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Abstract

We consider commuting in a congested urban area. While an efficient time-varying toll may eliminate queuing, a toll may not be politically feasible. We study the benefit of a substitute: a parking fee at the workplace. An optimal time-varying parking fee is charged at zero rate when there is queuing and eliminates queuing when the rate is non-zero. Within certain limits, inability to charge some drivers for parking does not reduce the potential welfare gain. Drivers who cannot be charged travel when there is queuing. In some cases, interaction between morning and evening commutes can be exploited to remove queueing completely.

Keywords: parking; dynamic; congestion; urban; traffic
JEL codes: D0; R4

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1 Introduction

Traffic congestion is an economically important problem affecting cities everywhere. An average American household travels annually about 20,000 miles on roads and spends about 15% of income on road transportation.[1] In 2011, congestion in the US caused around 5.5 billion hours of travel delay and 2.9 billion gallons of extra fuel consumption with a total cost of $121 billion (Schrank et al., 2012). Thus, policies to reduce the cost of mobility by car are of first order importance. This paper considers the possibility of using parking fees rather than congestion pricing to regulate urban congestion by influencing the timing of trips.

Ever since Dupuit (1844) and Pigou (1920), economists have advocated marginal cost pricing of road capacity as a means to improve efficiency. However, very few cities have actually implemented congestion tolls, notably Stockholm, Singapore, London and recently Gothenburg.[2] Congestion tolls have been proposed and then scrapped in many places, including New York, Hong Kong and Copenhagen. So there seems to be important political obstacles to congestion tolls and it is therefore of interest to look for alternative policies that can address road congestion.[3] It is natural to look at parking pricing, since parking is already priced almost everywhere. Another reason, noted by Shoup (2005), is that the technology needed to charge for parking is much simpler than that needed to charge for driving in congested traffic.

[1] http://nhts.ornl.gov/2009/pub/profile_2012.pdf and http://www.bls.gov/cex.

[2] The benefits from congestion tolls can be large. For Stockholm, an annual efficiency gain of 70 million Euro has been estimated, not accounting for investment costs; these would be recovered in 4-5 years (Eliasson, 2009). For London, an annual net efficiency gain of 100 million Euro has been estimated and this figure does account for system costs (Leape, 2006). In the case of Milan, an annual efficiency gain of 6 million Euro has been estimated; this figure does not account for system costs, but these would be recovered in less than two years (Rotaris et al., 2010).

[3] De Borger and Proost (2010) discuss the political economy of road pricing.

It is straightforward that the demand for trips to a city center is affected by the full price of the trip, including the price of parking. But the problem is not just the volume of traffic: the timing of demand is extremely important as is evident from the sharp demand peaks that characterize urban traffic. The physics of congestion implies that the amount of congestion delay is strongly dependent on the timing of trips. If only departures from home in the morning became more dispersed in time, then congestion delay could be much smaller while arrival times could be quite unaffected. So there is a large potential efficiency gain in the retiming of trips, even if the total traffic volume is unaffected. Congestion tolling aims to achieve such temporal dispersion by applying a toll that varies over time with the amount of congestion. The purpose of this paper is to explore the potential for time-varying parking pricing to achieve the same effect.

We use a generalized version of the Vickrey (1969) bottleneck model for this purpose (de Palma and Fosgerau, 2011a). The bottleneck model captures the essence of congestion dynamics, describing a continuum of drivers equipped with preferences regarding the timing of a trip to a common destination. This destination is located behind a bottleneck with a fixed capacity. If the rate at which drivers want to pass the bottleneck exceeds its capacity then delay results.[4] The delay is a pure loss and it could be reduced with no effect on arrival times if people could be induced to choose different departure times. A time-varying toll aims to induce such rescheduling. As long as it induces appropriate rescheduling of trips, it makes no difference where the toll is collected, it can be on any point of the trip.

In this paper we exploit that drivers park at the destination and pay a parking

[4]The bottleneck congestion technology is a means to represent city-wide congestion affecting all traffic and the bottleneck does not necessarily correspond to any single place in a city (Daganzo, 2007; Geroliminis and Daganzo, 2008).

fee. We will mainly consider a parking fee that accumulates at a non-negative time-varying rate. This restriction fundamentally distinguishes such parking fees from congestion tolls. Congestion tolls may vary freely up and down and may be lower on the shoulders of the peak and high in the middle. A parking fee charged at a positive rate during parking is always lower for later arrival times. As drivers differ in the time at which they pass the bottleneck, they differ also in the parking fee they pay. Therefore a parking fee can be used to induce rescheduling of trips but in a less flexible way than a toll.

