Least squares estimation of joint production functions by the Differential Evolution method of global optimization

Mishra, SK

North-Eastern Hill University, Shillong (India)

11 September 2007

Least Squares Estimation of Joint Production Functions by the Differential Evolution method of Global Optimization

SK Mishra
Department of Economics
North-Eastern Hill University
Shillong, Meghalaya (India)
E-mail: mishrasknehu@yahoo.com

Introduction: Although only occasionally dealt with in economic analysis and empirical econometrics, enterprises producing multiple or joint output with some common inputs are the most frequent empirical reality (Pfouts, 1961). Beginning with farming wherein not only wheat but also straw and chaff (that may have some market value) partake of the same inputs, animal husbandry wherein sheep yield wool and meat, manufacturing and service establishments, more often than not, produce a range of commodities applying the inputs most shared by all the products in unknown proportions. Then, estimation of a single production function of the leading product (or some weighted combination of different products) is subject to unknown type of biases intractably affecting efficiency, substitution and scale parameters and the subsequent economic implications thereof.

The economics of joint production often distinguishes between the two cases: the one in which a firm produces multiple products each produced under separate production process rarely using common variable inputs (but often using common fixed inputs or infrastructure), and the other "true joint production" where a number of outputs are produced from a single production process. In the second case all outputs of the process necessarily share all the common inputs without a clue to the share of any input allocated to different products. In the econometric practice the first case has often been dealt with by aggregation of individual production functions (for each product) into a gross or macro production function. The second case has often called for estimation of an implicit aggregate production function.

A Brief Literature Review: Econometric analysis of joint production perhaps dates back to the work of Klein (1947). Since then a number of studies have been carried out that deal with this topic. In particular, studies in agricultural economics have addressed this problem more frequently (see Chizmar and Zak, 1983; Just et al., 1983; Mundlak, 1963; Mundlak and Razin, 1971; Weaver, 1983). Methodologically those studies may be classified under four heads: those formulating process analysis models; those formulating simultaneous equations systems; those formulating composite macro function; those formulating composite implicit macro function. Some important works are briefly reviewed as follows.

Since the early work of Manne (1958) process analysis has amply exhibited its ability to deal with the economics of joint products. However, it requires a large database and solving large programming models. Further, it precludes the calculation of price and substitution elasticities that may have important policy implications. Griffin (1977) used a method similar to process analysis supplementing it with pseudo data to ascertain appropriate types of production frontier functions for different joint products of petroleum refinery. A pseudo data point shows the optimal input and output quantities corresponding to a vector of input and output prices. By repetitive solution of the process model for alternative price vectors, the shape of the production possibility frontier may be determined. However, as pointed out by Griffin himself, the efficiency of pseudo data approach to estimation of joint production functions ultimately rests on the quality of the engineering process model often difficult for an economist to build or evaluate. Even then, this approach does not rule out the possibilities of aggregation bias completely.

Just et al. (1983) formulated and estimated their multicrop production functions as a system of nonlinear simultaneous equation model. The methods of estimation were nonlinear two-stage and three-stage least squares. Chizmar and Zak (1983) discussed the appropriateness of simultaneous equation modeling of multiple products raised or manufactured simultaneously. However, they held that in case of joint products the implicit form single equation modeling would be appropriate.

Mundlak (1963) approached estimation of joint production function through aggregation. His method lies in specifying the individual micro production function for each (joint) product as well as the manner of aggregating them to an analogous macro production function. The macro production function is then estimated and its relationship with the micro production functions is investigated. However, the possibilities of establishing the relationship among the macro and micro production functions depend on availability of information on allocation of inputs used for different (joint) products. Mundlak also proposed formulation and estimation of a general implicit production function. This led to his further work (Mundlak, 1964) in which he formulated the problem of estimation of multiple/joint production functions as an exercise in estimation of an implicit function. If X are inputs and Y are output then the implicit function $g(f(X)-\varphi(Y))=0$ is expressed in terms of the composite input function $f(X)$ and the composite output function $\varphi(Y)$. Mundlak illustrated his approach by the transcendental specification (proposed by Halter, et al., 1957) of the composite functions $f(X)=a_0x_1^{a_1}x_2^{a_2}\exp(b_1x_1+b_2x_2)$, $f(Y)=y_1^{c_1}y_2^{c_2}\exp(d_1y_1+d_2y_2)$ and the simple implicit function $g(X,Y)=f(X)-\varphi(Y)=0$. It may be noted, however, that generally output is considered to contain errors due to specification of $f(X)$ such that any output vector $y_k=f(X)+u_k$ but inputs are considered non-stochastic. This consideration would lead to the specification $g(f(X)-\varphi(\hat{Y}))=\varepsilon$ where ε is the disturbance term. The least squares estimation of such functions has remained problematic. Mundlak and Razin (1971) also was basically an attempt to aggregation of micro functions to macro function.

