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Yang, Yi-You

Department of Applied Mathematics, Aletheia University

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Yi-You Yang*

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Abstract

In exchange economies with indivisible objects, the substitutability of agents' preferences is essential for the guaranteed existence of Walrasian equilibrium. In this paper, we analyze the ranges of variation for agents' preferences that will guarantee the existence of equilibrium when some agents' preferences are known to satisfy the substitutability condition. Our approach is based on a classification result that partitions the set of economies into disjoint weak similarity classes such that whenever a weak similarity class contains an economy with an equilibrium, each economy in this class also has an equilibrium. The links among economies in the same weak similarity class are established with the notion of monotonization and tax systems.

Keywords: Equilibrium; indivisibility; substitutability; free disposal; tax system.

*Department of Applied Mathematics, Aletheia University, New Taipei City 251, Taiwan. E-mail address: yyyang@mail.au.edu.tw

1 Introduction

Consider an exchange economy with heterogeneous indivisible objects and money. When some agents' preferences are known, a natural question is to analyze the ranges of variation for the other agents' preferences that will guarantee the existence of Walrasian equilibrium. To study the question, we introduce the notion of *t-substitutability* that extends the substitutability condition (Kelso and Crawford, 1982) by incorporating the effects of the availability of free disposal and a tax system t.

The free disposal condition is commonly imposed in economic models to ensure the monotonicity of agents' preferences. The intuition behind this argument is that once an agent is allowed to discard unwanted objects for free, possessing more objects will not actually hurt the agent's satisfaction. To address the influence of the availability of free disposal, we formulate the notion of *monotonization* of an agent's utility function that looks at the highest utility which can be achieved by dropping dislikes out of a given collection of objects. Then we use this notion, together with tax systems, to partition the set of economies into disjoint *weak similarity* classes and shows that whenever a weak similarity class contains an economy with an equilibrium, each economy in this class also has an equilibrium. This classification result will be essential in our analysis.

A tax system is represented by a *tax vector* $t = (t_a)$, i.e., a real-valued function over the set of objects. When such a system is employed, the amount of money paid for an object a will become the sum of its market price and t_a. Hence, the agent i's *after-tax utility* of consuming the object a is equal to i's original (before-tax) utility of consuming the combination of object a and $-t_a$ units of money.

An agent i's utility function is called *t-monotone* if i's after-tax utility function with respect to the tax vector t is monotone. An agent's utility function satisfies the

t*-substitutability* condition if objects are substitutes for the agent whenever free disposal of objects becomes available and the tax system t is employed. The notions of t-monotonicity and t-substitutability have transparent economic interpretations and are easily confirmed.[1] We prove that when an agent's utility function satisfies the substitutability condition and is t-monotone, the set consisting of all t-substitutability utility functions is a range for the other agents' utility functions for which the existence of equilibrium is guaranteed. This observation extends the existence result induced from the analysis of Kelso and Crawford (1982). In their seminal article, Kelso and Crawford study a job-matching market and prove that the core of their market, which coincides with the set of Walrasian equilibria, is non-empty if each firm's production technology satisfies the substitutability condition. Since the t-substitutability condition is weaker than the substitutability condition, our analysis generalizes Kelso and Crawford's existence result for economies with indivisible objects.

Furthermore, we show that as more agents reveal their preferences, the range for the other agents' preferences that can guarantee the existence of equilibrium might be further enlarged. In many practical economic situations, it is difficult to know precisely all the agents' preferences. Our analysis will then be useful when the range of agents' preferences is roughly known and a number of agents are willing to reveal their preferences truthfully. To better illustrate the point, consider a sequence of tax vectors $t^1, \ldots, t^l$ and an economy in which each agent's utility function is known to satisfy the t^i-substitutability condition for at least some $i \in \{1, \ldots, l\}$. If we can further verify that for each $i \in \{1, \ldots, l\}$, there exists an agent whose utility function satisfies the substitutability condition and is t^i-monotone, then our analysis can ensure the existence

[1] Reijnierse, Gellekom and Potters (2002) provide an easy way to verify substitutability for utility functions. In sight of their method, it is not difficult to verify whether a utility function satisfies the t-substitutability condition.

of equilibrium while the other agents' precise preferences are left unknown.

The substitutability for indivisible objects has received increasing attention in recent years. Gul and Stacchetti (1999) prove that both their *single improvement* (SI) condition and *no complementarities* (NC) condition are equivalent to the substitutability condition. Fujishige and Yang (2003) draw the equivalence between $M^{\natural}$-*concavity* and substitutability in the framework of discrete convex analysis. Various extensions of substitutability from Kelso and Crawford's job-matching market to matching models with multiple contract terms are respectively introduced by Hatfield and Milgrom (2005) and Hatfield and Kojima (2010).

The rest of the paper is organized as follows. We present the model in Section 2. In Section 3, we study the influence of the monotonization on the existence of Walrasian equilibrium and prove a structural result that classifies economies with the weak similarity relation. In Section 4, we generalize Kelso and Crawford's existence result with the notion of t-substitutability. Section 5 relates our analysis to a non-existence result by Gul and Stacchetti (1999) and a existence result by Sun and Yang (2006). Section 6 concludes and three proofs are presented in the Appendices.

2 The model

Consider an exchange economy with a finite set $N = \{1, \ldots, n\}$ of agents and a finite set $\Omega = \{a_1, \ldots, a_m\}$ of heterogeneous indivisible goods, and a perfectly divisible good called *money*. Each agent $i \in N$ has a quasi-linear utility function $u_i : 2^{\Omega} \times \mathbb{R} \to \mathbb{R}$, which is characterized by a *valuation* function $v_i : 2^{\Omega} \to \mathbb{R}$ such that the utility of agent i holding the bundle $A \in 2^{\Omega}$ and c units of money is $u_i(A, c) := v_i(A) + c$. Each agent $i \in N$ is initially endowed with a bundle Ω_i of goods and a sufficient amount of money

c_i such that $\Omega = \cup_{i \in N} \Omega_i$ and $c_i \geq v_i(A)$ for all $A \in 2^\Omega$. Under these assumptions, the initial endowment will be irrelevant to the efficient allocations and their supporting prices. Thus, we choose to leave the initial endowments of goods unspecified and simply represent this exchange economy by $E = \left(\Omega; (u_i)_{i \in N}\right)$.

