right)\right]e^{-rt}$$

In a vanilla call option, the strike price does not depend on the underlying asset price until maturity time T, then we decide or not to exercise the option according to the value of S_T. At time t $(0 < t < T)$, the option's holder does not expect any payment.

The closed-form of pricing a put option is written

$$\begin{aligned} P_{BSM}\left(S_0\ ,\ X\ ,\ 0,\ T\right) &= E\left(X - S_T\right)P\left(S_T \leq X\right) \\ &= Xe^{-rT} N\left(-d_{2,T}\right) - Se^{-dT} N\left(-d_{1,T}\right) \end{aligned}$$

2.2 Forward-start European call option

A forward-start European call option is option that will start in the future. To value this option, we rely on BSM pricing formula. At time t, the call price becomes ATM but expires at $(T - t)$. As noticed before, Rubinstein (1991) valued forward start call option at time 0 by the following

$$\begin{aligned} C_F\left(S_0\ ,\ X\ ,\ 0,\ T\right) &= E\left(S_T - X\right)P\left(S_T \geq X\right) \\ &= E\left(C\left(S_t\ ,\ S_t\ ,\ t,\ T\right)\right)e^{-rt} \\ &= E\left(S_t\right)\theta_t^C e^{-rt} \\ &= e^{-dt} C\left(S_t\ ,\ S_t\ ,\ t,\ T\right) \end{aligned} \tag{2}$$

where

- $\theta_t^C = e^{-d(T-t)} N\left(c_{1,\ T-t}\right) - e^{-r(T-t)} N\left(c_{2,T-t}\right)$
- $E\left(S_t\right) = e^{(r-d)t} S_t$
- $c_{1,\ T-t} = \frac{\left(r-d+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and $c_{2,\ T-t} = \frac{\left(r-d-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

To value forward-start European put option, we use

$$\begin{aligned} P_F\left(S_0\ ,\ X\ ,\ 0,\ T\right) &= E\left(S_t - S_T\right) P\left(S_T \leq X\right) \\ &= S e^{-dt} \theta_t^p \end{aligned}$$

where $\theta_t^p = e^{-r(T-t)} N\left(-c_{2,T-t}\right) - e^{-d(T-t)} N\left(-c_{1,\ T-t}\right)$.

Figure 1 compares the strike prices of the standard call and forward-start call options. Notice that the exercise price does not change over $[0,\ T]$ in the BSM pricing formula contrary to the forward-start one.

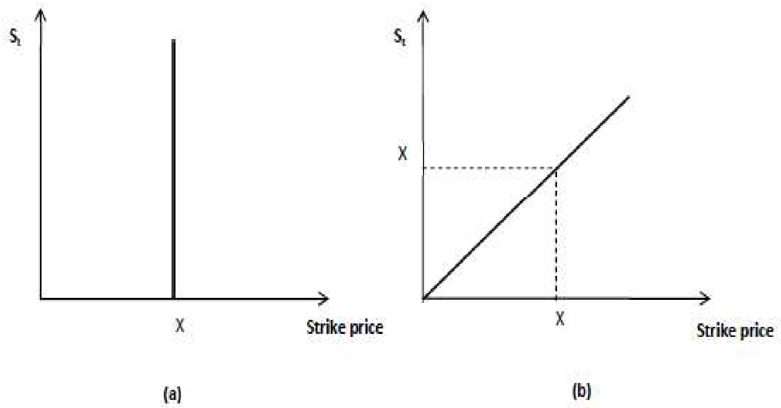

Figure 1: Sensitivity of strike prices of (a) BSM European call option and (b) a ATM European call option to changes in underlying asset price at date t.

2.3 Reset-out call option

As explained before, a reset call option protects investors amid declines in asset price through the reset of the strike price to the underlying asset price if the option becomes OTM at the reset date t. In other words, when $S_t < X$, it is replaced by an ATM call option with the same maturity. Notice that if $S_t \geq X$, the call option is ITM and does not need to be replaced. Figure 2 presents the payments of reset out call option.