Vinod (1968) addressed the problem of estimation of joint production function by Hotelling's canonical correlation analysis (Hotelling, 1936; Kendall and Stuart, 1968). Later he improved his method to take care of the estimation problem if the data on output (of different products) or inputs were collinear (Vinod, 1976). His method summarily lies, first, in transforming the input vectors (X) and the output vectors (Y) into two composite (weighted linear aggregate) vectors, $U=Xw$ and $V=Y\omega$ respectively where the weights, w and ω, are (mathematically derived) such as to maximize the squared (simple product moment) coefficient of correlation between U and V, and then transforming U and V back into X and Y respectively. He showed that the back transformation of the composite vectors U and V into X and Y poses no problem when the number of inputs is equal to or larger than the number of output. However, when that is not the case, one has to resort to some sort of least squares estimation (resulting from his suggested use of the least squares generalized inverse in the transformation process).

There were strong reactions to Vinod's method of estimation of joint production functions [Chetty (1969), Dhrymes and Mitchell (1969), Rao (1969)]. Rao pointed out that to be economically meaningful the production function must be convex and the transformation curve concave. However, the method proposed by Vinod did not yield composite output function (transformation function) that satisfied these requirements. Dhrymes and Mitchell (much like Chetty) pointed out that Vinod's formuation was partly erroneous and partly a "very complicated way of performing ordinary least squares." If the ordinary least squares method applied to estimate each production function separately and independently (ignoring the fact that they relate to joint products) were inconsistent then so would be the canonical correlation method. While acceding to the errors pointed out by the critics, in his reply Vinod (1969) disagreed on the inconsistency issue shown to exist in his method and argued that the critics (Dhrymes and Mtchell) had to establish the necessity and would not merely put up some particular cases thereof. It is interesting, however, to note that Vinod undermined the role of a single counterexample in demolishing the mathematical property of a method.

Apart from the problems pointed out above, Vinod's method cannot be useful when production functions are intrinsically nonlinear such that it is not possible to transform them (by some simple procedure such as log-linearization, etc) into linear equations. Secondly, it may not be correct to form the composite output function in Vinod's manner. Thirdly, it is not necessary that the specification of production functions is identical for all products. It is possible that while one of the products follows the CES, another follows the nested CES (Sato, 1967) and yet another follows the Diewert (1971) or any other specification.

The Present Study: Most of the studies relating to estimation of joint production functions have noted two difficulties: first that allocation of inputs to different outputs are not known, and the second that a method of estimation (such as the Least Squares) cannot have more than one dependent variable (output vector). Construction of a composite (macro) output function is at least partly motivated by the inability of the estimation methods to deal with multiple dependent variables and different forms of production function for different outputs. Of course, substitutability among different joint products also has been a motivating factor. We propose to deal with some of these problems here.

Our study is not based on any empirical data obtained from the real world. We have formulated some models and generated data accordingly. These data, with or without the random disturbances added to them, have been subjected to the joint Least Squares estimation of production functions directly. Since all the models formulated by us involve intrinsically nonlinear production functions, we have obtained the estimated parameters by nonlinear least squares method. To obtain the solutions we have used an algorithm based on the Differential Evolution (DE) method of global optimization (Mishra, 2007). The DE method is a population based stochastic search method that gives us immense flexibility to specify an optimand function as we desire.

The Results: In this study we have experimented with two models. The findings of our experiments are reported as follows:

Model I: In Model-I we specify two CES type production functions sharing the common inputs in unknown proportions.

$$PF_1 = A_1\left[\delta_1(\lambda_1 K)^{-\beta_1} + (1-\delta_1)\ (\mu_1 L)^{-\beta_1}\right]^{-(\rho_1/\beta_1)} + u_1$$

$$PF_2 = A_2\left[\delta_2(\lambda_2 K)^{-\beta_2} + (1-\delta_2)\ (\mu_2 L)^{-\beta_2}\right]^{-(\rho_2/\beta_2)} + u_2$$

$$\lambda_1 + \lambda_2 = 1;\ \mu_1 + \mu_2 = 1;\ A_j > 0;\ 0 < \delta_j < 1;\ \beta_j \geq -1;\ \rho_j > 0;\ j = 1,2$$

Here A_j is the scale parameter, δ_j is the distribution parameter, β_j is the substitution (among inputs) parameter and ρ_j is the returns to scale parameter. Additionally λ_1 and λ_2 are the allocation parameters of capital (K) and μ_1 and μ_2 are the allocation parameters of labour (L) between the two production functions PF_1 (product # 1) and PF_2 (product # 2). The allocation parameters are unobservable and over the two production functions they sum up to unity.