In summary, parking fees seem to be much easier to introduce than congestion tolls. Like congestion tolls, parking fees may be used to disperse demand over time in order to reduce congestion and gain efficiency, but the efficiency gain may be limited by the restrictions inherent in typical parking fees. The objective of this paper is to present an analysis of parking fees as a means to affect the timing of road use and as an alternative to congestion tolls.[5]

We initially make assumptions that allow us to ignore the influence of the time of unparking. We may think of the destination as the workplace, such that the model describes the morning commute. For the morning commute, we find that the imposition of a parking fee causes the departure interval to occur later than it would in the absence of policy. This shift compensates the early drivers who pay more for parking than later drivers. The optimal parking fee implements a situation where every morning there is first an interval with a demand peak that involves queueing just like an unregulated equilibrium except that it does not involve everybody who travels in the morning. The optimal parking fee rate is

[5]The US Federal Highway Administration has a series of parking pricing projects under their value pricing pilot program (in San Francisco, San Diego, and New York) that include time-varying parking fee rates.

zero during this interval such that the total parking fee is the same for all these drivers. The optimal parking fee rate becomes positive at the time when the queue has dissolved and is set such that zero queue is maintained during the remainder of the morning. The total parking fee decreases with later arrival because drivers who arrive later receive lower scheduling utility.

It is a recurring theme in the debate about charging for parking that some drivers cannot be charged since they have private parking available. In the current situation, it turns out they make no difference provided they can fit within the period where the optimal parking fee rate is zero and queueing occurs. Thus, within this limit, the existence of private parking does not affect the welfare gains that can be achieved from a parking fee.

Another way that drivers may escape the time-varying parking fee rate is through early bird specials, providing all day parking at a discounted price for drivers who arrive at a parking lot by a certain time such as 8 am. The paper characterizes the welfare maximizing combination of an early bird special with a time-varying parking fee rate.

After examining the morning peak, we show that the conclusions of the paper extend with few modifications to the evening commute, where parking is charged at the origin of the trip instead of at the destination.[6] The optimal parking fee affects the evening commute similarly to the morning commute, except that the order of the congested and uncongested intervals is reversed and the departure

[6]de Palma and Lindsey (2002) compare the morning and the evening commute, assuming that scheduling utility is additively separable in travel time and delay, where delay is defined in terms of arrival time for the morning commute and in terms of departure time for the evening commute. Here, we apply a general form of scheduling preferences that applies to both the morning and the evening commute. The difference in principle between the two commutes is whether the parking fee is charged at the origin or at the destination of the trip.

interval occurs earlier than it would in the absence of the parking fee.

The analysis so far ignores any interaction between the two commutes. The paper also analyzes a whole day with explicit interaction between the two commutes. Nonseparability between the morning and evening commutes implies that the morning commute can be affected via the evening parking fee and vice versa. It turns out the limitations involved in parking pricing as compared to freely time-varying congestion tolling can then be overcome, and in our stylized setting a parking fee scheme can be designed to remove congestion completely during both commutes simultaneously. This finding strengthens the case for using parking pricing to tackle urban road congestion.

The first to discuss regulation of parking in an economic context might be Vickrey (1954), who suggested time-varying parking fees as a means of regulating the use of parking space. Glazer and Niskanen (1992) present an analysis where parking fees are analyzed as a substitute for road pricing. They note that the idea rests on the assumption that an increase in the price of parking is equivalent to an increase in the price of a trip. However, this equivalence fails for people who can vary the length of time they park. Increasing the parking fee rate may induce drivers to park for a shorter time, thereby allowing more people to use parking spaces each day and thereby increasing traffic. However, Glazer and Niskanen (1992) do not consider congestion dynamics (see also Verhoef et al., 1995).