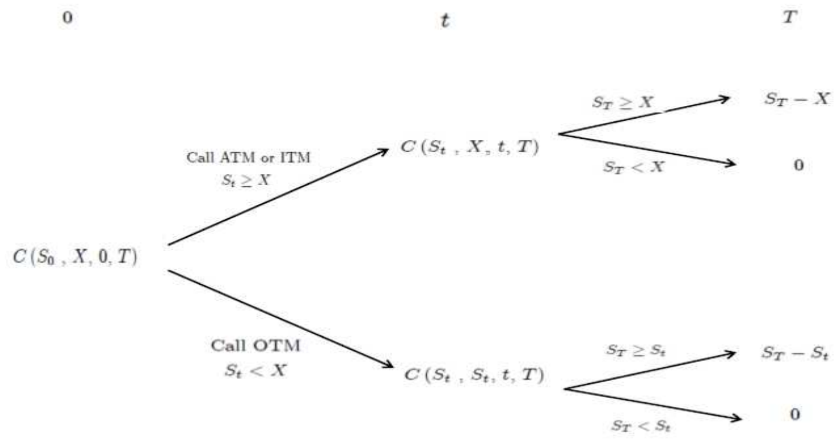

Figure 2: The pay offs of rest call option (Gray and Whaley, 1999).

Gray and Whaley (1999) derive a closed-form solution for the pricing of reset-in put option

$$\begin{aligned} P_{In}(S_0, X, 0, T) = \; & E(S_t - S_T)\Pr(S_t \leq X, S_T \leq S_t) + E(X - S_T)\Pr(S_t > X, S_T \leq X) \\ = \; & Se^{-rt}\theta_t^P N(-d_{1,t}) - Se^{-dT} M_2\left(d_{1,t}, \; -d_{1,T}\sqrt{t/T}\right) \\ & + Xe^{-rT} M_2\left(d_{2,t}, -d_{2,T}, \sqrt{t/T}\right) \end{aligned} \tag{3}$$

where $d_{1,i} = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - d + \frac{\sigma^2}{2}\right)i}{\sigma\sqrt{i}}$, $d_{2,i} = d_{1,i} - \sigma\sqrt{i}$, $i = t, T$ and $M_2\left(a, b, \sqrt{t/T}\right)$ is the bivariate cumulative normal distribution function with upper integral limits a and b and correlation coefficient $\sqrt{t/T}$ such that

$$\begin{aligned} M_2\left(a, -b, \sqrt{t/T}\right) & = N(-b) - M_1\left(-a, -b, \sqrt{t/T}\right) \\ & = N(a) - M_1\left(a, b, \sqrt{t/T}\right) \\ M_2\left(-a, b, \sqrt{t/T}\right) & = N(b) - M_1\left(a, b, \sqrt{t/T}\right) \\ & = N(-a) - M_1\left(-a, -b, \sqrt{t/T}\right) \end{aligned}$$

and $M_1\left(a, b, \sqrt{t/T}\right) = P(X > a, S_t > b)$.

We rely therefore on (3) to derive a closed-form solution for the pricing of reset-out call options

$$\begin{aligned} C_{Out}(S_0, X, 0, T) = \; & E(S_T - S_t)\Pr(S_t < X, S_T \geq S_t) + E(S_T - X)\Pr(S_t \geq X, S_T \geq X) \\ = \; & Se^{-dT} N(-d_{1,t}) N(c_{1,T-t}) - Se^{-dt} e^{-r(T-t)} N(-d_{1,t}) N(c_{2,T-t}) \\ & + Se^{-dT} M_1\left(d_{1,t}, d_{1,T}, \sqrt{t/T}\right) - Xe^{-rT} M_1\left(d_{2,t}, d_{2,T}, \sqrt{t/T}\right) \\ = \; & Se^{-rt}\theta_t^c N(-d_{1,t}) + Se^{-dT} M_1\left(d_{1,t}, d_{1,T}, \sqrt{t/T}\right) \\ & - Xe^{-rT} M_1\left(d_{2,t}, d_{2,T}, \sqrt{t/T}\right) \end{aligned} \tag{4}$$

The option price given by (4) has a fixed and variable price components that depend closely on the value of the strike price when it is not modified $\left(C_{Out}^{F}\right)$ and when it is adjusted $\left(C_{Out}^{V}\right)$. They are written:

$$\begin{aligned} C_{Out}^{V} &= E\left(S_T - S_t\right)\Pr\left(S_t < X,\, S_T \geq S_t\right) \\ C_{Out}^{F} &= E\left(S_T - X\right)\Pr\left(S_t \geq X,\, S_T \geq X\right) \end{aligned}$$

where the variable component C_{Out}^{V} comes from the adjustment of the strike price when it is OTM and replaced by an ATM one.

