An Easy Way to Teach First-order Linear Differential and Difference Equations with a Constant Term and a Constant Coefficient

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10 July 2013

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We present a simple method of solving first-order linear differential and difference equations with a constant term and a constant coefficient. When solving such equations standard books in mathematical economics resort to a particular integral and a complementary function without further explaining those to beginning undergraduate students. We use the derivative and the difference, respectively, which give rise to a number of parental functions whose time path is studied by economic dynamics. A derived function is "shared" by multiple parental functions, but a number of parental functions give rise to one derived function. The method is smooth and easy to understand. Instead of spending time on complicated theoretical math techniques, the professor teaching quantitative methods could emphasize substantive economic models applying such simple equations.

Keywords simple differential equations, simple difference equations, particular integral, complementary function, phase lines, Solow growth model

Jel codes A22, C02, C61, D2

Introduction

When covering advanced topics such as differential and difference equations introductory textbooks in mathematical economics start with the case of first-order, linear equations with a constant term and a constant coefficient and then continue with the case of a variable term and a variable coefficient. Starting with the simplest, textbooks usually discuss the first-order, linear equation with a constant term and a constant coefficient in the homogeneous case where the free term is equal to zero. Then the nonhomogeneous case follows where the constant term could be different from zero. Such an approach is quite logical and acceptable as it helps the course unfold smoothly and allows students to learn gradually and systematically. In all cases the solution in the literature is presented as the sum of the particular integral and the complementary function, the particular integral implying an all-time, that is, intertemporal equilibrium value for the function and the complementary function showing the deviation from or convergence to it with the passage of time. Obtaining the solution requires the knowledge of the more general case of a variable term and a variable coefficient, integration techniques, exact differential equations as well as integrating factors.

Chiang (1984) and Pemberton and Rau (2001) offer the most extensive coverage of differential and difference equations. Chiang (1984) dedicates two chapters to first-order equations and two

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to higher-order equations, respectively. Both books usher students into the solution with a complementary function and a particular integral without explaining what those are and how they obtain. Silbeberg (2000) and Intriligator (2002) cover only simple differential equations with no reference to difference ones. None of them solves the equation in the general form but moves directly to economic applications. Others are high level texts (Simon and Blume, 1994 and de la Fuente, 2000) covering both differential and difference equations. De la Fuente (2000) discusses both types of equations in detail but the text is at the doctoral level and, therefore, not suitable for undergraduate students of economics. With no exception all textbooks assume the solution with a complementary function and a particular integral. Of the problems books Todorova (2010) solves simple as well as higher-order differential and difference equations providing economic problems with solutions. Dowling (1992) accompanying Chiang (1984) takes the solution formulas as given and provides problems and solutions to economic examples. Theoretical mathematics textbooks on differential equations such as Rainville et al (1996) do not emphasize economic or social science applications, neither discuss the simple case of constant coefficients and constant terms. Except as an additional tool to a course in mathematical economics or economic dynamics, they cannot be used in such courses where the emphasis is on economic and other applications rather than pure math theory.

When taking an introductory course in mathematical economics or quantitative methods in the social sciences as part of program requirements students have never or rarely been exposed to differential or difference equations and the general solution to those of first-order, first-degree with a variable term and a variable coefficient. It is, therefore, particularly difficult for beginning students to understand the concept of the particular integral and the complementary function. When discussing the simplest types of equations with a constant term and a constant coefficient, textbooks assume that students know the method of solving the general case of equations and take that solution as ready-made. By use of a particular integral and a complementary function they try to provide students with a formula giving the time path of a function but to the student it is not clear how the formula obtains. More specifically, students do not understand how the particular integral and the complementary function come about and how the general solution is obtained.