A *price vector* $p = (p_a)_{a \in \Omega} \in \mathbb{R}^\Omega$ assigns a price for each good a in Ω. For each bundle $A \in 2^\Omega$, we use $p(A)$ to denote the sum of prices of those goods in A, i.e., $p(A) := \sum_{a \in A} p_a$; and for each $a \in \Omega$, let $e^a \in \mathbb{R}^\Omega$ denote the characteristic vector whose i-th coordinate is 1 if $a_i = a$ and 0 otherwise.

The *demand correspondence* $D_{u_i} : \mathbb{R}^\Omega \to 2^\Omega$ of an agent with the utility function u_i is defined by

$$D_{u_i}(p) = \left\{ A \in 2^\Omega : u_i(A, -p(A)) \geq u_i(B, -p(B)) \text{ for all } B \in 2^\Omega \right\}.$$

That is, $D_{u_i}(p)$ is the set of bundles that maximize i's utility at price level p.

An *allocation* for the economy $E = \left(\Omega; (u_i)_{i \in N}\right)$ is a partition of objects among all agents in N, that is, a set $\mathbf{X} = (X_1, \ldots, X_n)$ of mutually exclusive bundles that exhaust Ω, where X_i represents the set of objects consumed by agent i under the allocation $\mathbf{X}$. The possibility that $X_i = \emptyset$ for some i is allowed.

A *Walrasian equilibrium* for the economy $E = \left(\Omega; (u_i)_{i \in N}\right)$ is a pair $\langle \mathbf{X}, p \rangle$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is an allocation for E and $p \in \mathbb{R}^\Omega$ is a price vector such that for each agent $i \in N$, $u_i(X_i, -p(X_i)) \geq u_i(A, -p(A))$ for all $A \in 2^\Omega$, i.e., $X_i \in D_{u_i}(p)$. In this case, $\mathbf{X}$ is called an *equilibrium allocation* and p an *equilibrium price vector*.

A valuation function $v_i : 2^\Omega \to \mathbb{R}$ is said to be *monotone* if for all $B \subseteq A \subseteq \Omega$, $v_i(B) \leq v_i(A)$. The *monotonization* of a valuation function v_i is the valuation function $\widehat{v_i} : 2^\Omega \to \mathbb{R}$ given by $\widehat{v_i}(A) = \max_{B \subseteq A} v_i(B)$ for all bundles $A \in 2^\Omega$. Note that

v_i is monotone if and only if $v_i = \widehat{v_i}$. When a valuation v_i is monotone, the utility function u_i characterized by v_i is called a *monotone* utility function. Similarly, we define the *monotonization* $\widehat{u_i}$ of a utility function u_i by $\widehat{u_i}(A,c) = \widehat{v_i}(A) + c$ for all $(A,c) \in 2^{\Omega} \times \mathbb{R}$. Moreover, we define the *monotonization* $\widehat{E}$ of $E = \left(\Omega; (u_i)_{i \in N}\right)$ to be the economy obtained from E by replacing the utility function u_i by $\widehat{u_i}$ for all agents $i \in N$, i.e., $\widehat{E} := \left(\Omega; (\widehat{u_i})_{i \in N}\right)$.

The monotone condition on agents' utility functions is commonly employed in economic analyses.[2] An interpretation for this setting is the introduction of the notion of *free disposal* into the models. The intuition behind it is that once discarding unwanted objects turns to be costless, possessing more objects will not actually hurts an agent's satisfaction. Therefore, an agent's original utility function will be replaced by its monotonization when free disposal of unwanted objects becomes available to the agent. In the next section, we will study the influence of the availability of free disposal on the existence of equilibrium in terms of the notion of monotonization.

Another important factor in our analysis is the tax system characterized by a vector $t = (t_a)_{a \in \Omega} \in \mathbb{R}^{\Omega}$. When a tax system t is adopted, we define agent i's *after-tax utility function,* $u_i[t] : 2^{\Omega} \times \mathbb{R} \to \mathbb{R}$, by

$$u_i[t](A,c) = u_i(A, c - t(A)) \text{ for } (A,c) \in 2^{\Omega} \times \mathbb{R}, \tag{1}$$

and let $E[t] := \left(\Omega; (u_i[t])_{i \in N}\right)$ denote the *after-tax economy.* Note that $\langle \mathbf{X}, p \rangle$ is a Walrasian equilibrium for E if and only if $\langle \mathbf{X}, p - t \rangle$ is a Walrasian equilibrium for $E[t]$.

A utility function u_i is said to be *t-monotone* if the after-tax utility function $u_i[t]$

[2] See, for example, Bikhchandani and Mamer (1997), Ma (1998), Gul and Stacchetti (1999), Ausubel and Milgrom (2002), and Fujishige and Yang (2003) among others.

is monotone. Let $t^{u_i} = (t_a^{u_i})_{a\in\Omega} \in \mathbb{R}^{\Omega}$ denote the vector given by

$$t_a^{u_i} = \min\left\{v_i\left(A \cup \{a\}\right) - v_i\left(A\right) : A \subseteq \Omega \backslash \{a\}\right\} \text{ for } a \in \Omega. \tag{2}$$

Then u_i is t-monotone if and only if $t \leq t^{u_i}$.

3 A classification result

In this section, we will introduce the notion of *weak similarity*, which is an equivalence relation on the set of economies, and prove that for any two economies E and $\tilde{E}$ in the same weak similarity class, E has a Walrasian equilibrium if and only if $\tilde{E}$ has a Walrasian equilibrium.