However, Gray and Whaley's solution presents the same weaknesses of closed-form solutions, i.e. the lack of flexibility. It means that if payoffs change, we need to find a new solution-if it exists. This is why Haug and Haug (2001) consider an extension of the binomial tree of Cox et al. (1979) in the setting of Rendleman and Bartter (1980). They conclude that the value of a reset call option is equal to the sum of payoffs multiplied by the corresponding probabilities, discounted at the risk free interest rate such that the probability of going up or down is set equal to $\frac{1}{2}$. Let n denotes the number of time steps Δt to maturity, m is the number of time steps to reset time $(m < n)$, i the state at maturity and j the state at time step m.

$$C_{HH}\left(S_0,\, X,\, 0,\, T\right) = e^{-rT}\sum_{j=0}^{m}\sum_{i=j}^{n-m+j}\frac{m!\,(n-m)!}{j!\,(m-j)!\,(i-j)!\,(n-m-i+j)!}\left(\frac{1}{2}\right)^{n} g\left(Su^{i}d^{n-i}\ ,\, X_c\right)$$

where $u = e^{\left(r-d-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}}$, $d = e^{\left(r-d-\frac{\sigma^2}{2}\right)\Delta t-\sigma\sqrt{\Delta t}}$, $g\left(S\ ,\, X\right) = \max\left(S - X\ ,\, 0\right)$ and $X_c = \min\left(\delta S u^{i} d^{m-i}\ ,\, X\right)$. The constant δ indicates how much OTM or ITM the reset strike is.

It is straightforward to see that the price of a reset out call option is equal to the price of a BSM call option with strike price X at time 0. The alternative strategy would be to buy a BSM option with strike price X and to sell at t only if it becomes OTM. The potential gain will enable the investor to buy a more expensive BSM one but which is ATM.

Therefore, replacing OTM option by an ATM one is costly, in the sense, the option's holder has to pay fees in order to make his/her option more liquid by making a deposit in the clearing house. At time t, it costs

$$D_t^{out} = C_{BSM}\left(S_t,\, X,\, t,\, T\right) - C_{BSM}\left(S_t,\, S_t,\, t,\, T\right) \quad \text{, if } S_t < X \tag{5}$$

The investor can pay that deposit at time 0

$$D_0^{out} = e^{-rt} D_t^{out}$$

If the call option is deep OTM, in the sense $S_t < X - \beta$ where $0 < \beta < X$, the value of call option is given by:

$$\begin{aligned} C_{Out}(S_0, X, 0, T) = \;& E(S_T - S_t)\Pr(S_t < X - \beta, S_T \geq S_t) + E(S_T - X)\Pr(S_t \geq X - \beta, S_T \geq X - \beta) \\ =\;& Se^{-dT} N\left(-d_{1,t}^{\beta}\right) N(c_{1,T-t}) - Se^{-dt} e^{-r(T-t)} N\left(-d_{1,t}^{\beta}\right) N(c_{2,T-t}) \\ & + Se^{-dT} M_1\left(d_{1,t}^{\beta}, d_{1,T}, \sqrt{t/T}\right) - Xe^{-rT} M_1\left(d_{2,t}^{\beta}, d_{2,T}, \sqrt{t/T}\right) \\ =\;& Se^{-rt}\theta_t^c N\left(-d_{1,t}^{\beta}\right) + Se^{-dT} M_1\left(d_{1,t}^{\beta}, d_{1,T}, \sqrt{t/T}\right) \\ & - Xe^{-rT} M_1\left(d_{2,t}^{\beta}, d_{2,T}, \sqrt{t/T}\right) \end{aligned}$$

where $d_1^{\beta} = \frac{\ln\left(\frac{S_0}{X-\beta}\right)+\left(r-d+0{,}5\sigma^2\right)t}{\sigma\sqrt{t}}$ and $d_2^{\beta} = d_1^{\beta} - \sigma\sqrt{t}$. If $\beta > 0$, the reset-out call price decreases, while when β converges to X, $C_{Out}(S_0, X, 0, T)$ tends to the value of BSM call option.

