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Abstract: This article brings in two new discrete distributions: multivariate Binomial distribution and multivariate Poisson distribution. Those distributions were created in eventology as more correct generalizations of Binomial and Poisson distributions. Accordingly to eventology new laws take into account full distribution of events. Also, some properties and characteristics of these new multivariate discrete distributions are described.

Keywords and phrases: multivariate discrete distribution; multivariate Binomial distribution; multivariate Poisson distribution; eventology; dependence of events

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1 Introduction

Distribution of probabilities is one of principal idea in theory of probabilities and mathematical statistics. Its determination is tantamount to definition of all related stochastic events. But trials' results extremely rarely are expressed by one number and more frequently by system of numbers, vector or function. It is said about multivariate distribution if some regularity is described by several stochastic quantities, which are specified on the same probabilistic space. Thereby, it is involving for representation of behavior of the random vector, which serves a description of stochastic events, more or less near to reality. This work came in connection with appeared scientific necessity of assignment two new multivariate discrete distributions, which are naturally following from eventological principles and taking into account specific notion from probability theory namely arbitrary dependence of events, random quantities, tests.

2 Binomial multivariate distribution

Let there are finite sequence of n independent stochastic experiments. In the result of experiment i can ensue or not events from N-set $\mathfrak{X}^{(\mathrm{i})}$ of events $x^{(i)} \in \mathfrak{X}^{(\mathrm{i})}$. Eventological distributions of sets of events $\mathfrak{X}^{(\mathrm{i})}, i = 1, ..., n$ agree with the same eventological distribution $\{p(X), X \subseteq \mathfrak{X}\}$ of certain N-set $\mathfrak{X}$ of events $x \in \mathfrak{X}$, which aren't changing between experiments.

Such scheme of testing is called *multivariate (eventological) scheme of Bernoulli testing with producing set of events* $\mathfrak{X}$, and each of random quantities

$$\xi_x(\omega) = \sum_{i=1}^{n} \mathbf{1}_{x^{(i)}}(\omega), x^{(i)} \in \mathfrak{X}^{(\mathrm{i})}, x \in \mathfrak{X}$$

obey the Binomial distribution with parameters n, $p_x = \mathbf{P}(x)$, while random vector[3] $\hat{\xi} = (\xi_x, x \in \mathfrak{X})$ obey the *Binomial multivariate (N-variate) distribution* with parameters $(n, \{p(X),$
$\varnothing \neq X \subseteq \mathfrak{X}\})$.

Probabilities of *Binomial multivariate distribution, which is generated by N-set of events* $\mathfrak{X}$, are defined for every integer-valued collection $\hat{n} = (n_x, x \in \mathfrak{X}) \in [0, n]^N$ by the formula

$$b_{\hat{n}}\big(n; p(X), \varnothing \neq X \subseteq \mathfrak{X}\big) = \mathbf{P}(\hat{\xi} = \hat{n}) = \mathbf{P}(\xi_x = n_x, x \in \mathfrak{X}) = \sum_{\check{n}} m_{\check{n}}\big(n; \{p(X), X \subseteq \mathfrak{X}\}\big),$$

where

$$m_{\check{n}}\big(n; \{p(X), X \subseteq \mathfrak{X}\}\big) = \mathbf{P}(\check{\xi} = \check{n}) = \mathbf{P}\Big(\big(\xi(X), X \subseteq \mathfrak{X}\big) = \big(n(X), X \subseteq \mathfrak{X}\big)\Big) =$$

$$= \frac{n!}{\prod\limits_{X \subseteq \mathfrak{X}} n(X)!} \prod_{X \subseteq \mathfrak{X}} [p(X)]^{n(X)}$$

[3]In eventology, in general, and at this context, in particular, the notion ¡¡vector¿¿ is using in extended sense as *disordered* finite set or *disordered* finite collection of some elements.