We have generated one hundred points of (K, L, PF_1, PF_2) using the parameters as stated in Table-I.1-A. We will call them the true parameters. First, we have not added any disturbances (u_1 and u_2) to the outputs. Estimation of the parameters has been done jointly. The true and the estimated parameters are reported in Table-I.1-A and the measures of the goodness of fit in Table-I.1-B.

Table-I.1-A: True and Estimated Parameters of Model-I (Without Disturbances)

Parameters		A	δ	λ	β	$1-\delta$	μ	ρ
PF_1	True	750.00	0.40	0.60	0.80	0.60	0.30	1.00
	Estd	748.458562	0.260106954	0.351082944	0.800000012	0.739893046	0.390649162	1.000000000
PF_2	True	820.00	0.30	0.40	0.02	0.70	0.70	1.10
	Estd	777.068583	0.302621116	0.648917056	0.020000000	0.697378884	0.609350838	1.100000024

Table-I.1-B: Goodness of Fit of Model-I (Without Disturbances)

Goodness of Fit	R^2	S^2	RMS
PF_1	0.9999999999999960	0.0000223310014460	0.0004725568901860
PF_2	0.9999999999999990	0.0000221024877190	0.0004701328293060
Joint PF_1 and PF_2	0.9999999999999990	0.0000444334891650	0.0004713464180674

Then we have added disturbances to the outputs. We have two sets of disturbances – the one of $u_1 \sim N(0, 130)$, $u_2 \sim N(0, 800)$ and the other of $u_1 \sim N(0, 1200)$, $u_2 \sim N(0, 1000)$. We have truncated the value of output to zero in case adding of disturbances cause them to take on a negative value. Only one instance of the 100 cases was found negative in each set. The results of estimation are presented in Tables I.2-A I.2-B, I.3-A and I.3-B. Fig.-I.2 presents the results graphically.

Fig.-I.1: Model-I (Without Disturbances) - True and Estimated Output

True (Generated) Values	Estimated Values	True Vs. Estimated Values

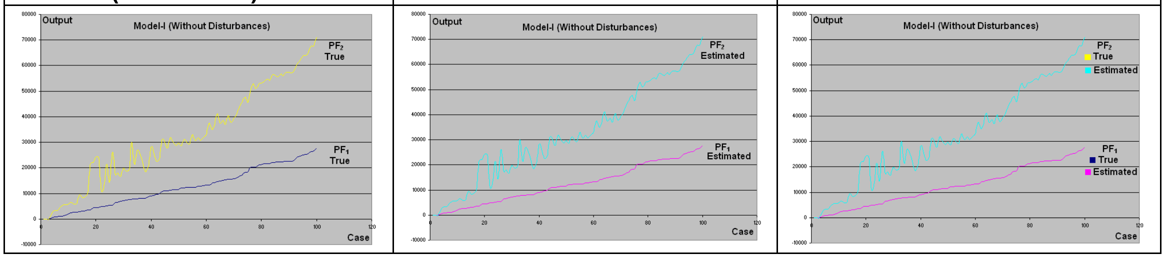


Table-I.2-A: Generated and Estimated Parameters of Model-I (With Disturbances)								
Parameters		A	δ	λ	β	$1-\delta$	μ	ρ
PF_1	True	750.00	0.40	0.60	0.80	0.60	0.30	1.00
	Estd	715.706313	0.27009126	0.37314968	0.77736639	0.72990874	0.39804099	1.00401693
PF_2	True	820.00	0.30	0.40	0.02	0.70	0.70	1.10
	Estd	794.960957	0.30241080	0.62685032	0.02053650	0.69758920	0.60195901	1.09944138

Table-I.2-B: Goodness of Fit of Model-I (With Disturbances)			
Goodness of Fit	R^2	S^2	RMS
PF_1	0.999998479	9082.70446	9.53032237
PF_2	0.999999813	6594.36565	8.12056996
Joint PF_1 and PF_2	0.999999620	15677.0701	8.85355016

Fig.-I.2: Model-I (With Disturbances) - Generated and Estimated Output

Generated with **u_1~ N(0, 130); u_2~ N(0, 80)**	**Generated with u_1~ N(0, 1200); u_2~ N(0, 1000)**