2.4 Reset-in call option

Unlike reset-out call option, reset-in call option (called also lock-in call option) enables to lock in the obtained profit of ATM option at a pre-specified time point. When $S_t > X$, the investor replaces ITM option with an ATM one at time t. The asset's holders have to meet their commitment at the option maturity T.

The value of this call option is given by

$$\begin{aligned} C_{In}(S_0, X, 0, T) = \;& E(S_T - S_t)\Pr(S_t \geq X, S_T \geq S_t) + E(S_T - X)\Pr(S_t < X, S_T \geq X) \\ =\;& Se^{-dT} N(d_{1,t}) N(c_{1,T-t}) - Se^{-dt} e^{-r(T-t)} N(d_{1,t}) N(c_{2,T-t}) \\ & + Se^{-dT} M_2\left(-d_{1,t}, d_{1,T}, \sqrt{t/T}\right) - Xe^{-rT} M_2\left(-d_{2,t}, d_{2,T}, \sqrt{t/T}\right) \\ =\;& Se^{-rt}\theta_t^c N(d_{1,t}) + Se^{-dT} M_2\left(-d_{1,t}, d_{1,T}, \sqrt{t/T}\right) \\ & - Xe^{-rT} M_2\left(-d_{2,t}, d_{2,T}, \sqrt{t/T}\right) \end{aligned} \tag{6}$$

Similarly, we derive the closed-form solution for the pricing of reset-out put option (called also lock-out put option):

$$\begin{aligned} P_{Out}(S_0, X, 0, T) = \;& = E(S_t - S_T)\Pr(S_t > X, S_T \leq S_t) + E(X - S_T)\Pr(S_t \leq X, S_T \leq X) \\ & = Se^{-rt}\theta_t^P N(d_{1,t}) - Se^{-dT} M_1\left(-d_{1,t}, -d_{1,T}, \sqrt{t/T}\right) \\ & + Xe^{-rT} M_1\left(-d_{2,t}, -d_{2,T}, \sqrt{t/T}\right) \end{aligned}$$

As noticed before, we distinguish fixed and variable parts in $C_{In}\left(S_0,\, X,\, 0,\, T\right)$ given respectively by

$$\begin{aligned} C_{In}^{V} = &\quad E\left(S_T - S_t\right) \Pr\left(S_t < X,\, S_T \geq S_t\right) \\ C_{In}^{F} = &\quad E\left(S_T - X\right) \Pr\left(S_t \geq X,\, S_T \geq X\right) \end{aligned}$$

such that the variable component comes from setting the ITM strike price to the then underlying asset price so that it is replaced by an ATM one.

In such case, the option's holder has a positive payoff

$$D_t^{In} = C_{BSM}\left(S_t,\, X,\, t,\, T\right) - C_{BSM}\left(S_t,\, S_t,\, t,\, T\right) \quad , \text{if } S_t > X$$

At time 0, the gain of replacing ITM option with an ATM one is

$$e^{-rt}\ D_t^{In}, \text{ if } S_t > X \tag{7}$$

The reset-in call price at time zero is equal to a vanilla call price with strike X. It can be implemented by buying a vanilla call at time zero and sell it when it becomes ITM at time t. The obtained gain could be used to acquire an ATM call option that matures at time T. The reset-in call price is equal to the BSM call price (with the strike X) diminished by the payment (7).

Figure 3 shows the change of strike prices in both cases with respect to changes in the underlying asset price at time t, S_t.