We hereby propose a method of solving the simple case of differential and difference equations with a constant coefficient and a constant term without having to go into the case of a variable term and a variable coefficient and take something for granted. This is particularly convenient for students taking the introductory, first-level course in mathematical economics who would never take the second level of higher-order differential or difference equations. Professors, who opt to go into brief coverage of simple differential and difference equations at the end of a semester of introductory mathematical economics, as required by any undergraduate economics program, and who wish to familiarize their students with the simplest type of differential and difference equations as they apply in economics, would be greatly facilitated by this method. It allows their students to have a glimpse of differential and difference equations without going into the jungle of sophisticated equations such as the more expansive case of a variable term and a variable coefficient, exact differential equations, separation of variables, integrating factors, etc. Since in the introductory course these topics appear at the end of the semester, there is rarely time to cover differential and difference equations extensively so the discussion usually stops on simple illustrations. Professors who do not wish to emphasize the topic of integration prior to differential equations could simply introduce the latter with applications in economics but without the need to discuss integration in depth or at all. Finally, being pressed by time at the end of the semester professors could conveniently use our technique to solve the equation and then swiftly apply it to economic models such as the dynamics of market price, the Solow growth model or phase lines in the case of the first-order, linear differential equation with a constant term and a constant coefficient or the cobweb model of market pricing in the similar case of difference equations. Thus, instead of burying students of economics into mathematical techniques and theory, professors could use the limited time more productively by covering essential economic models and illustrations.

Simple Differential Equations

It is habitual to start the chapter on simple differential equations with first-order linear differential equations with a constant coefficient and a constant term. Then the class discussion moves onto the more general case of first-order linear differential equations with a variable term and coefficient, and some special types of simple differential equations such as exact differential equations, integrating factors, separation of variables, Bernoulli equations, etc. Given a function $y(t)$, a differential equation is one that contains a derivative such as dy/dt. Not all, but most, differential equations resort to models in economics where the independent variable is time t. Hence, the time path of an economic function is obtained and analyzed. Solving the differential equation means finding the total function and dropping the derivative from the equation. If there is only a first-order derivative involved, the differential equation is said to be first-order. Higher-order derivatives result in higher-order differential equations and the order of the highest derivative gives the order of the differential equation. Thus, a first-order differential equation is one in which the highest derivative is first-order and a first-order linear differential equation takes the general form

$$\frac{dy}{dt} + u(t)y = v(t)$$

where $u(t)$ and $v(t)$ may be linear or nonlinear functions of t as well as constants. If they happen to be constants, the equation is said to be a first-order linear differential equation with a constant coefficient and a constant term. In the nonhomogeneous case we have

$$\frac{dy}{dt} + u(t)y = v(t) \qquad \text{where } v(t) \neq 0$$

The general solution to this first-order linear differential equation with a variable coefficient and a variable term is

$$y(t) = e^{-\int u dt}\left(A + \int v e^{\int u dt} dt\right)$$

where A is a constant that can be specified at the initial condition. The solution to this equation requires deeper knowledge of differential equations which beginning students lack when the topic is introduced. More specifically, in order to come up with this solution students should know integration techniques, exact differential equations as well as integrating factors which appear later in the chapter. In the Appendix we show how the solution is obtained. It can further be expressed as

$$y = e^{-\int u dt} A + e^{-\int u dt} \int v e^{\int u dt} dt = y_c + y_p$$

The first term involves the arbitrary constant A and is called the complementary function, while the second is the particular integral. The particular integral gives the intertemporal equilibrium value of the function $y(t)$, while the complementary function shows the deviation of the actual value of y from this equilibrium at any moment in time. Therefore, when y_c disappears with time, the time path of the function $y(t)$ shows dynamic stability. If y_c tends to grow as $t \to \infty$, we say that the time path of $y(t)$ from its initial value is dynamically unstable or divergent. Several observations follow. First, we can check that when u and v take the constant values a and b, respectively, i.e., we have the constant term and constant coefficient case, the general solution is exactly

$$y(t) = Ae^{-at} + \frac{b}{a} = y_c + y_p$$

where the complementary function is Ae^{-at} and the equilibrium value is given by the particular integral $\frac{b}{a}$. Thus, as can be expected, the case of a constant coefficient and a constant term is a special case of the general one with a variable term and a variable coefficient. Second, in the homogeneous case where v is zero, the intertemporal equilibrium is exactly zero and the solution consists only of the complementary function $y_c = e^{-\int u dt} A$ or $y_c = Ae^{-at}$.