are probabilities of 2^N-variate Polynomial distribution of a random vector $\hat{\xi} = \big(\xi(X),\ X \subseteq \mathfrak{X}\big)$ with parameters $(n; \{p(X), X \subseteq \mathfrak{X}\})$, which is generated by 2^N-set of terrace-events $\{ter(X),$
$X \subseteq \mathfrak{X}\}$, which has biunique correspondence to the given Binomial multivariate distribution; summation is made by all 2^N-variate sets $\hat{n} = \big(n(X), X \subseteq \mathfrak{X}\big) \in S^{2^N}$ from 2^N-vertex simplex S^{2^N}, i.e. such as

$$n = \sum_{X \subseteq \mathfrak{X}} n(X),$$

but which are meet the N equations

$$n_x = \sum_{x \in X} n(X), x \in \mathfrak{X}.$$

2.1 Binomial one-variate distribution

When $N = 1$ (i.e. *generating set* $\mathfrak{X} = \{x\}$ is a monoplet of events) Binomial one-variate distribution of a random quantity ξ_x coincides with the classical Binomial distribution with parameters $(n; p_x)$. In other words, probabilities of the *Binomial one-variate distribution* have classical format

$$b_{n_x}(n; p_x) = \mathbf{P}(\xi_x = n_x) = C_n^{n_x} p_x^{n_x} (1 - p_x)^{n - n_x},\ 0 \leqslant n_x \leqslant n.$$

2.2 Binomial two-variate distribution

When $N = 2$ (i.e. *generating set* $\mathfrak{X} = \{x, y\}$ is duplet of events) Binomial two-variate distribution of a random vector $\hat{\xi} = (\xi_x, \xi_y) = (\xi_x, x \in \mathfrak{X})$ is defined by four parameters $\big(n; p(x),$
$p(y), p(xy)\big)$, where[4]

$$p(x) = \mathbf{P}(x \cap y^c),\ p(y) = \mathbf{P}(x^c \cap y),\ p(xy) = \mathbf{P}(x \cap y).$$

Probabilities of *Binomial two-variate distribution* are calculating for any integer-valued vector $\hat{n} = (n_x, n_y) \in [0, n]^2$ by the formula

$$b_{\hat{n}}\big(n; p(x), p(y), p(xy)\big) = \mathbf{P}(\hat{\xi} = \hat{n}) = \mathbf{P}(\xi_x = n_x,\ \xi_y = n_y) =$$

$$= \sum_{n(xy)=\max\{0, n_x+n_y-n\}}^{\min\{n_x, n_y\}} m_{\check{n}}\big(n; p(\varnothing), p(x), p(y), p(xy)\big),$$

where

$$m_{\check{n}}\big(n; p(\varnothing), p(x), p(y), p(xy)\big) = \mathbf{P}(\check{\xi} = \check{n}) =$$

$$= \mathbf{P}\Big(\big(\xi(\varnothing), \xi(x), \xi(y), \xi(xy)\big) = \big(n(\varnothing), n(x), n(y), n(xy)\big)\Big) =$$

$$= \frac{n!}{n(\varnothing)! n(x)! n(y)! n(xy)!} [p(\varnothing)]^{n(\varnothing)} [p(x)]^{n(x)} [p(y)]^{n(y)} [p(xy)]^{n(xy)}$$

[4]Obviously, $p(\varnothing) = 1 - p(x) - p(y) - p(xy)$. Hereinafter are used next denominations: $p_x = \mathbf{P}(x) = p(x) + p(xy)$, $p_y = \mathbf{P}(y) = p(y) + p(xy)$, $\mathrm{Kov}_{xy} = p(xy) - p_x p_y$, $\sigma_x^2 = p_x(1 - p_x)$, $\sigma_y^2 = p_y(1 - p_y)$.

are probabilities of the Polynomial 4-variate distribution of random vector $\check{\xi} = \big(\xi(\varnothing), \xi(x), \xi(y), \xi(xy)\big)$ with parameters $\big(n; p(\varnothing), p(x), p(y), p(xy)\big)$; summation is made by all sets $\check{n} = \big(n(\varnothing), n(x), n(y), n(xy)\big)$ such as $n = n(\varnothing) + n(x) + n(y) + n(xy)$, for which are true 2 equations as well $n_x = n(x) + n(x,y)$, $n_y = n(y) + n(x,y)$, and can be turned into summation by the once parameter $n(x,y)$ within Frechet bounds, since when n_x and n_y are fixed, then all quantities $n(\varnothing), n(x), n(y), n(xy)$ can be expressed by one parameter, for instance, $n(xy)$:

$$n(x) = n_x - n(xy),\ n(y) = n_y - n(xy),\ n(\varnothing) = n - n_x - n_y + n(xy),$$

which is varying within Frechet bounds:

$$\max\{0, n_x + n_y - n\} \leqslant n(xy) \leqslant \min\{n_x, n_y\}.$$

Also, formula can be written in the next manner:

$$b_{\hat{n}}\big(n; p(x), p(y), p(xy)\big) = \mathbf{P}(\hat{\xi} = \hat{n}) =$$