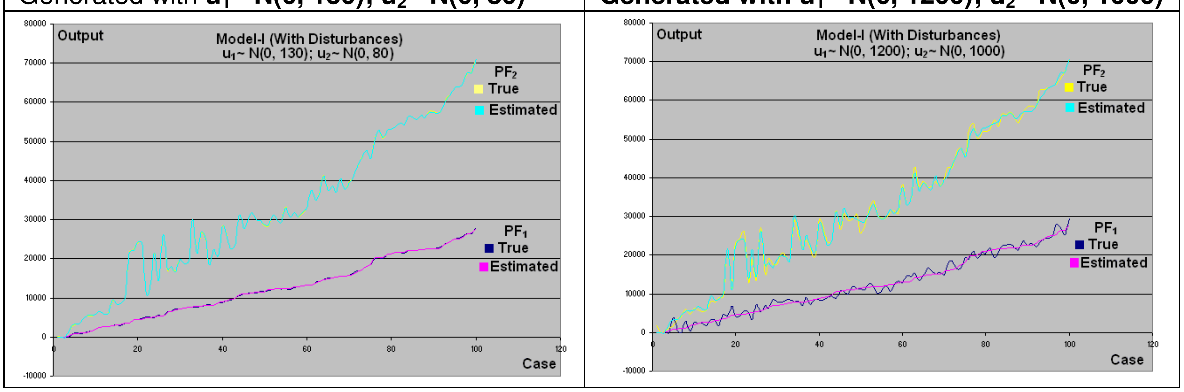


Table-I.3-A: Generated and Estimated Parameters of Model-I (With Disturbances)								
Parameters		A	δ	λ	β	$1-\delta$	μ	ρ
PF_1	True	750.00	0.40	0.60	0.80	0.60	0.30	1.00
	Estd	630.59953000	0.34837619	0.49053852	0.61077262	0.65162381	0.35196561	1.03701632
PF_2	True	820.00	0.30	0.40	0.02	0.70	0.70	1.10
	Estd	821.92067500	0.29948977	0.50946148	0.02674686	0.70051023	0.64803439	1.09301290

Table-I.3-B: Goodness of Fit of Model-I (With Disturbances)			
Goodness of Fit	R^2	S^2	RMS
PF_1	0.99998633	80924.810319	28.447286
PF_2	0.99999768	82432.573366	28.711073
Joint PF_1 and PF_2	0.99999606	163357.383685	28.579484

Some observations are worth noting. The values of R^2 are extremely high (near unity) in each case. The estimated values of substitution and returns to scale parameters (β and ρ) are very close to the true values. The value of efficiency parameter (A) also is not much different from the true one. However, the values of distribution and allocation parameters (δ, λ and μ) are not close to the (respective) true values. They satisfy the constraints ($\lambda_1 + \lambda_2 = 1;\ \mu_1 + \mu_2 = 1$) in each case. It appears, therefore, that many possible allocations of labour and capital (across the two production functions) yield the same output of each product. Invoking the economic argument one may say that unless price relatives of inputs and outputs are provided, it may not be possibly to determine the allocation of inputs across the production functions.

Let us now compare the results of our joint estimation with those obtained when we estimated the two production functions severally. In this formulation there are no allocation parameters. The output-1 as well as the output-2 is attributed to the entire quantities of inputs, capital and labour. The production functions are now the usual CES given as

$$PF_1 = A_1\left[\delta_1(K)^{-\beta_1} + (1-\delta_1)\ (L)^{-\beta_1}\right]^{-(\rho_1/\beta_1)} + u_1$$

$$PF_2 = A_2\left[\delta_2(K)^{-\beta_2} + (1-\delta_2)\ (L)^{-\beta_2}\right]^{-(\rho_2/\beta_2)} + u_2$$

$$A_j > 0;\ 0 < \delta_j < 1;\ \beta_j \geq -1;\ \rho_j > 0;\ j = 1, 2$$

Table-I.4: True and Estimated Parameters of Model-I (Without Disturbances)								
Parameters		A	δ	β	$1-\delta$	ρ	R^2	S^2
PF_1	True	750.00	0.40	0.80	0.60	1.00		
	Estd	284.1222	0.276882	0.800000	0.723118	1.000000	1.000	7.52E-12
PF_2	True	820.00	0.30	0.02	0.70	1.10		
	Estd	460.1553	0.302356	0.020000	0.697644	1.100000	1.000	7.6E-12

In Table-I.4 we have presented the results of CES fitting to the generated data. No disturbances have been added to outputs. The results reveal that the estimated values of substitution and returns to scale parameters (β and ρ) for both the functions are the true values, but the values of distribution parameters (δ) are quite at variance with the true values. They are not much different from what were obtained in joint estimation (ref Table-I.1-A). However, the efficiency parameters in both the functions are much below the true as well as the estimated values obtained by joint estimation (ref. Table-I.1-A). That is to say that the consequences of independent estimation of joint production function are critical for the efficiency parameters.