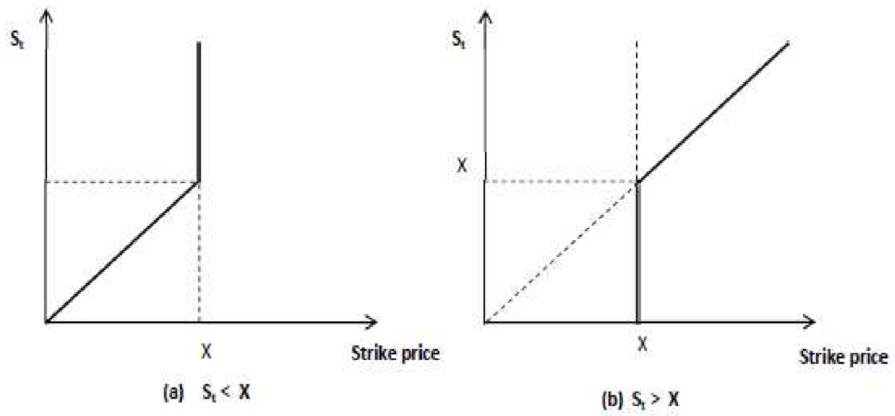

Figure 3: Sensitivity of strike prices of (1) reset out call option and (2) reset in call option to changes in S_t.

If the underlying price is significantly superior to the strike price, in the sense $S_t > X + \alpha$ where $\alpha > 0$, the ATM option is reset at a higher price $X + \alpha$. This is more advantageous for the option's holder than being paid a strike price X.

$$\begin{aligned}
C_{In}\left(S_0,\, X,\, 0,\, T\right) = \;& E\left(S_T - S_t\right)\Pr\left(S_t \geq X+\alpha,\, S_T \geq S_t\right) + E\left(S_T - X\right)\Pr\left(S_t < X+\alpha,\, S_T \geq X\right) \\
=\;& Se^{-dT} N\left(d_{1,t}^{\alpha}\right) N\left(c_{1,T-t}\right) - Se^{-dt}e^{-r(T-t)} N\left(d_{1,t}^{\alpha}\right) N\left(c_{2,T-t}\right) \\
& + Se^{-dT} M_2\left(-d_{1,t}^{\alpha}\;,\; d_{1,T}\;,\; \sqrt{t/T}\right) - Xe^{-rT} M_2\left(-d_{2,t}^{\alpha}\;,\; d_{2,T}\;,\; \sqrt{t/T}\right) \\
=\;& Se^{-rt}\theta_t^{c} N\left(d_{1,t}^{\alpha}\right) + Se^{-dT} M_2\left(-d_{1,t}^{\alpha}\;,\; d_{1,T}\;,\; \sqrt{t/T}\right) \\
& - Xe^{-rT} M_2\left(-d_{2,t}^{\alpha}\;,\; d_{2,T}\;,\; \sqrt{t/T}\right)
\end{aligned}$$

where

$$d_{1,t}^{\alpha} = \frac{\ln\left(\frac{S_0}{X+\alpha}\right) + \left(r - d + 0,5\sigma^2\right) i}{\sigma\sqrt{t}} \quad \text{and } d_{2,t}^{\alpha} = d_{1,t}^{\alpha} - \sigma\sqrt{t}$$

The option price depends closely on the value of α. If $\alpha > 0$, the value of reset-in call option increases dramatically. Otherwise, it becomes too close to the value of BSM call option.

3 Generalization of Gray and Whaley's reset option

3.1 Definition

In the following, we assume that:

$$\text{The call option is} \begin{cases} \text{ITM} & \text{if } S_t > X+\alpha \\ \text{OTM} & \text{if } S_t < X-\beta \qquad ; \ (\alpha\ ,\ \beta) \in \mathbb{R}_+^2 \\ \text{ATM} & \text{otherwise} \end{cases}$$

Consider now that at the time point t, the strike price is reset such that if the call is ITM or OTM, it becomes ATM[3]. Payoffs and call option prices at date t are summarized in figures 4 and 5.

[3] First, we consider that there is a single reset time t, $0 \leq t \leq T$. The general case with multiple strike reset dates will be discussed later.