$$= [p(\varnothing)]^n \, [\tau(x)]^{n_x} \, [\tau(y)]^{n_y} \sum_{n(xy)=\max\{0, n_x+n_y-n\}}^{\min\{n_x,n_y\}} \mathcal{C}_n^{n(x,y)}(\hat{n}) \, [\tau(x,y)]^{n(xy)} \,,$$

where

$$\mathcal{C}_n^{n(x,y)}(\hat{n}) = \frac{n!}{(n - n_x - n_y + n(xy))!(n_x - n(xy))!(n_y - n(xy))!n(xy)!}$$

is *two-variate Binomial coefficient*, and

$$\tau(x) = \frac{p(x)}{p(\varnothing)},\ \tau(y) = \frac{p(y)}{p(\varnothing)},\ \tau(x,y) = \frac{p(\varnothing)p(xy)}{p(x)p(y)}$$

is first and second degree *multicovariations* of events x and y.

Vector of mathematical expectations for Binomial two-variate random vector (ξ_x, ξ_y) is equal to $(\mathbf{E}\xi_x, \mathbf{E}\xi_x) = (np_x, np_y)$, and its covariation matrix can be expressed through covariation matrix of random vector $(\mathbf{1}_x, \mathbf{1}_y)$ of indicators of events from the *generating set* $\mathfrak{X} = \{x, y\}$ and is equal to

$$\begin{pmatrix} np_x(1-p_x) & n\mathrm{Kov}_{xy} \\ n\mathrm{Kov}_{xy} & np_y(1-p_y) \end{pmatrix} = n \begin{pmatrix} p_x(1-p_x) & \mathrm{Kov}_{xy} \\ \mathrm{Kov}_{xy} & p_y(1-p_y) \end{pmatrix}$$

Covariation matrix of the centered and normalized Binomial two-variate random vector

$$\left(\frac{\xi_x - np_x}{\sigma_x}, \frac{\xi_y - np_y}{\sigma_y} \right)$$

is expressed through covariation matrix of the random vector

$$\left(\frac{\mathbf{1}_x - p_x}{\sigma_x}, \frac{\mathbf{1}_y - p_y}{\sigma_y} \right)$$

centered and normalized indicators of events from $\mathfrak{X} = \{x, y\}$ and is equal to

$$\begin{pmatrix} n & n\rho_{xy} \\ n\rho_{xy} & n \end{pmatrix} = n \begin{pmatrix} 1 & \rho_{xy} \\ \rho_{xy} & 1 \end{pmatrix},$$

where $\rho_{xy} = \frac{\mathrm{Cov}_{xy}}{\sigma_x \sigma_y}$ is correlation coefficient for random quantities $\mathbf{1}_x$ and $\mathbf{1}_y$ (i.e. for indicators of events from $\mathfrak{X} = \{x, y\}$).

2.3 Characteristics of Binomial multivariate distribution

Vector of mathematical expectations for multivariate random vector $(\xi_x, x \in \mathfrak{X})$ is equal to $(\mathbf{E}\xi_x, x \in \mathfrak{X}) = (np_x, x \in \mathfrak{X})$, its covariation matrix is expressed through covariation matrix of random vector $(\mathbf{1}_x, x \in \mathfrak{X})$ of indicators of events from the *generating set* $\mathfrak{X}$ and is equal to

$$\begin{pmatrix} n\sigma_x^2 & \dots & n\mathrm{Kov}_{xy} \\ \dots & \dots & \dots \\ n\mathrm{Kov}_{xy} & \dots & n\sigma_y^2 \end{pmatrix} = n \begin{pmatrix} \sigma_x^2 & \dots & \mathrm{Kov}_{xy} \\ \dots & \dots & \dots \\ \mathrm{Kov}_{xy} & \dots & \sigma_y^2 \end{pmatrix},$$

where $\sigma_x^2 = p_x(1 - p_x)$, $\mathrm{Kov}_{xy} = -p_x p_y$ when $x \neq y$.