Table-I.5: True and Estimated Parameters of Model-I (With Disturbances)								
Parameters		A	δ	β	$1-\delta$	ρ	R^2	S^2
PF_1	True	750.00	0.40	0.80	0.60	1.00		
	Estd	245.3086	0.307911	0.615726	0.692089	1.032982	0.982207	105359560.62
PF_2	True	820.00	0.30	0.02	0.70	1.10		
	Estd	442.4014	0.303216	0.005898	0.696784	1.108731	0.997106	102748415.90

The consequences of independent estimation in presence of disturbances are presented in Table-I.5 [for u_1~ N(0, 1200); u_2~ N(0, 1000)]. The value of substitution parameter in PF_2 is much under-estimated (vis-à-vis the true value as well as the one obtained by joint estimation as reported in Table-I.3-A). The efficiency parameters remain subdued as in the case when disturbances were not added to the output data. The values of R^2 are lower than those obtained by joint estimation (ref. Table-I.3-A). Thus, overall, the results of joint estimation are better than those estimated severally (independently) in presence as well as absence of disturbances.

It may be noted that there exists some sort of collinearity, so to say, between the inputs (K and L) as well as between the outputs. Due to this collinearity (between inputs), the estimated distribution parameters are away from their true values irrespective of the method of estimation (joint or independent). Similarly, collinearity between the outputs has affected the estimation of allocation parameters. It appears, therefore, that our inability to obtain the estimated values of allocation and distribution parameters very close to their true values is due to the nature of data used and not due to the method of estimation. The joint estimation has given better R^2 and efficiency parameters.

Model II: In Model-II we specify a CES type production function for product-1 and a transcendental function of Halter, Carter and Hocking (HCH) for product-2 sharing the common inputs in unknown proportions.

$$PF_1 = A_1\left[\delta(\lambda_1 K)^{-\beta} + (1-\delta)\ (\mu_1 L)^{-\beta}\right]^{-(\rho/\beta)} +\ u_1$$

$$PF_2 = A_2[(\lambda_2 K)^{\alpha_1}(\mu_2 L)^{\alpha_2}]\exp(\gamma_1\lambda_2 K + \gamma_2\mu_2 L)\ +\ u_2$$

As it was specified before, the allocation parameters over the products sum up to unity.

We have generated one hundred points of (K, L, PF_1, PF_2) using the parameters as stated in Table-II.1. First, we have not added any disturbances (u_1 and u_2) to the outputs. Estimation of the parameters has been done jointly. The true and the estimated parameters as well as S^2 are reported in Table-II.1. For both functions we have $R^2 = 1.0$.

Table-II.1: Generated and Estimated Parameters of Model-II (Without Disturbances)									
Parameters		A_1	δ	λ_1	β	$1-\delta$	μ_1	ρ	S^2
PF_1	True	750.00	0.40	0.60	0.80	0.60	0.30	1.00	
	Estd	468.45836	0.333033	0.763968	0.800000	0.666967	0.548218	1.000000	3.99E-07
Parameters		A_2	α_1	λ_2	α_2	γ_1	μ_2	γ_2	S^2
PF_2	True	8.00	0.30	0.40	0.50	0.10	0.70	0.05	
	Estd	11.66545	0.300000	0.236032	0.500000	0.169469	0.451782	0.077471	1.67E-06

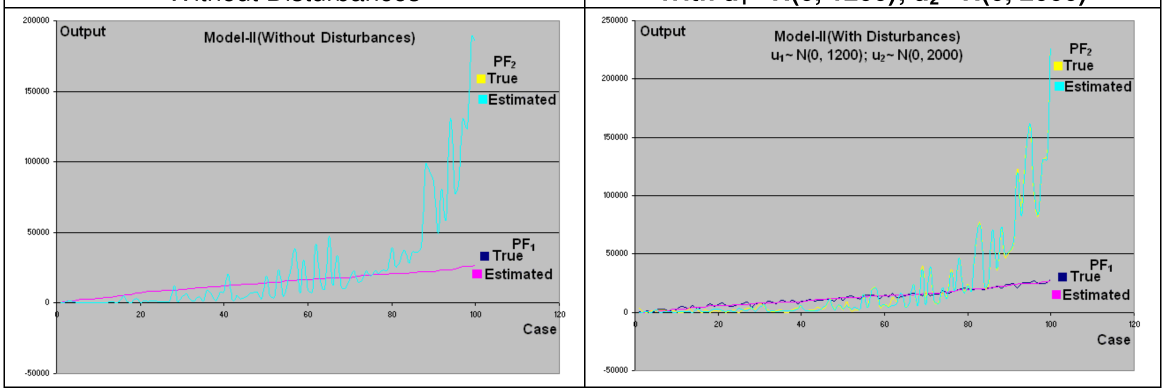
Then disturbances, $u_1 \sim N(0, 1200)$; $u_2 \sim N(0, 2000)$, were added to the outputs. If adding of disturbance led to a negative value of output, it was truncated to zero. The estimated parameters are presented in Table-II.2. Even with large disturbances the values of R^2 for both functions are very high. The values of S^2 for PF_1 and PF_2 are 1.33E08 and 3.09E08 respectively.

