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Abstract: This paper develops a non-equilibrium dynamic model (NEDyM) with Keynesian features (it allows for a disequilibrium between output and demand and it considers a constant marginal propensity to consume), but where production is undertaken under plain neoclassical conditions (a constant returns to scale production function, with the stocks of capital and labor fully employed, is assumed). The model involves only two endogenous / prognostic variables: the stock of physical capital per unit of labor and a goods inventory measure. The two-dimensional system allows for a careful analysis of local and global dynamics. Points of bifurcation and long-term cyclical motion are identified. The main conclusion is that the disequilibrium hypothesis leads to persistent fluctuations generated by intrinsic deterministic factors.

Keywords: NEDyM, Endogenous business cycles, Nonlinear growth, Keynesian macroeconomics, Cyclical dynamics and chaos.

JEL classification: E32, E12, O41, C62.

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1. Introduction

The long lasting debate on macroeconomics about the sources of business cycles has been built upon successive disagreements and also some consensus. The Keynesian tradition, opposed to the classical view of market clearing markets and external shocks over fundamentals, stresses the presence of disequilibria in the economic system. Firms and households, instead of choosing optimally, often use rules of thumb when deciding about price adjustments, how much to invest, how to distribute consumption over time or how to allocate time between work and leisure.

This paper analyzes a two dimensional macroeconomic model that combines classical and Keynesian features. The model is dynamic and purely deterministic. The main structure of the model is based on Hallegatte *et. al.* (2007) (hereafter HGDH), who present a problem designated as NEDyM (non-equilibrium dynamic model). Our aim, as the one in HGDH, is to obtain a long-term outcome where, depending on the particular economic scenario that is furnished by a given array of parameter values, we can have both a fixed-point balanced growth outcome (as in the neoclassical growth model) and endogenous fluctuations generated by the non-linear nature of the relation between endogenous variables (as in a Keynesian disequilibrium setup).

According to HGDH, a NEDyM is a growth model built upon a standard Solow (1956) model, but where multiple inefficiencies arise in the several markets that are considered. In this analysis, agents do not have perfect foresight and markets do not clear, and the main reason pointed out for such is the inertia that the economic system undergoes. Inertia implies a delay on the adjustment between production and demand, on one hand, and, on the other hand, a suboptimal investment process. Investment decisions are linked with short-run profits and these may give signs that differ from the reality attached to the long-term optimal scenario. Furthermore, the labor market is subject to relevant inefficiencies, which are translated into a Phillips curve that relates nominal wages with labor supply. Consumer decisions are not optimal, instead they depend on the available stock of real balances and on the Solow's constant rate of savings.

The HGDH model is, therefore, a large collection of Keynesian relations built upon a minimal classical growth structure; this consists just on a production function that fully employs available inputs and on a conventional capital accumulation difference equation. The authors are able to find a route to chaotic motion and, thus, for different parameter values, it is analytically possible to observe a fixed-point stable equilibrium or cycles of any periodicity and completely a-periodic cycles. Such a co-existence can be interpreted under the idea that, for certain arrays of parameters, classical economics dominate, while for others the inertia factors become sufficiently relevant in order to generate endogenous business cycles.

By modelling simultaneously the dynamics of the goods market, the labor market, the behavior of firms with investment as a function of profits and the behavior of households as a function of real balances, the problem proposed by HGDH becomes an 8-dimensional system with 8 endogenous variables (or prognostic variables, as the authors refer to them). Additionally, 11 other variables (diagnostic variables) are modelled as functions of the endogenous state variables. With such a high dimension, the problem cannot be analyzed in general terms; only through numerical particular examples one may infer about the behavior of the economy. Thus, what the authors gain in terms of completeness they evidently lose in what concerns tractability.

Here, the main distinction relatively to the analysis of HGDH, is that our model is more compact (it is just a two-dimensional model), allowing for the general analysis of local dynamics, as well as for the investigation of the long-term global asymptotic behavior of the assumed endogenous variables.

The features we maintain in this version of the NEDyM are, on one hand, the neoclassical production function and the capital accumulation process that is present in any growth optimization problem and, on the other hand, the most relevant Keynesian features; basically, we assume, as in the HGDH model, that an element of inertia is present in the goods market: production and demand are not always adjusted to one another, and thus a market disequilibrium persists in time. This implies the need to assume a goods inventory variable, which plays a fundamental role in the obtained results.

Differently from the HGDH model, investment and consumption decisions are not explicitly modelled; instead, consumption is given just as a constant share of income (the good old constant marginal propensity to consume is taken into account), while investment is obtained by default as the second component of demand; to the value of demand one arrives after analytically presenting relations concerning aggregate demand and aggregate supply. Aggregate demand is modelled exactly as in HGDH, while aggregate supply is a textbook Phillips curve defined in terms of inflation rate and output gap. The analysis of the labor market is neglected, by assuming that a fixed amount of labor is in every moment available to produce.

The framework that arises from the previous assumptions is a two dimensional deterministic system with physical capital (per unit of labor) and goods inventory (per unit of labor) as endogenous variables. Relatively to this model, one can address both local and global dynamics. Local analysis allows for perceiving that bifurcation points are eventually crossed, a necessary requirement to encounter long-term nonlinear motion. The global analysis, although less generic, confirms the generation of areas of endogenous cycles, that occur with a flip bifurcation. As in the HGDH problem, areas of fixed point stability can be interpreted as representing the balanced growth path that is characteristic of classical growth models, while regions where complex behavior is evidenced are the ones where the Keynesian features of the model (inertia, production-demand lack of alignment, constant propensity to consume, Phillips curve relation) become dominant. The main additional contribution that the present paper achieves is that it is able to obtain such a set of results without departing from a simple two-equation model, the dimension in which most of the classical models are explored (e.g., the Ramsey capital-consumption model).

The remainder of the paper is organized as follows. Section 2 highlights the relevance of business cycles as the main point of controversy between macroeconomic schools; this motivates the development of the model. Section 3 presents the basic structure of the NEDyM. Section 4 characterizes local dynamics. In section 5, specific functional forms for the neoclassical production function are proposed in order to obtain additional, more concrete, results. Section 6 explores global dynamics for a reasonable calibration of parameter values. Finally, section 7 concludes. Proofs of propositions are left to a final appendix.

2. General Overview of the Literature: Classical and Keynesian Macroeconomics

Since its birth, macroeconomic theory has evolved through the systematic debate among Keynesians and classics. In a brilliant survey, Mankiw (2006) describes the several stages of this debate. The early stages marked a clear disagreement in the way aggregate phenomena were understood; later on, some consensus was pursued. In order to better motivate the model we develop in the paper (a Keynesian model built upon a neoclassical growth structure), we begin by briefly reviewing this debate.

Early Keynesian theorists, as Hicks (1937), Modigliani (1944) and Samuelson and Solow (1960) have presented the fundamental structure of the Keynesian economic analysis; macro relations are driven by a set of disequilibrium equations that cannot be understood just by looking at the behavior of an average or a representative agent. Under this view, macroeconomics is a reality on its own, i.e., the established relations (e.g., IS, LM and Phillips curves) are understood only when the economy as a whole is considered. Prices do not adjust automatically, markets do not clear instantaneously; incomplete information, coordination failures and other inefficiencies give rise to relations between variables that depart from the appealing economic notions of equilibrium and efficiency. This view of the reality easily leads to the conclusion that business cycles are intrinsic to the economic performance of countries or regions. Early endogenous cycles' models built under the inspiration of Keynesian economics include Kalecki (1937), Harrod (1939), Kaldor (1940) and Goodwin (1951).

Modern (neo)classical macroeconomics has emerged around the beginning of the second half of the twentieth century. The influential work of Arrow and Debreu (1954) on the general equilibrium theory has influenced a whole new generation of economists, who gained access to the tools that allowed to question the macroeconomic science as it was born (as a Keynesian science). Optimization, market clearing and an overall idea of the invisible hand working at the aggregate level took over the mainstream economic thought. Lucas (1976) and Sargent and Wallace (1975) caused serious damage to Keynesian economics by pointing its inability to deal with policy analysis; at the heart of this critique is the rational expectations revolution. The impact of the classical macroeconomics was such that it led Lucas (1980) to announce the death of Keynesian economics.

As the Keynesian paradigm fell in disgrace in the early 1980s, a new paradigm became necessary to replace the previous view about how the economy behaves in the short-run. If cycles do exist, but macro relations should not be modelled as the result of market inefficiencies because these are not compatible with the optimizing behavior of well informed and rational agents, how can we find a reasonable explanation for observed economic fluctuations?

To this central question, Kydland and Prescott (1982) and Long and Plosser (1983) have answered with the theory of real business cycles (RBC). The RBC theory is clearly classical in its nature; prices adjust automatically, markets always clear and the basic problem consists on a representative household intertemporal optimization problem, where the utility withdrawn from consumption and leisure is maximized under an infinite horizon, given some rate of time preference. Cycles are triggered by stochastic disturbances on technology or public expenditures, and these induce optimal reallocations of the representative agent's time between work and leisure. Fluctuations are, thus, the result of the changes in the amount of available labor force that occur without the need of departing from market efficiency.

On their side, classical macroeconomists have always had the argument that Keynesian functional relations are established in an ad-hoc way (no micro fundamentals are addressed); this is the argument used by Friedman (1957) and Phelps (1968) to cause the first important damage on the Keynesian paradigm, when they criticized the empirical plausibility of the Keynesian consumption function and of the Phillips curve. Against classicals is the empirical lack of reasonability of the market clearing hypothesis, for instance in what concerns the labor market. Thus, Keynesian economics have reawakened in the mid 1980s, with the work of Calvo (1983), Mankiw (1985) and Akerlof and Yellen (1985), among others. This new work was a first approach in the direction of a consensus; while maintaining at the core of the analysis the idea that prices adjust sluggishly, now economists began the quest for micro explanations for such stickiness. Here we find the menu costs and efficiency wages theories, among others in which markets are modelled as non competitive entities.

While most of the contemporaneous macroeconomics continues to rely on optimization and the representative agent behavior, the referred consensus has evolved as well. The two visions of aggregate fluctuations have nowadays almost merged by leaving behind the most unappealing features and by integrating their strong points. A sign of this merger on thought is that while ones call the new paradigm 'the new neoclassical synthesis' [Goodfriend and King (1997)], others attribute to the same set of notions the designation of 'a new Keynesian perspective' [Clarida *et. al.* (1999)]. The new synthesis model is nothing more than a dynamic general equilibrium model with nominal rigidities, and this brings us back to what the birth of macroeconomics was all about: to explain how labor, monetary and goods markets interact in a world where market clearing is absent at the aggregate level.