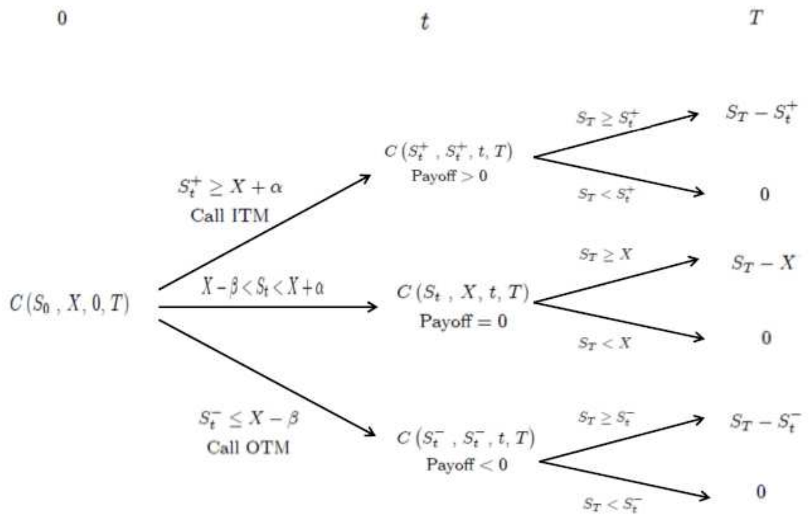

Figure 4: The pay offs of GR call option with respect to different cases ($\alpha \geq 0$, $\beta \geq 0$).

To deduce a closed-form solution for the pricing of the generalized call option, we rely on Gray and Whaley (1999) approach.

$$\begin{aligned} C_{GR}(S_0 , X , 0, T) = \; & E(S_T - S_t)\left[P(S_t \geq X+\alpha , S_T \geq S_t) + P(S_t \leq X-\beta , S_T \geq S_t)\right] \\ & + E(S_T - X)\,P(X-\beta < S_t < X+\alpha, S_T \geq X) \\ = \; & S_0 e^{-rt}\theta_t^C \left[N\left(d_{1,t}^{\alpha}\right) + N\left(-d_{1,t}^{\beta}\right)\right] + S_0 e^{-dT} M_3\left(d_{1,t}, d_{1,T} , \sqrt{t/T}\right) \\ & - X e^{-rT} M_3\left(d_{2,t}, d_{2,T} , \sqrt{t/T}\right) \end{aligned} \tag{8}$$

where

$$d_{1,i}^{\alpha} = \frac{\ln\left(\frac{S_0}{X+\alpha}\right) + \left(r - d + 0,5\sigma^2\right) i}{\sigma\sqrt{i}} \quad \text{and} \quad d_{1,i}^{\beta} = \frac{\ln\left(\frac{S_0}{X-\beta}\right) + \left(r - d + 0,5\sigma^2\right) i}{\sigma\sqrt{i}}, \; i = t, T$$

$$\begin{aligned} M_3\left(d_{j,t}, d_{j,T} , \sqrt{t/T}\right) = \; & M_2\left(-d_{j,t}^{\alpha}, d_{j,T} , \sqrt{t/T}\right) - M_2\left(-d_{j,t}, d_{j,T} , \sqrt{t/T}\right) \\ & + M_1\left(d_{j,t}^{\beta}, d_{j,T} , \sqrt{t/T}\right) - M_1\left(d_{j,t}, d_{j,T} , \sqrt{t/T}\right) \end{aligned} , \; j = 1, 2$$

Accordingly, the value of PR put option can be written

$$\begin{aligned} P_{GR}(S_0 , X , 0, T) = \; & E(S_t - S_T)\left[P(S_t \geq X+\alpha , S_T \leq S_t) + P(S_t \leq X-\beta , S_T \leq S_t)\right] \\ & + E(X - S_T)\,P(X-\beta < S_t < X+\alpha, S_T \leq X) \\ = \; & S e^{-rt}\theta_t^P \left[N\left(-d_{1,t}^{\alpha}\right) + N\left(d_{1,t}^{\beta}\right)\right] - S e^{-dT} M_3\left(d_{1,t}, d_{1,T} , \sqrt{t/T}\right) \\ & + X e^{-rT} M_3\left(d_{2,t}, d_{2,T} , \sqrt{t/T}\right) \end{aligned} \tag{9}$$