To establish links between economies in the same weak similarity class, we are interested in the question under which conditions the existence of equilibrium would be immune to the influence of the availability of free disposal.

Clearly, changes in the availability of free disposal could change agents' utility functions and, hence, possibly, the existence of Walrasian equilibrium. As an illustration, we consider the following two examples.

Example 1 *Consider the economy* $E = \left(\Omega; (u_i)_{i\in N}\right)$ *given by* $\Omega = \{a, b, c, d\}$, $N = \{1, 2, 3\}$, *and*

$$v_1\left(A\right) = \begin{cases} 9, & \textit{if } A = \{a, b\}, \\ 8, & \textit{if } A = \{a, b, c\}, \\ -6, & \textit{if } A = \{c\}, \\ 0, & \textit{otherwise,} \end{cases} \quad v_2\left(A\right) = v_3\left(A\right) = \begin{cases} 9, & \textit{if } A = \{a, d\} \textit{ or } \{b, d\}, \\ 4, & \textit{if } A = \{d\}, \\ -6, & \textit{if } c \in A, \\ 0, & \textit{otherwise.} \end{cases}$$

Note that E *has a Walrasian equilibrium* $\langle (\{a,b,c\},\{d\},\emptyset),(6,6,-5,4)\rangle$, *while no Walrasian equilibrium exists in* $\widehat{E}$.

The above example demonstrates that the existing equilibrium may be destroyed by the availability of free disposal. By contrast, the following example shows that sometimes making free disposal available to agents can help to generate an equilibrium.

Example 2 *Consider the economy* $E = \left(\Omega; (u_i)_{i\in N_1}\right)$ *given by* $\Omega = \{a,b,c\}$, $N = \{1,2\}$, *and*

$$v_1(A) = \begin{cases} 6, & \text{if } A = \{a,b,c\}, \\ 5, & \text{if } A = \{a\}, \\ 1, & \text{if } A = \{c\}, \\ 0, & \text{otherwise,} \end{cases} \qquad v_2(A) = \begin{cases} 7, & \text{if } A = \{a,b\}, \\ 5, & \text{if } A = \{a\} \text{ or } \{b\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to check that the economy has no Walrasian equilibrium, while it monotonization $\widehat{E}$ *has a Walrasian equilibrium* $\langle (\{a\},\{b,c\}),(4,4,0)\rangle$.

Examples 1 and 2 show that the existence of Walrasian equilibrium could be significantly influenced by the availability of free disposal. A feature shared by Examples 1 and 2 is that there are no agents with monotone preferences. This observation leads to the result of the following theorem, which shows that when there exists an agent whose utility function is monotone, the existence of equilibrium will be irrelevant to the availability of free disposal.

Theorem 3 *Let* $E = \left(\Omega; (u_i)_{i\in N}\right)$ *be an economy with an agent* $j \in N$ *whose utility function* u_j *is monotone.*

(a) *Each equilibrium allocation for* E *is an equilibrium allocation for* $\widehat{E}$.

(b) *Each equilibrium price vector for* $\widehat{E}$ *is an equilibrium price vector for* E.

(c) E *has a Walrasian equilibrium if and only if* $\widehat{E}$ *has a Walrasian equilibrium.*

Proof. See Appendix A. ■

In the following theorem, we try to characterize the influence of the availability of free disposal on the existence of Walrasian equilibrium by showing that allowing every agent to enjoy free disposal has the same effect for generating an equilibrium (or eliminating the existing equilibria) as making free disposal available to an arbitrary agent or as adding a fictitious agent 0 who values only money into the economy.

Theorem 4 *Let* $E = \left(\Omega; (u_i)_{i\in N}\right)$ *be an economy. Let* $E' = (\Omega; \widehat{u_1}, u_2, \cdots, u_n)$. *Let* $N_0 = N \cup \{0\}$ *and* $\mathring{E} = \left(\Omega; (u_i)_{i\in N_0}\right)$ *the economy constructed from* E *by introducing a fictitious agent,* 0, *whose utility function is given by* $u_0(A, c) = c$ *for* $(A, c) \in 2^{\Omega} \times \mathbb{R}$. *Then*

(a) $\widehat{E}$ *has a Walrasian equilibrium if and only if* E' *has a Walrasian equilibrium; and*

(b) $\widehat{E}$ *has a Walrasian equilibrium if and only if* $\mathring{E}$ *has a Walrasian equilibrium.*

Proof. See Appendix B. ■

Let $\mathcal{E}$ denote the set consisting of all economies with an agent having monotone preferences. Two economies E and $\tilde{E}$ are said to be *similar*, denoted by $E \sim \tilde{E}$, if there exist vectors $t, \tilde{t} \in \mathbb{R}^{\Omega}$ such that $E[t] \in \mathcal{E}$, $\tilde{E}\left[\tilde{t}\right] \in \mathcal{E}$, and $\widehat{E[t]} = \widehat{\tilde{E}\left[\tilde{t}\right]}$. Moreover, two economies E and $\tilde{E}$ are said to be *weakly similar* if there exists a sequence $\{E_k\}_{k=0}^{r}$ of economies such that $E = E_0, \tilde{E} = E_r$, and $E_{k-1} \sim E_k$ for $k = 1, \ldots, r$. The following result shows that whenever a weak similarity class contains an economy with an equilibrium, each economy in this class also has an equilibrium.

Theorem 5 *Let E be an economy that is weakly similar to another economy $\tilde{E}$. Then E has a Walrasian equilibrium if and only if $\tilde{E}$ has a Walrasian equilibrium.*

Proof. The proof is an immediate consequence of Theorem 3. ■

4 The t-substitutability condition

The substitutability condition is a sufficient condition on agents' preferences to guarantee the existence of Walrasian equilibrium (Kelso and Crawford, 1982, Theorem 2). A utility function u_i satisfies the *substitutability condition* if for any two price vectors $p, q \in \mathbb{R}^{\Omega}$ with $p \leq q$, and any bundle $A \in D_{u_i}(p)$, there exists $B \in D_{u_i}(q)$ such that $\{a \in \Omega : q_a = p_a\} \subseteq B$. Thus, the substitutability condition ensures that the demand for an object does not decrease when prices of some other objects increase. We denote by Γ the set of utility functions satisfying the substitutability condition.