The previous appointments about the macroeconomics debate allow us to clearly realize that the main point of controversy in this field relates to the sources of business cycles. The modern approaches to the study of cyclical motion are of various types. Monetary policy analysis has been an important field for the advancement of this debate. A simple aggregative model, involving a dynamic IS equation and a Phillips curve derived from microeconomic principles, is today a benchmark framework in the analysis of many macro policy related questions [the model is thoroughly developed in Woodford (2003)].

Other strand of literature that pursued compatibility between the two viewpoints is known as the endogenous business cycles (EBC) literature. This explores general equilibrium models to which it is possible to attach some kind of market inefficiency capable of producing nonlinear relations among variables, that in analytical terms translate in deterministic time series fluctuations that are perpetuated in the long-term.

The origins of this approach can be traced back to Medio (1979), Stutzer (1980), Benhabib and Day (1981), Day (1982), Grandmont (1985), Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986). The main concept had to do with the idea of competitive chaos; that is, using the benchmark optimization problem (intertemporal or with overlapping generations) and introducing some slight changes to the conventional presentation (e.g., by changing the shape of the production function), endogenous cycles were generated. These cycles corresponded, most of the times, to chaotic time series, i.e., time series exhibiting sensitive dependence on initial conditions.

The notion of competitive chaos has been further developed by a group of mathematical economists, who claim that nonlinearities can be found in conventional dynamic classical models without the need of considering any kind of inefficiency. In Nishimura *et. al.* (1994) and Nishimura and Yano (1994, 1995), among others, extreme conditions under which the competitive growth scenario can generate long term nonlinear motion are addressed (e.g., unrealistically high intertemporal discount rates).

The literature on EBC has gained an important new breath with the model by Christiano and Harrison (1999), who proved the existence of chaos in a standard deterministic RBC model with production externalities. This line of research, where a utility maximization control problem is taken into account (and where consumption and leisure are the arguments of the utility function), has been further developed by Schmitt-Grohé (2000), Guo and Lansing (2002), Goenka and Poulsen (2004), Coury and Wen (2005) and others. A similar strand of literature is the one that investigates the presence of bifurcation points and nonlinearities in overlapping generations growth models, which are also subject to production technologies showing increasing returns. This work is associated with the following references: Cazavillan *et. al.* (1998), Aloi *et. al.* (2000), Cazavillan and Pintus (2004) and Lloyd-Braga *et. al.* (2007), among others.

The models in the EBC literature are simple general equilibrium models, which assume that the individual firm faces a positive external effect from production in society; thus, the representative firm faces an increasing returns technology. This simple idea is capable of producing endogenous cycles, but it leaves unanswered many of the questions raised by Keynesian macroeconomics. Other authors search for additional features capable of introducing nonlinear dynamics in simple optimization models; one alternative to the benchmark model consists in departing from the idea of a representative agent. This is done in Goeree and Hommes (2000) and Onozaki *et. al.* (2000, 2003), who develop macro heterogeneous agents models. Other hypothesis has to do with learning mechanisms; Cellarier (2006) replaces the optimal planner problem by a constant gain learning mechanism that generates endogenous fluctuations.

Closer to a Keynesian setup is the analysis of Dosi *et. al.* (2006). These authors develop a model where endogenous fluctuations are the result of the way firms behave. In accordance with what empirical evidence shows, investment decisions are lumpy and constrained by the financial structure of firms; moreover, firms are boundedly rational when forming expectations about future events. Additional ingredients of Keynesian nature are added by Hallegatte *et. al.* (2007), who introduce the term NEDyM, and mix classical and Keynesian features in a way we explore further in the next sections.

3. The NEDyM: Basic Structure

Consider an economy populated by a large number of households and firms. Households consume, in each time moment t=0,1,…, a constant share of the available income, $c_t=b\cdot y_t$; variables $c_t\in\mathbb{R}_+$ and $y_t\in\mathbb{R}_+$ represent real per capita consumption and real per capita income, respectively. We assume that a constant amount of labor is available to produce (and, to simplify, that this coincides with total population); normalizing this quantity to unity, there is a coincidence between per capita values, per labor unit values and level values. We will, interchangeably, use any of these terms. Parameter $b\in(0,1)$ respects to a constant marginal propensity of consumption.

Output or income is generated by a neoclassical production function of the type $y_t=f(k_t)$, with $k_t\in\mathbb{R}_+$ physical capital per unit of labor. In section 5, particular results are derived for two specific functional forms of the production function. For now, we just postulate that this is a neoclassical production function, by assuming that:

a) f is a continuous and differentiable function ($f\in \mathbf{C}^2$) and it exhibits positive and diminishing marginal returns: $f'>0, f''<0$.

b) Inada conditions are satisfied: $\lim_{k\to 0} f' = +\infty$; $\lim_{k\to+\infty} f' = 0$.

The accumulation of capital is driven by a process of investment. Letting $i_t\in\mathbb{R}_+$ represent investment per labor unit and $\delta>0$ a capital depreciation rate, the process of capital accumulation is given by equation (1).

$$k_{t+1} - k_t = i_t - \delta \cdot k_t,\ k_0 \text{ given.} \tag{1}$$

Essential to the characterization of the capital accumulation process is the rule that establishes the evolution of investment over time. This will emerge from the assumption that the goods market does not clear, i.e., that a disequilibrium between output and demand persists over time. Here, we follow closely HGDH, who explain the misalignment between y_t and demand ($d_t = c_t + i_t$) through the introduction of a goods inventory variable, $h_t \in \mathbb{R}$ (this variable, as all the others, is defined in per capita terms).

The dynamics of the goods inventory is determined by the difference between production and demand and, therefore, it can assume both positive and negative values. In the case of a positive inventory, $h_t > 0$, there is a selling lag, i.e., temporary overproduction exists, which can be the result, for instance, of the time needed to sell the goods. A negative inventory, $h_t < 0$, indicates the presence of underproduction or a delivery lag, and can be interpreted as the time required for the consumer to get the goods she ordered.

Selling and delivery lags may be interpreted as a normal fact of economic activity, but additionally they can be thought as the result of the presence of inertia that turns difficult to change the productive capacity that exists in a given moment. An equilibrium situation will be the one in which $h_t = 0$, a scenario that characterizes a competitive market. In the developed model, equilibrium will not necessarily exist in the long-term, i.e., the system may converge to a steady state where despite the coincidence between output and demand, there is systematic under or overproduction.

As stated, changes in the aggregate level of inventories are the result of the difference between output and demand; this is expressed in equation (2),

$$h_{t+1} - h_t = y_t - d_t,\ h_0 \text{ given.} \tag{2}$$

Equation (2) just states that when output is above demand, inventories rise; they will fall in the opposite circumstance.

To complete the model, we need an aggregate demand and an aggregate supply relations. The aggregate demand equation is similar to the one in HGDH, that is, we assume that price changes are determined by the goods inventory per unit of demand:

$$p_{t+1} - p_t = -\theta \cdot \frac{h_t}{d_t} \cdot p_t \text{, } p_0 \text{ given, } \theta > 0. \tag{3}$$

The interpretation of expression (3) is straightforward: for a positive inventory there is a delay in the selling of production, meaning that market power is on the side of the consumers, who force prices ($p_t \in \mathbb{R}_+$) to decrease. If, otherwise, the goods inventory is a negative amount, then the lag is on delivery, making sellers to concentrate market power, and therefore producers can trigger a rise in prices. Defining $\pi_t \equiv \frac{p_{t+1} - p_t}{p_t}$, $\pi_t \in \mathbb{R}$, as the inflation rate, equation (3) is rewritten as,

$$\pi_t = -\theta \cdot \frac{h_t}{d_t} \tag{4}$$

On the supply side, we consider a trivial Phillips curve defined in terms of inflation and output gap,

$$\pi_t = \lambda \cdot x_t \tag{5}$$

In equation (5), the output gap, $x_t \in \mathbb{R}$, is the difference between the logarithm of effective output and the logarithm of potential output (or, similarly, the logarithm of the ratio between effective and potential output): $x_t \equiv \ln y_t - \ln y^*$. Potential output, y^*, will be assumed as the long-run value of output under optimality conditions (it will be interpreted, in sections 5 and 6, as the steady state value of income that is derived from a standard neoclassical optimization problem with intertemporal consumption utility maximization). Parameter $\lambda \in (0,1)$ describes a measure of price flexibility; if the value of λ is close to zero, prices are sticky or sluggish to adjust. At the light of the literature on new Keynesian monetary policy analysis [e.g., Woodford (2003)], equation (5) corresponds to a new Keynesian Phillips curve under two extreme simplifying assumptions: random disturbances upon supply are absent (recall that we are concerned with fluctuations produced under conditions of full determinism) and it is implicitly assumed that agents expect prices not to grow in the future (expectations about future inflation do not impact over the contemporaneous inflation value).

Combining equations (4) and (5), the following expression for demand is obtained:

$$d_t = -\frac{\theta}{\lambda} \cdot \frac{h_t}{\ln y_t - \ln y^*} \quad (6)$$

Positive levels of demand require one of the two following scenarios:

i) The goods inventory is positive and the effective level of output is smaller than the potential level;

ii) The goods inventory is negative and the effective level of output is larger than the potential level.

Thus, we are stating that periods of recession (defined as the time periods in which the output gap is negative) are periods of temporary overproduction: producers want to sell the generated goods, but demand is too low to cover such requirement; in this way, excess supply is characteristic of periods of recession. Periods of expansion, in which the output gap is positive, are periods of underproduction or negative goods inventory: people want to buy more, but the selling capacity is constrained, what originates delivery lags or, on other words, an excess demand.

Note, in equation (6), that parameters θ and λ are closely linked, that is, it is their ratio that determines the value of demand, and thus the impact of one of them is precisely symmetric to the impact of the other. Demand rises with a higher sensitivity of prices to inventory changes ($\theta\uparrow$) and with a lower degree of price flexibility ($\lambda\downarrow$).

Observe that market clearing exists when h_t=0 and x_t=0, and therefore the demand equilibrium level cannot be withdrawn from equation (6). In the hypothetical market clearing situation, because the goods inventory is constant and equal to zero, dynamic relation (2) states that demand is equal to the level of income (which coincides with output). Therefore, equation (6) is a disequilibrium relation, that allows to know the value of aggregate demand when effective and potential output differ (and, thus, a non zero inventory value holds). Consequently, by establishing a coincidence between the growth of the price level in the aggregate demand equation (4) and the aggregate supply equation (5) we are not stating an equilibrium condition; on the contrary, we are presenting a relation that measures the excess demand or excess supply in the market.

Note that the Walrasian equilibrium requires no price change ($\pi_t = 0$), since in this circumstance inventories are zero and potential and effective output are identical. A positive price change is found for a positive output gap and deflation will exist in scenarios of negative output gap. These remarks are just the interpretation of the Phillips equation in (5). A more realistic approach would require adding a constant positive level of inflation to the right hand side of the Phillips curve. In this way, a zero output gap would not mean zero inflation, and we could have a recession (negative output gap) without having necessarily a scenario of deflation. We omit this parameter, since it would add no new relevant information and it would just introduce an additional innocuous element to the structure of the model.