The strike price is reset to the underlying asset price. The amount of the deposit depends on how deep the call is OTM or ITM. If the underlying asset price S_t is significantly higher than the strike price, in the sense $S_t \geq X + \alpha$, the option's holder expects a positive payoff

$$C_{BSM}(S_t\ ,\ S_t\ ,\ t,\ T) - C_{BSM}(S_t\ ,\ X\ ,\ t,\ T)$$

In contrast, if it is significantly lower than X, in the sense $S_t \leq X - \beta$, the holder has to pay

$$C_{BSM}(S_t\ ,\ X\ ,\ t,\ T) - C_{BSM}(S_t\ ,\ S_t\ ,\ t,\ T)$$

to replace the OTM call option with an ATM call option. However, when $X-\beta < S_t < X+\alpha$, the option is near the money and the strike price does not depend on the asset price like in a standard BSM call option.

The alternative strategy could be to buy at time 0 a vanilla call option with strike X that expires at T. At time t, we sell the option only if it becomes ITM ($S_t \geq X + \alpha$) or OTM ($S_t \leq X - \beta$) and we use the obtained gain to buy an ATM option that matures at T.

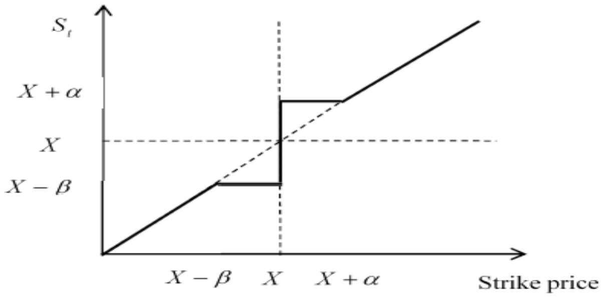

Figure 5: Sensitivity of GR call option to changes in the underlying asset price S_t.

Similarly, we derive the closed-form solution for pricing GR put option. It is written

$$\begin{aligned} P_{GR}(S_0\ ,\ X\ ,\ 0,\ T) = \ & E(S_T - S_t)\left[P(S_t \geq X + \alpha\ ,\ S_T \geq S_t) + P(S_t \leq X - \beta\ ,\ S_T \geq S_t)\right] \\ & + E(S_T - X)\, P(X - \beta < S_t < X + \alpha,\ S_T \geq X) \\ = \ & S_0 e^{-rt}\theta_t^C \left[N\left(d_{1,t}^{\alpha}\right) + N\left(-d_{1,t}^{\beta}\right)\right] + S_0 e^{-dT} M_3\left(d_{1,t},\ d_{1,T}\ ,\ \sqrt{t/T}\right) \\ & - X e^{-rT} M_3\left(d_{2,t}, d_{2,T}\ ,\ \sqrt{t/T}\right) \end{aligned} \tag{10}$$

This model is useful in many settings and covers formulae of the options discussed in the previous subsections. According to the values of α and β, we conclude the following:

- If $\alpha \longrightarrow +\infty$ and $\beta = X$, (8) becomes (1). Under these conditions, the PR call option becomes a standard BSM call option which implies that there is no rebate at the reset time t.
- If $\alpha = \beta = 0$, (8) is written (2). This means that the call option is a forward-start European call option and the option's holder can expect a positive or negative rebate.
- If $\alpha \longrightarrow +\infty$ and $\beta = 0$, the PR call option becomes a reset out call option (reset strike call option). Replacing OTM option at reset time t is costly for the option's holder. The cost is paid at time 0.
- If $\alpha = 0$ and $\beta = X$, this is a reset-in call option. As explained before, the profit is locked in when the option is ITM. This profit can be paid at the reset time t or until maturity T. [4]

3.2 Application

We adopt the binomial pricing approach to propose analytical pricing formula inspired by Cox et al. (1979) and Haug and Haug (2001). To overcome one of the weaknesses of this approach, we consider a large number of time steps $n = 5000$ time steps.

Tables 1 and 2 compare analytical pricing formulae and closed-form solutions for both call and put options in the settings discussed previously: BSM, forward-start, reset-out, reset-in and GR. The parameters used are $S = X = 1000$ euros, $r = 4\%$, $d = 2\%$, $\sigma = 30\%$,

[4] If we consider a put option,

- If $\alpha = 0$ and $\beta = X$, this a reset-out put and the assets' buyer has a gain at reset time t.
- If $\alpha \longrightarrow +\infty$ and $\beta = 0$, this a reset-in put option. The assets' buyer has to pay in order to reset the option ATM.