In a static simulation model, Calthrop et al. (2000) analyze the efficiency gains from parking fees and road pricing (a cordon toll). They find that these two policies are sub-additive: as roads are more efficiently priced, there is less need for pricing of parking. In contrast to us, they also find that second-best pricing of parking produces a higher welfare gain than a cordon charge around the simulated city. The explanation for this difference is that they consider the supply of parking but not congestion dynamics, where we take the supply of parking as given and consider how to exploit congestion dynamics using a time-varying parking fee.

Like us, Arnott et al. (1991) use the bottleneck model, but they consider a case where parking spaces are located between the bottleneck and the CBD, on a line away from the CBD and where the parking cost varies according to the distance to the CBD. In their analysis, the parking fee does not depend on the length of time the vehicle is parked. Arnott et al. (1991) find that optimal location-dependent parking fees do not eliminate queueing, but induce drivers to park in order of decreasing distance from the CBD, thereby concentrating arrival times closer to work start times. They find that for most reasonable parameter values, the optimal location-dependent parking fee is at least as efficient as the optimal time-varying road toll. In contrast, in the present setting where parking is located at the destination and with temporal but not spatial variation in the parking fee, only a smaller share of the efficiency gain from the optimal road toll can be realized by a parking fee. Qian et al. (2012) present an analysis similar to Arnott et al. (1991) but with parking capacity provided in two parking lots, where the capacity and parking fee may be regulated.

Arnott and Rowse (2009) focus on different aspects of parking. They analyze parking in a spatially homogeneous downtown area. Drivers choose between curbside and garage parking, and curbside parking is cheapest. Cruising for parking contributes to congestion and works to increase the full price for curbside parking until it equals the price of garage parking. Then increasing the curbside parking fee may generate an efficiency gain through reduction of cruising and the ensuing congestion and the efficiency gain may be large relative to the parking fee revenue. Other papers related to cruising include Douglas (1975), Arnott and Rowse (1999), Anderson and de Palma (2004), Arnott and Inci (2006), and Anderson and de Palma (2007). Van Ommeren et al. (2011) estimates the cost of cruising for the residents of Amsterdam. See also Proost and Van Dender (2008) and De Borger and Wuyts (2009).

Zhang et al. (2005) link the morning and evening commutes by treating the length of the work day as a decision variable in a model similar to ours. They do not analyze time-varying parking fees.

Section 2 introduces the model, Section 3 reviews the benchmark case of no policy, while Section 4 reviews the optimal time-varying toll at the bottleneck. Section 5 describes equilibrium under a parking fee, Section 6 considers the optimal parking fee, and Section 7 presents an example under specific assumptions about scheduling preferences. Section 8 considers the case when some drivers cannot be charged for parking and Section 9 characterizes social optimum including an early bird special. Section 10 discusses the evening commute while Section 11 considers the two commutes in combination. Section 12 concludes. Most proofs are relegated to the Appendix.

2 Model formulation

There is a continuum of mass $N > 0$ of drivers who all have to pass a congested bottleneck. They have identical preferences concerning the timing and cost of their trip expressed by the twice differentiable money metric utility $u(t, a) - \tau$, defined for all $t \leq a$ and τ, where t is the arrival time at the bottleneck, a is the exit time from the bottleneck and τ is the (monetary) cost of the trip. We speak of the length of the duration from t to a as the travel time or the bottleneck delay. We consider only costs in the form of a toll at the bottleneck or a parking fee at the destination. We refer to u as the scheduling utility.[7] Without loss of generality, t represents also the departure time and a the arrival time at the destination. It is also useful to define the schedule delay utility $v(t) \equiv u(t, t)$, which is the scheduling utility that is obtained when travel time is zero. Throughout this paper we make the following assumptions regarding the scheduling utility.

Assumption 1 *Marginal scheduling utility satisfies $u_1 > 0$ and $u_2 < 0$. Schedule delay utility $v(t)$ is strictly quasiconcave and attains maximum $v(t_*)$ at t_*.*

The assumption first requires that drivers always strictly prefer to depart later and to arrive earlier, no matter when they depart and arrive. The assumptions on v will ensure the uniqueness of equilibrium in the model.

The bottleneck has a capacity of ψ cars per time unit. Cars who have not yet been served wait before the bottleneck, which serves travelers in the sequence of arrival (first-in-first-out). The bottleneck capacity is always used if there are cars waiting before it. The physical extension of the queue has no consequences, we say the queue is vertical.