Table-II.2: Generated and Estimated Parameters of Model-II (With Disturbances)

Parameters		A_1	δ	λ_1	β	$1-\delta$	μ_1	ρ	R^2
PF_1	True	750.00	0.40	0.60	0.80	0.60	0.30	1.00	
	Estd	554.0777	0.283529	0.536040	0.942917	0.716471	0.502446	0.999876	0.974275
Parameters		A_2	α_1	λ_2	α_2	γ_1	μ_2	γ_2	R^2
PF_2	True	8.00	0.30	0.40	0.50	0.10	0.70	0.05	
	Estd	11.1687	0.260989	0.463960	0.518698	0.085312	0.497554	0.068174	0.99808

Fig.-II.1: Model-II (With or Without Disturbances) - Generated and Estimated Output

Without Disturbances	With $u_1 \sim N(0, 1200)$; $u_2 \sim N(0, 2000)$



We observe that when disturbances were not added to outputs, we obtained R^2 = 1.0 such that the estimated values completely covered the true values of outputs (left panel of Fig.-II.1). The outputs (generated and estimated) with disturbances are presented in Fig.-II.1 (Right panel). Further, as it was found in case of Model-I, the estimated values of distribution parameter of the CES function and allocation parameters of the CES as well as transcendental function were not very close to their true values. However, the substitution and returns to scale parameters of CES were accurately estimated. The two estimated parameters of the exponential factor of the transcendental function (γ_1 and γ_2) also were not far away from their true values.

Table-II.3: Generated and Severally Estimated Parameters of Model-II (With Disturbances)

Parameters		A_1	δ	β	$1-\delta$	ρ	R^2
PF_1	True	750.00	0.40	0.80	0.60	1.00	
	Estd	283.4606	0.283529	0.942917	0.716471	0.999876	0.974275
Parameters		A_2	α_1	α_2	γ_1	γ_2	R^2
PF_2	True	8.00	0.30	0.50	0.10	0.05	
	Estd	6.3637	0.260989	0.518698	0.039581	0.033920	0.99808

We severally (independently) estimated both production functions in the Model-II (with disturbances $u_1 \sim N(0, 1200)$; $u_2 \sim N(0, 2000)$). The results are presented in Table-II.3. Except that the efficiency parameters of both the functions and the parameters in the exponential factor of the transcendental function are depressed, there are no differences in other estimated parameters (ref. Table-II.2).

Model III: In Model-III we specify a Nerlove-Ringstad (NR) type production function (Nerlove, 1963; Ringstad, 1967) for product-1 and a Halter-Carter-Hocking (HCH) transcendental function for product-2 sharing the common inputs.

$$PF_1 = y^{c_3 \ln(y)} = A_1(\lambda_1 K)^{c_1}(\mu_1 L)^{c_2} + u_2$$

$$PF_2 = A_2[(\lambda_2 K)^{\alpha_1}(\mu_2 L)^{\alpha_2}]\exp(\gamma_1\lambda_2 K + \gamma_2\mu_2 L) \ + \ u_2$$

Table-III.1: Generated and Estimated Parameters of Model-III (Without Disturbances)