Consider an economy in which some agents' preferences are known to satisfy the substitutability condition. We are interested in analyzing the ranges of variation for the other agents' preferences that will guarantee the existence of equilibrium.

To study the issue, we introduce the notion of t-substitutability which incorporates the effects of monotonization and the tax system $t \in \mathbb{R}^{\Omega}$. A utility function u_i satisfies the *t-substitutability condition* if the monotonization $\widehat{u_i[t]}$ of the after-tax utility function $u_i[t]$ satisfies the substitutability condition. Let $\Gamma(t)$ denote the set of utility functions satisfying the t-substitutability condition.

Proposition 6 *Let $u_i : 2^{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a utility function and let* $\mathbf{0}$ *denote the zero vector in $\mathbb{R}^{\Omega}$.*

(a) *There exists a vector* $t \in \mathbb{R}^{\Omega}$ *such that* u_i *satisfies the* t*-substitutability condition, i.e.,* $u_i \in \bigcup_{t \in \mathbb{R}^{\Omega}} \Gamma(t)$.

(b) u_i *satisfies the substitutability condition if and only if* u_i *satisfies the* t*-substitutability condition for each* $t \in \mathbb{R}^{\Omega}$*, i.e.,* $\Gamma = \bigcap_{t \in \mathbb{R}^{\Omega}} \Gamma(t)$.

(c) $\widehat{u_i[t]} = \widehat{\widehat{u_i}[t]}$ *for each* $t \in \mathbb{R}^{\Omega}_{+}$.

(d) *If* u_i *satisfies the substitutability condition, then* $\widehat{u_i}$ *satisfies the substitutability condition. Moreover, for any* $t \in \mathbb{R}^{\Omega}_{+}$*, if* u_i *satisfies the* t*-substitutability condition, then* $\widehat{u_i}$ *satisfies the* t*-substitutability condition.*

(e) *If* u_i *satisfies the* t'*-substitutability condition for some* $t' \in \mathbb{R}^{\Omega}$ *and* $t \geq t'$*, then* u_i *satisfies the* t*-substitutability condition, i.e.,* $\Gamma(t) \supseteq \Gamma(t')$ *for any two vectors* $t, t' \in \mathbb{R}^{\Omega}$ *with* $t \geq t'$.

Proof. See Appendix C. ∎

Proposition 6 (a) and (b) respectively implies that the notion of t-substitutability is *general* enough to encompass all utility functions and is *restrictive* enough to characterize the substitutability condition. The result of Proposition 6 (d) shows that the substitutability (t-substitutability) of a utility function is immune to the influence of monotonization. This observation implies that when the existence of equilibrium is guaranteed by the substitutability condition, it cannot be destroyed by the availability of free disposal.

The following theorem generalizes Kelso and Crawford's existence result with the notion of t-substitutability.

Theorem 7 *Let $E = \left(\Omega; (u_i)_{i \in N}\right)$ be an economy such that u_1 satisfies the substitutability condition and is t-monotone for some $t \in \mathbb{R}^\Omega$. If u_i satisfies the t-substitutability condition for each $i \neq 1$, then E has a Walrasian equilibrium.*

Proof. Assume that u_i satisfies the t-substitutability condition for each $i \neq 1$. In other words, $\widehat{u_i [t]}$ satisfies the substitutability condition for each $i \neq 1$. Consider the after-tax economy $E[t] = \left(\Omega; (u_i [t])_{i \in N}\right)$ and its monotonization $\widehat{E[t]} = \left(\Omega; \left(\widehat{u_i [t]}\right)_{i \in N}\right)$. Since the utility function of each agent in $\widehat{E[t]}$ satisfies the substitutability condition, it follows that $\widehat{E[t]}$ has an equilibrium. Together with the fact that $u_1 [t]$ is monotone, we obtain that $E[t]$ has an equilibrium, and so does E. ■

When we try to analyze an economy in which agent 1's utility function is known to satisfy the substitutability condition. Based on our information about the vector t^{u_1}, the result of Theorem 7 can give ranges for the other agents' utility functions that can ensure the existence of equilibrium.[3] In case we know that $t^{u_1} \in T$ for some set of vectors $T \subseteq \mathbb{R}^\Omega$, there exists an equilibrium if $u_i \in \bigcap_{t \in T} \Gamma(t)$ for each $i \neq 1$. In case t^{u_1} is known precisely, then the range given by Theorem 7 can be enlarged up to $\Gamma(t^{u_1})$. Conversely, in case we have no idea about whether u_1 is t-monotone for any $t \in \mathbb{R}^\Omega$, the region given by Theorem 7 to ensure the existence of equilibrium will then reduce to $\Gamma = \bigcap_{t \in \mathbb{R}^\Omega} \Gamma(t)$.

In the next result, we analyze an economy in which a number of agents' preferences are given and satisfy the substitutability condition.

Theorem 8 *Let $E = \left(\Omega; (u_i)_{i \in N}\right)$ be an economy and let $\tilde{N}$ be a subset of N such that*

[3] Note that u_1 is t-monotone if and only if $t \leq t^{u_1}$.

u_i satisfies the substitutability condition for each $i \in \tilde{N}$. If

$$u_j \in \bigcup_{i \in \tilde{N}} \Gamma\left(t^{u_i}\right) \text{ for all } j \in N \backslash \tilde{N}, \tag{3}$$

then E has a Walrasian equilibrium.