We are now in conditions of stating the dynamic problem,

Definition 1. The NEDyM. The two-dimensional growth system, that combines Keynesian and classical features, is composed by equations (1) and (2). In equation (2), output originates on a neoclassical production function and demand can be obtained through the combination of equations (4) and (5) that describe aggregate market conditions. Investment, in equation (1), is given by the difference between demand, in equation (6), and consumption, which is defined as a constant share of income.

Relatively to the problem in the definition, note two things:

i) The described dynamical system is not only a two equations system, it also has only two endogenous variables: capital and the goods inventory;

ii) As referred in the introduction, there is a clear co-existence between Keynesian and classical elements. The first relate to the shape of the consumption function, the lack of equilibrium between output and demand and the consideration of a Phillips curve; the second are present in the shape of the production function and on how capital accumulation is modelled.

4. Results on Local Stability

The low dimensionality of the model allows for obtaining some generic local dynamic results. We begin by characterizing the steady state. This is defined as follows,

Definition 2. Steady state. A steady state or balanced growth path is a set $\{\bar{k},\bar{h},\bar{y},\bar{d},\bar{\pi},\bar{c},\bar{i}\}$ of constant values, which can be determined by imposing conditions $\bar{k} \equiv k_{t+1} = k_t$ and $\bar{h} \equiv h_{t+1} = h_t$ to equations (1) and (2).

By applying definition 2, it is straightforward to arrive to the following outcome,

Proposition 1. *The steady state exists, it is unique and it is characterized by the group of relations that follows: i)* $\bar{y}=\bar{d}$ *; ii)* $\frac{f(\bar{k})}{\bar{k}}=\frac{\delta}{1-b}$*; iii)* $\bar{\pi}=\lambda\cdot\ln\left(\frac{f(\bar{k})}{y^*}\right)$*; iv)* $\bar{h}=-\frac{\lambda}{\theta}\cdot\ln\left(\frac{f(\bar{k})}{y^*}\right)\cdot f(\bar{k})$*; v)* $\bar{c}=b\cdot f(\bar{k})$*; vi)* $\bar{i}=(1-b)\cdot f(\bar{k})$.

Proof: see appendix.

The steady state relations deserve some comments: first, note that independently of the long-term value of inventories, production and demand assume identical values; second, the average product of capital is constant in the steady state and it is as much higher as the larger are the values of the depreciation rate and of the marginal propensity to consume; third, prices rise in the long-run if a positive output gap persists and decline otherwise; fourth, the goods inventory is not only negative for a positive output gap, but it is also as more negative as the larger is the value of the effective output (a symmetric result can be established); fifth, because demand and income are identical in the long-run, investment can be expressed in the form of income times a constant marginal propensity to save (i.e., in the long-run, households' savings are integrally used by firms in their investment projects).

To study local dynamics, one needs to linearize the system in the vicinity of $\{\bar{k},\bar{h}\}$. The linearized system is

$$\begin{bmatrix} k_{t+1}-\bar{k} \\ h_{t+1}-\bar{h} \end{bmatrix} = \mathbf{J}\cdot\begin{bmatrix} k_t-\bar{k} \\ h_t-\bar{h} \end{bmatrix}, \text{ with } \mathbf{J}=\begin{bmatrix} 1-\delta-\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k}) & -\frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}} \\ \left(1+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k}) & 1+\frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}} \end{bmatrix} \quad (7)$$

with $\bar{x}=\ln\left(\bar{y}/y^*\right)$.

An important result regarding local stability is presented in proposition 2.

Proposition 2. *The existence of a negative output gap is a necessary condition for local asymptotic stability.*

Proof: see appendix.

In the chosen terminology, the expression 'local asymptotic stability' refers to any circumstance in which there is a coincidence between the stable eigenspace and the state space of the system. In other words, the term is associated to the case in which the two eigenvalues of the Jacobian matrix in (7) lie inside the unit circle. This result is independent of how the convergence to the steady state takes place: monotonically (if the two eigenvalues are real and positive), through improper oscillations (if the two eigenvalues are real and they are not both positive) or through a spiral movement with decreasing amplitude in time (if the eigenvalues are a pair of complex values). Proposition 3 makes the distinction between node stability and focus stability.

Proposition 3. *Assume that* $\bar{x} < 0$. *If a stable fixed-point exists, this corresponds to a stable node if the following condition is satisfied:*

$$\frac{1}{(2\bar{x})^2}\cdot\left(\frac{\theta}{\lambda}\right)^2-\left[1-\frac{\delta}{2}-\frac{1}{2}\cdot\left(b-\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})\right]\cdot\frac{1}{\bar{x}}\cdot\frac{\theta}{\lambda}$$
$$-1+\delta+\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})+\left[1-\frac{\delta}{2}-\frac{1}{2}\cdot\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})\right]^2>0$$

If the above inequality is of opposite sign, it becomes a necessary condition for the equilibrium point to be a stable focus (i.e., for the convergence to the stable equilibrium to occur in spiral).

Proof: see appendix.

Two remarks about proposition 3: first, we reemphasize that the presented condition is a necessary condition for the fixed point to be a stable node (we have not yet imposed additional conditions that ensure the presence of asymptotic stability). Second, the expression in the proposition was presented in such a way that it can be solved for the ratio θ/λ. Relatively to this quotient, the necessary stable node condition will be the area above some parabola.

Sufficient conditions for local asymptotic stability are the ones in proposition 4.

Proposition 4. *Local asymptotic stability holds if, besides* $\bar{x} < 0$, *the following inequalities are satisfied,* *i)* $\left[2-\delta+(1-b)\cdot f'(\bar{k})\right]\cdot\frac{1}{\bar{x}}\cdot\frac{\theta}{\lambda}+4-2\delta+2\cdot\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})>0$;

ii) $\left[1-\delta+(1-b)\cdot f'(\bar{k})\right]\cdot\frac{1}{\bar{x}}\cdot\frac{\theta}{\lambda}-\delta-\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})<0$.

Proof: see appendix.

Compiling the results in propositions 2 to 4, the stability result is the following: the fixed point is a stable node if all the displayed conditions in propositions 2 to 4 are satisfied; the fixed point is a stable focus if the inequality in proposition 3 is of opposite sign and the other referred conditions hold.

In what concerns the value of the ratio θ/λ, proposition 4 has the following corollary,

Corollary of proposition 4. *Consider again* $\bar{x}<0$. *Stability requires the ratio between the price-inventory sensitivity parameter and the price stickiness parameter to be bounded from below and from above:*

$$\frac{\theta}{\lambda}\in\left(\frac{\delta+\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})}{1-\delta+(1-b)\cdot f'(\bar{k})}\cdot\bar{x};-\frac{4-2\delta+2\cdot\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})}{2-\delta+(1-b)\cdot f'(\bar{k})}\cdot\bar{x}\right).$$

Evidently, the lower bound will be zero if the first value of the set is negative.

Proof: see appendix.

Regarding the absence of stability,

Proposition 5. *In the case* $\bar{x}<0$, *two additional local dynamic results are obtainable, besides asymptotic stability,*

- *Saddle-path stability, under* $\frac{\theta}{\lambda}>-\frac{4-2\delta+2\cdot\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})}{2-\delta+(1-b)\cdot f'(\bar{k})}\cdot\bar{x}$;
- *Instability, under* $\frac{\theta}{\lambda}<\frac{\delta+\left(b+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})}{1-\delta+(1-b)\cdot f'(\bar{k})}\cdot\bar{x}$.

Proof: see appendix.

The transition of regions of asymptotic stability to saddle-path stability or instability implies that bifurcation points are crossed. The point in which asymptotic stability gives place to saddle-path stability corresponds to a flip bifurcation point. In this case, one of the eigenvalues assumes the value -1, while the other remains inside the unit circle. A Neimark-Sacker bifurcation occurs in the transition between the stability area and the area in which the eigenvalues are complex with modulus higher than one. Note that, according to proposition 5 (or the corollary of proposition 4), the unique required condition for any of the bifurcations to occur is that the specified border values of θ/λ must be higher than zero.

Let us turn to the case in which the output gap is positive. In this case, one of the eigenvalues of **J** is always higher than 1 and, therefore, asymptotic stability is absent. Proposition 6 states the possible local dynamic results.

Proposition 6. *Let* $\bar{x} > 0$. *Saddle-path stability holds for a value of* θ/λ *inside the set presented in the corollary of proposition 4. For values of* θ/λ *outside the set, instability prevails.*

Proof: see appendix.

The instability result may correspond to two different time trajectories, depending on the stability condition that is violated. If $1 + Tr(\mathbf{J}) + Det(\mathbf{J}) < 0$, along with $1 - Tr(\mathbf{J}) + Det(\mathbf{J}) < 0$, then one of the eigenvalues is higher than 1 and the other lower than -1, and they are both real values. In this case, the trajectories will oscillate improperly as the system departs from the fixed-point. When the determinant of the Jacobian matrix is above unity, the divergence process is determined by the existence of an unstable focus fixed-point.

A better understanding of the previous set of results is achieved through a graphical illustration of the stability possibilities. Figure 1 is a diagram that relates the values of the trace and the determinant.

*** Figure 1 ***

In figure 1, we draw the three bifurcation lines; the area inside the inverted triangle formed by these three lines is the area of stability. The two bold lines represent the two cases in proposition 2: the one in which asymptotic stability is possible (to the left of the bifurcation line $1-Tr(\mathbf{J})+Det(\mathbf{J})=0$) and the one in which asymptotic stability is not admissible (to the right of this bifurcation line). In the first case, asymptotic stability can give place to a saddle-path result, if the flip bifurcation line is crossed; instability also arises for values of parameters such that the determinant of the Jacobian matrix becomes a value higher than 1. When the condition $1-Tr(\mathbf{J})+Det(\mathbf{J})>0$ is no longer verified, saddle-path stability holds as long as the other two stability conditions hold. Otherwise, if any of such conditions fails to hold, asymptotic instability will prevail according to what was established in proposition 6.

The stability case is straightforward to characterize from a dynamic analysis point of view. Independently of the initial state of the system (k_0,h_0), if this is in the vicinity of the steady state, then both variables will converge to the long-term steady state. Such result is coincident with the neoclassical growth outcome of a balanced growth path: given the decreasing returns to capital, the economy converges to a constant long term value of capital and output (and, consequently, constant levels of consumption and investment). The main difference relatively to the neoclassical model is that this outcome is achieved for a level of output below the optimal (this is an intuitive result if we recall that we have introduced a series of inefficiencies in our formulation) and for a steady state goods inventory that is above zero (some of the produced output is never sold, which is also a reflection of our model's inefficiencies).

