Interest rate paradox

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Abstract

System's properties are not always determined by properties of its elements. In this paper was made an attempt to analyze securities not isolated, but with respect to environment, i.e. participants' operations on a market, which results depend on securities. It was shown that risk-neutral probability density, implied in prices, depends on these operations. No arbitrage conditions were developed for this case. Using them it was shown that there are operations that make function that must be a probability density function not a probability density function. These operations are possible if there are assets with positive price and non-zero interest rate. Arbitrage is possible in this case and such market is inefficient.

Keywords: market efficiency, risk-neutral probability density, interest rate, arbitrage, efficiency conditions.

JEL classification codes: G10, G12

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1. Introduction

The theory of No Arbitrage plays a central role in Mathematical Finance. Development of pricing mechanisms (Black and Scholes 1973 and Merton 1973), understanding of market efficiency, no arbitrage conditions (Harrison and Kreps 1979, Harrison and Pliska 1981) and many other important themes, which highly influence nowadays markets, are close related to it. However, there are open questions, e.g. Fama (1997) concluded that existing anomalies require new behaviorally based theories of the stock market and we need to continue the search for better models of asset pricing.

In a modern world we use strategies and securities (e.g. CDOs) that become more and more complex. There are often cascade chains of operations between participant's account and elementary securities. However, most theories analyze elementary (basic) securities and extrapolate results to complex systems (markets). In this paper securities are analyzed using the traditional approach of no arbitrage, but with respect to systems complexity.

There is a class of securities, which price is determined by the next equation (variation of equation presented by Cox and Ross 1976):

$$P_{Po} = e^{-r\cdot T} \cdot \int_{-\infty}^{+\infty} d(S) \cdot Po(S) dS \tag{1}$$

where $Po(S)$ is a payoff function at the moment of expiration T; S is a parameter (often price), which becomes certain at exercising and determine value of $Po(S)$; $d(S)$ is a probability density function; $e^{-r\cdot T}$ is present value (price) of one unit of $Po(S)$.

If participants agree that premium is paid at the moment of expiration or use corresponding futures then $e^{-r\cdot T} = 1$.

If participants want to change basic asset of their accounts, but use securities with standard numeraire (e.g. dollars) then equation (1) can be transformed in the next way:

$$P_{Po} = \frac{e^{-r\cdot T}}{E(X_0)} \cdot \int_{-\infty}^{+\infty} d(S) \cdot Po(S) \cdot E(X) dS \tag{2}$$

where $E(X)$ is exchange function for payoff; X is some set of parameters, it may contain S; X_0 is expected value of X at the moment when premium is being paid.

After exercising participant transform payoff into some other preferable asset. To pay premium participant use inverse transformation $\frac{1}{E(X_0)}$.

This is not a numeraire change in classical sense (Jamshidian 1989). Securities are the same for different $E(X)$.

EUR/USD call options are examples of such securities. At least two cases are possible:

$$E(X)=1$$
$$E(X_0)=1 \tag{3}$$
$$Po(S)=S-K \text{ if } S>K$$

where K is a strike price.

Or

$$E(X)=\frac{1}{S}$$
$$E(X_0)=\frac{1}{S_0} \tag{4}$$
$$Po(S)=S-K \text{ if } S>K$$

where S_0 is initial price of underlying asset.

In second case payoff is paid in dollars and after that transformed in euro.

2. Main section

Proposition 1: Probability density function depends on $E(X)$.

Let $Po(S)$ be Dirac delta function $\delta(\mathrm{x-S})$. Then

$$P_\delta(S)=d(S)\cdot e^{-r\cdot T}\cdot\frac{E(X)}{E(X_0)} \tag{5}$$

Every other $Po(S)$ could be represented as a combination of $\delta(\mathrm{x-S})$. Consequently, $d(S)$ does not depend on $Po(S)$.

P_{Po} is independent from $E(X)$, because basic securities are the same for different $E(X)$. Consequently,

$$d_i(S) \cdot e^{-r_i \cdot T} \cdot \frac{E_i(X)}{E_i(X_0)} = d_j(S) \cdot e^{-r_j \cdot T} \cdot \frac{E_j(X)}{E_j(X_0)} \tag{6}$$

Except some particular cases like constant prices, $d(S)$ and/or $e^{-r \cdot T}$ depend on $E(X)$

Proposition 2: There are such $E_i(X)$ that $d_i(S)$ is not a probability density function.