Proof. Assume that condition (3) holds. Let N_0 denote the set of agents whose utility functions satisfy the substitutability condition. In case $N_0 = N$, the proof is done by Kelso and Crawford's existence result. If $N_0 \neq N$, we may choose $j_1 \in N \backslash N_0$. By (3), there exists $i_1 \in \tilde{N}$ such that u_{j_1} satisfies the $t^{u_{i_1}}$-substitutability condition. This implies that $\widehat{u_{j_1}\left[t^{u_{i_1}}\right]}$ satisfies the substitutability condition. Let $E_0 = E$ and let E_1 be the economy constructed from E_0 by replacing agent j_1's utility function with $\widehat{u_{j_1}\left[t^{u_{i_1}}\right]}\left[-t^{u_{i_1}}\right]$. Note that $E_0\left[t^{u_{i1}}\right] \in \mathcal{E}$, $E_2\left[t^{u_{i1}}\right] \in \mathcal{E}$, and $\widehat{E_0\left[t^{u_{i1}}\right]} = \widehat{E_1\left[t^{u_{i1}}\right]}$. Then we have $E_0 \sim E_1$. Let $N_1 = N_0 \cup \{j_1\}$. In case $N_1 = N$, the proof is done by the combination of Theorem 5 and Kelso and Crawford's existence result. If $N_1 \neq N$, we can choose $j_2 \in N \backslash N_1$ and construct the economy E_2 from E_1 by replacing agent j_2's utility function with $\widehat{u_{j_2}\left[t^{u_{i_2}}\right]}\left[-t^{u_{i_2}}\right]$ for some $i_2 \in \tilde{N}$ such that $\widehat{u_{j_2}\left[t^{u_{i_2}}\right]}\left[-t^{u_{i_2}}\right]$ satisfies the substitutability condition. It is easy to verify that E_1 is similar to E_2. Let $N_2 = N_1 \cup \{j_2\}$. Again, the proof is done if $N_2 = N$.

Since N is finite, we can inductively construct a sequence $\{E_k\}_{k=0}^{r}$ of economies such that $E_{k-1} \sim E_k$ for $k = 1, \ldots, r$, and the utility function of each agent in E_r satisfies the substitutability condition. Combining with Theorem 5 and Kelso and Crawford's existence result, we obtain the desired result. ■

Our existence results improves on Kelso and Crawford's existence result in two respects. First, our results can be applied to analyze economies in which the agents' utility functions might violate the substitutability condition. Second, in many economic situations, it is often difficult to verify that all agents' preferences satisfy the substitutability. Our analysis will then be useful when the range of agents' preferences is roughly known and a number of agents are willing to reveal their preferences truthfully. Moreover, as more agents reveal their preferences, the range for the other agents' preferences given by Theorem 8 to ensure the existence of equilibrium can be further enlarged.

5 Applications

5.1 Gul and Stacchetti's non-existence result

Gul and Stacchetti (1999) prove that there is no any weakening of the substitutability condition that can ensure the existence of equilibrium. This result can be considered as a converse to Kelso and Crawford's existence result and is recalled as follows.

Theorem 9 (See Gul and Stacchetti, 1999, Theorem 2) [4] *If agent 1's utility function u_1 violates the substitutability condition, then there exists an economy $E = \big(\Omega, (u_i)_{i\in N}\big)$ such that u_i satisfies the substitutability condition for $i = 2, \ldots, n$, but $E = \big(\Omega, (u_i)_{i\in N}\big)$ has no Walrasian equilibrium.*

Consider an economy $E = \big(\Omega, (u_i)_{i\in N}\big)$ in which the existence of Walrasian equilibrium is guaranteed by the substitutability condition. Gul and Stacchetti's Theorem 2 indicates that the existence of equilibrium seems vulnerable in the sense that changes in a *single* agent 1's utility function might violate its substitutability and then destroy the existence of equilibrium. An immediate application of Theorem 8 gives some new

[4] Gul and Stacchetti (1999) originally prove their Theorem 2 under the assumption that each utility function is monotone. However, it is not difficult to see that the non-existence result still holds when the monotonicity assumption is removed.

insights into the sensitivity of the existence of equilibrium to changes in agent 1's preferences. Namely, for any economy $\widetilde{E} = (\Omega, \widetilde{u_1}, u_2, \ldots, u_n)$ obtained from replacing agent 1's utility function by $\widetilde{u_1}$, there exists an equilibrium if $\widetilde{u_1} \in \bigcup_{i=2}^{n} \Gamma(t^{u_i})$.

Another interpretation for Gul and Stacchetti's Theorem 2 is that the set Γ of utility functions satisfying the substitutability condition is a largest set for which the existence of equilibrium is guaranteed. The following corollary gives an analogous result for the set $\Gamma(\mathbf{0})$ of utility functions satisfying the $\mathbf{0}$-monotonicity condition.

Corollary 10 *For economies with an agent having monotone preference, the set $\Gamma(\mathbf{0})$ is a largest set for which the existence of Walrasian equilibrium is guaranteed. In other words, we have*

(a) *for any economy $E = \left(\Omega, (u_i)_{i\in N}\right) \in \mathcal{E}$, there exists a Walrasian equilibrium if each agent i's utility function u_i satisfies the $\mathbf{0}$-substitutability condition; and*

(b) *for any utility function u_1 violating the $\mathbf{0}$-substitutability condition, there exists an economy $E = \left(\Omega, (u_i)_{i\in N}\right) \in \mathcal{E}$ such that u_i satisfies the $\mathbf{0}$-substitutability condition for $i = 2, \ldots, n$, but $E = \left(\Omega, (u_i)_{i\in N}\right)$ has no Walrasian equilibrium.*

Proof. (a) Assume that $E \in \mathcal{E}$ and u_i satisfies the $\mathbf{0}$-substitutability condition for each $i \in N$. Then there exists $j \in N$ such that u_j is $\mathbf{0}$-monotone and satisfies the substitutability condition. By Theorem 7, E has an equilibrium.