Covariation matrix of generated by partition centered and normalized Binomial multivariate random vector

$$\left(\frac{\xi_x - np_x}{\sigma_x}, x \in \mathfrak{X}\right)$$

is expressed through covariation matrix of random vector

$$\left(\frac{(\mathbf{1}_x - p_x)}{\sigma_x},\ x \in \mathfrak{X}\right)$$

of centered and normalized indicators of events from $\mathfrak{X}$ and is equal to

$$\begin{pmatrix} n\sigma_x^2 & \dots & n\rho_{xy} \\ \dots & \dots & \dots \\ n\rho_{xy} & \dots & n\sigma_y^2 \end{pmatrix} = n \begin{pmatrix} \sigma_x^2 & \dots & \rho_{xy} \\ \dots & \dots & \dots \\ \rho_{xy} & \dots & \sigma_y^2 \end{pmatrix},$$

where $\rho_{xy} = \frac{\mathrm{Cov}_{xy}}{\sigma_x \sigma_y} = -\frac{p_x p_y}{\sigma_x \sigma_y}$ is correlation coefficient of random quantities $\mathbf{1}_x$ and $\mathbf{1}_y$ (i.e. indicators of events from $\mathfrak{X}$).

2.4 Polynomial distribution is a particular case of Binomial multivariate distribution, when the latter is expressed by the partition of elementary events space

When the *generating N-set* $\mathfrak{X}$ consist of the events, which arising the partition $\Omega = \sum_{x \in \mathfrak{X}} x$, then *Binomial multivariate distribution* of a random vector $\hat{\xi} = (\xi_x, x \in \mathfrak{X})$ is defined by the N parameters[5] $(n; p_x, x \in \mathfrak{X})$, where $p_x = \mathbf{P}(x)$, $\sum_{x \in \mathfrak{X}} p_x = 1$, and it is Polynomial distribution with the given parameters.

Hence, probabilities of the *Binomial multivariate distribution, which is generated by the partition* Ω, is defined for every integer-valued vector $\hat{n} = (n_x, x \in \mathfrak{X})$ from the simplex $\mathcal{S}^N$ (because $\sum_{x \in \mathfrak{X}} n_x = n$) by the same formula as probabilities of corresponding Polynomial distribution

$$b_{\hat{n}}(n; p_x, x \in \mathfrak{X}) = \mathbf{P}(\hat{\xi} = \hat{n}) = \mathbf{P}(\xi_x = n_x, x \in \mathfrak{X}) = \frac{n!}{\prod\limits_{x \in \mathfrak{X}} n_x!} \prod_{x \in \mathfrak{X}} [p_x]^{n_x}.$$

[5]Since $\sum_{x \in \mathfrak{X}} p_x = 1$, there are only $N - 1$ independent probabilities among N.

2.5 Binomial N-variate distribution, which is generated by the set $\mathfrak{X}$, defines Polynomial 2^N-variate distribution, which is generated by the set of terrace-events $\{\mathrm{ter}(X),\ X \subseteq \mathfrak{X}\}$, but not visa versa

Multivariate (N-variate) Bernoulli testing scheme of n tests with the generating set of events $\mathfrak{X}$, which obey eventological distribution $\{p(X),\ X \subseteq \mathfrak{X}\}$, defines N random quantities

$$\xi_x(\omega) = \sum_{i=1}^{n} \mathbf{1}_{x^{(i)}}(\omega),\ x^{(i)} \in \mathfrak{X}^{(i)},\ x \in \mathfrak{X},$$

each of them has Binomial distribution with parameters $\big(n; p_x = \mathbf{P}(x)\big)$ and all together forms N-variate random vector $\hat{\xi} = (\xi_x,\ x \in \mathfrak{X})$, which is distributed by the *Binomial multivariate (N-variate) law* with 2^N parameters $\big(n; \{p(X), \varnothing \neq X \subseteq \mathfrak{X}\}\big)$, which contains amount of tests n and $2^N - 1$ probabilities from eventological distribution of the generating set of events $\mathfrak{X}$ (in other words, all 2^N probabilities $p(X)$ without $p(\varnothing)$).

The same Bernoulli multivariate testing scheme of n tests defines 2^N random quantities

$$\xi(X)(\omega) = \sum_{i=1}^{n} \mathbf{1}_{\mathrm{ter}(X^{(i)})}(\omega),\ X^{(i)} \subseteq \mathfrak{X}^{(i)},\ X \in \mathfrak{X},$$

each of them has Binomial distribution with parameters $\big(n; p(X) = \mathbf{P}(\mathrm{ter}(X))\big)$, and all together forms 2^N-variate random vector $\check{\xi} = \big(\xi(X),\ X \subseteq \mathfrak{X}\big)$, which is distributed by the *Polynomial multivariate (2^N-variate) law*, which is generated by the set of terrace-events $\{\mathrm{ter}(X),$
$X \subseteq \mathfrak{X}\}$ and which is defined by $2^N + 1$ parameters $(n; \{p(X),\ X \subseteq \mathfrak{X}\}$. Those parameters contains amount of tests n and all 2^N probabilities $p(X)$ from eventological distribution of the generating set $\mathfrak{X}$ of events.