Mathematical economics textbooks ordinarily start with the simplest case of a first-order, first-degree differential equation, the one with a constant coefficient and a constant term, or

$$\frac{dy}{dt} + ay = 0$$

which is known as the associated homogeneous equation where a is a constant and the free term b is zero. The usual method to solve this equation is rearranging it as

$$\frac{1}{y}\frac{dy}{dt} = -a$$

and integrating both sides of it.

$$\int \frac{1}{y}\frac{dy}{dt} dt = \int -a dt$$

$$\ln|y| + c_1 = -at + c_2 \qquad c = c_2 - c_1$$

$$\ln|y| = -at + c$$

Taking the antilog, we can write the general solution to the homogeneous equation

$$e^{\ln|y|} = e^{-at+c}$$

$y(t) = Ae^{-at}$ where $A = e^{c}$ (general solution)

Definitizing, i.e., finding the value of the A constant, yields the definite solution to the equation

$y(0) = Ae^{-a(0)}$ at $t = 0$ thus $A = y(0)$, or

$y(t) = y(0)e^{-at}$ (definite solution)

A short-cut way of solving this equation in class without the use of integration is the formula for rate of growth usually covered much earlier in the course. The rate of growth of the function $y(t) = Ae^{rt}$ used in continuous compounding or finding the future value of an asset can be expressed as $\frac{dy/dt}{y} = r$ where r is the interest rate. Thus, a differential equation of the form

$$\frac{1}{y}\frac{dy}{dt} = -a$$

implies a function of the type $y(t) = Ae^{-at}$ growing at the rate $-a$ with time. Professors, who do not want to resort to integration techniques or have not covered integration, could easily solve the homogeneous case of a simple differential equation using the concept of the rate of growth of a function. However this technique cannot be applied to the more general, nonhomogeneous case,

$$\frac{dy}{dt} + ay = b$$

Solving this equation through direct integration or rate of growth is not possible and this is where the standard literature on mathematical economics assumes the above solution for a simple differential equation with a variable term and a variable coefficient. Students have to take for granted the concept of the particular integral and the complementary function while a solid discussion of those in the introductory chapters is missing. Thus, the solution

$$y(t) = Ae^{-at} + \frac{b}{a} = y_c + y_p$$

which we already introduced, comes out of the blue and students have to borrow and apply it without any understanding. This is where our innovative method comes in handy. Instead of introducing students into the particular integral and the complementary function in the limited time available, we resort to a new technique. Both the general and definite solutions of a simple differential equation show the time path of a total function. From differentiation we know that a derived function is single but could have many parental functions. At the same time, by differentiation many parental, that is, total functions could give rise to only one derived, i.e., marginal function. Such is the example of labor force expressed as a function of time t as in

$$L(t) = t^{\frac{1}{2}} + c$$

where c is an arbitrary constant. The fact that c is arbitrary implies that no unique time path can be determined unless we know something about the initial or boundary condition of the labor force function. If we know that at the initial moment $t = 0$ the size of the labor force is $L(0) = 50$, then $c = 50$ gives the specific time path of the labor force function, that is,

$$L(t) = t^{\frac{1}{2}} + 50$$

In the general case when the constant of integration is not definitized, there are plenty of parental functions but only one marginal function or the rate of change of labor force with time is just