Cumulative departures are denoted $R(\cdot)$ and departures take place during an interval $[a_0, a_1]$. When $R(\cdot)$ is differentiable, we let $\rho(\cdot) = R'(\cdot)$ be the departure rate. If queueing begins at time a_0 and there is still queue at time t, then the queue length at time t is $R(t) - \psi(t - a_0)$ and the driver departing from home at time t

[7] A simple version of scheduling preferences has the so-called $\alpha - \beta - \gamma$ form formulated by Vickrey (1969), estimated by Small (1982), and used by numerous authors since.

exits the bottleneck at time

$$t + \frac{R(t) - \psi(t - a_0)}{\psi} = \frac{R(t)}{\psi} + a_0. \tag{1}$$

After passing the bottleneck, cars enter a parking space, which is vertical like the queue. Drivers pay a parking fee at a positive time-varying rate from the time of arrival at the parking lot until a time Ω which is the same for all drivers. The latter assumption allows us to focus attention on the interaction of the parking fee with the departure time and rules out any interaction with the later departure from the parking space. Specifically, it does not require that all cars have to leave the parking space at time Ω. It is sufficient if utility is a separable part of a more comprehensive utility that also describes preferences regarding times later than Ω, Later, in Section 11, we shall consider a case without separability between the two commutes.

We will not consider situations involving mass departures and so the cumulative departure rate will be invertible. For this reason and since the queue is first-in-first-out, we can make a change of variable and equivalently define the parking fee rate $\pi(\cdot) \geq 0$ in terms of the departure time t. The parking fee for a driver departing and arriving at the bottleneck at time t is then $P(t) = \int_t^{\Omega} \pi(s)\, ds$, where $P'(t) = -\pi(t)$.[8]

The analysis considers Nash equilibrium, which is defined by the property that, given the departure schedule R, no driver is able to strictly increase utility by unilaterally changing departure time. All drivers achieve the same utility in Nash equilibrium. The welfare measure employed is the equilibrium utility of drivers

[8]There is a slight mathematical issue here, since $-\pi$ could differ from P' on sets of measure zero; it is, however, not restrictive to assume they are equal.

times the number of drivers plus the revenue from any toll or parking fee. Since utility is the scheduling utility minus the monetary cost, the welfare measure is equal to the total scheduling utility obtained by drivers.

3 No policy equilibrium

Consider as an introduction the case of no policy. Nash equilibrium has arrivals at the bottleneck during an interval $[a_0, a_1]$, the endpoints of this interval are endogenous and determined in equilibrium. Equilibrium requires that there cannot be unused capacity during this interval, that there cannot be queue at the time of the last departure and that the utility of the first and last drivers to depart are the same.[9] Then the departure interval is uniquely determined by the conditions

$$v(a_0) = v(a_1),$$
$$a_1 = a_0 + N/\psi.$$

The conditions imply that $a_0 < t_* < a_1$, since v is strictly quasiconcave. There is always queue in the interior of $[a_0, a_1]$. The equilibrium is illustrated in Figure 1.

In equilibrium, the number of departures $R(t)$ that have occurred at time t can be determined using (1) by the equation

$$v(a_0) = u\left(t, \frac{R(t)}{\psi} + a_0\right). \tag{2}$$

This determines $R(t)$ since $a \to u(t, a)$ is invertible for all t. Moreover, differ-

[9]If there were unused capacity with departures before or after then some driver could move into the gap and gain. If there were queue at the time of the last departure, then the last driver could postpone departure without affecting arrival which would yield a gain.