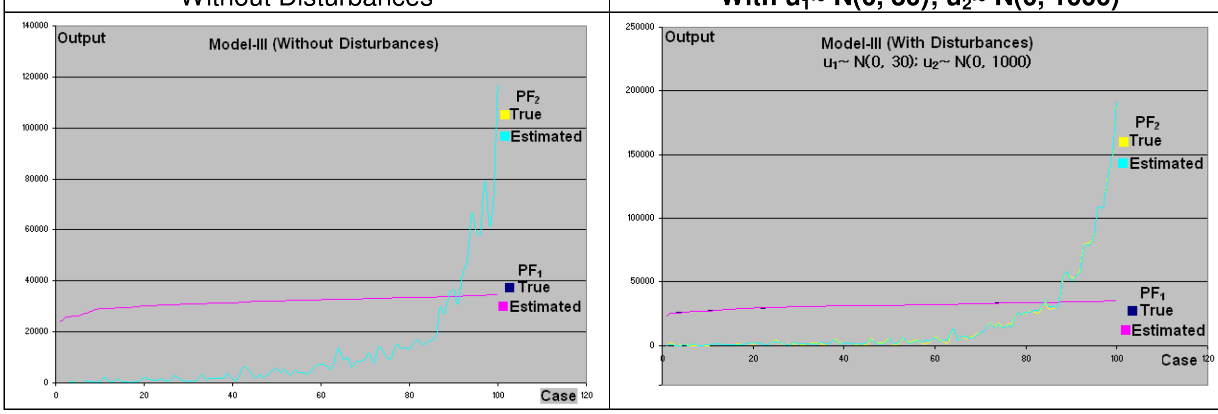
Parameters		A_1	c_1	λ_1	c_2	c_3	μ_1	--------	S^2
PF_1	True	40.00	0.40	0.60	0.50	0.30	0.30	----	
	Estd	39.9059	0.400062	0.477589	0.500077	0.300054	0.439330	-------	5.56E-06
Parameters		A_2	α_1	λ_2	α_2	γ_1	μ_2	γ_2	S^2
PF_2	True	5.00	0.30	0.40	0.50	0.10	0.70	0.05	
	Estd	5.1568	0.300000	0.522411	0.500000	0.076568	0.560670	0.062425	6.35E-06

Table-III.2: Generated and Estimated Parameters of Model-III (With Disturbances)

Parameters		A_1	c_1	λ_1	c_2	c_3	μ_1	--------	R^2
PF_1	True	40.00	0.40	0.60	0.50	0.30	0.30	----	
	Estd	29.2089	0.278818	0.627897	0.347726	0.194240	0.539557	-------	0.999889
Parameters		A_2	α_1	λ_2	α_2	γ_1	μ_2	γ_2	R^2
PF_2	True	5.00	0.30	0.40	0.50	0.10	0.70	0.05	
	Estd	10.7284	0.180765	0.372103	0.414727	0.111148	0.460443	0.077881	0.999356

Fig.-III.1: Model-III (With or Without Disturbances) - Generated and Estimated Output

Without Disturbances	With $u_1 \sim N(0, 30)$; $u_2 \sim N(0, 1000)$



We have generated one hundred points of (K, L, PF_1, PF_2) using the parameters as stated in Table-III.1. First, we have not added any disturbances (u_1 and u_2) to the outputs. Estimation of the parameters has been done jointly. The true and the estimated parameters as well as S^2 are reported in Table-III.1. Then we have added disturbances $u_1 \sim N(0, 30)$; $u_2 \sim N(0, 1000)$ and estimated the parameters jointly. The results are presented in Table-III.2.

We observe that when disturbances were not added to outputs the estimated parameters of the Nerlove-Ringstad (NR) as well the HCH function are very close to the true ones (R^2 = 1.0 for both functions). However, the estimated allocation parameters are not close to the true values. The value of λ_1 (use of capital by NR) is underestimated while λ_2 (use of capital by HCH) is overestimated. To compensate it, the allocation parameter of labour in NR (μ_1) is overestimated while that in HCH (μ_2) is underestimated.

When disturbances are added to the outputs we find that the functions are fitting well to the data (as the values of R^2 are almost equal to zero). However, the estimated values of parameters are generally far off from the true values. We have not gone in for computing the standard errors of estimates of the parameters. It is not possible, therefore, to state whether the estimated parameters are significantly different (in a statistical sense) from the true ones or not. However, the standard errors of estimates could be obtained by bootstrapping, which derives the sampling properties of empirical estimators using the sample data themselves (Efron and Tibshirani, 1993). To apply bootstrapping, repeated sub-samples are drawn from the given data set, parameters are estimated from the sub-samples and the standard errors of (the full sample parameters) are obtained from those estimated parameters (obtained from sub-samples).

Conclusion: The findings of our experiments here appear to be logical. Allocation parameters alter the scales of measurement of inputs. So, it should not affect the substitution parameter or the returns to scale parameter of a CES (Cobb-Douglas or Transcendental) production function. However, the change of scale should affect the efficiency parameter. In case of CES it may also affect the distribution parameter. This is what we observed in our experiments. However, we cannot generalize this conclusion over all functional specifications as to the extent and manner in which the estimated parameters would differ from the true ones, especially in presence of disturbances. Yet, instead of estimating joint production functions severally, it would be rewarding to estimate them jointly.