(b) Let u_1 be a utility function that violates the $\mathbf{0}$-substitutability condition. This implies that $\widehat{u_1} = \widehat{u_1[\mathbf{0}]}$ violates the substitutability condition. By Gul and Stacchetti's Theorem 2, there exists an economy $E' = (\Omega, \widehat{u_1}, u_2, \ldots, u_n) \in \mathcal{E}$ such that u_i satisfies the substitutability condition for $i = 2, \ldots, n$, but $E' = \left(\Omega, (u_i)_{i\in N}\right)$ has no Walrasian equilibrium. Let $E = (\Omega, u_1, \widehat{u_2}, \ldots, \widehat{u_n}) \in \mathcal{E}$. Since E is similar to E', it follows that

E has no Walrasian equilibrium by Theorem 5, but for $i = 2, \ldots, n$, $\widehat{u_i} \in \Gamma \subseteq \Gamma(\mathbf{0})$ by Proposition 6 (b) and (d). ■

5.2 The gross substitutability and complementarity

It should be noted that the classification approach given in Section 4 can be used to extend any general structure on preferences that is sufficient for the existence of equilibrium. To clarify the point, we will recall the gross substitutability and complementarity condition introduced by Sun and Yang (2006).

Sun and Yang study an economy $E = \left(\Omega; (u_i)_{i \in N}\right)$ in which the set Ω of objects can be divided into two disjoint groups S_1 and S_2, and show that if objects in the same group are substitutes and objects across these two groups are complements, then the economy E has a Walrasian equilibrium.

A utility function u_i satisfies the *gross substitutability and complementarity* (GSC) *condition* if for any price vector $p \in \mathbb{R}^{\Omega}$, $a \in S_j$, $\delta \geq 0$, and $A \in D_{u_i}(p)$, there exists $B \in D_{u_i}(p + \delta e^a)$ such that $[A \cap S_j] \setminus \{a\} \subseteq B \subseteq (A \cup S_j)$. When $S_1 = \emptyset$ or $S_2 = \emptyset$, the GSC condition reduces to the substitutability condition. However, it should be noted that when $\emptyset \neq S_1 \neq \Omega$, the GSC condition is logically independent from the substitutability condition.

Let $t \in \mathbb{R}^{\Omega}$ be a tax vector. A utility function u_i satisfies the *t-GSC condition* if $\widehat{u_i[t]}$ satisfies the GSC condition. Let $\bar{\Gamma}(t)$ denote the set of utility functions that satisfy the t-GSC condition. Theorem 3.1 of Sun and Yang (2006, p. 1388) shows that if each agent's utility function satisfies the GSC condition, then there exists a Walrasian equilibrium. This existence result, together with the notion of t-GSC condition and a proof similar to that of Theorem 8, naturally leads to the following theorem.

Theorem 11 *Let* $E = \left(\Omega; (u_i)_{i\in N}\right)$ *be an economy and let* $\tilde{N}$ *be a subset of* N *such that* u_i *satisfies the GSC condition and is* t^i*-monotone for each* $i \in \tilde{N}$*. If* $u_j \in \bigcup_{i\in\tilde{N}} \bar{\Gamma}(t^i)$ *for each* $j \in N \backslash \tilde{N}$*, then* E *has a Walrasian equilibrium.*

6 Concluding remarks

This paper contributes to the literature on economies with indivisible objects. We introduce the notion of t-substitutability to generalize the substitutability condition and characterize ranges of variation for agents' preferences that will guarantee the existence of Walrasian equilibrium when some agents' preferences are known and satisfy the substitutability condition. Since it is often difficult to obtain each agent's preferences precisely in practical economic situations, our results will then be useful when the range of agents' preferences is roughly known and a number of agents' preferences can be verified to satisfy the substitutability condition.

Our main results rely on a classification theorem that partitions the set of economies into weak similarity classes such that whenever a weak similarity class contains an economy with an equilibrium, each economy in this class also has an equilibrium. Finding similar classification results for other economic models may significantly extend the scope of existing results as demonstrated in this paper.

Appendix A. Proof of Theorem 3

The proof of Theorem 3 requires the following lemma.

Lemma 12 *Let* $E = \left(\Omega; (u_i)_{i\in N}\right)$ *be an economy and let* $j \in N$ *be an agent whose utility function* u_j *is monotone. Let* $\langle \mathbf{X}, p\rangle$ *be a Walrasian equilibrium for* E*, and let*

$p' \in \mathbb{R}_+^\Omega$ *be the price vector given by*

$$p_a^{'} = \begin{cases} p_a, & \text{if } p_a \geq 0, \\ 0, & \text{if } p_a < 0. \end{cases} \tag{4}$$

Then

(a) $\{a \in \Omega : p_a < 0\} \subseteq X_j$*; and*

(b) $\langle \mathbf{X}, p' \rangle$ *is a Walrasian equilibrium for* E

Proof. (a) Suppose that there exists $a \in \Omega \backslash X_j$ such that $p_a < 0$. Since u_j is monotone, we have

$$v_j (X_j \cup \{a\}) - p (X_j \cup \{a\}) \geq v_j (X_j) - p (X_j) - p_a > v_j (X_j) - p (X_j),$$

violating the fact that $\langle \mathbf{X}, p \rangle$ is a Walrasian equilibrium for E.

(b) Let $\bar{A} = \{a \in \Omega : p_a < 0\}$. In case $\bar{A} = \emptyset$, then $p' = p$ and we have done. In case $\bar{A} \neq \emptyset$, we have $\bar{A} \subseteq X_j$ by the result of (a). Together with the facts $p' \geq p$ and $X_i \in D_{u_i}(p)$, we have that for each agent $i \in N$ with $i \neq j$ and for each bundle $A \in 2^\Omega$,

$$v_i (X_i) - p' (X_i) = v_i (X_i) - p (X_i) \geq v_i (A) - p (A) \geq v_i (A) - p' (A).$$

Moreover, since $X_j \in D_{u_j}(p)$ and u_j is monotone, it follows that for any bundle $A \in 2^\Omega$,

$$\begin{aligned} v_j (X_j) - p' (X_j) &= v_j (X_j) - p (X_j) + p (\bar{A}) \geq v_j (A \cup \bar{A}) - p (A \cup \bar{A}) + p (\bar{A}) \\ &= v_j (A \cup \bar{A}) - p' (A \cup \bar{A}) \geq v_j (A) - p' (A \cup \bar{A}) \\ &= v_j (A) - p' (A). \end{aligned}$$

This completes the proof. ■

We are now ready to prove Theorem 3.