$$\frac{dL}{dt} = \frac{1}{2} t^{-\frac{1}{2}}$$

Analogously, we use the fact that in the simple homogeneous differential equation the form $y(t) = Ae^{-at}$ shows the general time path of a function $y(t)$. This is one of a few parental functions that have the derivative $dy/dt = -aAe^{-at}$. Hence, this derivative will be the marginal function for an infinite number of parental functions including those that satisfy the nonhomogeneous equation. We, thus, substitute the derivative dy/dt in the nonhomogeneous equation

$$\frac{dy}{dt} + ay = b$$

$-aAe^{-at} + ay = b$, which results in

$$y(t) = Ae^{-at} + \frac{b}{a} \qquad \text{(general solution)}$$

Similar to the homogeneous case this is the general solution to the nonhomogeneous differential equation. Definitizing the constant A,

$$y(0) = Ae^{o} + \frac{b}{a} = A + \frac{b}{a}$$

$$A = y(0) - \frac{b}{a}$$

Substituting in the general equation gives the definite solution

$$y(t) = \left[y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a} \qquad \text{(definite solution)}$$

It is easy to see from our method that the solutions to the homogeneous and nonhomogeneous equation, $y(t) = Ae^{-at}$ and $y(t) = Ae^{-at} + \frac{b}{a}$, respectively, have the same derivative, that is, $dy/dt = -aAe^{-at}$, which proves our line of thinking. Both they are parental functions to this derivative and differ only in their constant term. The method allows to obtain both solutions and then smoothly proceed with interesting economic illustrations such as the dynamic model of market price, the concept of the phase line or the essential model of economic growth advanced by Solow without further emphasizing math. At this point, we can simply mention to students that the intertemporal equilibrium value $\frac{b}{a}$ is called a particular integral, while the remaining exponential term, known as the complementary function, shows the deviation from this equilibrium of the total function with time. More specifically, at the intertemporal equilibrium the function $y(t)$ does not change and its derivative dy/dt is zero. Substituting $dy/dt = 0$ in the nonhomogeneous differential equation produces exactly the particular integral $y_p = \frac{b}{a}$. The homogeneous equation, on the other hand, gives the complementary function, $y_c = Ae^{-at}$, and, thus, the complete solution is the sum of the two. Such an explanation somewhat prepares students for the subsequent discussion of the variable term and variable coefficient case as well as higher-order differential equations, if those would be discussed in the semester.

Simple Difference Equations

Difference equations are the discrete-time equivalent of differential equations applied in continuous time. With discrete time the value of a variable y will change for each period, not at a point in time. The time factor here denotes a period so $t = 1$ means period 1, rather than the first moment, while t is allowed to take only integer values. The task is again to find a time path for the variable y over time. But the change is now $\frac{\Delta y}{\Delta t}$ which shows the difference in y between two consecutive periods so $\Delta t = 1$ and $\frac{\Delta y}{\Delta t} = \Delta y$. This change resembles the derivative $\frac{dy}{dt}$ in continuous time but is a discrete-time change. It is just this difference in y that gives the name of this group of equations. More specifically, the first difference of y can be defined as

$$\Delta y_t = y_{t+1} - y_t$$

where Δy_t is called a forward difference operator, y_t is the value of y in period t and y_{t+1} is its value in the following period. The pattern of change of y may be such that the difference between the values of y in two consecutive periods may be

$$\Delta y_t = y_{t+1} - y_t = c$$

This equation is equivalent to

$$y_{t+1} = y_t + c \qquad \text{or} \qquad y_{t+2} = y_{t+1} + c$$

First-order difference equations contain only a first difference Δy_t, that is, there are no several y terms such as y_{t+2}, y_{t+1} or y_t and there is one-period time lag only. Linear difference equations involve only y terms that come in the first power and are not multiplied by a y term of another period. Similar to differential equations, homogeneous difference equations have a right-hand side that is zero. Solving a difference equation implies finding a time path y_t which should be solely a function of t, given some initial condition, and should be free from any difference expressions of the type Δy_t or $y_{t+1} - y_t$. Given the equation,

$$y_{t+1} - ay_t = 0$$

where the initial condition is y_o we can find the time path of y. We rewrite it as