From the point of view of local analysis, the situation of saddle-path stability delivers some interesting results. Thus, let us suppose that $\bar{y} > y^*$ and that the condition in the corollary of proposition 4 holds.

Proposition 7. *If the system is saddle-path stable, the saddle trajectory is*

$$h_t - \bar{h} = -\frac{\left(1+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})}{1-\varepsilon_1+\frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}}}\cdot\left(k_t-\bar{k}\right) \tag{8}$$

with ε_1 the eigenvalue of $\mathbf{J}$ inside the unit circle.

Proof: see appendix.

If $\bar{x} > 0$, the stable trajectory in (8) is negatively sloped, meaning that if the convergence to the steady state is done through the saddle trajectory, then as the amount of capital rises, the goods inventory declines (or vice-versa).

The steady state may be disturbed by changes in any of the parameter values. For instance, if the prices become more sluggish (if λ falls), we know from proposition 1 that the steady state stock of capital remains unchanged, while the goods inventory becomes smaller. From (8), the slope of the stable trajectory decreases in absolute value, that is, the trajectory becomes flatter. Therefore, when prices become stickier, this will reduce the long-term level of inventories (that in the considered case are positive) and the impact over the convergence to the steady state is such that for a given change in the stock of capital, the change in the goods inventory will be less pronounced. Figure 2 illustrates the case.

*** Figure 2 ***

5. Specific Production Functions

To better understand the dynamics of the two dimensional NEDyM model, we now adopt two explicit functional forms for the production function. We also consider a specific value for the potential output. Potential output is defined as the steady state level of output that can be derived from an optimal control problem of utility maximization. We present this problem as

$$Max \sum_{t=0}^{+\infty} \beta^t \cdot U(c_t) \text{ subject to } k_{t+1} - k_t = f(k_t) - c_t - \delta \cdot k_t, k_0 \text{ given} \quad (9)$$

with $\beta \in (0,1)$ the discount factor.

Taking a simple logarithmic utility function, $U(c_t) = \ln c_t$, the computation of first order conditions of this Ramsey problem leads to the well known equation of motion for consumption $c_{t+1} = \beta \cdot (1 - \delta + f'(k_t)) \cdot c_t$. The evaluation of this equation in the steady state will give us the optimal long-term constant value of capital, which obeys to $f'(k^*) = 1/\beta - (1-\delta)$. Potential output is, then, defined as $y^* = f(k^*)$.

5.1 Cobb-Douglas Production Function

The first case we consider takes a Cobb-Douglas production function: $f(k_t) = A \cdot k_t^{\alpha}$. Parameter $A>0$ is a technological index, and $\alpha \in (0,1)$ is the output-capital elasticity. With this production function, the potential output is explicitly presentable as $y^* = A^{1/\alpha} \cdot \left(\frac{\alpha}{1/\beta - (1-\delta)} \right)^{\alpha/(1-\alpha)}$.

Now, the steady state results can all be given as functions of the assumed array of parameters. Recall from proposition 1 that, in the steady state, income and demand are identical, what allows for presenting the steady state stock of capital as $\bar{k} = \left(\frac{(1-b) \cdot A}{\delta} \right)^{1/(1-\alpha)}$. The long-term capital stock rises with the level of technology and with the output-capital elasticity and it falls as the marginal propensity to consume and the depreciation rate increase.

Consumption and investment are, respectively, given by $\bar{c} = b \cdot A^{1/(1-\alpha)} \cdot \left(\frac{1-b}{\delta} \right)^{\alpha/(1-\alpha)}$ and $\bar{i} = \left[(1-b) \cdot A \right]^{1/(1-\alpha)} \cdot \delta^{-\alpha/(1-\alpha)}$. Both consumption and investment steady state levels benefit from a better technology level and from a lower rate of capital depreciation. The impact of the propensity to consume over steady state investment is also unequivocal (a higher *b* damages the long-term capacity to invest), but it is not so straightforward in terms of long term consumption; computing the derivative of the steady state consumption level in order to *b*, one gets a positive value for $b<1-\alpha$; hence, we conclude that the marginal propensity to consume benefits long-run consumption only if this constant is lower than the output-labor elasticity. A higher *b* means that too many resources are withdrawn from the productive process in order to guarantee that consumption rises with an increasing share of consumption.

In what concerns goods inventories, the steady state becomes $\bar{h} = -\frac{\lambda}{\theta} \cdot \frac{\alpha}{1-\alpha} \cdot A^{1/(1-\alpha)} \cdot \left(\frac{1-b}{\delta} \right)^{\alpha/(1-\alpha)} \cdot \ln \left(\frac{(1-b) \cdot (1/\beta - (1-\delta))}{\alpha \cdot \delta} \right)$. Finally, we can look at the steady state inflation rate: $\bar{\pi} = \frac{\alpha \cdot \lambda}{1-\alpha} \cdot \ln \left(\frac{(1-b) \cdot (1/\beta - (1-\delta))}{\alpha \cdot \delta} \right)$. As in the general case, the most meaningful result regarding this steady state value is the fact that prices rise with a positive output gap and decline otherwise. Relatively to the last two steady state results, it is not straightforward to perceive the impact of some of the parameters over those results. Note, as an illustration, the role of the discount factor: the more intensely future is discounted (lower β) the higher is inflation (if this is positive) or the lower is deflation (for a negative price rise); similarly, a higher discount rate lowers the goods inventory level (if it is positive, it becomes closer to zero; if it is negative, it falls even more).

Concerning the sign of the output gap, we have the following result,

Proposition 8. *A positive steady state output gap requires* $b < \frac{(1-\beta)/\beta + \delta \cdot (1-\alpha)}{(1-\beta)/\beta + \delta}$.

Proof: see appendix.

According to the stability results in section 4, the sign of the output gap is of fundamental importance. Asymptotic stability requires a negative output gap and, thus, for constant values of β, δ and α, stability is found for a relatively high value of the marginal propensity to consume.

To address local dynamics, we should note that for the specific technology under appreciation, the steady state marginal product of capital is $f'(\bar{k}) = \frac{\alpha \cdot \delta}{1-b}$. Replacing this, and the several steady state values, in the propositions of section 4, we would obtain conditions for the characterization of local stability. Since this exercise does not add much information to the precedent generic results, we just remark that the relation in figure 1 between the trace and the determinant of matrix **J** is, with the Cobb-Douglas technology,

$$Det(\mathbf{J}) = Tr(\mathbf{J}) - 1 - \frac{\theta \cdot \delta \cdot (1-\alpha)^2}{\alpha \cdot \lambda \cdot \ln\left(\frac{(1-b)\cdot(1/\beta - (1-\delta))}{\alpha \cdot \delta}\right)} \quad (10)$$

A negative output gap will allow (10) to cross the stability area.

The exploration of a numerical example conducts to more tractable results. The calibration in table 1 is considered.

Parameter	Value	Source
A	1	Cellarier (2006)
α	1/3	Hallegate *et. al.* (2007)
δ	0.067	Guo and Lansing (2002)
β	0.962	Guo and Lansing (2002)
b	0.7; 0.9	[1]

Table 1 – Calibration in the Cobb-Douglas case.

We let the ratio θ/λ be any positive value, that is, we elect this ratio as the bifurcation parameter. With the above values, we compute steady states for the various variables. First note that the potential output is $y^* = 1.7691$. The steady state level of capital comes $\bar{k} = 9.4748$ for b=0.7 and $\bar{k} = 1.8234$ for b=0.9. To these capital levels, it corresponds the following output values: $\bar{y} = 2.116$ (b=0.7) and $\bar{y} = 1.2217$ (b=0.9). We confirm that the lower propensity to consume implies a positive output gap (and, thus, the impossibility of asymptotic stability), while the larger propensity to consume leads to a negative output gap.

The other steady state values are: $\bar{c} = 1.4812$ (b=0.7), $\bar{c} = 1.0995$ (b=0.9) (observe that the second steady state level of consumption is lower than the first, despite the fact that in the second case the propensity to consume is higher); $\bar{i} = 0.6348$ (b=0.7), $\bar{i} = 0.1222$ (b=0.9). The inflation rate comes $\bar{\pi} = 0.1791 \cdot \lambda$ (b=0.7), $\bar{\pi} = -0.3702 \cdot \lambda$ (b=0.9); as we should expect, inflation exists when the output gap is positive, and deflation arises for a negative output gap. Finally, concerning inventories, we get $\bar{h} = -0.3789 \cdot \dfrac{\lambda}{\theta}$ (b=0.7), $\bar{h} = 0.4523 \cdot \dfrac{\lambda}{\theta}$ (b=0.9).

To address local dynamics, it is possible to present the Jacobian matrices of the system, considering each one of the propensities to consume. These matrices are:

$$\mathbf{J}_{(b=0.7)} = \begin{bmatrix} 0.4651 & -5.5847 \cdot \dfrac{\theta}{\lambda} \\ 0.4902 & 1 + 5.5847 \cdot \dfrac{\theta}{\lambda} \end{bmatrix}; \; \mathbf{J}_{(b=0.9)} = \begin{bmatrix} 1.3352 & 2.701 \cdot \dfrac{\theta}{\lambda} \\ -0.3799 & 1 - 2.701 \cdot \dfrac{\theta}{\lambda} \end{bmatrix}$$

The computation of the eigenvalues of the above matrices is straightforward:

$$\varepsilon_{1,2} = 0.7325 + 2.7924 \cdot \frac{\theta}{\lambda} \pm 0.5 \cdot \sqrt{31.189 \cdot \left(\frac{\theta}{\lambda}\right)^2 - 4.976 \cdot \frac{\theta}{\lambda} + 0.2861} \quad (b\text{=0.7});$$

[1] b needs to be higher than 0.7907 to exist a region of stability (proposition 8); therefore, we consider two values that produce two different outcomes.

$$\varepsilon_{1,2} = 1.1676 - 1.3505 \cdot \frac{\theta}{\lambda} \pm 0.5 \cdot \sqrt{7.2954 \cdot \left(\frac{\theta}{\lambda}\right)^2 - 2.2937 \cdot \frac{\theta}{\lambda} + 0.1124} \quad (b=0.9).$$

Asymptotic stability requires both eigenvalues to be inside the unit circle. For the first set of eigenvalues this does not happen. One of the eigenvalues is always above 1 for positive values of the ratio θ/λ. The other eigenvalue lies above minus one for any positive value of θ/λ, and below 1 for a θ/λ below 1,007.6. Since it does not make much sense to assume such a huge difference between the values of the demand function parameter and the price flexibility / stickiness parameter, we can guarantee that for b=0.7 saddle-path stability is found (one of the eigenvalues inside and the other one outside the unit circle).