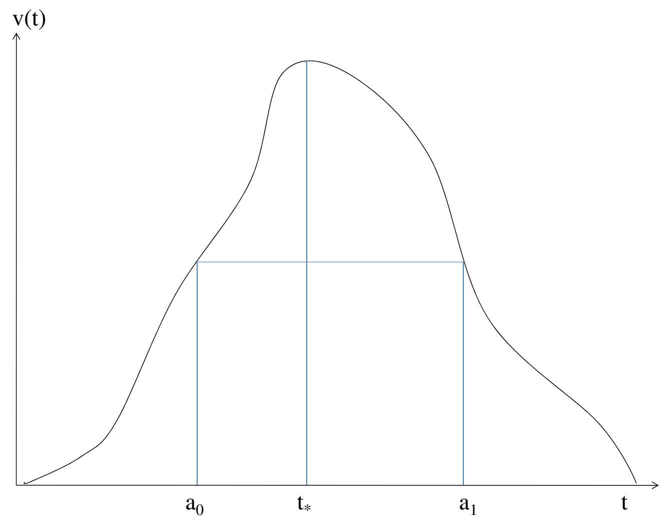

Figure 1: Schedule delay utility and no policy equilibrium

entiating (2), the departure rate is given by

$$\rho(t) = -\psi \frac{u_1\left(t, \frac{R(t)}{\psi} + a_0\right)}{u_2\left(t, \frac{R(t)}{\psi} + a_0\right)} > 0.$$

Here and later, the departure rate is determined almost everywhere.

Figure 2 shows the cumulative departures R as well as the number of cars that have passed the bottleneck $\psi(t - a_0)$. The vertical distance between the two curves corresponds to the length of the queue and the horizontal distance corresponds to the delay in the queue.

4 The optimal time-varying toll at the bottleneck

It is well known that a time varying toll can achieve maximum efficiency by removing the incentive to queue (Vickrey, 1969, 1973; Arnott et al., 1993; de Palma

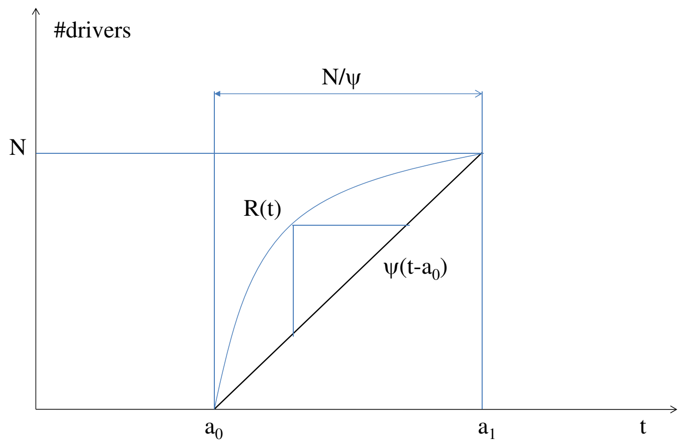

Figure 2: No policy equilibrium

and Fosgerau, 2011b). The efficient toll is charged at the bottleneck at the time varying rate $\tau(t)$. Since total demand is assumed to be completely inelastic, we can set $\tau(a_0) = 0$ at no loss of generality. Efficiency requires $v(a_0) = v(a_1)$ so the efficient toll leaves the departure interval $[a_0, a_1]$ unchanged relative to the no policy equilibrium while maintaining the departure rate at $\rho(t) = \psi$. This requires $\tau(t) = v(t) - v(a_0)$. It follows that the efficient toll inherits strict quasiconcavity from v. Moreover, $a_0 < t_* < a_1$, and the efficient toll is increasing on $[a_0, t_*]$ and decreasing on $[t_*, a_1]$. The revenue from the efficient toll is

$$TR = \psi \int_{a_0}^{a_1} (v(t) - v(a_0))\, dt.$$

Drivers achieve the same utility in equilibrium as under no policy and hence the revenue from the efficient toll is equal to the welfare gain.

5 Parking fee equilibrium

Consider now a parking fee $P(t) = \int_t^{\Omega} \pi(s)\, ds$, where Ω is larger than any departure time. By definition it is decreasing as a function of arrival time (since $P'(t) = -\pi(t)$) and hence it cannot replicate the efficient toll, which is increasing early in the peak.

Some basic properties of equilibrium are given in the following theorem. The proof is included here in the main text since it is helpful in motivating the conditions of the theorem.

Theorem 1 *Consider a parking fee schedule* $P(\cdot)$ *with*

$$
\begin{aligned}
b_0 &< t_* < b_1 && (3)\\
v(t) - P(t) &\geq v(b_0) - P(b_0) \Leftrightarrow t \in [b_0, b_1] && (4)\\
b_1 &= b_0 + \frac{N}{\psi} && (5)\\
\pi(t) + u_2(t,t) &< 0. && (6)
\end{aligned}
$$

Then $\Delta \equiv P(b_0) - P(b_1) \leq \Delta^* \equiv v(t_*) - v(a_0)$ *and there exists a unique departure time equilibrium solution defined on* $[b_0, b_1]$. b_0 *increases strictly as a function of* Δ *as* Δ *ranges over* $[0, \Delta^*]$.

