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Abstract. We characterize essential stability of Cournot-Nash equilibria for generalized games with a continuum of players. As application, we rationalize the active participation of politically engaged individuals as the unique essential equilibrium in an electoral game with a continuum of Cournot-Nash equilibria.

Keywords. Essential equilibria - Essential sets and components - Large games
JEL Classification: C62, C72, C02.

1. Introduction

In this study we focus on essential stability of Cournot-Nash equilibria for large generalized games, analyzing how equilibrium allocations change when some characteristics of the game are perturbed. We allow for any kind of perturbation, provided that it can be defined through a continuous parametrization over a complete metric space of parameters.

In large generalized games, strategy profiles may affect players' objective functions and admissible strategies. There is a continuum set of non-atomic players, and a finite number of atomic players. Atomic players' strategies may directly affect decisions of other individuals, while decisions of non-atomic players impact others participants only through aggregate information. Indeed, strategy profiles of non-atomic players are codified and aggregated, generating messages to other participants to the game. Under mild conditions on the characteristics of the generalized game, pure strategy Cournot-Nash equilibrium always exists (cf. Balder (1999, 2002), Riascos and Torres-Martínez (2012), Carmona and Podczeck (2013)).

In this context, it is natural to ask how equilibrium strategies of atomic players and equilibrium messages induced by decisions of non-atomic players—the pieces of information that fully determine the strategic behavior of players—change when the characteristics of the generalized game are perturbed. The approach is on essential stability: under which conditions Cournot-Nash equilibria of a large game can be approximated by equilibria of perturbed games.

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We depart our analysis of essential stability assuming that games are continuous[1] and any characteristic can be perturbed, i.e., objective functions, action sets or correspondences of admissible strategies. In this context, our first result ensures that for a dense residual subset of the space of generalized games, messages and atomic players' strategies associated to Cournot-Nash equilibria are stable to perturbations (Theorem 1).[2] Also, uniqueness of equilibrium messages and actions for atomic players is a sufficient condition for stability. We analyze stability of subsets of equilibrium messages and actions, obtaining results analogous to those ensured in the literature for convex games with finitely many players: every generalized game have essential subsets of Cournot-Nash equilibria (Theorem 2).

Stability results above are extended to allow for specific perturbations, that we capture through parametrizations of the set of generalized games. We prove that, if the set of parameters constitutes a complete metric space and the mapping associating parameters with large generalized games is continuous, then stability results previously described still holds (Theorem 3) and essential sets are stable too (Theorem 4). As byproduct, we obtain stability results for convex continuous generalized games with finitely many players, extending the literature to enable a great variety of admissible perturbations and dependence of admissible strategies on other players actions (cf. Wu and Jian (1962), Jiang (1963), Yu (1999), Yu, Yang and Xiang (2005), Zhou, Yu and Xiang (2007), Yu (2009)).

After that, we extend our analysis to allow for discontinuities on objective functions and correspondences of admissible strategies. To guarantee equilibrium existence, we follow the results of Carmona and Podczeck (2013), who recently generalize the model of Balder (2002) to the discontinuous case. We concentrate in the case of large generalized games with upper semicontinuous payoff functions and upper hemicontinuous correspondences of admissible strategies. In this context, we prove that the collection of generalized payoff secure large games (cf. Carmona and Podczeck (2013, Definition 4)) is a complete metric space. Thus, when perturbations on payoff functions can be captured through continuous parametrization defined on complete metric space of parameters, Cournot-Nash equilibria are generically essential and any large game have essential subsets of equilibria that are stable (Theorem 5). Since the model captures finite-player convex games as a particular case, our findings about stability of discontinuous games complements the previous results obtained by Yu(1999), Carbonell-Nicolau (2010), and Scalzo (2012).

To obtain our results about essential stability, regardless of whether the game is continuous or discontinuous, we prove that the compact-valued correspondence that associates generalized games with sets of equilibrium messages-actions, referred as Cournot-Nash correspondence, has closed graph. To guarantee this property, we use the fact that the set of non-atomic players has finite measure and their strategies are transformed into finite-dimensional codes (which are integrated to obtain messages). Indeed, under these conditions, we can ensure the closed graph property of the Cournot-Nash correspondence by applying the multidimensional Fatou's Lemma (cf. Hildenbrand (1974, Lemma 3, page 69)).

[1]That is, for every player, objective functions and correspondences of admissible strategies are continuous.

[2]A subset of a metric space is residual if it contains the intersection of a countable family of dense and open sets.

Essential stability of equilibrium messages and atomic player actions have relevant implications in applied game theory. Our results ensure that in models based in large generalized games, small errors in the estimation or calibration of some parameters does not necessarily affect player's decisions. In addition, essential stability can also be used as a refinement criteria in the presence of multiple equilibria. We illustrate this last possibility through an electoral games with the aim to give a rationale for electoral participation of politically engaged individuals.

The rest of the paper is organized as follows: Section 2 is devoted to discuss the related literature. In Section 3 we describe the space of large continuous generalized games. In Section 4 and 5 we analyze essential stability properties of Cournot-Nash equilibria. In Section 6 we apply our results to an electoral game. In Section 7 we extend our model to include discontinuous games. The proofs of our results are given in the Appendix.

2. Related Literature

The concept of essential stability has its origins in the mathematical analysis literature, where it was introduced as a natural property of fixed points of functions and correspondences. In a seminal paper, Fort (1950) introduces the concept of essential fixed point of a continuous function: a fixed point is essential if it can be approximated by fixed points of functions close to the original one. In addition, a continuous function is essential if it has only essential fixed points. Considering the set of continuous functions from a compact metric space to itself, Fort (1950) ensures that the set of essential functions is dense. He also proves that continuous functions that have only one fixed point are essential. These concepts and properties have natural extensions to multivalued mappings, as shown by Jiang (1962). However, not all mappings are essential and, therefore, it is natural to analyze the stability of subsets of fixed points. With this aim, Kinoshita (1952) introduces the concept of essential component of the set of fixed points of a function: a maximal connected set that is stable to perturbations on the characteristics of the function. He proves that any continuous mapping has at least one essential component. Jiang (1963) and Yu and Yang (2004) extend these results to multivalued mappings. They prove that compact-valued upper hemicontinuous correspondences have at least one essential component, although fixed points of these correspondences may not be essential. These results are complemented by Yu, Yang, and Xiang (2005) who also analyze how essential components change when mappings are perturbed.

This literature motivates the study of equilibrium stability in games. Since in every non-cooperative game the set of Nash equilibria coincides with the set of fixed points of the aggregate best response correspondence, techniques described above allow to analyze how the equilibria of a game change when payoffs and action sets are perturbed. In this direction, essential stability of Nash equilibria of games with finitely many players is studied by Wu and Jiang (1962), Yu (1999), Yu, Yang, and Xiang (2005), Zhou, Yu and Xiang (2007), Yu (2009), Carbonell-Nicolau (2010) and Scalzo (2012).

More precisely, Wu and Jiang (1962) address stability of the set of Nash equilibria for finite games. They ensure that any game can be approximated by a game whose equilibria are all essential. Yu (1999) formalizes and extends these results for convex games with a finite number of players and infinite many strategies, analyzing perturbations in payoffs, action sets and correspondences of admissible strategies. Jiang (1963), Yu, Yang, and Xiang (2005) and Yu (2009) analyze the existence of essential components of the set of Nash equilibria for games and generalized games. Zhou, Yu and Xiang (2007) study the notion of essential stability for mixed-strategy equilibria in games with compact sets of pure strategies and finitely many players. They also compare the concept of essential stability with strategic stability, a notion studied by Kohlberg and Mertens (1986), Hillas (1990), and Al-Najjar (1995). Allowing for discontinuities on objective functions, Yu (1999), Carbonell-Nicolau (2010) and Scalzo (2012) analyze the essential stability of Nash equilibria for games with finitely many players.

As we describe in the introduction, our goal is to contribute to this growing literature by addressing essential stability properties of Cournot-Nash equilibria in large generalized games. However, results of essential stability for games with finitely many players take advantage of the fact that the equilibrium correspondence[3] is closed, with non-empty and compact values. Actually, with these properties, the equilibrium correspondence is generically lower hemicontinuous, which in turn implies generic stability. In our case, under mild conditions on the characteristics of the generalized game, a pure strategy Cournot-Nash equilibrium always exists.[4] However, even when a large generalized game is continuous, the equilibrium correspondence may not have compact values (see footnote 7). Therefore, the traditional analysis of essential stability can not be directly implemented in our context.

Nevertheless, associated to any Cournot-Nash equilibrium of a large generalized game there is a vector of messages (generated by strategy profiles of non-atomic players) and a vector of optimal strategies of atomic players. These messages-actions vectors constitute all the relevant information that a player takes into account to make optimal decisions. In addition, the correspondence that associates games with the set of equilibrium messages and atomic players' profiles has closed graph and compact values (see Theorem 1 and Theorem 5). Hence, we focus our analysis on the stability of equilibrium messages-actions to perturbations on the characteristics of the generalized game.

3. The Space $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ of Continuous Generalized Games

We introduce continuous large generalized games, as those studied by Riascos and Torres-Martínez (2013). Through our model some characteristics of the games are fixed and summarized by a vector $(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$. The set of non-atomic players T_1 is a non-empty and compact subset of a metric space and there is a σ-algebra $\mathcal{A}$ such that, for some finite measure μ, $(T_1, \mathcal{A}, \mu)$ is a complete atomless measure space. The set of atomic players, denoted by T_2, is non-empty and finite. $\widehat{K}$ is a non-empty and compact metric space where non-atomic players' strategies belongs. For any atomic player t, let $\widehat{K}_t$ be the non-empty set of actions, that we assume to be a compact subset of a metrizable and locally convex topological vector space. Finally, non-atomic players' strategies are

[3]That is, the correspondence that associates games with the set of its pure strategy equilibria

[4]See Schmeidler (1973) for continuous large games, Balder (1999, 2002) and Riascos and Torres-Martínez (2013) for continuous generalized games, and Carmona and Podczeck (2013) for the discontinuous case.

codified by a function $H : T_1 \times \widehat{K} \to \mathbb{R}^m$, which is continuous with respect to the product topology induced by the metrics of T_1 and $\widehat{K}$.

In a game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$, each $t \in T_1$ has associated a closed and non-empty action space $K_t \subseteq \widehat{K}$, while each $t \in T_2$ has a closed, convex and non-empty action space $K_t \subseteq \widehat{K}_t$. A strategy profile of players in T_1 is given by a function $f : T_1 \to \widehat{K}$ such that $f(t) \in K_t$, for any $t \in T_1$. Any vector $a = (a_t; t \in T_2) \in \prod_{t\in T_2} K_t$ constitutes a strategy profile for players in T_2. For each $i \in \{1, 2\}$, let $\mathcal{F}^i((K_t)_{t\in T_i})$ be the space of strategy profiles for agents in T_i. In addition, for any $t \in T_2$, let $\mathcal{F}^2_{-t}((K_j)_{j\in T_2\setminus\{t\}})$ be the set of vectors $a_{-t} \in \Pi_{j\in T_2\setminus\{t\}} K_j$.

Each participant to the game considers aggregated information about strategies taken by players in T_1. Thus, if non-atomic players choose a strategy profile $f \in \mathcal{F}^1((K_t)_{t\in T_1})$, then its relevant characteristics are coded by the function H. Also, each player only take into account, for strategic purposes, aggregated information about these available characteristics through a message $m(f) := \int_{T_1} H(t, f(t))d\mu$. For this reason, we concentrate our attention only on those profiles of strategies for which messages are well defined, by considering profiles $f \in \mathcal{F}^1((K_t)_{t\in T_1})$ such that $H(\cdot, f(\cdot))$ is a measurable function from T_1 to $\mathbb{R}^m$.[5] Therefore, the set of messages associated with strategy profiles of non-atomic players is given by

$$M((K_t)_{t\in T_1}) = \left\{ \int_{T_1} H(t, f(t))d\mu : \quad f \in \mathcal{F}^1((K_t)_{t\in T_1}) \ \wedge \ H(\cdot, f(\cdot))\, \text{is measurable} \right\}.$$

Let $\widehat{M} = M((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^1 = \mathcal{F}^1((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^2 = \mathcal{F}^2((\widehat{K}_t)_{t\in T_2})$, and $\widehat{\mathcal{F}}^2_{-t} = \mathcal{F}^2((\widehat{K}_s)_{s\in T_2\setminus\{t\}})$. Assume that messages and strategy profiles may restrict players admissible strategies. Hence, the set of strategies available for a player $t \in T_1$ is determined by a correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow K_t$ with non-empty and compact values, where for every $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ the correspondence associating to any $t \in T_1$ the set $\Gamma_t(m, a)$ is measurable. Analogously, the set of strategies that $t \in T_2$ can choose is determined by a correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2_{-t} \twoheadrightarrow K_t$ with non-empty, compact and convex values.

Given a set S, let $\mathcal{U}(S)$ be the collection of bounded functions $u : S \to \mathbb{R}$ endowed with the sup norm topology. We assume that the map $U : T_1 \times \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2 \to \mathbb{R}$ given by $U(t, x, m, a) = u_t(x, m, a)$ belongs to $\mathcal{U}(T_1 \times \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ and the mapping associating to any $t \in T_1$ the function u_t is measurable. Each atomic player $t \in T_2$ has an objective function $u_t \in \mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ which is quasi-concave in its own strategy a_t (we refer to this subset of $\mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ as $\mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2)$).

Definition 1. *A Cournot-Nash equilibrium of* $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ *is given by a strategy profile* $(f^*, a^*) \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$, *with* $m(f^*) \in \widehat{M}$, *such that,*
(i) For almost all $t \in T_1$, $f^*(t) \in \Gamma_t(m(f^*, a^*)$ *and*

$$u_t(f^*(t), m(f^*), a^*) \geq u_t(f(t), m^*, a^*), \quad \forall f(t) \in \Gamma_t(m(f^*), a^*).$$

(ii) For any $t \in T_2$, $a^*_t \in \Gamma_t(m(f^*), a^*_{-t})$ *and*

$$u_t(m)f^*), a^*) \geq u_t(m(f^*), a_t, a^*_{-t}), \quad \forall a_t \in \Gamma_t(m(f^*), a^*_{-t}).$$

[5] That is, for any Borelian set $E \subseteq \mathbb{R}^m$, $\{t \in T_1 : H(t, f(t)) \in E\}$ belongs to $\mathcal{A}$.