On the other case, b=0.9, one of the eigenvalues is always inside the unit circle, while the other eigenvalue lies inside the unit circle as long as $\theta/\lambda<0.8843$. When the ratio reaches this value, a flip bifurcation occurs (the eigenvalue assumes the value -1). When the ratio is above 0.8843, then local dynamics are characterized by saddle-path stability. This result is confirmed with the global analysis of the following section. For the assumed parameterization, the Neimark-Sacker bifurcation does not occur under any positive value of θ/λ. We can state an additional result by recovering proposition 3. The fixed-point is a stable node for values of θ/λ obeying $7.2954 \cdot \left(\frac{\theta}{\lambda}\right)^2 - 2.2937 \cdot \frac{\theta}{\lambda} + 0.1124 > 0$, that is, for $\theta/\lambda<0.0674 \wedge 0.2537<\theta/\lambda<0.8843$. Any other value of the ratio in which there is asymptotic stability corresponds to a stable focus equilibrium.

Let us return to the case b=0.7 in order to obtain the expression of the stable trajectory in the saddle-path case. Recalling equation (8), that gives us the saddle-trajectory, one has in the present case: $h_t = -0.3789 \cdot \frac{\lambda}{\theta} + \frac{4.6445}{\kappa} - \frac{0.4902}{\kappa} \cdot k_t$, with $\kappa = 0.2675 + 2.7923 \cdot \frac{\theta}{\lambda} + 0.5 \cdot \sqrt{31.189 \cdot \left(\frac{\theta}{\lambda}\right)^2 - 4.976 \cdot \frac{\theta}{\lambda} + 0.2861}$. Note that $\kappa<0$ means that the slope of the stable trajectory is positive, being negative in the symmetric case. Since $\kappa>0$, $\forall \theta/\lambda$, under the imposed conditions and calibration, the stable trajectory is negatively sloped; as the stock of capital rises towards equilibrium, the goods inventory falls.

5.2 CES Production Function

In this section, we consider an alternative neoclassical production function (as the Cobb-Douglas function, it exhibits positive and diminishing marginal returns and the Inada conditions hold). This is a constant elasticity of substitution (CES) production function, and we present it as in Barro and Sala-i-Martin (1995),

$$f(k_t) = A \cdot \left[a \cdot (m \cdot k_t)^{\psi} + (1-a) \cdot (1-m)^{\psi}\right]^{1/\psi} \tag{11}$$

In production function (11), $A>0$ is again the technology index, and $0<a<1$, $0<m<1$, $\psi \in (-\infty, 1) \backslash \{0\}$. The elasticity of substitution between capital and labor is $1/(1-\psi)$. The CES function has, as limit cases, other shapes of production functions. When $\psi \to 0$, the elasticity of substitution approaches 1, and the production function approaches a Cobb-Douglas form. When $\psi=1$, the production function becomes linear (the elasticity of substitution is infinite). Finally, when $\psi \to -\infty$, we approach a Leontief production function with a fixed-proportions technology (the elasticity of substitution is zero).

The CES production function is more demanding to deal with analytically. In the appendix (A10), we compute the potential output as defined earlier. The outcome is $y^* = A \cdot \dfrac{(1-a) \cdot (1-m)^{\psi} \cdot z}{z - a \cdot m^{\psi}}$, with $z \equiv \left(\dfrac{1/\beta - (1-\delta)}{A \cdot a \cdot m^{\psi}}\right)^{\psi/(1-\psi)}$.

The proposed model implies, as a generic result, that in the steady state income and demand are equal, and therefore it is once again straightforward to obtain the long-term stock of capital from the capital accumulation equation. This is given by $\bar{k} = \dfrac{(1-a)^{1/\psi} \cdot (1-m)}{\left[\left(\dfrac{\delta}{(1-b) \cdot A}\right)^{\psi} - am^{\psi}\right]^{1/\psi}}$. Observe that, as we should expect, the impact of parameters A, b and δ over the steady state capital stock is qualitatively the same as in the Cobb-Douglas case. The steady state level of output is $\bar{y} = A \cdot \left(\dfrac{(1-a) \cdot (1-m)^{\psi} \cdot \omega}{\omega - a \cdot m^{\psi}}\right)^{1/\psi}$, with $\omega \equiv \left(\dfrac{\delta}{(1-b) \cdot A}\right)^{\psi}$.

Steady state values of consumption and investment are, respectively, $\bar{c} = bA \cdot \left(\dfrac{(1-a) \cdot (1-m)^{\psi} \cdot \omega}{\omega - a \cdot m^{\psi}}\right)^{1/\psi}$ and $\bar{i} = (1-b) \cdot A \cdot \left(\dfrac{(1-a) \cdot (1-m)^{\psi} \cdot \omega}{\omega - a \cdot m^{\psi}}\right)^{1/\psi}$.

The steady state goods inventory and inflation rate are dependent on the output gap and, as discussed in the general case, we observe that a positive output gap implies a negative inventory level and a positive inflation rate.

Proposition 9. *With a CES technology, the necessary condition for stability* $\bar{x} < 0$ *implies the following constraint on b,*

$$b > \frac{\vartheta^{1/\psi} \cdot A - \delta}{\vartheta^{1/\psi} \cdot A}, \text{ with } \vartheta \equiv \frac{a \cdot m^{\psi} \cdot z^{\psi}}{z^{\psi} - (1-a)^{1-\psi} \cdot (1-m)^{\psi \cdot (1-\psi)} \cdot (z - a \cdot m^{\psi})^{\psi}}.$$

Proof: see appendix.

Proposition 9 shows that, similarly to the Cobb-Douglas technology case, a lower bound is imposed on the marginal propensity to consume in order to asymptotic stability to be feasible.

An example illustrates the CES case. The values of parameters A, δ and β are the ones considered in the Cobb-Douglas example, and we take a=0.4 and m=0.7. The elasticity of substitution between capital and labor is 0.9 (a value near the Cobb-Douglas case); this elasticity of substitution means that ψ=-1/9. Once again, the ratio θ/λ is left to be the bifurcation parameter. To choose a value for b, we first look at proposition 9 under this particular example. For the selected array of parameters, z=1.146, and y^*=1.077. It is also true that ϑ=1.3942. Computation implies the following necessary stability condition: b>0.0074.

Thus, in the case in appreciation, the steady state output gap is negative for all values of the propensity to consume, except extremely low values, which from an empirical plausibility point of view are negligible. This is a significant departure from the Cobb-Douglas case. Despite the chosen elasticity of substitution in the CES case being close to the one in the Cobb-Douglas scenario, the value of the propensity to consume required to find stability can be significantly different. Because in the present case any reasonable propensity to consume implies a long-term state where output is below potential, we select a reasonable value for b; this is b=0.7.

With the selected array of parameters, one computes ω=1.1812 and $\bar{y} = 0.597$. We confirm that the output gap is negative, meaning that we should encounter an area of stability for a given interval of values of θ/λ. Additionally, a flip bifurcation will be identified.

The Jacobian matrix is, for the system under appreciation (this is directly computed from (7), with $f'(\bar{k})$=0.0787),

$$\mathbf{J}=\begin{bmatrix} 1.0113 & 1.6949\cdot\frac{\theta}{\lambda} \\ -0.0547 & 1-1.6949\cdot\frac{\theta}{\lambda} \end{bmatrix}$$

The eigenvalues of the Jacobian matrix are:

$$\varepsilon_{1,2}=1.0057-0.8474\cdot\frac{\theta}{\lambda}\pm 0.5\cdot\sqrt{2.8727\cdot\left(\frac{\theta}{\lambda}\right)^2-0.3325\cdot\frac{\theta}{\lambda}+0.0001}$$

One of the eigenvalues is inside the unit circle for θ/λ<21,005, that is, it lies inside the unit circle for any reasonable parameterization. The other eigenvalue lies inside the unit circle for θ/λ<1.213. When θ/λ=1.213, the system crosses a flip bifurcation and, consequently, the possibility of endogenous fluctuations arises. The stable fixed-point respects to a stable node equilibrium when $2.8727\cdot\left(\frac{\theta}{\lambda}\right)^2-0.3325\cdot\frac{\theta}{\lambda}+0.0001>0$, i.e., for 0.0003<$\theta/\lambda$<0.1154. A stable focus will mean that 0<θ/λ<0.0003 or 0.1154<θ/λ<1.213.

In the case of saddle-path stability (θ/λ>1.213), one may compute the expression of the stable trajectory. As an illustration, assume that θ=1.5; for this value, the eigenvalue above -1 is ε_1=0.9557 (the other is ε_2=-1.4866). Recovering the stable arm in (8), this comes $h_t-\bar{h}=-0.2782\cdot\left(k_t-\bar{k}\right)$. In this example, assuming that the stable path is followed, a one point increase in the stock of capital occurs simultaneously with a 0.2782 points decrease in the goods inventory, as the convergence to the steady-state eventually takes place.

In an overall evaluation, and despite the difference found about the constraint bounding parameter *b* in order to separate the cases of positive and negative steady state output gap, we find similar results when comparing the dynamics of the model when to its structure underlie two different production technologies. In both cases, stability requires θ/λ to be lower than a bifurcation point, that once crossed leads to saddle-path stability.

6. Comparing Global Dynamic Results

In this section, we resort to the numerical examples one as presented earlier to make a graphical evaluation of global dynamics. We find that, for both types of production functions, the flip bifurcation gives place to a period doubling route to chaos, such that one may identify the presence of endogenous cycles for certain arrays of parameter values.

The graphical analysis includes the presentation of a bifurcation diagram, long-term attractors, time series of the most relevant variables and the computation of Lyapunov characteristic exponents (LCEs). LCEs are a well accepted measure of sensitive dependence on initial conditions, a feature that constitutes one of the main properties of chaotic systems.[2] We begin by analyzing the Cobb-Douglas case, under the parameterization in table 1.

Relevant global dynamic results only exist for b>0.7907, the case in which the condition $1 - Tr(\mathbf{J}) + Det(\mathbf{J}) > 0$ is satisfied. Thus, we work with b=0.9. Recall that for this propensity to consume, local dynamics has pointed to stability under θ/λ<0.8843 and saddle-path stability otherwise. Figure 3 displays the bifurcation diagram of variable k_t as we change the value of the ratio θ/λ.[3]

*** Figure 3 ***

The bifurcation diagram furnishes a visual confirmation of the existence of a stability area to the left of the bifurcation point (the steady state value of the capital variable that one has computed in section 5, $\bar{k} = 1.8234$, is obtained) and, once the bifurcation takes place, it is possible to observe that cycles of growing periodicity arise as the value of the ratio θ/λ rises. Chaotic motion is found for values of this ratio slightly above one. This means that endogenous irregular cycles are present for a value of the price coefficient in the demand relation slightly above the value of the price stickiness parameter. The presence of chaos is confirmed with the presentation of LCEs in figure 4.

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[2] See Alligood, Sauer and Yorke (1997), Lorenz (1997) or Medio and Lines (2001) for detailed analysis of chaotic systems and respective applications to economics.