(a) Assume that $\langle \mathbf{X}, p \rangle$ is a Walrasian equilibrium for E. Let p' be the price vector given by (4), by Lemma 12 (b), we obtain that $\langle \mathbf{X}, p' \rangle$ is a Walrasian equilibrium for E. In what follows, we are going to show that $\langle \mathbf{X}, p' \rangle$ is also a Walrasian equilibrium for $\hat{E}$.

We first verify that

$$\hat{v}_i (X_i) = v_i (X_i) \ \ \forall i \in N. \tag{5}$$

Since u_j is monotone, we have $\hat{v}_j (X_j) = v_j (X_j)$. Suppose that there exists some agent $i \in N \backslash \{j\}$ such that $\hat{v}_i (X_i) > v_i (X_i)$. Then there exists a proper subbundle B of X_i such that $\hat{v}_i (X_i) = v_i (B) = \hat{v}_i (B)$. By Lemma 12 (a), we have $p (X_i) \geq p (B)$. This implies $v_i (B) - p (B) > v_i (X_i) - p (B) \geq v_i (X_i) - p (X_i)$, violating the fact $X_i \in D_{u_i} (p)$.

We suppose on the contrary that $\langle \mathbf{X}, p' \rangle$ is not a Walrasian equilibrium for $\hat{E}$. Then there exists an agent $i \in N \backslash \{j\}$ such that $\hat{v}_i (X_i) - p' (X_i) < \hat{v}_i (C) - p' (C)$ for some bundle $C \in 2^{\Omega}$. Since $\langle \mathbf{X}, p' \rangle$ is a Walrasian equilibrium for E, together with (5), we have

$$v_i (C) - p' (C) \leq v_i (X_i) - p' (X_i) = \hat{v}_i (X_i) - p' (X_i) < \hat{v}_i (C) - p' (C) , \tag{6}$$

and hence $v_i (C) < \hat{v}_i (C)$. This implies that there exists a proper subbundle C' of C such that $\hat{v}_i (C) = v_i (C')$. Combining with (6), we have

$$v_i (X_i) - p' (X_i) < \hat{v}_i (C) - p' (C) = v_i (C') - p' (C) \leq v_i (C') - p' (C') ,$$

violating the fact $X_i \in D_{u_i} (p')$.

(b) Assume that $\langle \mathbf{X}, p \rangle$ is a Walrasian equilibrium for $\hat{E}$. Since each agent in $\hat{E}$ has a monotone utility function, the result of Lemma 12 (a) implies $p \geq 0$. We are going to show that there exists a Walrasian equilibrium $\langle \mathbf{Y}, p \rangle$ for E such that $Y_i \subseteq X_i$ and $\hat{v}_i (X_i) = v_i (Y_i) = \hat{v}_i (Y_i)$ for each agent $i \in N \setminus \{j\}$, and $Y_j = \left(\cup_{i \in N \setminus \{j\}} (X_i \setminus Y_i) \right) \cup X_j$. Let i be an agent with $i \neq j$. We consider two cases.

Case I. $v_i (X_i) = \hat{v}_i (X_i)$. Let $Y_i = X_i$. Then for any bundle $A \in 2^{\Omega}$,

$$v_i (Y_i) - p (Y_i) = \hat{v}_i (X_i) - p (X_i) \geq \hat{v}_i (A) - p (A) \geq v_i (A) - p (A) . \tag{7}$$

Case II. $v_i (X_i) < \hat{v}_i (X_i)$. Then there exists a proper subbundle Y_i of X_i such that $\hat{v}_i (X_i) = v_i (Y_i) = \hat{v}_i (Y_i)$. Together with the fact $X_i \in D_{\hat{u}_i} (p)$, we have

$$\hat{v}_i (X_i) - p (X_i) \geq \hat{v}_i (Y_i) - p (Y_i) = \hat{v}_i (X_i) - p (Y_i) ,$$

and hence $p_a = 0$ for all $a \in X_i \setminus Y_i$. It follows that (7) holds for any bundle $A \in 2^{\Omega}$.

Let $Y_j = \left(\cup_{i \in N \setminus \{j\}} (X_i \setminus Y_i) \right) \cup X_j$. Since u_j is monotone and $p_a = 0$ for all $a \in Y_j \setminus X_j$, it follows that for any bundle $A \in 2^{\Omega}$,

$$\begin{aligned} v_j (Y_j) - p (Y_j) &\geq v_j (X_j) - p (X_j) = \hat{v}_j (X_j) - p (X_j) \\ &\geq \hat{v}_j (A) - p (A) = v_j (A) - p (A) , \end{aligned}$$

and the proof of (b) is done.

Finally, the result of (c) is an immediate consequence of the combination of (a) and (b). This completes the proof.

Appendix B. Proof of Theorem 4

(a) Since $E' \in \mathcal{E}$ and $\hat{E}' = \hat{E}$, in sight of Theorem 3, it follows that $\hat{E}$ has a Walrasian equilibrium if and only if E' has a Walrasian equilibrium.

(b) By the result of (a), it suffices to show that E' has a Walrasian equilibrium if and only if $\mathring{E}$ has a Walrasian equilibrium.