Riascos and Torres-Martínez (2013, Theorem 1) ensure that, for any large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ satisfying assumptions described above, the set of Cournot-Nash equilibria $\mathrm{CN}(\mathcal{G})$ is non-empty provided that the following hypotheses hold:

(A1) *For any $t \in T_1 \cup T_2$, the objective function u_t is continuous.*

(A2) *For any $t \in T_1 \cup T_2$, the correspondence of admissible strategies Γ_t is continuous.*[6]

Let $\mathbb{G} = \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$ be the collection of large generalized games satisfying the hypotheses previously described. We endow the set $\mathbb{G}$ with the following metric:

$$\begin{aligned}
\rho(\mathcal{G}_1, \mathcal{G}_2) \;=\; & \sup_{t \in T_1} \sup_{(x,m,a) \in \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2} |u_t^1(x,m,a) - u_t^2(x,m,a)| \\
& + \sup_{t \in T_1} \sup_{(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2} d_H(\Gamma_t^1(m,a), \Gamma_t^2(m,a)) + \sup_{t \in T_1} d_H(K_t^1, K_t^2) \\
& + \max_{t \in T_2} \sup_{(m,x,a_{-t}) \in \widehat{M} \times \widehat{K}_t \times \widehat{\mathcal{F}}^2_{-t}} |u_t^1(m,x,a_{-t}) - u_t^2(m,x,a_{-t})| \\
& + \max_{t \in T_2} \sup_{(m,a_{-t}) \in \widehat{M} \times \widehat{\mathcal{F}}^2_{-t}} d_{H,t}(\Gamma_t^1(m,a_{-t}), \Gamma_t^2(m,a_{-t})) + \max_{t \in T_2} d_{H,t}(K_t^1, K_t^2),
\end{aligned}$$

where $\mathcal{G}_i = \mathcal{G}_i((K_t^i, \Gamma_t^i, u_t^i)_{t \in T_1 \cup T_2})$, d_H denotes the Hausdorff distance induced by the metric of $\widehat{K}$ over the collection of its non-empty and compact subsets, and for every $t \in T_2$ the Hausdorff distance induced by the metric of $\widehat{K}_t$ is denoted by $d_{H,t}$. Since $(T_1, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}))$ are compact sets, T_2 is finite, and $\widehat{M}$ is a non-empty and compact set (see Riascos and Torres-Martínez (2013, Theorem 1, Step 1)), it follows that $(\mathbb{G}, \rho)$ is a complete metric space (see the Appendix for detailed arguments).

Our main objective is the study of stability properties of Cournot-Nash equilibria for a generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ when parameters $(K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2}$ change. Note that, the correspondence that associates the parameters that define the generalized game with the set of Cournot-Nash equilibria is not necessarily compact valued,[7] a property that was required by the previous literature of essential stability in games with finitely many players. However, given any Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, the pair $(m(f^*), a^*)$ contains all the information that players require

[6]Continuity of correspondences $(\Gamma_t)_{t \in T_1}$ requires that, for every $t \in T_1$, $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow K_t$ be both upper hemicontinuous and lower hemicontinuous. Upper hemicontinuity is satisfied at $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ when for any open set $A \subseteq K_t$ such that $\Gamma_t(m,a) \subseteq A$, there exists an open set $U \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $\Gamma_t(m',a') \subseteq A$ for every $(m',a') \in U$. Lower hemicontinuity is satisfied at $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ when for any open set $A \subseteq K_t$ such that $\Gamma_t(m,a) \cap A \neq \emptyset$, there exists an open set $U \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $\Gamma_t(m',a') \cap A \neq \emptyset$ for every $(m',a') \in U$. Same definitions apply for the correspondences of admissible strategies associated to atomic players $(\Gamma_t)_{t \in T_2}$.

[7]For instance, consider an electoral game with a continuum of non-atomic players, $T_1 = [0,1]$, which vote for a party in $\{a,b\}$. Let x_t be the action of player $t \in T_1$, and assume that his objective function, u_t, only takes into account the benefits that he receives for any party $\{v_t(a), v_t(b)\}$ weighted by the support that each party has in the population, i.e. $u_t \equiv v_t(a)\mu(\{s \in T_1 : x_s = a\}) + v_t(b)(1 - \mu(\{s \in T_1 : x_s = a\}))$, where μ denotes the Lebesgue measure in $[0,1]$. That is, the utility level of a player $t \in T_1$ in unaffected by his own action and, therefore, any measurable profile $x : [0,1] \to \{a,b\}$ constitutes a Nash equilibrium of the game. Hence, the set of Nash equilibria is not compact. However, if we consider that each player receives as a message the support that party a has in the population, $m = \mu(\{s \in T_1 : x_s = a\})$, then the set of equilibrium messages is equal to $[0,1]$, which is a compact set.

to take their decisions. Thus, we can focus our analysis of stability in the effects that perturbations on the characteristics of a game have on messages and strategies of atomic players.[8]

Definition 2. *The Cournot-Nash correspondence of* $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ *is given by the multivalued function* $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ *that associates to any* $\mathcal{G} \in \mathbb{G}$ *the set of messages and actions* $(m^*, a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ *such that, for some* $f^* \in \widehat{\mathcal{F}}^1$*, we have* $m^* = m(f^*)$ *and* $(f^*, a^*) \in CN(\mathcal{G})$*.*

4. Essential Stability of Equilibria in $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$

We analyze how the set of Cournot-Nash equilibria of a generalized game changes when the parameters that define the game are modified. Our analysis is based on the concept of *essential stability*, that was introduced in the literature by Fort (1950), for single valued mappings, and by Jiang (1962), for the case of correspondences.

Definition 3. *Let* $\mathbb{G}' \subseteq \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$*. Given a large generalized game* $\mathcal{G}_0 \in \mathbb{G}'$*,* $(f^*, a^*) \in CN(\mathcal{G}_0)$ *is an essential equilibrium of* $\mathcal{G}_0$ *with respect to* $\mathbb{G}'$ *when, for any open set* $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ *such that* $(m(f^*), a^*) \in O$*, there exists* $\epsilon > 0$ *such that* $\Lambda(\mathcal{G}) \cap O \neq \emptyset$*, for any* $\mathcal{G} \in \mathbb{G}'$ *that satisfies* $\rho(\mathcal{G}_0, \mathcal{G}) < \epsilon$*. A generalized game* $\mathcal{G}_0$ *is essential with respect to* $\mathbb{G}'$ *if its Cournot-Nash equilibria are essential with respect to* $\mathbb{G}'$*.*

Hence, a large generalized game $\mathcal{G}_0 \in \mathbb{G}'$ is essential with respect to $\mathbb{G}' \subseteq \mathbb{G}$ if and only if messages and atomic players strategies associated to a Cournot-Nash equilibrium of $\mathcal{G}_0$ can be approximated by equilibrium messages and strategies of generalized games in $\mathbb{G}'$ close to it. Note that, if $\mathcal{G}_0$ is essential with respect to $\mathbb{G}'$, then it is essential with respect to any non-empty set $\mathbb{G}'' \subseteq \mathbb{G}'$ that contains it. Unfortunately, as the following example shows, not all games in $\mathbb{G}$ are essential.

Example. Suppose that $T_1 = [0, 1]$, $T_2 = \{\alpha\}$, $\widehat{K} = \{0, 1\}$, $\widehat{K}_\alpha = [0, 1]$. Consider a generalized game $\mathcal{G}$ where for each $t \in T_1$, $(K_t, \Gamma_t) \equiv (\widehat{K}, \widehat{K})$, $(K_\alpha, \Gamma_\alpha) \equiv (\widehat{K}_\alpha, \widehat{K}_\alpha)$, and $H(\cdot, x) \equiv x$. In addition, $u_\alpha(m, x) = -\|m - x\|^2$ and, for every $t \in T_1$, $(u_t(0, m, a_\alpha), u_t(1, m, a_\alpha)) = (0.5, 0.5)$.

Then, there is a continuum of Cournot-Nash equilibria and $\Lambda(\mathcal{G}) = \{(\lambda, \lambda) \in \mathbb{R}^2 : \lambda \in [0, 1]\}$. On the other hand, given $\epsilon > 0$, let $\mathcal{G}_\epsilon$ be the generalized game obtaining from $\mathcal{G}$ by changing the objective functions of non-atomic players to $(u^\epsilon_t(0, m, a_\alpha), u^\epsilon_t(1, m, a_\alpha)) = (0.5(1 + \epsilon), 0.5)$, for any $t \in T_1$. It follows that $\mathcal{G}_\epsilon$ has only one Cournot-Nash equilibrium and $\Lambda(\mathcal{G}_\epsilon) = \{(0, 0)\}$. Since ϵ is arbitrary and $\rho(\mathcal{G}, \mathcal{G}_\epsilon) < \epsilon$, we conclude that $\mathcal{G}$ is not essential with respect to $\mathbb{G}$. □

Theorem 1. *Let* $\mathbb{G}'$ *be a closed subset of* $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$*. Then, the collection of generalized games that are essential with respect to* $\mathbb{G}'$ *is a dense residual subset of* $\mathbb{G}'$*.*
Given $\mathcal{G} \in \mathbb{G}'$*, if* $\Lambda(\mathcal{G})$ *is a singleton, then* $\mathcal{G}$ *is essential with respect to* $\mathbb{G}'$*.*

[8]Since action profiles are coded using the function H, there may exist several Cournot-Nash equilibria that induce a same message. Even that, this indetermination does not have *real effects* on players utility levels.

The proof is given in the Appendix.

It follows that the set of generalized games that are essential with respect to a closed set $\mathbb{G}' \subseteq \mathbb{G}$ is dense and contains the intersection of a sequence of dense and open subsets of $\mathbb{G}'$. In particular, given $\mathcal{G}_0 \in \mathbb{G}'$, for any $\epsilon > 0$ there exists an essential generalized game $\mathcal{G} \in \mathbb{G}'$ such that $\rho(\mathcal{G}_0, \mathcal{G}) < \epsilon$.

Furthermore, even unessential generalized games may have subsets of Cournot-Nash equilibria that are stable. To formalize this property, we introduce concepts of stability for subsets of equilibrium points.

DEFINITION 4. *Let $\mathbb{G}' \subseteq \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$. Given $\mathcal{G}_0 \in \mathbb{G}'$, a subset $e(\mathcal{G}_0) \subseteq \Lambda(\mathcal{G}_0)$ is essential with respect to $\mathbb{G}'$ if it is non-empty, compact, and for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$,*

$$[e(\mathcal{G}_0) \subset O] \Longrightarrow [\exists \epsilon > 0 : \mathcal{G} \in \mathbb{G}',\ \ \rho(\mathcal{G}_0, \mathcal{G}) < \epsilon \Longrightarrow \Lambda(\mathcal{G}) \cap O \neq \emptyset]\,.$$

An essential set $e(\mathcal{G}_0)$ is minimal if it is a minimal element ordered by set inclusion.
A set $e(\mathcal{G}_0)$ is a component of $\Lambda(\mathcal{G}_0)$ if there is $(m^*, a^*) \in \Lambda(\mathcal{G}_0)$ such that, $e(\mathcal{G}_0)$ is the union of all connected subsets of $\Lambda(\mathcal{G}_0)$ containing (m^*, a^*).

This definition adapts to our framework the concepts of essential set and essential component that were introduced by Jiang (1963) and Yu and Yang (2004) in the context of stability of fixed point of multivalued mappings. These concepts were also addressed by Zhou, Yu, and Xiang (2007) to study stability of mixed strategy equilibria in non-convex finite-player games.

Since the Cournot-Nash correspondence Λ is non-empty valued and upper hemicontinuous (see the proof of Theorem 1), it follows from the topological characterization of upper hemicontinuity that, given $\mathbb{G}' \subseteq \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$, for any $\mathcal{G} \in \mathbb{G}'$ the set $\Lambda(\mathcal{G})$ is essential with respect to $\mathbb{G}'$. Moreover, given $A \subset B \subseteq \Lambda(\mathcal{G})$, if A is essential with respect to $\mathbb{G}'$ and B is compact, then B is essential with respect to $\mathbb{G}'$ too.[9] Thus, we focus the attention on the existence of minimal essentials sets.

In this direction, given a closed set $\mathbb{G}' \subseteq \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$, some results can be inferred from Theorem 1:

(i) If for some $\mathcal{G} \in \mathbb{G}'$ there is an essential Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, then $\{(m(f^*), a^*)\}$ is a minimal essential subset of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$. Therefore, it follows from Theorem 1 that *there exists a dense residual collection of generalized games with at least one minimal essential subset that is also connected.*

(ii) Since for any $\mathcal{G} \in \mathbb{G}'$ the set $\Lambda(\mathcal{G})$ is compact, any component of $\Lambda(\mathcal{G})$ is non-empty, connected and compact.[10] Hence, when $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$ is essential, the component associated to $\{(m(f^*), a^*)\}$

[9]Indeed, B is non-empty and compact. Also, for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $B \subset O$ we have that $A \subset O$. Thus, the essentiality of A with respect to $\mathbb{G}'$ ensures that B is essential too.

[10]By definition, components are non-empty. Since a component is the union of connected sets with at least one common element, it is connected too. Since the closure of a connected set is connected, components of compact sets are closed and, therefore, compact (for more details, see Berge (1997, page 98)).

is an essential subset of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$ (because it is compact and contains the essential set $\{(m(f^*), a^*)\}$). Therefore, it follows from Theorem 1 that *there exists a dense residual subset of* $\mathbb{G}'$ *in which any generalized game has at least one essential component.*

The following result ensures that the two properties above hold for *every* generalized game when spaces of strategies are Banach.

THEOREM 2. *Let $\mathbb{G}'$ be a closed subset of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$. For each $\mathcal{G} \in \mathbb{G}'$ there is a minimal essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$. In addition, if $\Lambda(\mathcal{G})$ has a connected essential set with respect to $\mathbb{G}'$, then it has an essential component.*

If $\widehat{K}$ and $\widehat{K}_t$, where $t \in T_2$, are convex subsets of Banach spaces with metrics induced by norms, then every minimal essential set of $\Lambda(\mathcal{G})$ is connected.

The proof is given in the Appendix.