[3] This figure, and all the following, are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

*** Figure 4 ***

In a two-dimensional system, two LCEs can be computed. If one of them is a positive value, then there is exponential divergence of nearby orbits, that is, time series are sensitive to their initial values (a small difference in the initial values means, for a chaotic system, completely different trajectories over time). Thus, an LCE above zero is synonymous of the presence of chaotic motion. We observe that the contents of figure 4 confirm, in fact, the information furnished by figure 3. In particular, one of the LCEs assumes a positive value for most of the interval $\theta/\lambda \in (1;1.06)$.

Figure 5 presents the long-run attractor of the relation between the two endogenous variables, for a value of θ/λ under which chaotic motion exists ($\theta/\lambda=1.05$). Note that, although we have chosen to work with the case in which the output gap is negative and inventories are positive, since this is the case that allows for stability and for a bifurcation that generates endogenous cycles, we observe in the figure that the goods inventory can assume negative values, as variable h_t fluctuates in a region bounded above by 2.3 and below by -0.3 (approximately). Thus, although the inventories are, on average, around 1.3, fluctuations will imply that the goods inventory can fall below zero, even in the circumstance one is considering of a negative output gap. Another curious and relevant feature in figure 5 is the negatively sloped shape of the attractor. This seems to make sense if one thinks that more capital directly leads to increased output, and with more output the higher is also the value of the output gap (recall that the potential output is modelled as a constant); therefore, the information in the figure is in accordance with the inverse relation one has established between the output gap and the goods inventory.

*** Figure 5 ***

Figures 6 and 7 display the long term time series (the first 10,000 observations are excluded) of the physical capital and goods inventory variables for the same value of the ratio θ/λ that allowed for drawing the previous attractor. Now, one directly observes the presence of endogenous fluctuations, that we have interpreted earlier as the result of a prevalence of the Keynesian features of the model, relatively to the neoclassical properties, which in turn dominate in the balanced growth case, found for lower values of the quotient θ/λ.

*** Figures 6 and 7 ***

One final figure is presented for the Cobb-Douglas case. This calls the attention for the need of selecting initial values of the endogenous variables that allow for convergence to the long-run state (being this a fixed-point, any periodic point or a chaotic attracting set). As we see in the basin of attraction of figure 8, not all combinations of initial values are feasible. If one starts from a point in the dark area (outside the basin of attraction), the system just diverges to infinity.

*** Figure 8 ***

Relatively to the CES case, the qualitative results are not significantly different from the ones just obtained for the case with a Cobb-Douglas production function. To save in space, we just present the bifurcation diagram, similar to the one in figure 3, and the attractor, which has also a same shape as the one in figure 5.

To present the bifurcation diagram in figure 9, we take the same set of parameters used in the local dynamics example. In this, asymptotic stability was guaranteed under $\theta/\lambda < 1.213$. Then, a flip bifurcation occurs and, locally, saddle-path stability sets in. The figure confirms these results, and it reveals that also in this case, the flip bifurcation originates a process of cyclical motion with increasing periodicity and where a region of chaos is observable.

*** Figure 9 ***

Comparing figures 3 and 9, one realizes that differences are eminently quantitative; for the selected parameter values, the steady state stock of capital is larger in the CES case, and, also in this case, the flip bifurcation occurs for a higher value of the ratio θ/λ.

Observing figure 9, we see that, for instance, for $\theta/\lambda=1.5$ there is chaotic motion. Figure 10 presents, for this value, the long-term attractor (once again, the first 10,000 observations are withdrawn). As one would expect, the similarities with the attractor in figure 5 exist. What one has said about negative values for the goods inventory and for the negative relation that is established in the long-term, applies to the CES case as well.

*** Figure 10 ***

It is possible to conclude that the type of the production function does not change the main dynamic properties of the model under a global analysis point of view, because both production functions are neoclassical in nature.

7. Conclusions

Keynesian economics can be characterized as the analysis of non equilibrium situations in aggregate market relations. Following recent literature on the theme, we have developed a NEDyM with only two dynamic equations, one respecting to capital accumulation and the other to the adjustment of output and demand over the goods inventory. Behind this reduced form there is a set of neoclassical (market clearing) and Keynesian (non equilibrium) assumptions.

We were able to address patterns of growth and to realize that, by combining neoclassical growth features with Keynesian disequilibrium elements, a multitude of long-term results can be found, ranging from balanced growth stability to cycles of any periodicity and completely a-periodic cycles. While the classical components pull in the direction of the stable outcome, the several inefficiencies that were introduced led to the possibility of endogenous business cycles. The main advantage of this approach relatively to other models in the area is that the used low dimensionality allowed for finding some relevant generic results, namely concerning local analysis.

A meaningful result concerns the idea that stability is possible only for a negative output gap. This is intuitive if one takes in consideration the set of inefficiencies that were considered; the benchmark case is the neoclassical growth model (the potential output is the steady state level of output computed when assessing an optimal control utility maximization problem), thus, by introducing non equilibrium components to the model, it seems obvious that the balanced growth path that one can find must correspond to a long-run output level below the optimal one. Furthermore, the assumptions of the model imply that along with a negative output gap, goods inventories are positive, i.e., in each time moment (and, in this case, in every time moment of the long-run outcome) there are goods that are produced but not sold. Thus, periods of recession (negative output gap) are periods of overproduction (demand is below the level of available goods). This is also an intuitive result.

The most relevant conclusion is that the non equilibrium features that are attached to the neoclassical growth model are such that they introduce nonlinear relations between variables, which are capable of generating endogenous cycles for admissible parameter values. This may be used as an argument to justify the relevance of Keynesian economics, under which no external shock is necessary to trigger fluctuations.

Appendix

A1 – Proof of proposition 1.

Just apply the conditions mentioned in definition 2 to arrive to the group of relations in the proposition. The uniqueness of the steady state is guaranteed by the concave shape of the neoclassical production function, which makes the average product of capital (which is a decreasing function in all of its domain) to intersect the constant value $\delta/(1\text{-}b)$ in a single point■

A2 – Proof of proposition 2.

The trace and the determinant of matrix **J** in (7) are, respectively,

$$Tr(\mathbf{J}) = 2 - \delta - \left(b + \frac{1}{\bar{x}}\right)\cdot f'(\bar{k}) + \frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}}$$

$$Det(\mathbf{J}) = 1 - \delta - \left(b + \frac{1}{\bar{x}}\right)\cdot f'(\bar{k}) + (1-\delta)\cdot\frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}} + \frac{\theta}{\lambda}\cdot(1-b)\cdot\frac{1}{\bar{x}}\cdot f'(\bar{k})$$

From the above expressions, one withdraws a relation between trace and determinant,

$$Det(\mathbf{J}) = Tr(\mathbf{J}) - 1 + \frac{\theta}{\lambda}\cdot\left[(1-b)\cdot f'(\bar{k}) - \delta\right]\cdot\frac{1}{\bar{x}}.$$

One of the necessary conditions for asymptotic stability is $1 - Tr(\mathbf{J}) + Det(\mathbf{J}) > 0$. This condition will require expression $\frac{\theta}{\lambda}\cdot\left[(1-b)\cdot f'(\bar{k}) - \delta\right]\cdot\frac{1}{\bar{x}}$ to correspond to a positive value. Note that the expression may be presented as $\frac{\theta\cdot(1-b)}{\lambda}\cdot\left[f'(\bar{k}) - \frac{f(\bar{k})}{\bar{k}}\right]\cdot\frac{1}{\bar{x}}$. This is positive if $\bar{y} > y^* \wedge f'(\bar{k}) > \frac{f(\bar{k})}{\bar{k}}$ or $\bar{y} < y^* \wedge f'(\bar{k}) < \frac{f(\bar{k})}{\bar{k}}$. This set of conditions can be restricted by recalling the neoclassical nature of the production function. In this function, marginal returns are positive but diminishing. This means that introducing additional capital implies getting progressively smaller increments on output. Therefore, the marginal product of capital will be lower than the average product of capital for any admissible value of this variable. Thus, by stating that $f'(\bar{k}) < \frac{f(\bar{k})}{\bar{k}}$, we restrict the possibility of asymptotic stability to the case in which the steady state output level is below the corresponding potential level■

A3 – Proof of proposition 3.

The parabola $Det(\mathbf{J}) = (Tr(\mathbf{J})/2)^2$ defines the case in which the two eigenvalues of **J** are identical and equal to $Tr(\mathbf{J})/2$. Above this parabola [$Det(\mathbf{J}) > (Tr(\mathbf{J})/2)^2$] the eigenvalues are complex, and below it [$Det(\mathbf{J}) < (Tr(\mathbf{J})/2)^2$] they are two real values. Assuming that asymptotic stability prevails, the last inequality defines the condition under which a stable node exists. Applying this condition to the specific Jacobian matrix in (7) and resorting to the trace and determinant expressions computed in the proof of proposition 2, we get the expression in this proposition■

A4 – Proof of proposition 4.

The two eigenvalues of **J** lie inside the unit circle if the following three conditions are simultaneously satisfied: $1 - Tr(\mathbf{J}) + Det(\mathbf{J}) > 0$; $1 + Tr(\mathbf{J}) + Det(\mathbf{J}) > 0$; $1 - Det(\mathbf{J}) > 0$. The first condition was applied to arrive to the result in proposition 2. The other two correspond, respectively, to conditions i) and ii) in the proposition■

A5 – Proof of the corollary of proposition 4.

The expressions in proposition 4 establish two bounds on the ratio θ/λ; thus, we just have to rearrange the expressions in the proposition to get the boundaries of the set in the corollary. The main issue resides in identifying which one is the lower bound and which one is the upper bound. To reach this result, observe that condition $0 < f'(\bar{k}) < \frac{f(\bar{k})}{\bar{k}}$ holds and that the steady state average product of capital is the one derived in proposition 1. The above condition implies that the terms that multiply by θ/λ in the two conditions of proposition 4 are negative values (keep in mind that the output gap is negative); thus, when solving the inequalities in the proposition in order to θ/λ, the first one gives a value of the ratio below some combination of parameters, while the second gives a value of the ratio above some other combination of parameters. If the first quantity is higher than the second, asymptotic stability is guaranteed for any value of θ/λ inside the presented set. In the opposite case, asymptotic stability is absent from the possible steady state results■

A6 – Proof of proposition 5.

The conditions in the proposition are the ones that imply that one of the eigenvalues becomes lower than -1 ($1+Tr(\mathbf{J})+Det(\mathbf{J})<0$) and that the two eigenvalues become a pair of complex conjugate values ($1-Det(\mathbf{J})<0$), respectively■

A7 – Proof of proposition 6.