$y_{t+1} = ay_t$, which implies

$$y_1 = ay_o$$
$$y_2 = ay_1 = a(ay_o) = a^2 y_o$$
$$y_3 = ay_2 = a(a^2 y_o) = a^3 y_o,$$ hence,

$$y_t = a^t y_o$$

This last equation shows the time path of y at any period t and is free from any difference expressions. Thus, it represents the solution to the difference equation. This solution is obtained using a rough method known as the iterative method or iteration. A more general method of solving is needed, both in the homogeneous and nonhomogeneous case. The first-order difference equation

$$y_{t+1} + by_t = c$$

is nonhomogeneous with a constant coefficient b and a constant term c. Its reduced form is

$$y_{t+1} + by_t = 0$$

which represents the associated homogeneous equation. Note the resemblance with the differential equation

$\frac{dy}{dt} + ay = b$ and

$$\frac{dy}{dt} + ay = 0$$

in the nonhomogeneous and homogeneous case, respectively. Rewriting the homogeneous difference equation,

$$y_{t+1} + by_t = 0$$
$$y_{t+1} = -by_t$$

Assuming a solution of the type $y_t = Aa^t$ similar to $y_t = y_o a^t$ we obtained previously for the form of a nonlinear function we also have

$$y_{t+1} = Aa^{t+1}$$

Substituting for y_t and y_{t+1} in the difference equation,

$Aa^{t+1} = -bAa^t$ or

$$a = -b$$

gives the solution to the homogeneous equation

$$y_t = A(-b)^t$$

How to find the general solution to the nonhomogeneous equation? When providing a solution, standard textbooks such as Chiang (1984) do not even explain about the particular integral and the complementary function, as they do with differential equations. This time the student is directly thrown into a general formula for a particular integral of a discrete-time function. The student, therefore, has to guess how the formula is obtained and it is not before the last chapter on higher-order difference equations that the formula gets explained. Again, for students who never get to that chapter in the introductory course the solution of a first-order, first-degree difference equation is a mystery.

To solve this type of equation elegantly and without reference to the particular integral and the complementary function which the beginning student is unaware of, we can use the fact that the difference Δy_t will take the same values in the homogeneous and nonhomogenous case. This is similar to our logic of differential equations and the fact there the derivative is the same, both in the homogeneous and nonhomogeneous case. Expressing the difference in the two cases and equating the two results yields

$$y_{t+1} = c - by_t$$
$$y_{t+1} - y_t = c - by_t - y_t$$
$$\Delta y_t = c - y_t(b+1)$$

Similarly from the solution to the homogeneous equation,

$$y_{t+1} = A(-b)^{t+1} \qquad \text{and}$$
$$y_t = A(-b)^t$$

we obtain a second expression for the difference

$$y_{t+1} - y_t = A(-b)^{t+1} - A(-b)^t$$
$$\Delta y_t = -A(-b)^t(b+1)$$

Equating the two expressions gives the general solution to the nonhomogeneous equation

$$c - y_t(b+1) = -A(-b)^t(b+1)$$

$$y_t = A(-b)^t + \frac{c}{b+1} \qquad \text{where } b \neq -1$$

Definitizing A at moment $t = 0$,

$$y_o = A + \frac{c}{b+1}$$

$$A = y_o - \frac{c}{b+1}$$

and substituting for A we get the definite solution of the nonhomogeneous equation

$$y_t = \left(y_o - \frac{c}{b+1} \right)(-b)^t + \frac{c}{b+1} = y_c + y_p \qquad \text{where } b \neq -1$$