Suppose that $\widehat{K}$ and $\widehat{K}_t$, with $t \in T_2$, are convex subsets of Banach spaces with metrics induced by norms. It follows from Theorem 2 that, given a closed set $\mathbb{G}' \subseteq \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ and $\mathcal{G} \in \mathbb{G}'$, if $\Lambda(\mathcal{G})$ is a finite set, then at least one Cournot-Nash equilibrum of $\mathcal{G}$ is essential with respect to $\mathbb{G}'$.[11] In particular, if $\mathcal{G} \in \mathbb{G}'$ has a finite number of Cournot-Nash equilibria, then it has at least one essential equilibrium with respect to $\mathbb{G}'$.

5. ESSENTIAL STABILITY FOR PARAMETERIZATIONS OF $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$

In this section, we discuss stability of Cournot-Nash equilibria when only some characteristics of the large generalized game are perturbed.

DEFINITION 5. *A parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ is given by a complete metric space $(\mathbb{X}, \tau)$ of parameters and a continuous function $\kappa : \mathbb{X} \to \mathbb{G}$ that associates parameters with generalized games.*

DEFINITION 6. *Let $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ be a parametrization of the space $\mathbb{G}$.*
(i) Given a parameter $\mathcal{X}_0 \in \mathbb{X}$, a Cournot-Nash equilibrium $(f^*, a^*) \in CN(\kappa(\mathcal{X}_0))$ is essential with respect to the set of parameters $\mathbb{X}$ under κ, if for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $(m(f^*), a^*) \in O$, there exists $\epsilon > 0$ such that $\Lambda(\kappa(\mathcal{X})) \cap O \neq \emptyset$, for any parameter $\mathcal{X} \in \mathbb{X}$ that satisfies $\tau(\mathcal{X}_0, \mathcal{X}) < \epsilon$.
(ii) A generalized game $\mathcal{G}_0 \in \mathbb{G}$ is $\mathcal{T}$-essential if there exists $\mathcal{X}_0 \in \mathbb{X}$ such that $\mathcal{G}_0 = \kappa(\mathcal{X}_0)$, and all of its Cournot-Nash equilibrium are essential with respect to $\mathbb{X}$ under κ.
(iii) Given a generalized game $\mathcal{G}_0 \in \mathbb{G}$, a subset $e(\mathcal{G}_0) \subseteq \Lambda(\mathcal{G}_0)$ is $\mathcal{T}$-essential—or essential with respect to $\mathbb{X}$ under κ—if there exists a parameter $\mathcal{X}_0 \in \mathbb{X}$ such that: (a) $\mathcal{G}_0 = \kappa(\mathcal{X}_0)$; (b) $e(\mathcal{G}_0)$ is non-empty and compact; and (c) for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ with $e(\mathcal{G}_0) \subset O$ there exists $\epsilon > 0$ such that, if $\mathcal{X} \in \mathbb{X}$ and $\tau(\mathcal{X}_0, \mathcal{X}) < \epsilon$, then $\Lambda(\kappa(\mathcal{X})) \cap O \neq \emptyset$.

[11] Theorem 2 ensures that $\Lambda(\mathcal{G})$ has a minimal essential set which is connected. Since $\Lambda(\mathcal{G})$ is finite, the only possibility is that minimal essential sets be equal to a singleton.

Some remarks:

(i) A generalized game $\mathcal{G}$ is essential with respect to $\mathbb{G}' \subseteq \mathbb{G}$ if and only if it is $\mathcal{T}$-essential for any parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ such that, for some $\mathcal{X} \in \mathbb{X}$, $\mathcal{G} = \kappa(\mathcal{X})$.[12] As a consequence, generically for games $\mathcal{G} \in \mathbb{G}$ and for any complete metric space $(\mathbb{X}, \tau)$ there is at least one parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ such that $\mathcal{G}$ is $\mathcal{T}$-essential. Indeed, it is sufficient to consider $\kappa(\cdot) = \mathcal{G}$.

(ii) Assume that $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ satisfies $\mathbb{X} \subseteq \mathbb{G}$, $\tau = \rho$, and κ is the immersion of $\mathbb{X}$ on $\mathbb{G}$. Then, for any $\mathcal{X} \in \mathbb{X}$, $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential if and only if $\mathcal{X}$ is essential with respect to $\mathbb{X}$.

The following result states stability properties of Cournot-Nash equilibria when admissible perturbations are determined by a parametrization of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$. Hence, we obtain stability results of Cournot-Nash equilibria when some but not necessarily all characteristics that define a generalized game are allowed to change.[13]

Theorem 3. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$, the collection of parameters $\mathcal{X} \in \mathbb{X}$ for which $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential is a dense residual subset of $\mathbb{X}$.*
Furthermore, for any $\mathcal{X} \in \mathbb{X}$ we have that:
(i) If $\Lambda(\kappa(\mathcal{X}))$ is a singleton, then $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential.
(ii) There is a minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$.
(iii) Any $\mathcal{T}$-essential and connected set $m(\mathcal{X}) \subseteq \Lambda(\kappa(\mathcal{X}))$ is contained in a $\mathcal{T}$-essential component.
(iv) Suppose that $(\mathbb{X}, \tau)$ is a convex metric space contained in a Banach space, where τ is a metric induced by a norm. Then, every minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is connected.

Proof. By assumptions $\kappa : \mathbb{X} \to \mathbb{G}$ is continuous and $(\mathbb{X}, \tau)$ is a complete metric space. In addition, Theorem 1 ensures that Λ is a closed correspondence that has non-empty and compact values. Thus, the set-valued mapping $\Lambda \circ \kappa : \mathbb{X} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ has closed graph with non-empty and compact values. Therefore, the first two properties follow from identical arguments to those made in the proof of Theorem 1. Furthermore, properties (ii)-(iv) can be obtained by analogous arguments of those made in the proof of Theorem 2, changing $(\mathbb{G}, \rho, \Lambda)$ by $(\mathbb{X}, \tau, \Lambda \circ \kappa)$. Q.E.D.

If we fix a generalized game in which strategies chosen by non-atomic players have no effects on other agents decisions, then equilibrium strategies associated to atomic players are Cournot-Nash equilibria of a convex generalized game where they are the only participants. Therefore, stability

[12] A direct consequence of the (τ, ρ)-continuity of $\kappa : \mathbb{X} \to \mathbb{G}$.

[13] For instance, when only objective functions or sets of admissible strategies can be perturbed. Or even, when there are personalized perturbations on players characteristics. As an example, fix a game $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2}) \in \mathbb{G}$. Given $i \in \{1, 2\}$, let $T_i^a, T_i^b, T_i^c \subseteq T_i$ be, respectively, the subsets of players in T_i for which we allow perturbations on objective functions, on strategy sets, and on correspondences of admissible strategies. Let $\mathbb{G}_{\mathcal{G}}((T_i^a, T_i^b, T_i^c)_{i \in \{1,2\}}) \subseteq \mathbb{G}$ be the set of generalized games $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}((\widetilde{K}_t, \widetilde{\Gamma}_t, \widetilde{u}_t)_{t \in T_1 \cup T_2})$ such that: (1) for any $t \in (T_1 \setminus T_1^a) \cup (T_2 \setminus T_2^a)$, $\widetilde{u}_t = u_t$; (2) for any $t \in (T_1 \setminus T_1^b) \cup (T_2 \setminus T_2^b)$, $\widetilde{K}_t = K_t$; and (3) for any $t \in (T_1 \setminus T_1^c) \cup (T_2 \setminus T_2^b)$, $\widetilde{\Gamma}_t = \Gamma_t$. Since $\mathbb{G}_{\mathcal{G}}((T_i^u, T_i^s, T_i^a)_{i \in \{1,2\}})$ is ρ-closed, it follows that $(\mathbb{G}_{\mathcal{G}}((T_i^u, T_i^s, T_i^a)_{i \in \{1,2\}}), \rho)$ is a complete metric space. Therefore, as the immersion $\iota : \mathbb{G}_{\mathcal{G}} \hookrightarrow \mathbb{G}$ is continuous, $((\mathbb{G}_{\mathcal{G}}, \rho), \iota)$ is a parametrization of $\mathbb{G}$.

properties of equilibria in continuous and convex generalized games with a finite number of players can be obtained as a particular case of our results above.

We close this section with results about stability of essential sets and essential components. The following result ensures that essential sets varies continuously when parameters are perturbed.

Given $\epsilon > 0$ and $A \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$, the ϵ-neighborhood of A is defined by

$$B[\epsilon, A] = \left\{(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2 : \exists\,(m',a') \in A,\ \widehat{\sigma}((m,a),(m',a')) \leq \epsilon\right\},$$

where $\widehat{\sigma}$ is the metric associated to the product topology of $\mathbb{R}^m \times \prod_{t \in T_2} \widehat{K}_t$.

Definition 7. *Fix a parametrization $\mathcal{T} = ((\mathbb{X},\tau),\kappa)$ and $\mathcal{X} \in \mathbb{X}$.*
(i) The set $E \in \Lambda(\kappa(\mathcal{X}))$ is stable if for every $\epsilon > 0$ there is $\delta > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X},\mathcal{X}') < \delta$, there exists a minimal $\mathcal{T}$-essential set $E' \in \Lambda(\kappa(\mathcal{X}'))$ for which $E' \subseteq B[\epsilon, E]$.
(ii) The set $E \in \Lambda(\kappa(\mathcal{X}))$ is strongly stable if for every $\epsilon > 0$ there is $\delta > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X},\mathcal{X}') < \delta$, there exists a $\mathcal{T}$-essential component $E' \in \Lambda(\kappa(\mathcal{X}'))$ for which $E' \subseteq B[\epsilon, E]$.

Note that, any set $E' \in \Lambda(\kappa(\mathcal{X}))$ which contains a (strongly) stable set, is (strongly) stable too.

Theorem 4. *Fix a parametrization $\mathcal{T} = ((\mathbb{X},\tau),\kappa)$ and let $\mathcal{X} \in \mathbb{X}$.*
(i) Every $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is stable.
(ii) Suppose that $(\mathbb{X},\tau)$ is a convex metric space contained in a Banach space, where τ is a metric induced by a norm. Then, every $\mathcal{T}$-essential component of $\Lambda(\kappa(\mathcal{X}))$ is strongly stable.

The proof is given in the Appendix.

6. Essential Stability as a Rationale for Electoral Participation

In a recent paper Barlo and Carmona (2011) introduce the refinement concept of strategic equilibria in large games. Intuitively, a Nash equilibrium of a large game is strategic if it is the limit of equilibria in abstract perturbed games, where players believe that have a positive impact on the social choice.[14] As an application of their results, they give a rationale to explain why electors vote for their favorite candidate. Introducing a large game with proportional voting, they show that there is a continuum of Cournot-Nash equilibria, but only one strategic equilibrium: that in which electors vote by their favorite party (see Barlo and Carmona (2011, Example 2.1)).

Inspired by this result, we analyze a large generalized electoral game where electors have different degrees of political interest. The Cournot-Nash equilibrium where only politically engaged players

[14]More precisely, departing from a large game $\mathcal{G}$ with only non-atomic players, for any $\epsilon > 0$ define an ϵ-perturbed game $\mathcal{G}_\epsilon$ as a game in which every player perceives that he, but no other, has a positive small impact on the social choice. Then, following our notation, $(f,a) \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ is a *strategic equilibrium* for a game $\mathcal{G}$ if there exists $\{\epsilon_k\}_{k\in\mathbb{N}} \subset (0,1)$ decreasing to zero, and a sequence $\{(f_k,a_k)\}_{k\in\mathbb{N}} \subset \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ converging to (f,a), such that (f_k,a_k) is a Cournot-Nash equilibrium for $\mathcal{G}_{\epsilon_k}$ for any $k \in \mathbb{N}$.

vote and support their favorite party appears as the unique $\mathcal{T}$-essential equilibrium of our electoral game, for some parametrization $\mathcal{T}$.

Given a set of parties $P = \{1, ..., \overline{p}\}$ and a parameter $\mu \geq 0$, consider an electoral game $\mathcal{E}_\mu(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H), (K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$, where for any non-atomic player $t \in T_1 := [0,1]$ the action space is given by $K_t = \widehat{K} := \left\{(x_1, \ldots, x_{\overline{p}}) \in \mathbb{Z}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} x_p \leq 1\right\}$, and strategies of other players do not affect her admissible allocations, i.e., $\Gamma_t \equiv \widehat{K}$. Thus, any $t \in T_1$ can either vote for a party $p \in P$ by choosing $x \in K_t$ such that $x_p = 1$, or she can not vote by choosing $(x_1, \ldots, x_{\overline{p}}) = 0$.

Each $t \in T_1$ gives an importance $v_t(p) \geq 0$ to party $p \in P$ and has a favorite party $p_t^* \in P$, i.e. $v_t(p_t^*) > v_t(p)$ for all $p \in P \setminus \{p_t^*\}$. Her objective function is given by the weighted average of the utilities getting from individual parties, and a component that reflect the private level of satisfaction associated to her action, that is, for any $x = (x_1, \ldots, x_{\overline{p}}) \in K_t$,

$$u_t^\mu(x,a) = \sum_{p=1}^{\overline{p}} v_t(p) a_p + \mu \sum_{p=1}^{\overline{p}} (v_t(p) - \eta_t) x_p,$$

where a_p is the probability that party p has to win the election, and the coefficient $\eta_t \geq 0$ measures the electoral engagement of player t. Indeed, when $\mu > 0$, as greater η_t less interested in the election would be player t. We assume that for any $t \in T_1$ either $\eta_t > v_t(p_t^*)$ or $\eta_t < v_t(p_t^*)$. The set of politically engaged players is defined as $T_1^* = \{t \in T_1 : \eta_t < v_t(p_t^*)\}$, and we assume that it is a subset of T_1 with positive measure.

On the other hand, there is an atomic player $T_2 = \{e\}$ whose objective is to determine the probabilities $(a_1, \ldots, a_{\overline{p}})$ that parties have to win. These probabilities are taken as given by non-atomic players. Hence, $\widehat{K}_e = K_e = \Gamma_e = \{(z_1, \ldots, z_{\overline{p}}) \in \mathbb{R}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} z_p = 1\}$, and

$$u_e(m,a) = -\sum_{p=1}^{\overline{p}} \left(a_p \sum_{p'=1}^{\overline{p}} m_{p'} - m_p \right)^2,$$

where $m = (m_1, \ldots, m_{\overline{p}})$ is the message obtained from non-atomic players votes, assuming that $H(t,x) = x$. In other words, when a positive measure of players vote, probabilities are given by the proportion of issued votes that each party receives.