In the case where a positive output gap exists, condition $1-Tr(\mathbf{J})+Det(\mathbf{J})>0$ is violated (see proof of proposition 2) and, therefore, one of the eigenvalues of **J** is higher than 1. Thus, at best we will have a one stable dimension. This stable dimension exists if the other two stability conditions hold ($1+Tr(\mathbf{J})+Det(\mathbf{J})>0$ and $1-Det(\mathbf{J})>0$). In our specific system, these are the conditions that allow reaching the interval in the corollary of proposition 4, according to the proof of such proposition. If saddle-path stability does not hold, no eigenvalue with modulus lower than 1 is determined, implying instability or divergence relatively to the fixed-point, independently of initial conditions■

A8 – Proof of proposition 7.

Assume that matrix **J** in (7) has, as eigenvalues, $|\varepsilon_1|<1$ and $|\varepsilon_2|>1$. In this case, a unique stable trajectory exists and this is given by expression $h_t-\bar{h}=\frac{p_2}{p_1}\cdot\left(k_t-\bar{k}\right)$, with p_1 and p_2 the elements of an eigenvector associated with ε_1. The eigenvector $P=\begin{bmatrix}p_1 & p_2\end{bmatrix}^T$ may be determined resorting to one of the lines of **J**. Taking the second line in consideration, the following relation applies: $\left(1+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})\cdot p_1+\left(1-\varepsilon_1+\frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}}\right)\cdot p_2=0$. Choosing p_1=1, the eigenvector is

$$P=\begin{bmatrix}1 & -\dfrac{\left(1+\frac{1}{\bar{x}}\right)\cdot f'(\bar{k})}{1-\varepsilon_1+\frac{\theta}{\lambda}\cdot\frac{1}{\bar{x}}}\end{bmatrix}$$

. From P, we withdraw the elements necessary to present the slope of the stable arm, as displayed in the proposition■

A9 – Proof of proposition 8.

The steady state output gap is $\bar{x} = \frac{\alpha}{1-\alpha} \cdot \ln\left(\frac{(1-b)\cdot(1/\beta-(1-\delta))}{\alpha\cdot\delta}\right)$. This is a positive value if the expression inside the logarithm is higher than 1; by rearranging this condition, one arrives to the inequality in the proposition■

A10 - Derivation of the potential output in the CES case.

The potential output was defined as the steady state value of output for an optimal growth problem with a logarithmic utility function. Thus, after computing first-order conditions, one arrives to the standard steady state relation $f'(k^*) = 1/\beta - (1-\delta)$.

The marginal product of capital is, in the steady state, $f'(k^*) = Aam^{\psi} \cdot \left[am^{\psi} + (1-a)\cdot(1-m)^{\psi} \cdot k^{*-\psi}\right]^{(1-\psi)/\psi}$. The relation between the potential output and the steady state capital stock is given by the production function: $y^* = A\cdot\left[a\cdot(m\cdot k^*)^{\psi} + (1-a)\cdot(1-m)^{\psi}\right]^{1/\psi}$, which can be rewritten in order to k^*, $k^* = \left[\frac{(y^*/A)^{\psi} - (1-a)\cdot(1-m)^{\psi}}{a\cdot m^{\psi}}\right]^{1/\psi}$. Replacing this value of k^* in the marginal product expression, the steady state condition comes:

$$Aam^{\psi} \cdot \left[am^{\psi} + (1-a)\cdot(1-m)^{\psi} \cdot \frac{a\cdot m^{\psi}}{(y^*/A)^{\psi} - (1-a)\cdot(1-m)^{\psi}}\right]^{(1-\psi)/\psi} = 1/\beta - (1-\delta).$$

Solving this last equation in order to the potential level of output one obtains $y^* = A\cdot\frac{(1-a)\cdot(1-m)^{\psi}\cdot z}{z - a\cdot m^{\psi}}$, with $z \equiv \left(\frac{1/\beta-(1-\delta)}{A\cdot a\cdot m^{\psi}}\right)^{\psi/(1-\psi)}$.

A11 – Proof of proposition 9.

This proof is just a matter of analytical calculation. The steady state negative output gap condition, $\bar{y} < y^*$, writes in the CES case as $A\cdot\left(\frac{(1-a)\cdot(1-m)^{\psi}\cdot\omega}{\omega - a\cdot m^{\psi}}\right)^{1/\psi} < A\cdot\frac{(1-a)\cdot(1-m)^{\psi}\cdot z}{z-a\cdot m^{\psi}}$. Solving in order to ω, $\omega > \frac{a\cdot m^{\psi}\cdot z^{\psi}}{z^{\psi} - (1-a)^{1-\psi}\cdot(1-m)^{\psi\cdot(1-\psi)}\cdot(z-a\cdot m^{\psi})^{\psi}}$. To simplify notation, denote the right hand side of the previous inequality by ϑ. Thus, given the definition of ω, it comes $b > \frac{\vartheta^{1/\psi} \cdot A - \delta}{\vartheta^{1/\psi} \cdot A}$ ■

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Figures

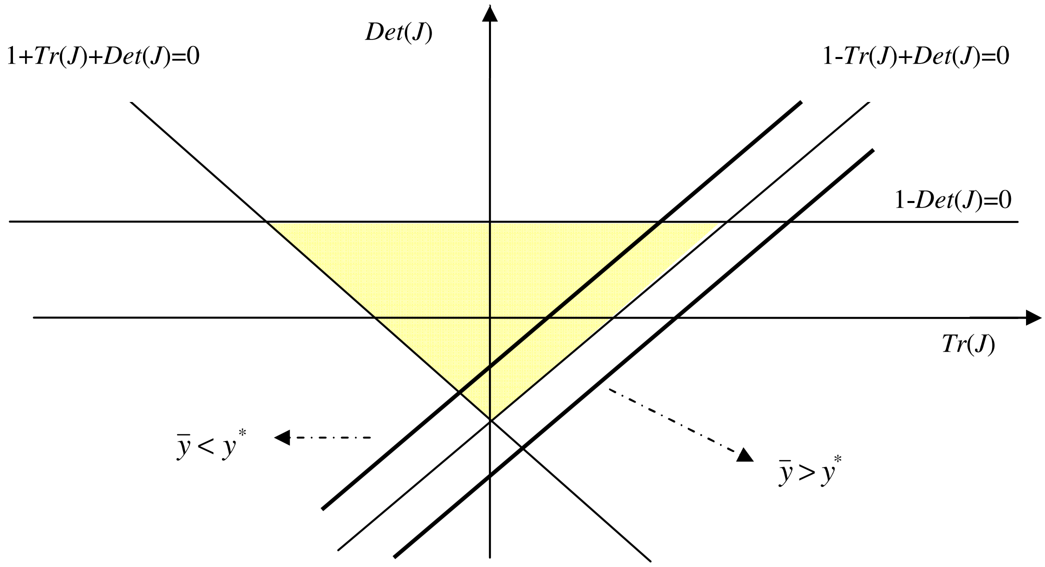

Figure 1 – Characterization of local dynamics. Trace-determinant diagram.

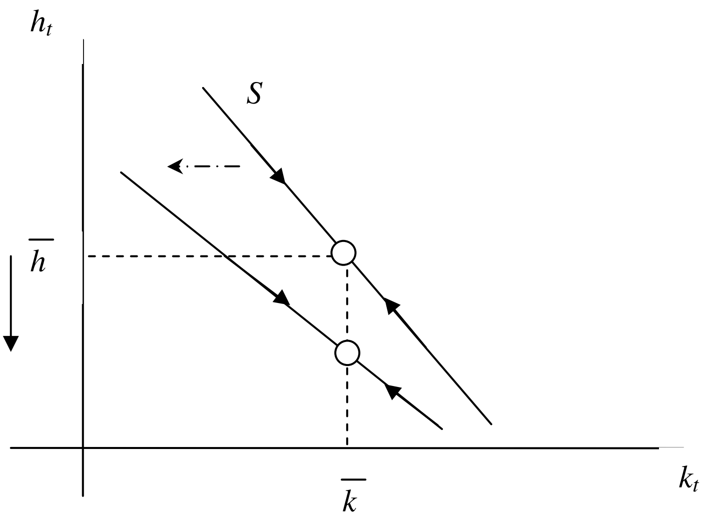

Figure 2 – Saddle-path trajectory. The effect of stickier prices.

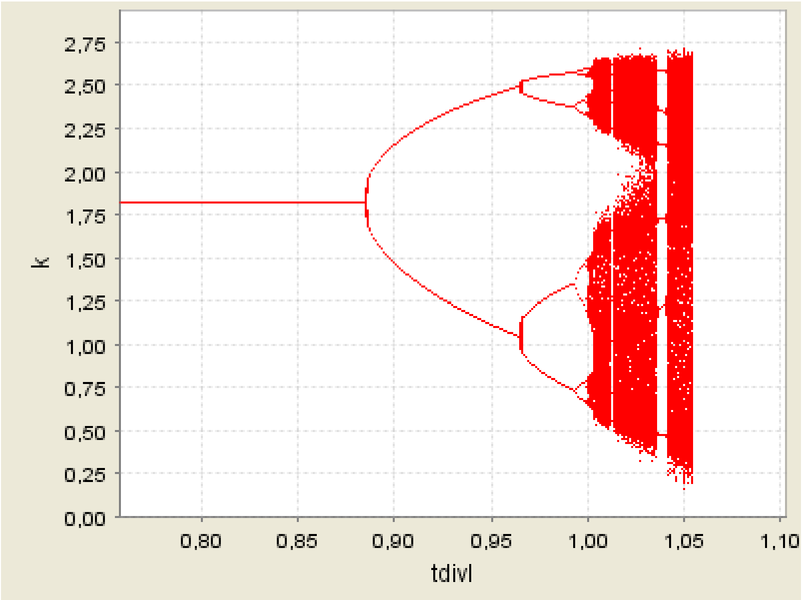

Figure 3 – Bifurcation diagram [Cobb-Douglas technology] (k_t,θ/λ).

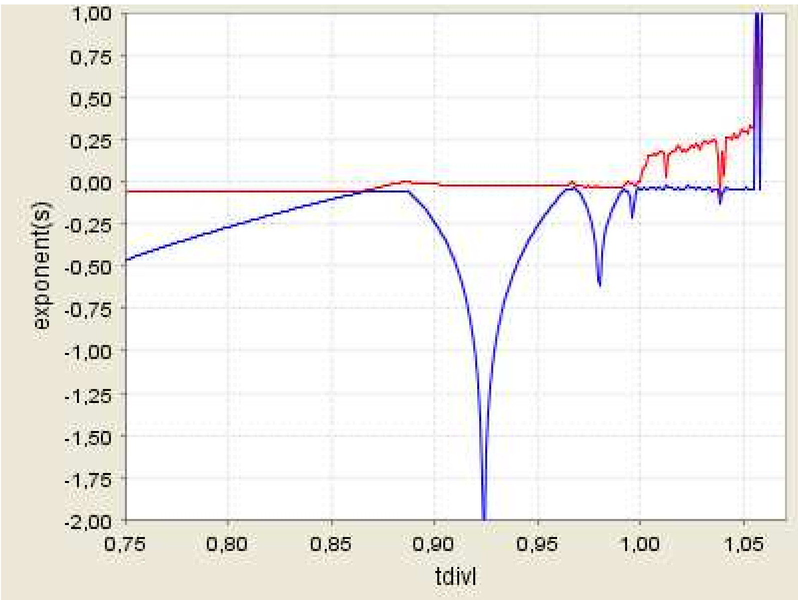

Figure 4 – Lyapunov characteristic exponents [Cobb-Douglas technology] (0.75<θ/λ<1.06).