Probabilities of the given *Binomial and Polynomial multivariate distributions* are bound for every N-variate collections of nonnegative numbers $\hat{n} = \{n_x, x \in \mathfrak{X}\} \in [0, n]^N$ by the formula $\mathbf{P}(\hat{\xi} = \hat{n}) = \sum_{\check{n}} \mathbf{P}(\check{\xi} = \check{n})$ where summation is made by the all 2^N-variate sets of nonnegative numbers $\check{n} = \big(n(X),\ X \subseteq \mathfrak{X}\big) \in \mathcal{S}^{2^N}$ for which $n = \sum_{X \subseteq \mathfrak{X}} n(X)$ and also $n_x = \sum_{x \in X} n(X),\ x \in \mathfrak{X}$.

Remark. For any Binomial N-variate, which is generated by the set of events $\mathfrak{X}$, there is *unique* Polynomial 2^N-variate distribution, which is generated by the set of terrace-events $\{\mathrm{ter}(X), X \subseteq \mathfrak{X}\}$. The contrary is not true, i.e. for arbitrary Polynomial 2^N-variate distribution, which is generated by the 2^N-set of events (those events form partition of Ω), there are, generally speaking, $(2^N)!$ Binomial N-variate distributions, which is generated by the N-sets of terrace-events $\mathfrak{X}$ (appreciably depends on the way of partition events' labelling as subsets $X \subseteq \mathfrak{X}$ and total amount of such ways is equal to $(2^N)!$).

3 Poisson multivariate distribution

Poisson multivariate distribution is a discrete distribution of probabilities of a random vector $\hat{\xi} = (\xi_x,\ x \in \mathfrak{X})$, which have values $\hat{n} = (n_x,\ x \in \mathfrak{X})$ with the probabilities

$$\mathbf{P}(\hat{\xi} = \hat{n}) = \mathbf{P}(\xi_x = n_x,\ x \in \mathfrak{X}) = \pi_{\hat{n}}\big(\lambda(X),\ \varnothing \neq X \subseteq \mathfrak{X}\big) =$$

$$= e^{-\lambda} \sum_{\check{n}} \prod_{X \neq \varnothing} \frac{[\lambda(X)]^{n(X)}}{n(X)!},$$

where summation is made by collections of such nonnegative integer-valued numbers $\check{n} = \big(n(X),$
$\varnothing \neq X \subseteq \mathfrak{X}\big)$, for which there are N equations $n_x = \sum_{x \in X} n(X),\ x \in \mathfrak{X}$, and $\{\lambda(X),\ \varnothing \neq X \subseteq \mathfrak{X}\}$ with parameters: $\lambda(X)$ is an average number of coming of the terrace-event

$$ter(X) = \bigcap_{x \in X} x \bigcap_{x \in X^c} x^c,$$

in other words, average number of coming of the all events from X and there are not events from X^c.

$$\lambda = \sum_{X \neq \varnothing} \lambda(X)$$

is an average number of coming at least one event from $\mathfrak{X}$, in other words, average number of coming of event $\bigcup_{x \in \mathfrak{X}} x$ (union of all events from $\mathfrak{X}$).

For example, when $\hat{n} = (0, \dots, 0)$ then

$$\mathbf{P}\big(\hat{\xi} = (0, \dots, 0)\big) = \mathbf{P}(\xi_x = 0,\ x \in \mathfrak{X}) = e^{-\lambda},$$

when $\hat{n} = (0, \dots, 0, n_x, 0, \dots, 0),\ x \in \mathfrak{X}$ then

$$\mathbf{P}\big(\hat{\xi} = (0, \dots, 0, n_x, 0, \dots, 0)\big) = \mathbf{P}(\xi_x = n_x,\ \xi_y = 0, y \neq x) = e^{-\lambda} \frac{[\lambda(x)]^{n(x)}}{n(x)!}.$$

If the vector $\hat{n}$ has one fixed component n_x and other items are arbitrary:

$$\hat{n} = (\cdot, \dots, \cdot, n_x, \cdot, \dots, \cdot),\ x \in \mathfrak{X},$$

then

$$\mathbf{P}\big(\hat{\xi} = (\cdot, \dots, \cdot, n_x, \cdot, \dots, \cdot)\big) = \mathbf{P}(\xi_x = n_x) = e^{-\lambda_x} \frac{[\lambda_x]^{n_x}}{n_x!}$$

is a formula of Poisson one-variate distribution with parameter λ_x of the random quantity ξ_x, where $\lambda_x = \sum_{x \in X} \lambda(X)$ is defined for each $x \in \mathfrak{X}$ by the parameter of the Poisson one-variate distribution.