In any generalized game $\mathcal{E}_\mu$, with $\mu \geq 0$, the strategy chosen by a non-atomic player does not affect the social choice. However, the vote of a non-atomic player $t \in T_1$ affects her own utility level when she gives a private value to her actions, i.e., when $\mu > 0$.

Consider the case where non-atomic players do not give importance to their strategies, i.e., $\mu = 0$. Then, given a measurable action profile $x : T_1 \to \widehat{K}$ and a strategy $a \in \widehat{K}_e$, the vector

$$\begin{cases} \left(x, \left(\dfrac{\int_{T_1} x_p(t)dt}{\sum_{s=1}^{\overline{p}} \int_{T_1} x_s(t)dt} \right)_{p\in P} \right), & \text{if } \int_{T_1} x(t)dt \neq 0; \\ (x,a), & \text{if } \int_{T_1} x(t)dt = 0; \end{cases}$$

constitutes a Cournot-Nash equilibrium for $\mathcal{E}_0$. Thus, when electors do not give any private value to electoral participation, there is a continuum of equilibria.

On the other hand, for any $\mu > 0$ the generalized game $\mathcal{E}_\mu$ has only one Cournot-Nash equilibrium. Indeed, any player $t \in T_1^*$ votes for his favorite party, while any player in $T_1 \setminus T_1^*$ does not vote. As T_1^* has positive measure, the equilibrium vector of probabilities is well defined. Hence, it follows from Theorem 1 that $\mathcal{E}_\mu$ is an essential generalized game for any $\mu > 0$.

Since the space $([0,1], |\cdot|)$ is complete and $\kappa : [0,1] \to \mathbb{G}$ given by $\kappa(\mu) = \mathcal{E}_\mu$ is continuous, $\mathcal{T} = (([0,1], |\cdot|), \kappa)$ is a parametrization of $\mathbb{G}$, in the sense of Definition 5. Therefore, we conclude that $\mathcal{E}_0$—the electoral game where players do not give any value to their private strategies—has a unique $\mathcal{T}$-essential Cournot-Nash equilibrium, the one in which only politically engaged players vote supporting their favorite party. That is, we obtain a rationale for electoral participation of politically engaged agents using essential stability as a refinement concept of Cournot-Nash equilibria.

Note that, under alternative perturbations we can still ensure that the only essential equilibrium is that where only politically engaged players vote. It is sufficient that only non-atomic player's payoff functions suffer perturbations, and the importance level that players give to the result of the election be small enough to maintain the same preferences over alternatives.[15]

7. On Essential Equilibria of Discontinuous Large Games

In this section we extend previous result of essential stability to a complete metric space that includes discontinuous generalized games.

Definition 9. *Given a large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ and an open set $U \subset \widehat{M} \times \widehat{\mathcal{F}}^2$, $(\varphi_t)_{t \in T_1 \cup T_2}$ are selectors of strategies supported on U when, for every $t \in T_1 \cup T_2$, $\varphi_t : U \twoheadrightarrow K_t$ is a closed correspondence with non-empty values, and the following properties hold for each $(m, a) \in U$:*
(i) For each $(t,k) \in T_1 \times T_2$, $\varphi_t(m,a) \times \varphi_k(m,a) \subseteq \Gamma_t(m,a) \times \Gamma_k(m, a_{-k})$.
(ii) The correspondence associating to each $t \in T_1$ the set $\varphi_t(m,a)$ is measurable.
(iii) For any $t \in T_2$, $\varphi_t(m,a)$ is convex.

The following definition, that generalizes the notion of *continuous security* of Barelli and Meneghel (2012), was introduced by Carmona and Podczeck (2013) as a key element to ensure equilibrium existence in discontinuous large generalized games.

Definition 10 (Continuous Security). *A large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ satisfies continuous security if for every $(m,a) \notin \Lambda(\mathcal{G})$ there is an open set $U \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ containing (m,a) such that, for some selectors of strategies supported on U, $(\varphi_t)_{t \in T_1 \cup T_2}$, and for some measurable function $\alpha : T_1 \cup T_2 \to [-\infty, +\infty]$, the following properties hold:*
(i) For every $(m', a') \in U$ there exists a full measure set $T_1' \subseteq T_1$ satisfying

[15] Perturbations on actions sets for non-atomic players, or on any characteristic of the atomic player, may change the underline institutional structure, destroying the electoral dimension of the game. However, a natural perturbation in action set is to forbid voluntary vote, by changing K_t to $\left\{(x_1, \ldots, x_{\overline{p}}) \in \mathbb{Z}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} x_p = 1\right\}$. In this case, in any Cournot-Nash equilibrium for $\mathcal{E}_0$, all voters participate on the election. In addition, the $\mathcal{T}$-essential Cournot-Nash equilibria of $\mathcal{E}_0$ are those in which politically engaged players vote supporting their favorite party.

$u_t(x, m', a') \geq \alpha(t),\ \forall t \in T_1', \forall x \in \varphi_t(m', a'),$

$u_t(m', x, a'_{-t}) \geq \alpha(t),\ \forall t \in T_2, \forall x \in \varphi_t(m', a').$

(ii) Fix $(f', a') \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ *such that* $(m(f'), a') \in U$*,* $f'(t) \in \Gamma_t(m(f'), a')$ *for almost all* $t \in T_1$*, and* $a'_t \in \Gamma_t(m(f'), a'_{-t})$ *for all* $t \in T_2$*. Then, either there exists a positive measure set* $T_1' \subseteq T_1$ *such that* $u_t(f'(t), m(f'), a') < \alpha(t), \forall t \in T_1'$*, or there exists* $t \in T_2$ *such that* $u_t(m(f'), a'_t, a'_{-t}) < \alpha(t)$*.*

As was shown by Carmona and Podczeck (2013, Theorem 3 and Example 2), under the hypothesis described on Section 3, continuous security is weaker than (A1)-(A2) and, therefore, it is satisfied by any large generalized game in $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$. Furthermore, Carmona and Podczeck (2013, Theorem 1) guarantee that any large generalized game satisfying continuous security has a pure strategy Nash equilibrium.

To characterize essential stability of equilibria, we strength *continuous security* to ensure that the set of large generalized games is a complete metric space. The following concept was introduced by Carmona and Podczeck (2013) as a natural extension to large generalized games of *generalized payoff security*, a property introduced for finite-player games by Barelli and Soza (2009).

Definition 11 (Generalized Payoff Security). *A large generalized game* $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ *satisfies generalized payoff security if for every* $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ *and* $\epsilon > 0$ *there exists an open set* $U \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ *containing* (m, a) *such that, for some selectors of strategies supported on* U*,* $(\varphi_t)_{t \in T_1 \cup T_2}$*, and for some measurable function* $\alpha : T_1 \cup T_2 \to [-\infty, +\infty]$*, we have:*

(i) For every $(m', a') \in U$ *there exists a full measure set* $T_1' \subseteq T_1$ *satisfying*

$u_t(x, m', a') \geq \alpha(t),\ \forall t \in T_1', \forall x \in \varphi_t(m', a'),$

$u_t(m', x, a'_{-t}) \geq \alpha(t),\ \forall t \in T_2, \forall x \in \varphi_t(m', a').$

(ii) For any atomic player $t \in T_2$ *we have that* $\alpha(t) + \epsilon \geq \sup_{x \in \Gamma_t(m, a_{-t})} u_t(m, x, a_{-t})$*. In addition, the set* $\{t \in T_1 : \alpha(t) + \epsilon \geq \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a)\}$ *has a measure greater than or equal to* $\mu(T_1) - \epsilon$*.*

Definition 12. *A large generalized game* $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ *is upper semicontinuous when for any* $t \in T_1 \cup T_2$ *both* u_t *is upper semicontinuous and* Γ_t *is upper hemicontinuous.*[16]

Any large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ that is generalized payoff secure and upper semicontinuous satisfies continuous security—see the proof in the Appendix—and, therefore, has a non-empty set of Cournot-Nash equilibria.

Note that, allowing for perturbations on actions sets or on correspondences of admissible strategies, the collection of generalized payoff secure and upper semicontinuous games is not necessarily a ρ-closed set, as the following example illustrates.

Example. Suppose that $T_1 = [0, 1]$, $\widehat{K} = [0, 1]$, and $H(t, x) = x$. Thus, $\widehat{M} = [0, 1]$. For any $n \in \mathbb{N}$, let $\mathcal{G}_n$ a large generalized game with only non-atomic players, characterized by $K_t^n = \left[0, 1 - \frac{1}{n}\right]$,

[16] Given a topological space X, $u : X \to \mathbb{R}$ is upper semicontinuous if $\{x \in X : u(x) \geq a\}$ is open for any $a \in \mathbb{R}$.

$\Gamma_t^n(m) = \left[0, \min\{m, 1-\frac{1}{n}\}\right]$, and $u_t^n(f(t), m) = v(f(t))$, where $v : [0,1] \to \{0,1\}$ is such that $v(x) = 1$ if and only if $x = 1$. Hence, $\mathcal{G}_n$ is generalized payoff secure and upper semicontinuous.[17]

Let $\overline{\mathcal{G}}$ be the large generalized game characterized by $K_t = [0,1]$, $\Gamma_t(m) = [0, \min\{m, 1\}]$, and $u_t(f(t), m) = v(f(t))$. It follows that $\rho(\mathcal{G}_n, \overline{\mathcal{G}})$ converges to zero as n goes to infinity. However, although $\overline{\mathcal{G}}$ is upper semicontinuous, it is no generalized payoff secure.[18]

Therefore, if we consider perturbations on action sets or on admissible strategies, we can not ensure that the collection of large generalized games that are generalized payoff secure and upper semicontinuous is a complete metric space under the distance ρ. $\square$

Taking as given actions spaces and correspondences of admissible strategies $(K_t, \Gamma_t)_{t \in T_1 \cup T_2}$, let $\mathbb{G}_d = \mathbb{G}_d(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H), (K_t, \Gamma_t)_{t \in T_1 \cup T_2})$ be the collection of large generalized games $\mathcal{G}((u_t)_{t \in T_1 \cup T_2})$ that satisfies the hypothesis described in Section 3 but instead of Assumption (A1)-(A2) are both *generalized payoff secure* and *upper semicontinuous*.

Since $(\mathbb{G}_d, \rho)$ is a complete metric space—see the proof in the Appendix—we can adapt our previous arguments to ensure the following properties of essential stability.

Theorem 5. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}_d$, the collection of parameters $\mathcal{X} \in \mathbb{X}$ for which $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential is a dense residual subset of $\mathbb{X}$.*
Furthermore, for any $\mathcal{X} \in \mathbb{X}$ we have that:
(i) If $\Lambda(\kappa(\mathcal{X}))$ is a singleton, then $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential.
(ii) There is a minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$.
(iii) Any $\mathcal{T}$-essential and connected set $m(\mathcal{X}) \subseteq \Lambda(\kappa(\mathcal{X}))$ is contained in a $\mathcal{T}$-essential component.
(iv) Every $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is stable.

The proof is given in the Appendix.

Our findings complement the previous results of stability for finite-player discontinuous games (see Yu (1999), Carbonell-Nicolau (2010), and Scalzo (2012)). Indeed, our model includes finite-player convex games as a particular case.[19] Therefore, although we focus on a particular set of hypotheses—generalized payoff security and upper semicontinuity of games—we contribute to this growing literature analyzing essential stability for generalized games, under general types of perturbations on objective functions, and including results of existence and stability for essential sets.

8. Concluding Remarks

In this paper, we use the stability theory of fixed points developed by Fort (1950) and Jiang (1962) to address the essential stability of Cournot-Nash equilibria in large generalized games.

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[17]For every $m \in \widehat{M}$ and $\epsilon > 0$, generalized payoff security holds by choosing $\alpha \equiv 0$.

[18]Taking $m = 1$ and $\epsilon \in (0,1)$, if generalized payoff security holds for $\overline{\mathcal{G}}$, then Definition 11(i) implies that $\alpha(t) \leq 0, \forall t \in T_1$. On the other hand, Definition 11(ii) ensures that there exists a positive measure set $T' \subseteq T_1$ such that $\alpha(t) + \epsilon \geq \sup_{x \in \Gamma_t(1)} u_t(x, 1)$, which in turn implies that $\epsilon \geq 1$. A contradiction.

[19]If non-atomic players' strategies have no effects on other agents decisions, equilibrium actions of atomic players are a Cournot-Nash equilibrium for the game in which they are the only participants.

We guarantee that essential stability is a generic property in the space of continuous large generalized games. Essential equilibria are still generic in a space of discontinuous large games provided that only payoff perturbations be allowed. Furthermore, all games have essential subsets of the set of equilibria which varies continuously.

Our results are compatible with general types of perturbations on the characteristics of generalized games. Indeed, stability properties still hold when (i) admissible perturbations can be captured by a continuous parametrization of the set of generalized games; and (ii) the set of parameters constitutes a complete metric space.

Appendix

Completeness of $(\mathbb{G},\rho)$.

Given a metric space (S,d), consider the sets $A(S)=\{K\subseteq S: K$ is non-empty and compact$\}$, and $A_c(S)=\{C\in A(S): C$ is convex$\}$. Denote by d_H the Hausdorff metric induced by the metric of S. If S is compact, then $(A(S),d_H)$ is a complete metric space. Also, when S is compact and convex, $(A_c(S),d_H)$ is complete.[20]

Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ a Cauchy sequence on $(\mathbb{G},\rho)$, where $\mathcal{G}_n=\mathcal{G}_n((K_{n,t},\Gamma_{n,t},u_{n,t})_{t\in T_1\cup T_2})$. By the definition of $\mathbb{G}$ and ρ it follows that, for any non-atomic player $t\in T_1$, $\{K_{n,t}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $(A(\widehat{K}),d_H)$. Also, for any atomic player $s\in T_2$, $\{K_{n,s}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $(A_c(\widehat{K}_s),d_{H,s})$. Hence, there are sets $\{\overline{K}_t\}_{t\in T_1\cup T_2}$ such that: (i) $(\overline{K}_t,\overline{K}_s)\in A(\widehat{K})\times A_c(\widehat{K}_s)$, $\forall (t,s)\in T_1\times T_2$; and (ii) for any $(t,s)\in T_1\times T_2$, we have that

$$\lim_{n\to+\infty} d_H(K_{n,t},\overline{K}_t)=\lim_{n\to+\infty} d_{H,s}(K_{n,s},\overline{K}_s)=0.$$

The definition of the metric ρ ensures that, for any $t\in T_1$ and $(m,a)\in\widehat{M}\times\widehat{\mathcal{F}}^2$, the sequence $\{\Gamma_{n,t}(m,a)\}_{n\in\mathbb{N}}\subseteq A(\widehat{K})$ is Cauchy and, therefore, there exists a set $K_t(m,a)\in A(\widehat{K})$ such that $d_H(\Gamma_{n,t}(m,a),K_t(m,a))$ converges to zero as n goes to infinity. Let $\overline{\Gamma}_t:\widehat{M}\times\widehat{\mathcal{F}}^2\twoheadrightarrow\widehat{K}$ be the set-valued mapping defined by $\overline{\Gamma}_t(m,a)=K_t(m,a)$. Correspondences $(\overline{\Gamma}_t)_{t\in T}$ are continuous.[21] By analogous arguments, we can ensure that for any $s\in T_2$ there is a continuous correspondence $\overline{\Gamma}_s:\widehat{M}\times\widehat{\mathcal{F}}^2_{-s}\twoheadrightarrow\widehat{K}_s$ such that, for each $(m,a_{-s})\in\widehat{M}\times\widehat{\mathcal{F}}^2_{-s}$ both $\overline{\Gamma}_s(m,a_{-s})\in A_c(\widehat{K}_s)$ and $d_{H,s}(\Gamma_{n,s}(m,a_{-s}),\overline{\Gamma}_s(m,a_{-s}))$ converges to zero as n increases.