In this paper we have limited ourselves to experiments with the joint estimation of production functions by introducing into them the allocation parameters only. But these experiments do not limit the powers of our method of estimation that can jointly estimate several production functions with different specifications. In our scheme we do not need constructing composite functions. However, we have not experimented with production functions with some common but other product specific inputs. We have not experimented with common parameters across the production functions. These experiments are feasible and can be accomplished by the method proposed by us.

Note: FORTRAN Codes of the program may be obtained from the author on request (contact: mishrasknehu@yahoo.com).

References

- Chetty, VK (1969) "Econometrics of Joint Production: A Comment", *Econometrica*, 37(4), p. 731.
- Chizmar, JF and Zak, TA (1983) "Modeling Multiple Outputs in Educational Production Functions" , *The American Economic Review*, 73(2), pp. 18-22.
- Dhrymes, PJ and Mitchell, BM (1969) "Estimation of Joint Production Functions", *Econometrica*, 37(4), pp. 732-736.
- Diewert, W.E. (1971) "An Application of the Shepherd Duality Theorem: A Generalized Leontief Production Function", *The Journal of Political Economy*, 79(3), pp. 481-507.
- Efron, B and Tibshirani, R (1993) *An Introduction to the Bootstrap*, Chapman and Hall, New York.
- Griffin, JM (1977) "The Economics of Joint Production: Another Approach", *The Review of Economics and Statistics*, 59(4), pp. 389-397.
- Halter, AN, Carter, HO and Hocking, JG (1957) "A Note on the Transcendental Production Function", *Journal of Farm Economics*, 29, pp. 966-974.
- Hotelling, H (1936) "Relations Between Two Sets of Variates", *Biometrica*, 28, pp. 321-377.
- Just, RE, Zilberman, D and Hochman, E (1983) "Estimation of Multicrop Production Functions", *American Journal of Agricultural Economics*, 65(4), pp. 770-780.
- Kendall, MG and Stuart, A (1968) *The Advanced Theory of Statistics*, Vol. 3, Charles Griffin & Co. London.
- Klein, LR (1947) *The Use of Cross-Section Data in Econometrics with Application to a Study of Production of Railroad Services in the United States*, (Mimeographed) National Bureau of Economic Research, Washington, DC.
- Manne, AS (1958) "A Linear Programming Model of the U.S. Petroleum Refinery Industry", *Econometrica*, 26, pp. 67-106.
- Mishra, SK (2007) "Performance of Differential Evolution Method in Least Squares Fitting of Some Typical Nonlinear Curves", *SSRN* http://ssrn.com/abstract=1010508
- Mundlak, Y (1963) "Specification and Estimation of Multiproduct Production Functions" (mimeographed), paper read at the meetings of the Econometric Society and the American Farm Economic Association, Pittsburgh, USA (Dec. 1962), summarized in the *Journal of Farm Economics*, 45 pp. 433-443.
- Mundlak, Y (1964) "Transcendental Multiproduct Production Functions", *International Economic Review*, 5(3), pp. 273-284.
- Mundlak, Y and Razin, A (1971) "On Multistage Multiproduct Production functions", *American Journal of Agricultural Economics*, 53(3), pp. 491-499.
- Nerlove, M (1963) "Returns to Scale in Electricity Supply", in Christ, CF et al. (Eds) *Measurement in Econometrics: Studies in Mathematical Economics and Econometrics in Memory of Yehuda Grunfeld*, Stanford Univ. Press, Stanford.
- Pfouts, RW (1961) "The Theory of Cost and Production in the Multiproduct Firm", *Econometrica*, 39, pp. 65-68.
- Rao, P (1969) "A Note on Econometrics of Joint Production" , *Econometrica*, 37(4), pp. 737-38.
- Ringstad, V (1967) "Econometric Analysis based on a Production Function with Neutrally Variable Scale Elasticity", *Swedish Journal of Economics*, 69, pp. 115-133.
- Sato, K. (1967) "A Two-Level Constant-Elasticity-of-Substitution Production Function", *Review of Economic Studies,* 43, pp. 201-218.

- Vinod, HD (1968) "Econometrics of Joint Production", *Econometrica*, 36(2), pp. 322-336.
- Vinod, HD (1969) "Econometrics of Joint Production – A Reply", *Econometrica*, 37(4), pp. 739-740.
- Vinod, HD (1976) "Canonical Ridge and Econometrics of Joint Production", *Journal of Econometrics*, 4(2), pp. 147-166.
- Weaver, RD (1983) "Multiple Input, Multiple Output Production Choices and Technology in the U.S. Wheat Region", *American Journal of Agricultural Economics*, 65(1), pp. 45-56.