3.1 Eventological interpretation

Let there are countable sequence of n independent stochastic experiments. In the result of experiment n can ensue or not events from $\mathfrak{X}$. Possibilities $p_x = \mathbf{P}(x)$ of events $x \in \mathfrak{X}$ are small, i.e. possibilities $p(X),\ \varnothing \neq X \subseteq \mathfrak{X}$ of generated by them terrace-events ter$(X),\ \varnothing \neq X \subseteq \mathfrak{X}$ are small too, and when $n \to \infty$ then $np(X) \to \lambda(X),\ \varnothing \neq X \subseteq \mathfrak{X}$. Then random vector

$$\hat{\xi} = (\xi_x,\ x \in \mathfrak{X}) = \sum_{n=1}^{\infty} \mathbf{1}_{x(n)},\ x \in \mathfrak{X}$$

obeys multivariate (N-variate) Poisson distribution with parameters $\{\lambda(X),\ \varnothing \neq X \subseteq \mathfrak{X}\}$.

Remark. It's incorrect to imagine that possibilities so tend to zero that only in n-th test $np(X) = \lambda(X), \quad X \subseteq \mathfrak{X}$. In truth, it's rather to believe that tending of possibilities to zero like that this equation is true *for all first n tests*. Thus, stochastic experiment consists in the *sequence of n-series of independent tests (series of n tests)*, and this equation is true for all tests from n-series. Then n-series defines Binomial multivariate (N-variate) distribution with parameters $\big(n; p(X), \varnothing \neq X \subseteq \mathfrak{X}\big)$, which by $n \to \infty$ tends to the Poisson multivariate (N-variate) with parameters $\big(\lambda(X), \varnothing \neq X \subseteq \mathfrak{X}\big)$.

3.2 Characteristics of the Poisson multivariate distribution

Vector of mathematical expectations of the Poisson multivariate distribution is $(\mathbf{E}\xi_x, \ x \in \mathfrak{X}) = (\lambda_x, \ x \in \mathfrak{X})$, where $\lambda_x = \sum_{x \in X} \lambda(X), \ x \in \mathfrak{X}$. Since $\mathrm{Cov}(\xi_x, \xi_y) = \lambda_{xy}$, where $\lambda_{xy} = \sum_{\{x,y\} \subseteq X} \lambda(X)$,
$\{x, y\} \subseteq \mathfrak{X}$, so covariance matrix is equal to

$$\begin{pmatrix} \lambda_x & \dots & \lambda_{xy} \\ \dots & \dots & \dots \\ \lambda_{xy} & \dots & \lambda_y \end{pmatrix}.$$

In the case of two dimensions, when $\mathfrak{X} = \{x, y\}$, summation is making by the one parameter $n(xy) = n(\{x, y\})$, which is changing in a Frechet bounds:

$$\mathbf{P}(\hat{\xi} = \hat{n}) = \mathbf{P}(\xi_x = n_x, \xi_y = n_y) = \pi_{\hat{n}}\big(\lambda(x), \lambda(y), \lambda(xy)\big) =$$

$$= e^{-\lambda} \sum_{n(xy)=0}^{\min\{n_x, n_y\}} \frac{[\lambda(x)]^{n(x)}}{n(x)!} \frac{[\lambda(y)]^{n(y)}}{n(y)!} \frac{[\lambda(xy)]^{n(xy)}}{n(xy)!},$$

where $n(x) = n_x - n(xy)$, $n(y) = n_y - n(xy)$, and $\lambda = \lambda(x) + \lambda(y) + \lambda(xy)$.

Vector of mathematical expectations Poisson two-variate distribution $(\mathbf{E}\xi_x, \ \mathbf{E}\xi_y) = (\lambda_x, \ \lambda_y)$, where $\lambda_x = \lambda(x) + \lambda(xy)$ and $\lambda_y = \lambda(y) + \lambda(xy)$, covariance matrix is equal to

$$\begin{pmatrix} \lambda_x & \lambda(xy) \\ \lambda(xy) & \lambda_y \end{pmatrix},$$

because in the case of two dimensions $\lambda_{xy} = \lambda(xy)$.