Similar to differential equations, the free term $\frac{c}{b+1}$ gives the particular integral y_p of y, while the first shows the complementary function y_c. Thus, the particular integral $y_p = \frac{c}{b+1}$ represents the intertemporal equilibrium of the function. In the special homogeneous case when $c = 0$ this equilibrium value is zero. Similar to the continuous-time case, the complementary function gives the deviation of y from the equilibrium in any period. When the difference is zero, that is, $\Delta y_t = 0$ or $y_{t+1} = y_t$, then the nonhomogeneous equation gives rise to the particular integral $y_p = \frac{c}{b+1}$. The deviation comes from the solution to the homogeneous difference equation which is already found as $y_c = A(-b)^t$ and the complete solution is the sum of the two results. For $|b| < 1$ the term tends to disappear as $t \to \infty$ and the time path of the function y_t is convergent. In the opposite case the time path is divergent. Depending on the sign of b we have an oscillating or non-oscillating time path. Once the solution to this simple difference equation with a constant term and a constant coefficient is obtained, the professor can explain the concept of the particular integral and the complementary function to prepare students for advanced difference equations or another semester of quantitative methods focused on economic dynamics, economic growth, applied differential and difference equations, etc. Furthermore, he can conveniently apply the solution to the interesting cobweb model of market price.

Conclusion

Our purpose as teachers is to make math less intimidating and the subject of mathematical economics more enjoyable for students. We should, therefore, search for innovative and easy methods of familiarizing students with some mathematical tools. By introducing a quicker and easier way of solving simple differential and difference equations such as those with a constant term and a constant coefficient the professor is making this challenging material accessible for undergraduates of economics and other social sciences. After solving the equations easily the professor can move to very substantive and interesting illustrations such as the dynamics of market price model, the Solow growth model or the cobweb model. This can be done without the need to bury students into sophisticated math theory and intimidating methods of solving advanced differential and difference equations. Such innovative tools help foster student learning. Furthermore, they popularize the economic profession and make economics attractive and lovable even for students who are not so mathematically oriented.

Appendix

We now show the solution to the first-order linear differential equation with a variable coefficient and a variable term, or

$$\frac{dy}{dt} + uy = v$$

which can alternatively be written as

$$dy + (uy - v)dt = 0$$

Using $e^{\int udt}$ as an integrating factor, we obtain

$$e^{\int udt}dy + e^{\int udt}(uy - v)dt = 0$$

which can be solved as an exact equation. To test it for exactness we need $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$.

$$\underbrace{e^{\int udt}}_{M}dy + \underbrace{e^{\int udt}(uy - v)}_{N}dt = 0$$

$$\frac{\partial M}{\partial t} = ue^{\int udt} \qquad \frac{\partial N}{\partial y} = ue^{\int udt}$$

so the condition for exactness of the equation is met. We continue solving through the method of exact differential equations.

$$F(y,t) = \int Mdy + n(t) = \int e^{\int udt}dy + n(t) = ye^{\int udt} + n(t)$$

$$\frac{\partial F}{\partial t} = yue^{\int udt} + n'(t) = N = e^{\int udt}(uy - v)$$

$$n'(t) = -ve^{\int udt}$$

$$n(t) = \int -ve^{\int udt}dt = -\int ve^{\int udt}dt$$

Substituting in $F(y,t)$,

$$F(y,t) = ye^{\int udt} - \int ve^{\int udt}dt = A \qquad A = const.$$

$$ye^{\int udt} = A + \int ve^{\int udt}dt$$

$$y = e^{-\int udt}\left(A + \int ve^{\int udt}dt\right)$$

which is the general solution we have already seen. It can further be expressed as

$$y = e^{-\int udt}A + e^{-\int udt}\int ve^{\int udt}dt = y_c + y_p$$

In the special case when u and v take the constant values a and b, respectively, i.e., we have the constant coefficient and constant term case, the general solution becomes

$$y(t) = Ae^{-at} + \frac{b}{a} = y_c + y_p$$

where the complementary function is again Ae^{-at} and the equilibrium value is given by the particular integral $\frac{b}{a}$. In the homogeneous case where v is zero, the intertemporal equilibrium is also zero and the solution consists only of the complementary function $y_c = e^{-\int u dt} A$ or $y_c = Ae^{-at}$.