($\Leftarrow$) Let $\langle \mathbf{X}, p \rangle$ be a Walrasian equilibrium for $\mathring{E}$ and let p' be the price vector given by (4). Since $X_0 \in D_{u_0}(p)$, we have $p_a \leq 0$ for all $a \in X_0$. By Lemma 12 and the proof of Theorem 3 (a), it follows that

$$p'_a = \begin{cases} p_a \geq 0, & \text{if } a \in \Omega \backslash X_0, \\ 0, & \text{if } a \in X_0, \end{cases}$$

and $\langle \mathbf{X}, p' \rangle$ is a Walrasian equilibrium for $\mathring{E}$. In what follows, we are going to show $X_0 \cup X_1 \in D_{\hat{u}_1}(p')$, which implies that $\langle (X_0 \cup X_1, X_2, \cdots, X_n), p' \rangle$ is a Walrasian equilibrium for E'. For each bundle $A \in 2^\Omega$, there exists a subbundle A' of A such that $\hat{v}_1(A) = v_1(A')$ and hence

$$\begin{aligned} \hat{v}_1(X_0 \cup X_1) - p'(X_0 \cup X_1) &\geq \hat{v}_1(X_1) - p'(X_1) \geq v_1(X_1) - p'(X_1) \\ &\geq v_1(A') - p'(A') \geq \hat{v}_1(A) - p'(A). \end{aligned}$$

($\Rightarrow$) Let $\langle \mathbf{X}, p \rangle$ be a Walrasian equilibrium for E'. In sight of Lemma 12, we may assume that $p \geq 0$ without loss of generality. Since $X_1 \in D_{\hat{u}_1}(p)$, there exists $Y_1 \subseteq X_1$ such that $\hat{v}_1(X_1) = v_1(Y_1) = \hat{v}_1(Y_1)$ and $p_a = 0$ for all $a \in X_1 \backslash Y_1$. This implies that for any bundle $A \in 2^{\Omega}$,

$$v_1(Y_1) - p(Y_1) = \hat{v}_1(X_1) - p(X_1) \geq \hat{v}_1(A) - p(A) \geq v_1(A) - p(A),$$

i.e., $Y_1 \in D_{u_1}(p)$. Let $Y_0 = X_1 \backslash Y_1$ and $Y_i = X_i$ for $i = 2, \ldots, n$. Then $\langle \mathbf{Y}, p \rangle$ is a Walrasian equilibrium for $\mathring{E}$.

Appendix C. Proof of Proposition 6

Proof. (a) Clearly, there exists $t \in \mathbb{R}_{+}^{\Omega}$ such that $u_i[t](A, 0) \leq 0$ for each $A \in 2^{\Omega}$. Then $\widehat{u_i[t]}(A, c) = c$ for each pair $(A, c) \in 2^{\Omega} \times \mathbb{R}$. This implies that $\widehat{u_i[t]}$ satisfies the substitutability condition.

(c) Let $t \in \mathbb{R}_{+}^{\Omega}$ and let $(A, c) \in 2^{\Omega} \times \mathbb{R}$. It suffices to prove $\max_{C \subseteq A} u_i[t](C, c) = \max_{B \subseteq A} \widehat{u}_i[t](B, c)$. Let $C^* \in \arg\max_{C \subseteq A} u_i[t](C, c)$ and let $B^* \in \arg\max_{B \subseteq A} \widehat{u}_i[t](B, c)$. Then there exists a subbundle B' of B^* such that $\widehat{v}_i(B^*) = v_i(B') = \widehat{v}_i(B')$, and hence

$$\begin{aligned} \widehat{u}_i[t](B^*, c) &= \widehat{u}_i(B^*, c) - t(B^*) = \widehat{v}_i(B^*) + c - t(B^*) \leq v_i(B') + c - t(B') \\ &= u_i[t](B', c) \leq u_i[t](C^*, c) = u_i(C^*, c) - t(C^*) \leq \widehat{u}_i(C^*, c) - t(C^*) \\ &= \widehat{u}_i[t](C^*, c) \leq \widehat{u}_i[t](B^*, c). \end{aligned}$$

(d) Suppose that u_i satisfies the substitutability condition, but $\widehat{u}_i$ violates the substitutability condition. By Gul and Stacchetti's Theorem 2, there exists an economy $E' = (\Omega; u_1, \ldots, \widehat{u}_i, \ldots, u_n)$ such that u_j satisfies the substitutability condition for $j \in N \backslash \{i\}$, but E' has no Walrasian equilibrium. This implies that the economy $\mathring{E} = (\Omega; u_0, u_1, \ldots, u_n)$, where u_0 is defined by $u_0(A, c) = c$ for $(A, c) \in 2^{\Omega} \times \mathbb{R}$, has no equilibrium by Theorem 4, contradicting the the fact that each agent's utility function in $\mathring{E}$ satisfies the substitutability condition.

In case u_i satisfies the t-substitutability condition for some $t \in \mathbb{R}_+^\Omega$, then $\widehat{\widehat{u_i}[t]}$ satisfies the substitutability condition. Together with the result of (c), we obtain that $\widehat{u_i}$ satisfies the t-substitutability condition.

(e) Let $t, t' \in \mathbb{R}_+^\Omega$ such that $t \geq t'$. Assume that u_i satisfies the t'-substitutability condition. Let $u'_i = u_i[t']$. Then $\widehat{u'_i}$ satisfies the substitutability condition and $u_i[t] = u'_i[t - t']$. Since $t - t' \geq \mathbf{0}$, we have $\widehat{u_i[t]} = \widehat{u'_i[t-t']} = \widehat{\widehat{u'_i}[t-t']}$ by (c). This implies that u_i satisfies the t-substitutability condition.

(b) ($\Rightarrow$) Assume that u_i satisfies the substitutability condition. Let $t \in \mathbb{R}^\Omega$. Then $u_i[t]$ satisfies the substitutability, and so does $\widehat{u_i[t]}$ by (d). ($\Leftarrow$) Assume that $\widehat{u_i[t]}$ satisfies the substitutability condition for each $t \in \mathbb{R}^\Omega$. Since $u_i[t^{u_i}]$ is monotone, $u_i[t^{u_i}] = \widehat{u_i[t^{u_i}]}$ satisfies the substitutability condition and so does u_i. ■

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