3.3 Poisson multivariate approximation

If amount of independent experiments n is large-scale and possibilities $p_x = \mathbf{P}(x)$ of events $x \in \mathfrak{X}$ is small (i.e. possibilities $p(X), \ \varnothing \neq X \subseteq \mathfrak{X}$ of generated by them terrace-events $\mathrm{ter}(X), \varnothing \neq X \subseteq \mathfrak{X}$ is small too) then for each collection of integer-valued numbers $\hat{n} = (n_x, \ x \in \mathfrak{X}) \in [0, n]^N$ Binomial possibilities is expressed in the rough by terms of the *Poisson multivariate distribution*:

$$b_{\hat{n}}\big(n; p(X), \ \varnothing \neq X \subseteq \mathfrak{X}\big) \approx e^{-n \sum_{X \neq \varnothing} p(X)} \sum \prod_{X \neq \varnothing} \frac{[np(X)]^{n(X)}}{n(X)!},$$

where summation is applied to such sets $\big(n(X), \varnothing \neq X \subseteq \mathfrak{X}\big)$, for which $n \geqslant \sum\limits_{X \subseteq \mathfrak{X}} n(X)$ and N equations $n_x = \sum\limits_{x \in X} n(X),\ x \in \mathfrak{X}$ are true.

In the case of two dimensions, when $\mathfrak{X} = \{x, y\}$, summation is making by the one parameter $n(xy) = n(\{x, y\})$, which is changing in so-called Frechet bounds:

$$b_{\hat{n}}\big(n; p(x), p(y), p(xy)\big) \approx$$

$$\approx e^{-n(p(x)+p(y)+p(xy))} \sum_{n(xy)=0}^{\min\{n_x, n_y\}} \frac{[np(x)]^{n(x)}}{n(x)!} \frac{[np(y)]^{n(y)}}{n(y)!} \frac{[np(xy)]^{n(xy)}}{n(xy)!},$$

where $n(x) = n_x - n(xy), n(y) = n_y - n(xy)$.

Poisson theorem (multivariate case). Let $p_x \to 0,\ x \in \mathfrak{X}$ when $n \to \infty$, and $np(X) \to \lambda(X)$ for all nonempty subsets $\varnothing \neq X \subseteq \mathfrak{X}$ as that. Then for any collection of integer-valued numbers $\hat{n} = (n_x, x \in \mathfrak{X}) \in [0, n]^N$ (when $n \to \infty$)

$$b_{\hat{n}}\big(n; p(X), \varnothing \neq X \subseteq \mathfrak{X}\big) \to \pi_{\hat{n}}\big(\lambda(X), \varnothing \neq X \subseteq \mathfrak{X}\big),$$

where

$$\pi_{\hat{n}}\big(\lambda(X), \varnothing \neq X \subseteq \mathfrak{X}\big) = e^{-\sum_{X \neq \varnothing} \lambda(X)} \sum_{\check{n}} \prod_{X \neq \varnothing} \frac{[\lambda(X)]^{n(X)}}{n(X)!},$$

is *Poisson multivariate possibility*, and summation is applied to such sets $\check{n}$, for which $n_x = \sum_{x \in X} n(X),\ x \in \mathfrak{X}$.

Proof. Because when n is large while $n \geqslant \sum_{X \subseteq \mathfrak{X}} n(X)$ is true for any fixed $n(X), X \subseteq \mathfrak{X}$, then summation in formulas of Binomial

$$b_{\hat{n}}\big(n; p(X), \varnothing \neq X \subseteq \mathfrak{X}\big) = \sum_{\check{n}} m_{\check{n}}\big(n; \{p(X), X \subseteq \mathfrak{X}\}\big)$$

and Poisson

$$\pi_{\hat{n}}\big(\lambda(X), \varnothing \neq X \subseteq \mathfrak{X}\big) = e^{-\sum_{X \neq \varnothing} \lambda(X)} \sum_{\check{n}} \prod_{X \neq \varnothing} \frac{[\lambda(X)]^{n(X)}}{n(X)!}$$

possibilities is applied to the same sets $\check{n}$, for which $n_x = \sum_{x \in X} n(X),\ x \in \mathfrak{X}$.