Since $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is Cauchy on $(\mathbb{G},\rho)$, there is a function $\overline{U}\in\mathcal{U}(T_1\times\widehat{K}\times\widehat{M}\times\widehat{\mathcal{F}}^2)$ such that, for every $t\in T_1$, $\overline{U}(t):=\overline{u}_t$ is a continuous function and $\{u_{n,t}\}_{n\in\mathbb{N}}\subseteq\mathcal{U}(\widehat{K}\times\widehat{M}\times\widehat{\mathcal{F}}^2)$ converges to

[20] Since (S,d) is a compact metric space, it follows from Aliprantis and Border (2006, Theorem 3.85-(3) and Theorem 3.88-(2), pages 116 and 119) that $A(S)$ is a complete metric space under the Hausdorff distance induced by d. When the space is restricted to $A_c(S)$, $(A_c(S),d_H)$ remains a complete metric space, since the Hausdorff limit of a sequence of convex sets is still a convex set.

[21] Since $\widehat{M}\times\widehat{\mathcal{F}}^2$ is compact and $(A(\widehat{K}),d_H)$ is complete, for every $t\in T_1$ the continuity of the correspondence $\overline{\Gamma}_t$ follows from the completeness of the space of continuous functions $\nu:\widehat{M}\times\widehat{\mathcal{F}}^2\to A(\widehat{K})$ under the uniform metric induced by the Haussdorf distance. Indeed, any correspondence $\Gamma:\widehat{M}\times\widehat{\mathcal{F}}^2\twoheadrightarrow\widehat{K}$ with non-empty and compact values can be identified with the function $B_\Gamma:\widehat{M}\times\widehat{\mathcal{F}}^2\to A(\widehat{K})$ given by $B_\Gamma(m,a)=\Gamma(m,a)$, in such form that Γ is continuous if and only if B_Γ is continuous (see Aliprantis and Border (2006, Lemma 3.97 and Theorem 17.15, pages 124 and 563)).

it. Analogously, for any $s \in T_2$, the Cauchy sequence $\{u_{n,s}\}_{n\in\mathbb{N}} \subseteq \mathcal{U}_s(\widehat{M} \times \widehat{\mathcal{F}}^2)$ converges to some continuous function $\overline{u}_s \in \mathcal{U}_s(\widehat{M} \times \widehat{\mathcal{F}}^2)$.

Let $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{K}_t, \overline{\Gamma}_t, \overline{u}_t)_{t\in T_1\cup T_2})$. It follows from arguments above that $\lim_{n\to+\infty} \rho(\mathcal{G}_n, \overline{\mathcal{G}}) = 0$. Thus, to prove that $(\mathbb{G}, \rho)$ is complete, it is sufficient to guarantee that:

(i) the function $\overline{U} : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ defined by $\overline{U}(t) = \overline{u}_t$ is measurable;

(ii) for any $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the correspondence $t \in T_1 \twoheadrightarrow \overline{\Gamma}_t(m, a)$ is measurable.

The definition of ρ ensures that measurable functions $U_n : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ defined by $U_n(t) = u_{n,t}$ converge to $\overline{U}$. Since T_1 is a measurable space and $\mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ is a metric space, $\overline{U}$ is measurable (see Aliprantis and Border (2006, Lemma 4.29, page 142)). Hence, item (i) holds.

Fix $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$. Given $n \in \mathbb{N}$, the correspondence that associates to any $t \in T_1$ the set $\Gamma_{n,t}(m, a)$ is measurable. Thus, it follows form Aliprantis and Border (2006, Theorem 18.10, page 598) that the function $\Theta_{n,(m,a)} : T_1 \to A(\widehat{K})$ defined by $\Theta_{n,(m,a)}(t) = \Gamma_{n,t}(m, a)$ is Borel measurable. Also, the sequence $\{\Theta_{n,(m,a)}\}_{n\in\mathbb{N}}$ converges to $\overline{\Theta}_{(m,a)} : T_1 \to A(\widehat{K})$, where $\overline{\Theta}_{(m,a)}(t) = \overline{\Gamma}_t(m, a)$. By Aliprantis and Border (2006, Lemma 4.29), $\overline{\Theta}_{(m,a)}$ is a Borel measurable function. Thus, $t \in T_1 \twoheadrightarrow \overline{\Gamma}_t(m, a)$ is measurable (cf. Aliprantis and Border (2006, Theorem 18.10)). Q.E.D

Proof of Theorem 1.

The proof of the theorem is a direct consequence of the following steps.

Step 1. The correspondence $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ is upper hemicontinuous with compact values.

Since $\widehat{M} \times \widehat{\mathcal{F}}^2$ is compact and non-empty, we only need to prove that Graph(Λ) is closed, where Graph$(\Lambda) := \Big\{(\mathcal{G}, (m, a)) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2 : (m, a) \in \Lambda(\mathcal{G})\Big\}$.

Let $\{(\mathcal{G}_n, (m_n, a_n))\}_{n\in\mathbb{N}} \subset$ Graph(Λ) such that $(\mathcal{G}_n, (m_n, a_n)) \to (\overline{\mathcal{G}}, (\overline{m}, \overline{a})) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2$, where $\mathcal{G}_n = \mathcal{G}_n((K^n_t, \Gamma^n_t, u^n_t)_{t\in T_1\cup T_2})$ and $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{K}_t, \overline{\Gamma}_t, \overline{u}_t)_{t\in T_1\cup T_2})$. To prove that Graph($\Lambda$) is closed it is sufficient to ensure that $(\overline{m}, \overline{a}) \in \Lambda(\overline{\mathcal{G}})$.

Since $(m_n, a_n) \in \Lambda(\mathcal{G}_n)$, for almost all $t \in T_1$ there exists $f_n(t) \in \Gamma^n_t(m_n, a_n)$ such that,

$$m_n = \int_{T_1} H(t, f_n(t))d\mu, \qquad\qquad u^n_t(f_n(t), m_n, a_n) = \max_{x\in\Gamma^n_t(m_n,a_n)} u^n_t(x, m_n, a_n),$$

where the function $g_n(\cdot) = H(\cdot, f_n(\cdot))$ is measurable.

Claim A. For any $t \in T_1$ there exists $\overline{f}(t) \in \widehat{K}$ such that $\overline{m} = \int_{T_1} H(t, \overline{f}(t))d\mu$.

Proof. Since H is continuous, T_1 is compact and, for almost all $t \in T_1$, $f_n(t) \in \widehat{K}$, it follows that the sequence $\{g_n\}_{n\in\mathbb{N}}$ is uniformly integrable (see Hildenbrand (1974, page 52)). In addition, $\{\int_{T_1} g_n(t)d\mu\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ converges to $\overline{m}$ as n goes to infinity and, therefore, the Fatou's Lemma in m-dimension (see Hildenbrand (1974, page 69)) guarantees that there is $g : T_1 \to \mathbb{R}^m$ integrable such that,[22]

(1) $\lim_{n\to\infty} \int_{T_1} g_n(t)d\mu = \int_{T_1} g(t)d\mu$

[22]Although functions $\{g_n\}_{n\in\mathbb{N}}$ can take negative values, they are uniformly bounded from below (since H is continuous and $\{\widehat{K}, T_1\}$ are compact sets). Thus, as T_1 has finite Lebesgue measure, we can apply the Fatou's Lemma.

(2) There exists $\tilde{T}_1 \subseteq T_1$ such that, for any $t \in \tilde{T}_1$, $g(t) \in L_S(g_n(t))$, where $L_S(g_n(t))$ is the set of cluster points of $\{g_n(t)\}_{n\in\mathbb{N}}$ and $T_1 \setminus \tilde{T}_1$ has zero measure.[23]

Fix $t \in \tilde{T}_1$. Then there is a subsequence $(g_{n_k}(t))_k$ converging to $g(t)$. Since $\{f_{n_k}(t)\}_{k\in\mathbb{N}} \subseteq \widehat{K}$, taking a subsequence again if it is necessary, we can ensure that there exists $f(t) \in \widehat{K}$ such that both $f_{n_k}(t) \to f(t)$ and $g(t) = \lim_{k\to\infty} H(t, f_{n_k}(t)) = H(t, f(t))$.

Let $\overline{f} : T_1 \to \widehat{K}$ such that

$$\overline{f}(t) \in \begin{cases} \{f(t)\}, & \text{if } t \in \tilde{T}_1, \\ \operatorname*{argmax}_{x\in\overline{\Gamma}_t(\overline{m},\overline{a})} \overline{u}_t(x,\overline{m},\overline{a}), & \text{if } t \in T \setminus \tilde{T}_1. \end{cases}$$

By definition $\overline{m} = \lim_{n\to\infty} m_n$ and for any $n \in \mathbb{N}$ we have that $m_n = \int_{T_1} g_n(t)d\mu$, where $g_n(t) = H(t, f_n(t))$. Then, it follows that

$$\overline{m} = \lim_{n\to\infty} \int_{T_1} g_n(t)d\mu = \int_{T_1} g(t)d\mu = \int_{T_1} H(t, f(t))d\mu = \int_{T_1} H(t, \overline{f}(t))d\mu,$$

where the last equality follows from the fact that $T_1 \setminus \tilde{T}_1$ has zero measure. □

Claim B. For any $t \in T_1$, $\overline{f}(t) \in \overline{\Gamma}_t(\overline{m}, \overline{a})$.

Proof. The results follows by definition for any $t \in T_1 \setminus \tilde{T}_1$. Thus, fix $t \in \tilde{T}_1$ and let $\{f_{n_k}(t)\}_{k\in\mathbb{N}}$ the sequence that was obtained in the previous claim and that converges to $\overline{f}(t)$. We known that, for any $k \in \mathbb{N}$, $f_{n_k}(t) \in \Gamma_t^{n_k}(m_{n_k}, a_{n_k})$ and, therefore,

$$\begin{aligned} d(\overline{f}(t), \overline{\Gamma}_t(\overline{m}, \overline{a})) &\leq \widehat{d}(\overline{f}(t), f_{n_k}(t)) + d(f_{n_k}(t), \Gamma_t^{n_k}(m_{n_k}, a_{n_k})) + d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(m_{n_k}, a_{n_k})) \\ &\qquad + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})) \\ &\leq \widehat{d}(\overline{f}(t), f_{n_k}(t)) + \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})), \end{aligned}$$

where $\widehat{d}$ denotes the metric of the compact metric space $\widehat{K}$. Since $\overline{\Gamma}_t$ is continuous, by taking the limit as k goes to infinity, we obtain the result. □

Claim C. For any $t \in T_1$, $\overline{f}(t) \in \operatorname*{argmax}_{x\in\overline{\Gamma}_t(\overline{m},\overline{a})} \overline{u}_t(x, \overline{m}, \overline{a})$.

Proof. As in the previous claim, the case $t \in T_1 \setminus \tilde{T}_1$ follows from definition. With the same notation used in the previous claim, we have that

$$d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})) \quad \leq \quad \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})), \quad \forall t \in \tilde{T}_1.$$

Then $\Gamma_t^{n_k}(m_{n_k}, a_{n_k}) \longrightarrow_k \overline{\Gamma}_t(\overline{m}, \overline{a})$. Since $u_t^{n_k}$ converges uniformly to $\overline{u}_t$, it follows from Yu (1999, Lemma 2.5) and Aubin (1982, Theorem 3, page 70) that,

$$u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) = \max_{x\in\Gamma_t^{n_k}(m_{n_k},a_{n_k})} u_t^{n_k}(x, m_{n_k}, a_{n_k}) \longrightarrow_k \max_{x\in\overline{\Gamma}_t(\overline{m},\overline{a})} \overline{u}_t(x, \overline{m}, \overline{a})$$

On the other hand,

$$|u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})| \quad \leq \quad \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + |\overline{u}_t(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})|.$$

Taking the limit as k goes to infinity, we obtain that $u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) \to \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})$. Hence, it follows from Claim B that $\overline{f}(t) \in \operatorname*{argmax}_{x\in\overline{\Gamma}_t(\overline{m},\overline{a})} \overline{u}_t(x, \overline{m}, \overline{a})$. □

[23]In other words, for any $t \in \tilde{T}_1$ there is at least one subsequence of $\{g_n(t)\}_{n\in\mathbb{N}}$ converging to $g(t)$.

Claim D. For any $t \in T_2$, $\overline{a}_t \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$.

Proof. For any $(t,n) \in T_2 \times \mathbb{N}$, $a_{n,t} \in \Gamma_t^n(m_n, a_{n,-t})$ and, therefore,

$$\begin{aligned} d(\overline{a}_t, \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) &\leq \widehat{d}_t(\overline{a}_t, a_{n,t}) + d(a_{n,t}, \Gamma_t^n(m_n, a_{n,-t})) + d_{H,t}(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(m_n, a_{n,-t})) \\ &\quad + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \\ &\leq \widehat{d}_t(\overline{a}_t, a_{n,t}) + \rho(\mathcal{G}_n, \overline{\mathcal{G}}) + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})), \end{aligned}$$

where $\widehat{d}_t$ denotes the metric of $\widehat{K}_t$. Taking the limit as n goes to infinity, we obtain the result. □

Claim E. For any $t \in T_2$, $\overline{a}_t \in \underset{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})}{\operatorname{argmax}} \ \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$.