For the case of first-order difference equations we recall that they involve terms like y_{t+1} and y_t where the difference in each period is given. Thus, knowing some initial value y_o we can determine the time path of the y function as the time factor t changes. A simple second-order difference equation is

$$y_{t+2} + b_1 y_{t+1} + b_2 y_t = c$$

To find the particular integral in the simplest case we can take a solution of the form $y_t = k$ where in every period y is the constant k

$$k + b_1 k + b_2 k = c \qquad \text{and}$$

$$y_p = k = \frac{c}{1 + b_1 + b_2} \qquad \text{where} \quad b_1 + b_2 \neq -1$$

In the special case of a first-order difference equation where $b_1 = 1$ and $y_{t+2} = 0$ the particular integral is $y_p = k = \frac{c}{1+b}$. With first-order difference equations we found that the expression $y_t = Aa^t$ describes well the general solution of such an equation and we try it to find the complementary function. This implies that $y_{t+1} = Aa^{t+1}$ and $y_{t+2} = Aa^{t+2}$ which upon substitution in

$$y_{t+2} + b_1 y_{t+1} + b_2 y_t = 0 \qquad \text{yields}$$

$$Aa^{t+2} + b_1 Aa^{t+1} + b_2 Aa^t = 0, \text{ or}$$

$$a^2 + b_1 a + b_2 = 0$$

This characteristic equation has roots $a_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2}$. Hence, for the complementary function we have three possibilities:

1. Distinct real roots. If $b_1^2 > 4b_2$, then both roots are real and different so the complementary function is

$$y_c = y_1 + y_2 = A_1 a_1^t + A_2 a_2^t$$

2. Single real root. If $b_1^2 = 4b_2$, there is only one real root $a = -\frac{b_1}{2}$ and the complementary function is

$y_c = y_1 + y_2 = A_1 a^t + A_2 t a^t$

3. Complex roots. When $b_1^2 < 4b_2$, again a pair of conjugate complex numbers $a_{1,2} = m \pm ni$ obtains where $m = -\frac{b_1}{2}$ and $n = \frac{\sqrt{4b_2 - b_1^2}}{2}$. The complementary function is

$y_c = A_1 a_1^t + A_2 a_2^t = A_1 (m+ni)^t + A_2 (m-ni)^t$

From the De Moivre's theorem it follows that $(m \pm ni)^t = R^t (\cos\theta t \pm i \sin\theta t)$ where

$R = \sqrt{m^2 + n^2} = \sqrt{\frac{b_1^2 + 4b_2 - b_1^2}{4}} = \sqrt{b_2}$

Here θ is measured in radians and $\cos\theta = \frac{m}{R} = -\frac{b_1}{2\sqrt{b_2}}$ and $\sin\theta = \frac{n}{R} = \sqrt{1 - \frac{b_1^2}{4b_2}}$. Hence, the complementary function is

$y_c = A_1 R^t (\cos\theta t + i \sin\theta t) + A_2 R^t (\cos\theta t - i \sin\theta t) = R^t (B_1 \cos\theta t + B_2 \sin\theta t)$

where the multiplicative factor R^t substitutes the natural exponential term e^{mt} used in differential equations. Since in the special case of a first-order difference equation we have $b_1 = 1$, $b_2 = b$ and from $a^2 + b_1 a + b_2 = 0$ we have only one root $a = -b$, the complementary function is $y_c = Aa^t = A(-b)^t$. Thus, the general solution of a first-order, first-degree difference equation with a constant term and a constant coefficient of the type $y_{t+1} + by_t = c$ is

$y_t = A(-b)^t + \frac{c}{b+1} = y_c + y_p$

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