Now we show Poisson approximation for Polynomial possibilities

$$m_{\check{n}}\big(n; \{p(X), X \subseteq \mathfrak{X}\}\big) = \frac{n!}{\prod\limits_{X \subseteq \mathfrak{X}} n(X)!} \prod_{X \subseteq \mathfrak{X}} [p(X)]^{n(X)}. \tag{1}$$

It should be pointed out, that for any fixed $n(X), X \subseteq \mathfrak{X}$ and sufficiently large n there are follow equations:

$$\frac{m_{(n(\varnothing), n(Z), \{n(X), Z \neq X \subseteq \mathfrak{X}\})}\big(n; \{p(X), X \subseteq \mathfrak{X}\}\big)}{m_{(n(\varnothing)+1, n(Z)-1, \{n(X), Z \neq X \subseteq \mathfrak{X}\})}\big(n; \{p(X), X \subseteq \mathfrak{X}\}\big)} = \frac{p(Z)\big(n(\varnothing)+1\big)}{n(Z)p(\varnothing)},$$

where $Z \subseteq \mathfrak{X}$. By multiplying and dividing numerator and denominator by n and in consideration of $\frac{n(\varnothing)+1}{n} \approx 1$ and $p(\varnothing) \approx 1$, where $\approx$ signify approximate equality with precision up to n^{-1}, we obtain

$$\frac{p(Z)\big(n(\varnothing)+1\big)}{n(Z)p(\varnothing)}\frac{n}{n} = \frac{np(Z)}{n(Z)} \cdot \frac{n(\varnothing)+1}{n} \cdot \frac{1}{p(\varnothing)} \approx \frac{np(Z)}{n(Z)}.$$

By the data of the theorem $np(Z) \to \lambda(Z)$, therefore

$$\frac{m_{(n(\varnothing),n(Z),\{n(X),Z\neq X\subseteq\mathfrak{X}\})}\big(n;\{p(X),X\subseteq\mathfrak{X}\}\big)}{m_{(n(\varnothing)+1,n(Z)-1,\{n(X),Z\neq X\subseteq\mathfrak{X}\})}\big(n;\{p(X),X\subseteq\mathfrak{X}\}\big)} \approx \frac{\lambda(Z)}{n(Z)} \tag{2}$$

When $n(X) = 0, \varnothing \neq X \subseteq \mathfrak{X}$, then

$$m_{(n(\varnothing),0,\dots,0)}\big(n;\{p(X),X\subseteq\mathfrak{X}\}\big) = [p(\varnothing)]^n = \left(1 - \sum_{X\subseteq\mathfrak{X}} \frac{\lambda(X)}{n}\right)^n = \left(1-\frac{\lambda}{n}\right)^n,$$

where $\lambda = \sum_{X\subseteq\mathfrak{X}} \lambda(X)$. After finding the logarithm of both parts of the equation and factorizing into Maclaurin series[6] we have

$$ln\big[m_{(n(\varnothing),0,\dots,0)}\big(n;\{p(X),X\subseteq\mathfrak{X}\}\big)\big] = n \cdot ln\left(1-\frac{\lambda}{n}\right) = -\lambda - \frac{\lambda^2}{2n} - \dots$$

When n is large we conclude that

$$m_{(n(\varnothing),0,\dots,0)}\big(n;\{p(X),X\subseteq\mathfrak{X}\}\big) \approx e^{-\lambda}. \tag{3}$$

By the sequentially applying equation (2) to approximation (3) we come to

$$m_{(n(\varnothing),\{n(X),X\subseteq\mathfrak{X}\})}\big(n;\{p(X),X\subseteq\mathfrak{X}\}\big) \approx e^{-\lambda} \prod_{\varnothing\neq X\subseteq\mathfrak{X}} \frac{[\lambda(X)]^{n(X)}}{n(X)!},$$

i.e. Poisson approximation of the Polynomial possibility (1), from where the assertion of the theorem follows directly.

4 Conclusion

The multivariate generalizations of Binomial and Poisson distribution offered in the paper allow to consider any structure of dependences of the fixed set of events in sequence of independent multivariate tests and include as very special variant for the Polynomial distribution long time known in probability theory. Offered multivariate discrete distributions pay increase in set of parameters for this unique opportunity.

[6] $ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$

References

[1] Vorobyev O.Yu. *Eventology.* — Krasnoyarsk. — Siberian Federal University. — 2007. — 435p.; ISBN 978-5-7638-0741-7

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