Proof. Following the same arguments of Claim C, we have that

$$d_{H,t}(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \quad \leq \quad \rho(\mathcal{G}_n, \mathcal{G}) + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})),$$

which implies that $\Gamma_t^n(m_n, a_{n,-t})$ converges to $\overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$ as n goes to infinity. Hence, Yu (1999, Lemma 2.5) ensures that,

$$u_t^n(m_n, a_n) = \max_{x \in \Gamma_t^n(m_n, a_{n,-t})} u_t^n(m_n, x, a_{n,-t}) \longrightarrow \max_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t}).$$

Since $\lim_{n \to +\infty} u_t^n(m_n, a_n) = \overline{u}_t(\overline{m}, \overline{a})$,[24] it follows that $\overline{a}_t \in \underset{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})}{\operatorname{argmax}} \ \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$. □

It follows from Claims A, C and E that $(\overline{m}, \overline{a}) \in \Lambda(\overline{\mathcal{G}})$. Thus, we ensure that Λ is an upper hemicontinuous correspondence with compact values.

Step 2. There is a dense residual set $Q \subseteq \mathbb{G}'$ where Λ is lower hemicontinuous.

As $\mathbb{G}'$ is a closed subset of $\mathbb{G}$, it follows that $(\mathbb{G}', \rho)$ is a complete metric space and, therefore, $\mathbb{G}'$ is a Baire space. Since the correspondence Λ is non-empty, compact-valued and upper hemicontinuous, it follows from Lemmas 5 and 6 in Carbonell-Nicolau (2010) (see also Fort (1949) and Jiang (1962)) that there exists a dense residual subset Q of $\mathbb{G}'$ in which Λ is lower hemicontinuous.

Step 3. If $\mathcal{G} \in \mathbb{G}'$ is a point of lower hemicontinuity of Λ, then $\mathcal{G}$ is essential with respect to $\mathbb{G}'$.

Fix an equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$. Then, for any open set $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $(m(f^*), a^*) \in O$ we have $\Lambda(\mathcal{G}) \cap O \neq \emptyset$ and, therefore, the lower inverse $\{\mathcal{G}' \in \mathbb{G}' : \ \Lambda(\mathcal{G}') \cap O \neq \emptyset\}$ contains a neighborhood of $\mathcal{G}$ in $\mathbb{G}'$. That is, there is $\epsilon > 0$ such that, for any $\mathcal{G}' \in \mathbb{G}'$ such that $\rho(\mathcal{G}', \mathcal{G}) < \epsilon$, we have $\Lambda(\mathcal{G}') \cap O \neq \emptyset$. Hence, all Cournot-Nash equilibrium of $\mathcal{G}$ are essential with respect to $\mathbb{G}'$.

It follows from Steps 2 and 3 that any generalized game in the dense residual set Q is essential. Finally, suppose that for a game $\mathcal{G} \in \mathbb{G}'$ the set $\Lambda(\mathcal{G})$ is a singleton. Then, Λ s continuous at this point. Using Step 3, we conclude that $\mathcal{G}$ is an essential generalized game with respect to $\mathbb{G}'$. Q.E.D.

Proof of Theorem 2.

[24]It is a direct consequence of the fact that, for any $n \in \mathbb{N}$, we have

$$|u_t^n(m_n, a_n) - \overline{u}_t(\overline{m}, \overline{a})| \quad \leq \quad \rho(\mathcal{G}_n, \mathcal{G}) + |\overline{u}_t(m_n, a_n) - \overline{u}_t(\overline{m}, \overline{a})|.$$

(i) Fix $\mathcal{G} \in \mathbb{G}'$. Let $\mathcal{S}$ be the family of essential subsets of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$ ordered by set inclusion. Since Λ is upper hemicontinuous, we have that $\Lambda(\mathcal{G}) \in \mathcal{S}$ and, hence, $\mathcal{S} \neq \emptyset$. As any element of $\mathcal{S}$ is compact, any totally ordered subset of $\mathcal{S}$ has a lower bounded element. By Zorn's Lemma, $\mathcal{S}$ has a minimal element, and by definition of $\mathcal{S}$, its minimal element is an essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$.

(ii) Suppose that there is a connected essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$, denoted by $c(\mathcal{G})$. Since by definition $c(\mathcal{G})$ is non-empty, fix $(\widehat{m}, \widehat{a}) \in c(\mathcal{G})$ and consider the set $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ defined as the union of all connected subsets of $\Lambda(\mathcal{G})$ that contains $(\widehat{m}, \widehat{a})$. By definition, $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is a component of $\Lambda(\mathcal{G})$. As the closure of a connected set is connected and $\Lambda(\mathcal{G})$ is compact, it follows that $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is compact. Hence, the essentiality of $c(\mathcal{G}) \subset \Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ with respect to $\mathbb{G}'$ ensures that the component $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is also an essential subset of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$.

(iii) Suppose that $\widehat{K}$ and $\widehat{K}_t$, where $t \in T_2$, are convex subsets of Banach spaces with metrics induced by the norm of the associated spaces. Fix a minimal essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$, denoted by $m(\mathcal{G})$. We want to prove that $m(\mathcal{G})$ is connected.

By contradiction, if $m(\mathcal{G})$ is not connected, then there are closed and non-empty subsets of $\Lambda_{\mathbb{G}'}(\mathcal{G})$, A_1 and A_2 such that $A_1 \cap A_2 = \emptyset$ and $m(\mathcal{G}) = A_1 \cup A_2$. Also, there are open sets V_1, V_2 such that $A_1 \subset V_1$, $A_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since $m(\mathcal{G})$ is minimal, neither A_1 nor A_2 are essentials with respect to $\mathbb{G}'$.

Fix $i \in \{1, 2\}$. Since A_i is not essential with respect to $\mathbb{G}'$, there exists an open set O_i such that $A_i \subset O_i$ and for all $\epsilon > 0$ there exists $\mathcal{G}_i \in \mathbb{G}'$ such that $\rho(\mathcal{G}, \mathcal{G}_i) < \epsilon$ and $\Lambda(\mathcal{G}_i) \cap O_i = \emptyset$. Since A_i is compact, there exists an open set U_i such that $A_i \subset U_i \subset \overline{U}_i \subset V_i \cap O_i$.

Therefore, $m(\mathcal{G}) \subset U_1 \cup U_2$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. As $m(\mathcal{G})$ is essential with respect to $\mathbb{G}'$, there exists $\nu > 0$ such that for every $\mathcal{G}' \in \mathbb{G}'$ with $\rho(\mathcal{G}, \mathcal{G}') < \nu$, we have $\Lambda(\mathcal{G}') \cap (U_1 \cup U_2) \neq \emptyset$.

On the other hand, given $i \in \{1, 2\}$, as $U_i \subset O_i$, there exists $\mathcal{G}'_i \in \mathbb{G}'$ such that $\rho(\mathcal{G}, \mathcal{G}'_i) < \frac{\nu}{3}$ and $\Lambda(\mathcal{G}'_i) \cap U_i = \emptyset$. Let $G : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \mathbb{G}'$ be the correspondence

$$G(m, a) = \lambda(m, a)\mathcal{G}'_1 + (1 - \lambda(m, a))\mathcal{G}'_2, \quad \forall (m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2,$$

where $\lambda : \widehat{M} \times \widehat{\mathcal{F}}^2 \to [0, 1]$ is the continuous function given by,

$$\lambda(m, a) = \frac{d((m, a), \overline{U}_2)}{d((m, a), \overline{U}_1) + d((m, a), \overline{U}_2)}.$$

Notice that, $(m, a) \in \overline{U}_i$ if and only if $G(m, a) = \mathcal{G}'_i$.

In addition, for any $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, we have that,

$$\begin{aligned}
\rho(G(m, a), \mathcal{G}'_1) &= \rho\left(\lambda(m, a)\,\mathcal{G}'_1 + (1 - \lambda(m, a))\,\mathcal{G}'_2, \lambda(m, a)\,\mathcal{G}'_1 + (1 - \lambda(m, a))\,\mathcal{G}'_1\right) \\
&\leq \rho(\,\mathcal{G}'_2, \mathcal{G}'_1) \leq \rho(\,\mathcal{G}'_2, \mathcal{G}) + \rho(\,\mathcal{G}, \mathcal{G}'_1) < \frac{2\nu}{3},
\end{aligned}$$

which implies that,

$$\rho(\mathcal{G}, G(m, a)) \leq \rho(\,\mathcal{G}, \mathcal{G}'_1) + \rho(\,\mathcal{G}'_1, G(m, a)) < \nu,$$

and, therefore, for each $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, $\Lambda(G(m,a)) \cap (U_1 \cup U_2) \neq \emptyset$.[25]

Claim. There exists $(\overline{m},\overline{a}) \in U_1$ such that, $(\overline{m},\overline{a}) \in \Lambda(G(\overline{m},\overline{a}))$.

Proof. Let $\tilde{A}_1 \subset U_1$ be a compact, convex and non-empty set. Define $\Theta : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1 \times \tilde{A}_1$ by $\Theta((m_1,a_1),(m_2,a_2)) = \left(\Phi_{G(m_1,a_1)}(m_2,a_2) \cap \tilde{A}_1\right) \times \{(m_1,a_1)\}$, where for every $\mathcal{G} \in \mathbb{G}$ the correspondence $\Phi_{\mathcal{G}} : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ is given by $\Phi_{\mathcal{G}}(m,a) = \left(\Omega^{\mathcal{G}}(m,a), (B_t^{\mathcal{G}}(m,a_{-t}))_{t\in T_2}\right)$ with

$$\begin{aligned}
\Omega^{\mathcal{G}}(m,a) &= \int_{T_1} H(t, B_t^{\mathcal{G}}(m,a))d\mu;\\
B_t^{\mathcal{G}}(m,a) &= \underset{x_t\in\Gamma_t(m,a)}{\operatorname{argmax}}\; u_t(x_t,m,a), \quad \forall t \in T_1;\\
B_t^{\mathcal{G}}(m,a_{-t}) &= \underset{x_t\in\Gamma_t(m,a_{-t})}{\operatorname{argmax}}\; u_t(x_t,m,a_{-t}), \quad \forall t \in T_2.
\end{aligned}$$

It follows from Riascos and Torres-Martínez (2013, Theorem 1) that for every game $\mathcal{G} \in \mathbb{G}$ the correspondence $\Phi_{\mathcal{G}}$ is upper hemicontinuous with non-empty, compact and convex values. Thus, by Kakutani's Fixed Point Theorem, the set of fixed points of $\Phi_{\mathcal{G}}$ is non-empty and compact. Furthermore, (f^*,a^*) is a Cournot-Nash equilibrium of $\mathcal{G}$ if and only if $(m^*,a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ is a fixed point of $\Phi_{\mathcal{G}}$, where $m^* = \int_{T_1} H(t,f^*(t))d\mu$.

Therefore, if $\Theta_1 : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1$ given by $\Theta_1((m_1,a_1),(m_2,a_2)) = \Phi_{G(m_1,a_1)}(m_2,a_2) \cap \tilde{A}_1$ has closed graph, then the correspondence Θ is upper hemicontinuous and has non-empty, compact and convex values. Thus, applying the Kakutani's Fixed Point Theorem we can find $(\overline{m},\overline{a}) \in \tilde{A}_1 \subset U_1$ such that, $(\overline{m},\overline{a}) \in \Lambda(G(\overline{m},\overline{a}))$.

Thus, let $\{(z_1^n, z_2^n, (m^n,a^n))\}_{n\in\mathbb{N}} \subset \text{Graph}(\Theta_1)$ a sequence that converges to $(\tilde{z}_1,\tilde{z}_2,(\tilde{m},\tilde{a})) \in \tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1$. We want to prove that $(\tilde{m},\tilde{a}) \in \Theta_1(\tilde{z}_1,\tilde{z}_2)$.

Fix $t \in T_2$ and let $\gamma_t : (\widehat{M} \times \widehat{\mathcal{F}}^2_{-t}) \times \tilde{A}_1 \twoheadrightarrow \widehat{K}_t$ the correspondence characterized by

$$\gamma_t((m,a_{-t}),z) = \underset{x\in\Psi((m,a_{-t}),z)}{\operatorname{argmax}}\; v_t(x,(m,a_{-t}),z),$$

where

$$\begin{aligned}
\Psi((m,a_{-t}),z) &= \lambda(z)\Gamma_t^1(m,a_{-t}) + (1-\lambda(z))\Gamma_t^2(m,a_{-t}),\\
v_t(x,(m,a_{-t}),z) &= \lambda(z)u_t^1(m,x,a_{-t}) + (1-\lambda(z))u_t^2(m,x,a_{-t}),
\end{aligned}$$

and, for each $i \in \{1,2\}$, $\mathcal{G}_i' = \mathcal{G}_i'((K_t^i,\Gamma_t^i,u_t^i)_{t\in T_1\cup T_2})$.[26] Since $\mathcal{G}_1',\mathcal{G}_2' \in \mathbb{G}'$ and λ is continuous, it follows that γ_t is upper hemicountinuous with non-empty and compact values. Therefore, the correspondence $\gamma : (\widehat{M}\times\widehat{\mathcal{F}}^2)\times\tilde{A}_1 \twoheadrightarrow \Pi_{t\in T_2}\widehat{K}_t$ given by $\gamma((m,a),z_2) = \prod_{t\in T_2}\gamma_t((m,a_{-t}),z)$ is upper hemicontinuous with compact and non-empty values. In particular, γ has closed graph. Therefore, as for any $n \in \mathbb{N}$, $(z_1^n,z_2^n,a^n) \in \text{Graph}(\gamma)$, it follows that $\tilde{a} \in \gamma(\tilde{z}_1,\tilde{z}_2)$.

On the other hand, for each $n \in \mathbb{N}$ there exists $f_n : T_1 \to \widehat{K}$ such that, $m_n = \int_{T_1} H(t,f_n(t))d\mu$ and, for any $t \in T_1$, $f_n(t) \in \xi_t(z_1^n,z_2^n) := \underset{x\in\Psi(z_1^n,z_2^n)}{\operatorname{argmax}}\; v_t(x,z_1^n,z_2^n)$, where we use analogous notations

[25] The additional assumptions about metric spaces $\widehat{K}$ and $\{\widehat{K}_t\}_{t\in T_2}$ ensure that for any $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ both $G(m,a)$ is a well defined generalized game and $\rho(G(m,a),\mathcal{G}_1') \leq \rho(\mathcal{G}_2',\mathcal{G}_1')$.