For every i:

$$\int_{-\infty}^{\infty} d_i(S) dS = 1 \tag{7}$$

Form equation (2) follows that equation (7) reflects possibility to obtain one unit of preferable asset as a payoff and pay for this the same one unit of preferable asset at the moment of expiration as premium. It is apparent that arbitrage is possible otherwise. It should be noted that equation (7) may represent not a single security, but a complex combination of securities. By these reasons equation (7) is a strong no arbitrage condition.

Using equation (6) no arbitrage condition can be transformed in the next two:

$$\int_{-\infty}^{\infty} d_i(S) \cdot \frac{E_i(X)}{E_j(\mathrm{X})} dS = e^{-(r_j - r_i) \cdot T} \cdot \frac{E_i(X_0)}{E_j(\mathrm{X}_0)} \tag{8}$$

$$\int_{-\infty}^{\infty} d_j(S) \cdot \frac{E_j(X)}{E_i(\mathrm{X})} dS = e^{-(r_i - r_j) \cdot T} \cdot \frac{E_j(X_0)}{E_i(\mathrm{X}_0)} \tag{9}$$

There have to be no such $E_i(X)$ and $E_j(X)$ that make equations (8) and (9) false. Otherwise arbitrage opportunities exist.

Assume that S is not expected to be constant and

$$\begin{aligned} &E_1(X) = S, E_1(X_0) = S_0 \\ &E_2(X) = 1, E_2(X_0) = 1 \\ &E_3(X) = \frac{1}{S}, E_3(X_0) = \frac{1}{S_0} \end{aligned} \tag{10}$$

Premiums are paid at the moment of expiration. Consequently, $e^{-r_i \cdot T} = 1$.

No arbitrage conditions:

$$\int_{-\infty}^{\infty} d_1(S)dS = 1$$
$$\int_{-\infty}^{\infty} d_2(S)dS = 1 \tag{11}$$
$$\int_{-\infty}^{\infty} d_3(S)dS = 1$$

Then

$$\int_{-\infty}^{\infty} d_2(S)dS = \int_{-\infty}^{\infty} d_1(S)\cdot\frac{\mathrm{S}}{S_0}dS = \int_{-\infty}^{\infty} d_1(S)dS + \int_{-\infty}^{\infty} d_1(S)\cdot\frac{S-S_0}{S_0}dS =$$
$$1 + \int_{-\infty}^{\infty} d_1(S)\cdot\frac{S-S_0}{S_0}dS = 1 \tag{12}$$

Consequently,

$$\int_{-\infty}^{\infty} d_1(S)\cdot(S-S_0)dS = 0 \tag{13}$$

At the same time

$$\int_{-\infty}^{\infty} d_2(S)\cdot(S-S_0)dS = \int_{-\infty}^{\infty} d_1(S)\cdot\frac{S}{S_0}\cdot(S-S_0)dS = \int_{-\infty}^{\infty} d_1(S)\cdot(S-S_0)d(S) +$$
$$+\int_{-\infty}^{\infty} d_1(S)\cdot\frac{(S-S_0)^2}{S_0}dS = \int_{-\infty}^{\infty} d_1(S)\cdot\frac{(S-S_0)^2}{S_0}dS \tag{14}$$

All multipliers are above zero. Consequently,

$$\int_{-\infty}^{\infty} d_2(S)\cdot(S-S_0)dS \neq 0 \tag{15}$$

At the same time

$$\int_{-\infty}^{\infty} d_3(S)dS = \int_{-\infty}^{\infty} d_2(S)\cdot\frac{S}{S_0}dS = \int_{-\infty}^{\infty} d_2(S)dS + \int_{-\infty}^{\infty} d_2(S)\cdot\frac{S-S_0}{S_0}dS =$$
$$1 + \int_{-\infty}^{\infty} d_2(S)\cdot\frac{S-S_0}{S_0}dS = 1 \tag{16}$$

Consequently,

$$\int_{-\infty}^{\infty} d_2(S)\cdot(S-S_0)dS = 0 \tag{17}$$

Equations (15) and (17) contradict each other. Consequently, at least one of equations (11) is not true. Arbitrage is possible.

Proposition 3: If some asset has a non-zero interest rate then some of $d_i(S)$ is not a probability density function.

There are two securities: S_1 and S_2. S_2 at initial moment consists of some amount a_1 of S_1.

S_1 has non-zero interest rate $r_1(0,t')$. S_2 at some moment t' consists of $e^{r_1(0,t')\cdot t'}\cdot a_1 - \Delta a$ of S_1

where Δa is a managed parameter, not negative, equal to zero before t', constant after t' and is well known to participants at initial moment of time.