$t = 0,25$ and $T = 1$ (year) and $\alpha = \beta = 100$ for PR options.

Call option	Closed-form solution CFS	Analytical solution AS	$\frac{CFS-AS}{AS}$
BSM	$125,6770$	$125,6712$	$0,005$ %
Forward-start	$108,0199$	$108,0200$	$0,000$ %
Reset-out	$144,2763$	$144,2680$	$0,006$ %
Reset-in	$89,4206$	$89,4232$	$-0,003$ %
GR	$108,3568$	$108,5477$	$-0,176$ %

Table 1: Call models comparison

Put option	Closed-form solution CFS	Analytical solution AS	$\frac{CFS-AS}{AS}$
BSM	$106,6277$	$106,2619$	$0,005$ %
Forward-start	$93,4267$	$93,4267$	$0,000$ %
Reset-out	$69,6581$	$69,6613$	$0,007$ %
Reset-in	$130,0363$	$130,0274$	$-0,005$ %
GR	$95,4858$	$95,3582$	$0,134$ %

Table 2: Put models comparison

In both cases the percentage of error does not exceed $0,15\%$. One explanation could be errors generated by the estimation of bivariate cumulative normal distribution, specifically in the presence of correlation between the strike and underlying asset prices (the correlation coefficient is given by $\sqrt{\frac{t}{T}}$).

3.2.1 When to reset the strike price?

We analyze the sensitivity of GR call and put options to several variations of reset time point (see table 3).We consider analytical and closed-form solutions for the following reset dates $t_1 = 0,25$, $t_1 = 0,50$ and $t_1 = 0,75$. The two approaches provide very close results:

the difference is estimated to less than $0,2$ %.

Reset date	GR call option		GR put option	
	CFS	AS	CFS	AS
0,25	108,357	108,548	95,486	95,358
0,5	87,758	87,915	79,378	79,288
0,75	61,606	61,753	57,883	57,822

Table 3: Sensitivity of GR option prices to reset date.

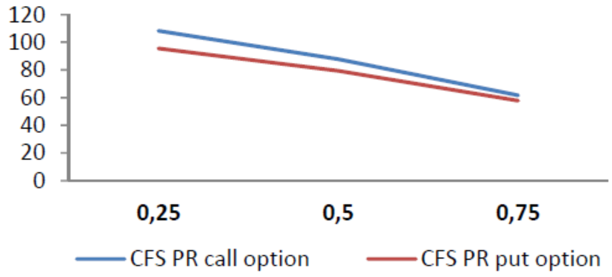

Figure 6: Sensitivity of GR call and put options to reset time.

Unlike reset options, the price of GR option decreases when the reset time becomes close to maturity (see figure 6). However, when the option is OTM, in the sense $S_t \leq X - \beta$, considering multiple reset dates to begin each subperiod with an ATM option is not value-enhancing. Only the last adjustment will determine the cost to be paid at time 0 to replace OTM option with an ATM one. When the option is ITM, a multiple reset dates is advantageous for the option's holder as it enables him to lock in the gains until maturity even if the option is not going to be exercised at T.

As the value of option contract is equal to the sum of the current values of the gains (for the holders of ITM options) and costs (for the holders of OTM options), it does depend on the number of reset dates.

The option value has, however an effect on the gain generated by the option. For instance, when this value is positive (respectively negative), the strike price is increased (respectively diminished) which decreases (respectively increases) the probability of exercising the option and the gain expected from buying the option is then reduced (respectively raised).

4 Conclusion

We derived general analytical pricing formula to value PR option inspired by among others, Gray and Whaley (1999), Haug and Haug (2001) and Cox et al. (1973). Comparison with BSM, Reset-in and reset-out, provides quite satisfying results.

It would be interesting to provide empirical validation of this generalized reset option in options' market to analyze the liquidity effect of resetting automatically the strike price option to the underlying asset price at a preagreed time point.

In the current paper, we focused on the particular case of European rest option. In future work, we will be glad to propose a generalized American reset option.

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