[26] Remember that $\lambda(z) = \frac{d(z,\overline{U}_2)}{d(z,\overline{U}_1)+d(z,\overline{U}_2)}$.

to those described above. Thus, as in the case of γ_t, the correspondences $(\xi_t; t \in T_1)$ have closed graph.

Since $m^n \rightarrow \tilde{m}$, analogous arguments to those made in Claim A of Theorem 1 ensure that, applying the multidimensional Fatou's Lemma (see Hildenbrand (1974, page 69)), there exists a zero-measure set $\dot{T}_1 \subset T_1$ and a function $\overline{f} : T_1 \rightarrow \widehat{K}$ such that,

(i) For any $t \in \dot{T}_1$, $\overline{f}(t) \in \xi_t(\tilde{z}_1, \tilde{z}_2)$;

(ii) For any $t \in T_1 \setminus \dot{T}_1$, there is a subsequence of $\{f_n(t)\}_{n \in \mathbb{N}}$ that converges to $\overline{f}(t)$;

(iii) $\tilde{m} = \int_{T_1} H(t, \overline{f}(t)) d\mu$.

As for any $t \in T_1 \setminus \dot{T}_1$, the correspondence ξ_t is closed, it follows from item (ii) above that $\overline{f}(t) \in \xi_t(\tilde{z}_1, \tilde{z}_2)$. By items (i) and (iii), jointly with the fact that $\tilde{a} \in \gamma(\tilde{z}_1, \tilde{z}_2)$, we have that $(\tilde{m}, \tilde{a}) \in \Theta_1(\tilde{z}_1, \tilde{z}_2)$. This concludes the proof of the claim. □

Since $(\overline{m}, \overline{a}) \in U_1$, $G(\overline{m}, \overline{a}) = \mathcal{G}_1'$ and, therefore, by the definition of $\mathcal{G}_1'$ we have that $\Lambda(G(\overline{m}, \overline{a})) \cap U_1 = \Lambda(\mathcal{G}_1') \cap U_1 = \emptyset$. A contradiction, since both $(\overline{m}, \overline{a}) \in U_1$ and $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$. Thus, the set $m(\mathcal{G})$ is connected. Q.E.D.

Proof of Theorem 4.

This result follows from Theorem 3 and Yu, Yang and Xiang (2005, Theorems 4.1 and 4.2).

(i) It follows from the definition of stability that it suffices to guarantee that minimal essential sets are stable. Let $\Lambda_m(\mathcal{T}, \mathcal{X})$ be the set of minimal $\mathcal{T}$-essential subsets of $\Lambda(\kappa(\mathcal{X}))$. Suppose, by contradiction, that there is $A \in \Lambda_m(\mathcal{T}, \mathcal{X})$ and $\epsilon_0 > 0$ such that, for any $\delta > 0$ there is $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$ and $A' \cap B[\epsilon_0, A]^c \neq \emptyset$, $\forall A' \in \Lambda_m(\mathcal{T}, \mathcal{X}')$.

Since A is $\mathcal{T}$-essential, there is $\delta_0 > 0$ such that, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_0$ we have that $\Lambda(\kappa(\mathcal{X}')) \cap B(\epsilon_0, A) \neq \emptyset$, where $B(\epsilon_0, A) = \left\{ (m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2 : \inf_{(m', a') \in A} \widehat{\sigma}((m, a), (m', a')) < \epsilon \right\}$. Fix $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_0$. It follows that $\Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]$ is a non-empty and closed set contained in $B[\epsilon_0, A]$ and, therefore, it is not and essential subset of $\Lambda(\kappa(\mathcal{X}'))$—a direct consequence of the property stated in the previous paragraph.

Hence, there exists $\epsilon_1 > 0$ such that, for any $n \in \mathbb{N}$ there is $\mathcal{X}_n \in \mathbb{X}$ with $\tau(\mathcal{X}', \mathcal{X}_n) < \frac{\delta_1}{n}$ and $B(\epsilon_1, \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]) \cap \Lambda(\kappa(\mathcal{X}_n)) = \emptyset$, where $\delta_1 > 0$ satisfies $\tau(\mathcal{X}'', \mathcal{X}') < \delta_1 \Longrightarrow \tau(\mathcal{X}'', \mathcal{X}) < \delta_0$. The last property ensures that $\tau(\mathcal{X}, \mathcal{X}_n) < \delta_0$ for any $n \in \mathbb{N}$, which implies that $\Lambda(\kappa(\mathcal{X}_n)) \cap B(\epsilon_0, A)$ is non-empty. Take a sequence $\{(m_n, a_n)\}_{n \in \mathbb{N}}$ such that $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}_n)) \cap B(\epsilon_0, A)$, $\forall n \in \mathbb{N}$. Without loss of generality, there is $(m_0, a_0) \in B[\epsilon_0, A]$ such that $(m_n, a_n) \rightarrow_n (m_0, a_0)$. The upper hemicontinuity of $(\Lambda \circ \kappa)$ ensures that $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}'))$. That is, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]$.

However, as for any $n \in \mathbb{N}$, $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}_n))$ and $B(\epsilon_1, \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]) \cap \Lambda(\kappa(\mathcal{X}_n)) = \emptyset$, it follows that $(m_n, a_n) \notin B(\epsilon_1, \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A])$, $\forall n \in \mathbb{N}$. A contradiction, because $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]$.

(ii) Since $\Lambda(\kappa(\mathcal{X}))$ is a compact and locally connected metric space, it has a finite number of connected components (see Berge (1997, Theorem 2, page 100)). Thus, there always exists $\pi > 0$

such that, $B[\pi, A] \cap B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$ for any $A \in \Lambda_c(\mathcal{T}, \mathcal{X})$, where $\Lambda_c(\mathcal{T}, \mathcal{X})$ is the set of $\mathcal{T}$-essential components of $\Lambda(\kappa(\mathcal{X}))$.

Let $A \in \Lambda_c(\mathcal{T}, \mathcal{X})$. It follows from the proof of Theorem 2 (item (i)) that there is $A_m \in \Lambda_m(\mathcal{T}, \mathcal{X})$ such that $A_m \subseteq A$. By the previous item, for each $\epsilon > 0$ there is $\delta_1 > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_1$, there exists $A'_m \in \Lambda_m(\mathcal{T}, \mathcal{X}')$ for which $A'_m \subseteq B[\epsilon, A_m] \subseteq B[\epsilon, A]$. By Theorem 3(iv), minimal essential sets are connected and, therefore, following analogous arguments to those made in the proof of Theorem 2 we can ensure that for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_1$ there is an essential component $A' \in \Lambda_c(\mathcal{T}, \mathcal{X}')$ which contains A'_m.

Since the correspondence $\Lambda \circ \kappa$ is upper hemicontinuous, there is $\delta_2 > 0$ such that for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_2$ we have that $\Lambda(\kappa(\mathcal{X}')) \subset B(\epsilon, \Lambda(\kappa(\mathcal{X}))) \subset B[\epsilon, A] \bigcup B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A]$.

Notice that $\Lambda(\kappa(\mathcal{X})) \setminus A$ is a compact set.[27] Let $\delta = \min\{\delta_0, \delta_1\}$ and fix $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$. If $A' \cap B[\epsilon, A]^c \neq \emptyset$, then $A' \cap B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A] \neq \emptyset$ and $A' \cap B[\epsilon, A] \neq \emptyset$. In addition, when $\epsilon < \pi$ it follows that $B[\epsilon, A] \bigcap B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$. Since A and $\Lambda(\kappa(\mathcal{X})) \setminus A$ are compact sets, it follows that $B[\epsilon, A]$ and $B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A]$ are closed sets. Thus, we obtain a partition of the connected set A' into two non-empty and disjoint closed sets, $A' \cap B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A]$ and $A' \cap B[\epsilon, A]$, which is a contradiction. Therefore, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$ we have that $A' \subset B[\epsilon, A]$. Q.E.D.

Proposition 1. *Let $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ be a generalized payoff secure and upper semicontinuous game. Then, $\mathcal{G}$ satisfies continuous security.*

Proof. Given $(m, a) \notin \Lambda(\mathcal{G})$, generalized payoff security guarantees that, for any $\epsilon > 0$ there exists $(U^\epsilon, (\varphi_t^\epsilon)_{t \in T_1 \cup T_2}, \alpha^\epsilon)$ satisfying the requirements imposed by item (i) on Definition 10. Thus, to guarantee that $\mathcal{G}$ is continuous secure, it is sufficient to prove that $(U^\epsilon, \alpha^\epsilon)$ satisfies Definition 10(ii) for some $\epsilon > 0$. Suppose, by contradiction, that for any $n \in \mathbb{N}$ there is $(f_n, a_n) \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ satisfying,

(a) $(m(f_n), a_n) \in U^{\frac{1}{n}}$,
(b) $f_n(t) \in \Gamma_t(m(f_n), a_n)$ for almost all $t \in T_1$,
(c) $a_{n,t} \in \Gamma_t(m(f_n), a_{n,-t})$ for all $t \in T_2$,
(d) for almost all $t \in T_1$, $u_t(f_n(t), m(f_n), a_n) \geq \alpha^{\frac{1}{n}}(t)$,
(e) for any $t \in T_2$, $u_t(m(f_n), a_{n,t}, a_{n,-t}) \geq \alpha^{\frac{1}{n}}(t)$.

Since we can assume that $\bigcap_n U^{\frac{1}{n}} = \{(m, a)\}$, it follows from (a) that $(m(f_n), a_n) \to_n (m, a)$. Conditions (b)-(c) guarantee that, using analogous arguments to those made in the proof of Claim A on Theorem 1, we can find a strategy profile $f \in \mathcal{F}^1((K_t)_{t \in T_1})$ such that $m = m(f)$ and, for almost all player $t \in T_1$, $f(t) \in L_S(f_n(t))$. In addition, as correspondences of admissible strategies have closed graph, it follows that: (i) for almost all $t \in T_1$, $f(t) \in \Gamma_t(m(f), a)$; (ii) for all $k \in T_2$, $a_k \in \Gamma_k(m(f), a_{-k})$.

[27]Indeed, since $(\Lambda(\kappa(\mathcal{X})) \setminus A) \subset \Lambda(\kappa(\mathcal{X}))$, it is sufficient to ensure that it is closed. Let $\{(m_n, a_n)\}_{n \in \mathbb{N}} \subset (\Lambda(\kappa(\mathcal{X})) \setminus A)$ be a sequence that converges to $(m_0, a_0) \in \widehat{M} \times \widehat{\mathcal{F}}_2$. For any $n \in \mathbb{N}$, $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}))$ and $(m_n, a_n) \notin A$. Thus, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}))$. Furthermore, if $(m_0, a_0) \in A$, then for n large enough $(m_n, a_n) \in B[\pi, A]$, a contradiction with $B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] \cap B[\pi, A] = \emptyset$. Therefore, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X})) \setminus A$.

Hence, as $(m,a) \notin \Lambda(\mathcal{G})$, there exists $\delta > 0$ such that either for a positive measure set $T' \subseteq T_1$,

$$u_t(f(t), m, a) + \delta < \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a), \quad \forall t \in T',$$

or for some $t \in T_2$,

$$u_t(m, a_t, a_{-t}) + \delta < \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}).$$

Since $\mathcal{G}$ is upper semicontinuous and $(m(f_n), a_n) \to_n (m,a)$, it follows from the definition of f that for any $n \in \mathbb{N}$ large enough we have that either for all t in a positive measure set $T'' \subseteq T'$,

$$u_t(f_n(t), m(f_n), a_n) + 0.5\,\delta < \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a), \tag{1}$$

or for some $t \in T_2$,

$$u_t(m(f_n), a_{n,t}, a_{n,-t}) + 0.5\,\delta < \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}). \tag{2}$$

On the other hand, it follows from conditions (d)-(e) above and Definition 11(ii) that for any $n \in \mathbb{N}$ there exists $T'_n \subseteq T_1$ with $\mu(T'_n) \geq \mu(T_1) - \frac{1}{n}$ such that, or any $t \in T'_n$,

$$u_t(f_n(t), m(f_n), a_n) \geq \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a) - \frac{1}{n},$$

and for every atomic player $t \in T_2$,

$$u_t(m(f_n), a_{n,t}, a_{n,-t}) \geq \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}) - \frac{1}{n}.$$

Thus, $\underline{\lim}_n u_t(f_n(t), m(f_n), a_n) \geq \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a)$ for almost all non-atomic player $t \in T_1$, and $\underline{\lim}_n u_t(m(f_n), a_{n,t}, a_{n,-t}) \geq \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t})$ for each atomic player $t \in T_2$. These properties contradict conditions (1) and (2). Q.E.D.

Completeness of $(\mathbb{G}_d, \rho)$.

Since $(K_t, \Gamma_t)_{t \in T_1 \cup T_2}$ does not change, $(\mathbb{G}_d, \rho)$ can be consider as a subset of the space of bounded functions $\mathcal{B} := \mathcal{U}(T_1 \times \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2) \times \prod_{t \in T_2} \mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2)$. Note that $(\mathcal{B}, \rho)$ is a complete metric space and, therefore, it is sufficient to ensure that $\mathbb{G}_d$ is a closed subset of $\mathcal{B}$.

Fix a sequence $\{\mathcal{G}_n\}_{n \in \mathbb{N}} \subset \mathbb{G}_d$, with $\mathcal{G}_n = \mathcal{G}_n((u^n_t)_{t \in T_1 \cup T_2})$ for any $n \in \mathbb{N}$, which converges to $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{u}_t)_{t \in T_1 \cup T_2}) \in \mathcal{B}$. We want to prove that $\overline{\mathcal{G}} \in \mathbb{G}_d$.