$\Delta a \cdot P_{S_1}(t')$ are dividends paid by manager for one unit of S_2 .

Price of S_2 at t' is

$$P_{S_2}(t') = P_{S_1}(t') \cdot (e^{r_1(0,t')\cdot t'} \cdot a_1 - \Delta a) \tag{18}$$

Risk-free investing in S_1 and S_2 must have equal profitability.

$$\frac{1}{P_{S_1}(0)} \cdot e^{r_1(0,t')\cdot t'} \cdot F_{S_1} = \frac{1}{P_{S_2}(0)} \cdot e^{r_2(0,t')\cdot t'} \cdot F_{S_2} \tag{19}$$

where F_{S_1} and F_{S_2} are prices of futures on S_1 and S_2; $r_1(0,t')$ and $r_2(0,t')$ are interest (growth) rates of S_1 and S_2.

According to eqaution (18):

$$F_{S_2} = f(\Delta a) \tag{20}$$

Interest rate of S_2 for period (t_i, t_j) is also a managed parameter.

$$r_2(t_i, t_j) = f(\Delta a) \tag{21}$$

From equation (19) follows:

$$\begin{gathered} \lim_{\Delta a \to a_1(\mathrm{t}')} r_2 = \infty \\ \lim_{\Delta a \to 0} r_2 = 0 \end{gathered} \tag{22}$$

Interest rate of S_2 can vary from 0 to ∞.

Let S_3, S_4 and S_5 be futures on S_2 with expiration at t_3, t_4 and t_5, $t_3 < t_4 < t_5, t_5 - t_4 = t_4 - t_3 = \Delta t$.

Let $r_2(t_3, t_4) = r_2(t_4, t_5) = r$.

Let there is a security, priced in the way of equation (2), with underlying asset S_5 and numeraire S_4. Payoff depends on price of S_5 in S_4, which is after expiration transformed into S_5 or S_3.

Price of S_5 in S_4 is

$$P_{S_5,S_4} = \frac{1}{E_3(X)} = e^{-r_2(t_4,t_5)\cdot(t_5-t_4)} = e^{-r\cdot\Delta t} \tag{23}$$

Price of S_3 in S_4 is

$$P_{S_3,S_4} = \frac{1}{E_1(X)} = \frac{1}{e^{-r_2(t_3,t_4)\cdot(t_4-t_3)}} = \frac{1}{e^{-r\cdot\Delta t}} \tag{24}$$

Conditions of *proposition 2* are satisfied. Consequently, $d_3(\mathrm{S})$, $d_4(\mathrm{S})$ or $d_5(\mathrm{S})$ is not a probability density function. It allows arbitrage and making risk-free profit.

Consequently, efficient state of market is when there are no non-zero interest rates on it. Otherwise arbitrage opportunities exist.

3. Conclusion

Well known asset pricing formula was generalized to the case when participant instead of performing one operation with elementary security perform chain of operations. Prices in these operations are connected to each other.

It was shown that risk-neutral probability density, implied in price of basic security, depends on other operations in chain. Different densities are connected to each other. This property was used to obtain no arbitrage conditions for this case.

The main result is that existence of non-zero interest rate means existence of arbitrage opportunities, because function that has to be a probability density function becomes not a probability density function. This makes such market inefficient.

This leads to:

1. Martingale prices.
2. Shares that do not pay dividends.
3. No risks represented as interest rates.
4. No discounting.
5. Many others.

It sounds rather paradoxical and even absurdly. However, if it true then most principles and theories of modern market economy must be revised.

From the practical point of view it should be noted that differences between probability density functions should be tiny. By that reason lack of liquidity and transaction costs could be barriers in usage of found arbitrage opportunities. However, found inefficiency is fundamental. So could be a profit.

References

Black, F., Scholes M. (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy 81 (3): 637–654.

Cox, J. and Ross S. (1976). The valuations of options for alternative stochastic processes. Journal of Financial Economics 3: 145-166.

Fama, E. (1998). Market Efficiency, Long-Term Returns and Behavioural Finance. Journal of Financial Economics 49: 283–306.

Harrison J.M. and Kreps D.M. (1979). Martingales and arbitrage in multiperiod arbitrage markets. Journal of Economic Theory 20: 381-408.

Harrison, J. M and Pliska, S. R. (1981). Martingales and Stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications 11 (3): 215–260.

Jamshidian, F. (1989). An Exact Bond Option Pricing Formula. The Journal of Finance 44: 205–209.

Merton, R. C. (1973). Theory of Rational Option Pricing. Bell Journal of Economics and Management Science 4 (1): 141–183.