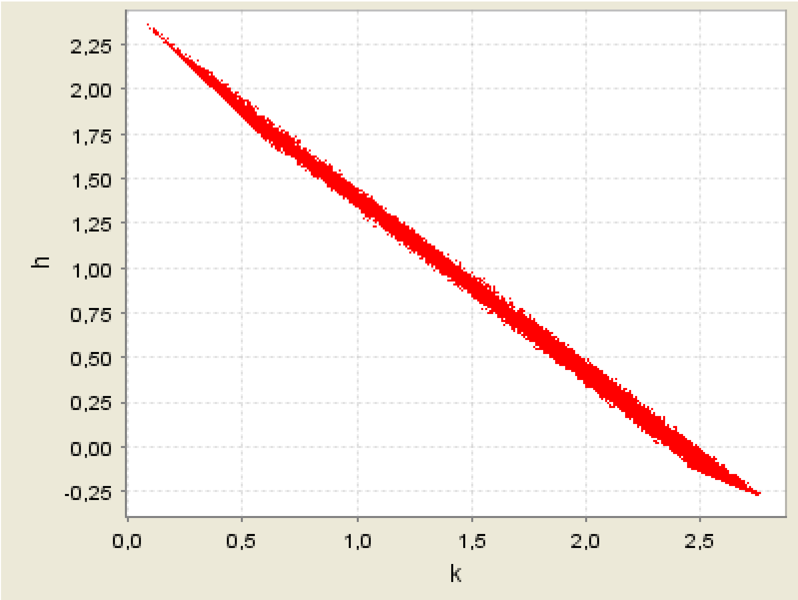

Figure 5 – Attracting set [Cobb-Douglas technology] (k_t, h_t); θ/λ=1.05.

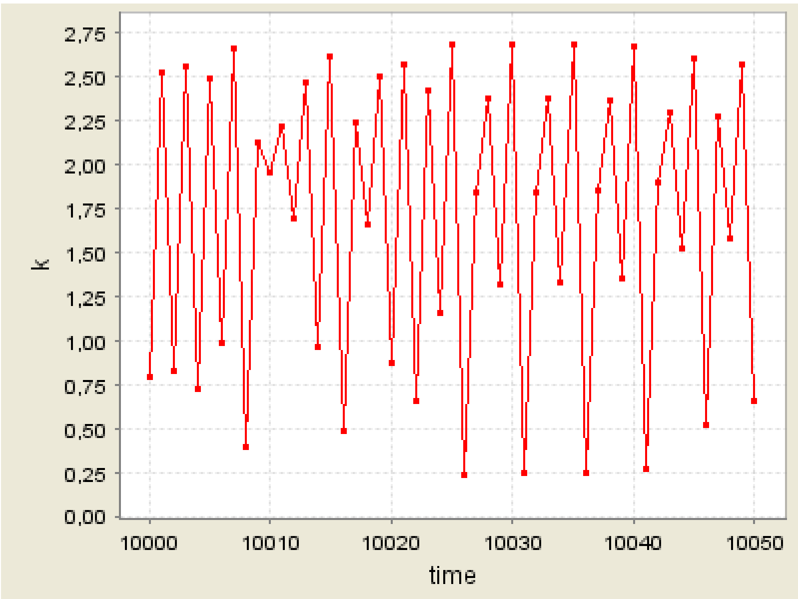

Figure 6 – Time series of k_t [Cobb-Douglas technology]; θ/λ=1.05.

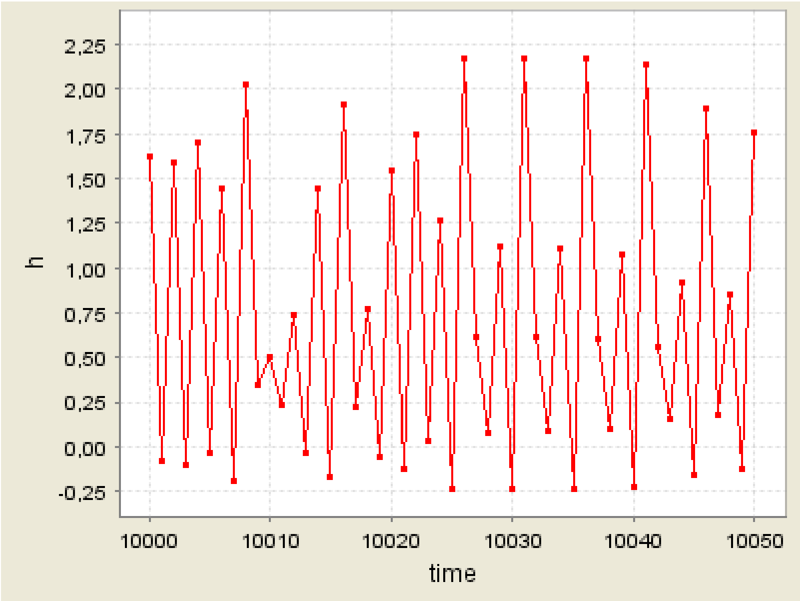

Figure 7 – Time series of h_t [Cobb-Douglas technology]; θ/λ=1.05.

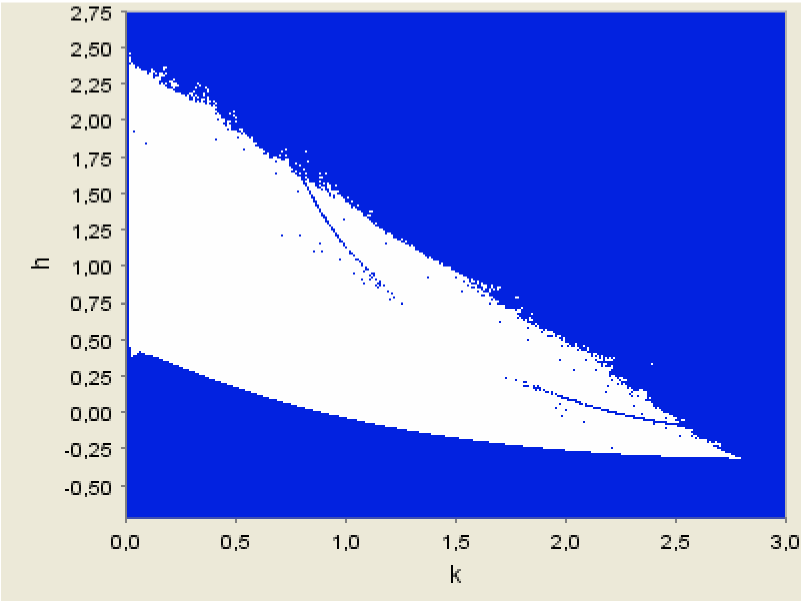

Figure 8 – Basin of attraction [Cobb-Douglas technology]; θ/λ=1.05.

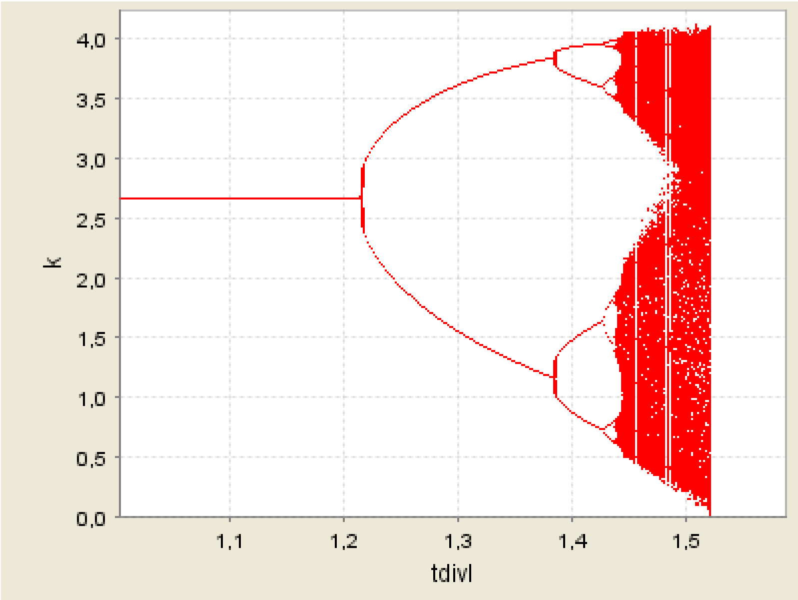

Figure 9 – Bifurcation diagram [CES technology] (k_t,θ/λ).

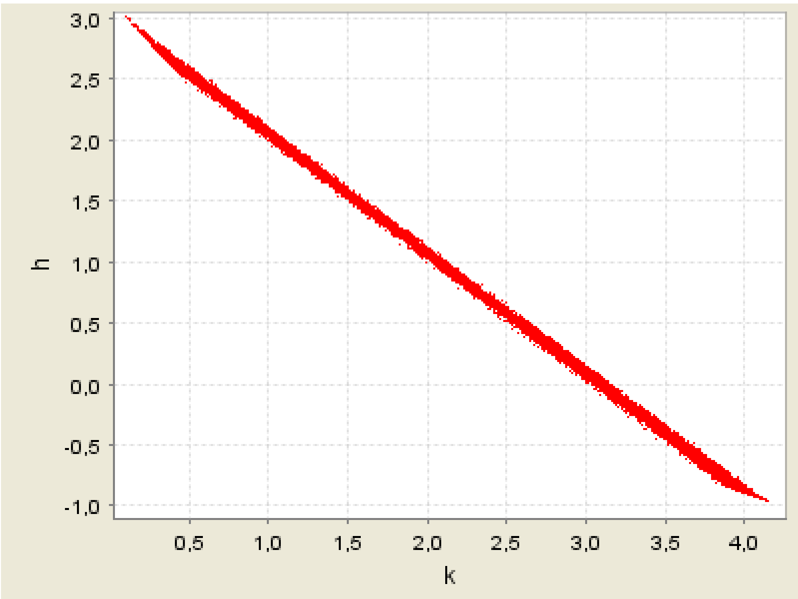

Figure 10 – Attracting set [CES technology] (k_t, h_t); θ/λ=1.5.