Claim. $\overline{\mathcal{G}}$ *is generalized payoff secure.*

Given $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ and $\epsilon > 0$, fix $(\delta_1, \delta_2) \gg 0$ such that $\delta_1 + \delta_2 < \epsilon$. Generalized payoff security of $\mathcal{G}_n$ at $((m,a), \delta_1)$ implies that there exists $(U^{n,\delta_1}, (\varphi^{n,\delta_1}_t)_{t \in T_1 \cup T_2}, \alpha^{n,\delta_1})$ such that, for almost all $t \in T_1$, for all $k \in T_2$, and for every $(m', a') \in U^{n,\delta_1}$ the following properties hold,

$u^n_t(x, m', a') > \alpha^{n,\delta_1}(t) - \delta_2, \ \forall x \in \varphi^{n,\delta_1}_t(m', a'),$

$u^n_k(m', x, a'_{-t}) > \alpha^{n,\delta_1}(k) - \delta_2, \ \forall x \in \varphi^{n,\delta_1}_k(m', a').$

Thus, for n large enough, for almost all $t \in T_1$, and for all $k \in T_2$,

$\overline{u}_t(x, m', a') > \alpha^{n,\delta_1}(t) - 0.5\,\delta_2, \ \forall x \in \varphi^{n,\delta_1}_t(m', a'),$

$\overline{u}_k(m', x, a'_{-t}) > \alpha^{n,\delta_1}(k) - 0.5\,\delta_2, \ \forall x \in \varphi^{n,\delta_1}_k(m', a').$

Furthermore, Definition 11(ii) ensures that, for n large enough, and for all $k \in T_2$,

$$\left(\alpha^{n,\delta_1}(k) - 0.5\,\delta_2\right) + \epsilon > \sup_{x\in\Gamma_t(m,a_{-k})} u_t^n(m,x,a_{-k}) + 0.5\,\delta_2 \quad > \sup_{x\in\Gamma_t(m,a_{-k})} \overline{u}_t(m,x,a_{-k}).$$

Analogously, for n large enough,

$$\mu\left\{t \in T_1 : \left(\alpha^{n,\delta_1}(t) - 0.5\,\delta_2\right) + \epsilon \geq \sup_{x\in\Gamma_t(m,a)} \overline{u}_t(x,m,a)\right\} \geq \mu(T_1) - \delta_1.$$

Since $\delta_1 < \epsilon$, taking n large enough and choosing $(U^{n,\delta_1}, (\varphi_t^{n,\delta_1})_{t\in T_1\cup T_2}, \alpha^{n,\delta_1} - 0.5\,\delta_2)$, we ensure that $\overline{\mathcal{G}}$ is generalized payoff secure at $((m,a),\epsilon)$. □

It is a direct consequence of Carbonell-Nicolau (2010, Lemma 1, page 425) that $\overline{\mathcal{G}}$ is upper semicontinuous and atomic players' objective functions $(\overline{u}_t)_{t\in T_2}$ are quasi-concave. In addition, the same arguments made in the proof of the completeness of $(\mathbb{G},\rho)$ guarantee that the map associating to each $t \in T_1$ the function u_t is measurable. This concludes the proof. Q.E.D.

Proof of Theorem 5.

Since $(\mathbb{G}_d,\rho)$ is complete, it follows from the proofs of Theorems 1-4 that all the properties on Theorem 5 hold provided that Λ still has a closed graph when its domain is extended to $\mathbb{G}_d$.

Thus, let $\{(\mathcal{G}_n,(m_n,a_n))\}_{n\in\mathbb{N}} \subset \text{Graph}(\Lambda)$ such that $(\mathcal{G}_n,(m_n,a_n)) \to (\overline{\mathcal{G}},(\overline{m},\overline{a})) \in \mathbb{G}_d \times \widehat{M} \times \widehat{\mathcal{F}}^2$, where $\mathcal{G}_n = \mathcal{G}_n((u_t^n)_{t\in T_1\cup T_2})$ and $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{u}_t)_{t\in T_1\cup T_2})$. We want to prove that $(\overline{m},\overline{a}) \in \Lambda(\overline{\mathcal{G}})$.

Since $(m_n,a_n) \in \Lambda(\mathcal{G}_n)$, for almost all $t \in T_1$ there exists $f_n(t) \in \Gamma_t(m_n,a_n)$ such that,

$$m_n = \int_{T_1} H(t,f_n(t))d\mu, \qquad\qquad u_t^n(f_n(t),m_n,a_n) = \sup_{x\in\Gamma_t(m_n,a_n)} u_t^n(x,m_n,a_n),$$

where the function $g_n(\cdot) = H(\cdot,f_n(\cdot))$ is measurable. Therefore, using analogous arguments to those made in the proof of Claim A on Theorem 1, we can find a strategy profile $\overline{f} \in \mathcal{F}^1((K_t)_{t\in T_1})$ such that $\overline{m} = m(\overline{f})$ and, for almost all player $t \in T_1$, $\overline{f}(t) \in L_S(f_n(t))$. The closed graph property of correspondences of admissible strategies ensures that: (i) for almost all $t \in T_1$, $\overline{f}(t) \in \Gamma_t(\overline{m},\overline{a})$; (ii) for all $k \in T_2$, $\overline{a}_k \in \Gamma_k(\overline{m},\overline{a}_{-k})$.

If we suppose that $(\overline{m},\overline{a}) \notin \Lambda(\overline{\mathcal{G}})$, then as $\overline{\mathcal{G}}$ is upper semicontinuous we can obtain, by identical arguments to those made in the proof of Proposition 1 to ensure conditions (1)-(2), that there exists $\delta > 0$ such that for n large enough either for all t in a positive measure set $T' \subseteq T$,

$$\overline{u}_t(f_n(t),m_n,a_n) + \delta < \sup_{x\in\Gamma_t(\overline{m},\overline{a})} \overline{u}_t(x,\overline{m},\overline{a}),$$

or for some $t \in T_2$,

$$\overline{u}_t(m_n,a_{n,t},a_{n,-t}) + \delta < \sup_{x\in\Gamma_t(\overline{m},\overline{a}_{-t})} \overline{u}_t(\overline{m},x,\overline{a}_{-t}).$$

Thus, as $\rho(\mathcal{G}_n,\overline{\mathcal{G}}) \to_n 0$,for n large enough and either for all t in a positive measure set $T'' \subseteq T'$,

$$u_t^n(f_n(t),m_n,a_n) + 0.5\,\delta < \sup_{x\in\Gamma_t(\overline{m},\overline{a})} \overline{u}_t(x,\overline{m},\overline{a}), \tag{3}$$

or for some $t \in T_2$,

$$u_t^n(m_n,a_{n,t},a_{n,-t}) + 0.5\,\delta < \sup_{x\in\Gamma_t(\overline{m},\overline{a}_{-t})} \overline{u}_t(\overline{m},x,\overline{a}_{-t}). \tag{4}$$

On the other hand, since $\overline{\mathcal{G}}$ is a generalized payoff secure large game, for every $\epsilon > 0$ there exists $(U^\epsilon, (\varphi_t^\epsilon)_{t\in T_1\cup T_2}, \alpha^\epsilon)$ satisfying Definition 11. In particular, as $(m_n, a_n) \to_n (\overline{m}, \overline{a})$, there exists a set $T'_\epsilon \subseteq T_1$ with $\mu(T'_\epsilon) \geq \mu(T) - \epsilon$ such that, for any n large enough we have $(m_n, a_n) \in U^\epsilon$ and the following properties hold for every $(t,k) \in T'_\epsilon \times T_2$:

$$\begin{aligned}
\sup_{x\in\Gamma_t(m_n,a_n)} \overline{u}_t(x, m_n, a_n) &\geq \sup_{x\in\varphi_t^\epsilon(m_n,a_n)} \overline{u}_t(x, m_n, a_n) \geq \sup_{x\in\Gamma_t(\overline{m},\overline{a})} \overline{u}_t(x, \overline{m}, \overline{a}) - \epsilon, \\
\sup_{x\in\Gamma_k(m_n,a_{n,-k})} \overline{u}_k(m_n, x, a_{n,-k}) &\geq \sup_{x\in\varphi_k^\epsilon(m_n,a_{n,-k})} \overline{u}_k(m_n, x, a_{n,-k}) \geq \sup_{x\in\Gamma_k(\overline{m},\overline{a}_{-k})} \overline{u}_k(\overline{m}, x, \overline{a}_{-k}) - \epsilon.
\end{aligned}$$

As objective functions are bounded, the uniform convergence of $\mathcal{G}_n$ to $\overline{\mathcal{G}}$ ensures that, for n large enough, for almost all $t \in T_1$, and for all $k \in T_2$,

$$\begin{aligned}
u_t^n(f_n(t), m_n, a_n) + \epsilon &= \sup_{x\in\Gamma_t(m_n,a_n)} u_t^n(x, m_n, a_n) + \epsilon \geq \sup_{x\in\Gamma_t(m_n,a_n)} \overline{u}_t(x, m_n, a_n), \\
u_k^n(m_n, a_{n,k}, a_{n,-k}) + \epsilon &= \sup_{x\in\Gamma_k(m_n,a_{n,-k})} u_k^n(m_n, x, a_{n,-k}) + \epsilon \geq \sup_{x\in\Gamma_k(m_n,a_{n,-k})} \overline{u}_k(m_n, x, a_{n,-k}).
\end{aligned}$$

Therefore, for every $\epsilon > 0$, there exists $T''_\epsilon \subseteq T'_\epsilon$ with $\mu(T''_\epsilon) \geq \mu(T_1) - \epsilon$ such that, for every player $(t,k) \in T''_\epsilon \times T_2$ we have

$$\begin{aligned}
\underline{\lim}_n\, u_t^n(f_n(t), m_n, a_n) + \epsilon &\geq \sup_{x\in\Gamma_t(\overline{m},\overline{a})} \overline{u}_t(x, \overline{m}, \overline{a}) - \epsilon, \\
\underline{\lim}_n\, u_k^n(m_n, a_{n,k}, a_{n,-k}) + \epsilon &\geq \sup_{x\in\Gamma_k(\overline{m},\overline{a}_{-k})} \overline{u}_k(\overline{m}, x, \overline{a}_{-k}) - \epsilon.
\end{aligned}$$

Taking the limit as ϵ goes to zero, we contradict conditions (3)-(4), concluding the proof. Q.E.D.

References

[1] Aubin, J.P. (1982): *Mathematical Methods of Games and Economic Theory,* North-Holland, Amsterdam.

[2] Aliprantis, C. and K. Border (1999): *Infinite Dimensional Analysis,* Springer-Verlag, Berlin, Heidelberg.

[3] Al-Najjar, N. (1995): "Strategically stable equilibria in games with infinitely many pure strategies," *Mathematical Social Sciences*, volume 29, pages 151-164.

[4] Balder, E.J. (1999):"On the existence of Cournot-Nash equilibria in continuum games," *Journal of Mathematical Economics* 32, pages 207-223.

[5] Balder, E.J. (2002): "A unifying pair of Cournot-Nash equilibrium existence results," *Journal of Economic Theory* 102, pages 437-470.

[6] Barelli, P. and I. Meneghel (2012): "A note on the equilibrium existence problem in discontinuous games," *Econometrica*, forthcoming.

[7] Barelli, P. and I. Soza (2009): "On the existence of Nash equilibria in discontinuous and qualitative games," mimeo, University of Rochester.

[8] Barlo, M. and G. Carmona (2011): "Strategic behaviour in non-atomic games," working paper, Munich Personal REPEC Arquive, http://mpra.ub.uni-muenchen.de/35549

[9] Berge, C. (1997): "Topological Spaces," Dover Publications, New York.

[10] Carbonell-Nicolau, O. (2010): "Essential equilibria in normal-form games," *Journal of Economic Theory*, volume 145, pages 421-431.

[11] Carmona, G. and K. Podczeck (2013): "Existence of Nash equilibrium in games with a measure space of players and discontinuous payoff functions," Munich Personal REPEC Arquive, http://mpra.ub.uni-muenchen.de/44263

[12] Fort, M.K. (1949): "A unified theory of semi-continuity," *Duke Mathematical Journal*, volume 16, pages 237-246.

[13] Fort, M.K. (1950): "Essential and non-essential fixed points," *American Journal of Mathematics*, volume 72, pages 315-322.

[14] Hildenbrand, W. (1974): *Core and Equilibria in a Large Economy,* Princeton University Press, Princeton, New Jersey.

[15] Hillas, J. (1990): "On the definition of the strategic stability of equilibria," *Econometrica*, volume 58, pages 1365-1390.

[16] Jiang, J.H. (1962): "Essential fixed points of the multivalued mappings," *Scientia Sinica,* volume XI. pages 293-298.

[17] Jiang, J.H. (1963): "Essential component of the set of fixed points of the multivalued mappings and its application to the theory of games," *Scientia Sinica,* volume XII. pages 951-964.

[18] Kinoshita, S. (1952): "On essential components of the set of fixed points," *Osaka Mathematical Journal,* volume 4, pages 19-22.

[19] Kohlberg, E. and J.F. Mertens (1986): "On the strategic stability of equilibrium point," *Econometrica,* volume 54, pages 1003-1037.

[20] Rath, K.P. (1992): "A direct proof of the existence of pure strategy equilibria in games with a continuum of players", *Economic Theory*, Volume 2, pages 427-433.

[21] Riascos, A.J. and J.P. Torres-Martínez (2013):"On pure strategy equilibria in large generalized games," Munich Personal REPEC Arquive, http://mpra.ub.uni-muenchen.de/46840

[22] Scalzo, V. (2012): "Essential equilibria of discontinuous games." *Economic Theory*, doi:10.1007/s00199-012-0726-y

[23] Schmeidler, D. (1973): "Equilibrium point of non-atomic games," *Journal of Statistical Physics*, volume 17, 295-300.

[24] Yu, J. (1999): "Essential equilibrium of n-person noncooperative games," *Journal of Mathematical Economics*, volume 31, pages 361-372.

[25] Yu, J., and H. Yang (2004): "Essential components of the set of equilibrium points for set-valued maps," *Journal of Mathematical Analysis and Applications*, volume 300, pages 334-342.

[26] Yu, J., H. Yang, and S. Xiang (2005): "Unified approach to existence and stability of essential components," *Nonlinear Analysis*, volume 63, pages 2415-2425.

[27] Yu, X. (2009): "Essential components of the set of equilibrium points for generalized games in the uniform topological space of best reply correspondences," *International Journal of Pure and Applied Mathematics*, volume 55, pages 349-357.

[28] Zhou, Y., J.Yu, and S.Xiang (2007): "Essential stability in games with infinitely many pure strategies," *International Journal of Game Theory*, volume 35, pages 493-503.

[29] Wu, W., and J.H. Jiang (1962): 'Essential equilibrium points of n-person noncooperative games," *Scientia Sinica,* volume XI, pages 1307-1322.