Proof. That $\Delta \leq \Delta^*$ follows from (3) and the quasiconcavity of v. Condition (4) ensures that nobody will want to depart outside $[b_0, b_1]$ and condition (5) ensures that all cars fit within this interval with capacity utilized throughout. Existence and uniqueness of equilibrium then follows if there exists a unique departure rate maintaining constant utility for departures in $[b_0, b_1]$. Condition (4) ensures

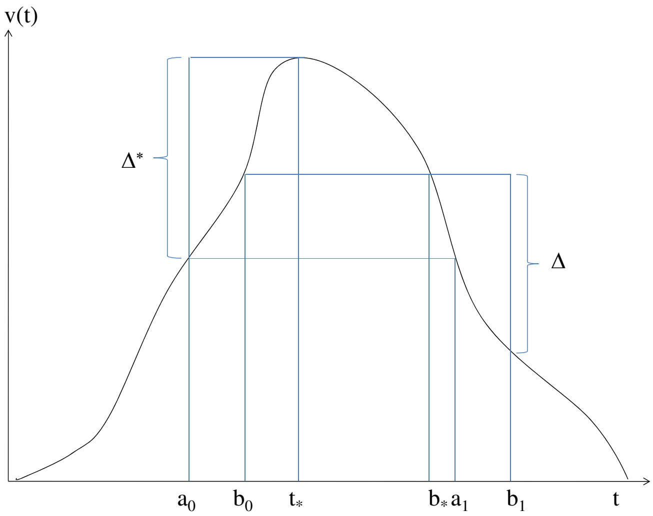

Figure 3: Equilibrium with parking fee

that utility can be constant in equilibrium for departures within $[b_0, b_1]$ with non-negative queue length and this ensures that capacity is fully utilized during $[b_0, b_1]$. The equilibrium queue length exists uniquely and then so does the equilibrium departure rate from home. Condition (6) ensures that the equilibrium departure rate from home is strictly positive. The final conclusion of the theorem follows from the strict quasiconcavity of v. ■

Define for convenience b_* as the unique time $b_* > t_*$ where $v(b_0) = v(b_*)$. The equilibrium is illustrated in Figure 3.

6 Optimal parking fee

Fixing the difference Δ at some value and finding the corresponding departure interval $[b_0, b_1]$, welfare is maximized for a parking fee that extracts maximal revenue while satisfying the condition (4).

Find the unique $b_* > t_*$ with $v(b_0) = v(b_*)$ (see Figure 3). Let $P(t) = P(b_0)$ for $t \in [b_0, b_*]$. This satisfies the conditions of Theorem 1. It is also true that $R(b_*) = \psi(b_* - b_0)$, such that the queue is exactly gone at time b_*.

During the remaining time $[b_*, b_1]$ let $P(t) = v(t) - v(b_0) + P(b_0)$. This also satisfies the conditions of Theorem 1.

With this fee, utility is constant during $[b_*, b_1]$ so there can be no queue. Therefore it is not possible to extract further revenue during this interval. We have therefore established the optimal parking fee conditional on a value of Δ.

Assume without loss of generality that $P(b_1) = 0$. The welfare function defined in terms of Δ is

$$W(\Delta) = \psi(b_* - b_0)\, v(b_0) + \psi \int_{b_*}^{b_1} v(t)\, dt. \tag{7}$$

We can find the optimal value of Δ as given in the following theorem. All proofs of this and theorems following below are given in the Appendix.

Theorem 2 *The optimal parking fee rate is*

$$\pi(t) = \begin{cases} 0, & t \in [b_0, b_*], \\ -v'(t), & t \in]b_*, b_1]. \end{cases}$$

Assume further that $v(\cdot)$ is concave. Then the welfare function $W(\cdot)$ is quasiconcave on $]0, \Delta^*[$, the welfare maximizing value of Δ exists, is unique and satisfies $\Delta = (b_* - b_0)\, v'(b_0) \in]0, \Delta^*[$.

The first statement of this theorem is that the optimal parking fee rate is zero during the interval $[b_0, b_*]$, which is the interval where there is queue under the

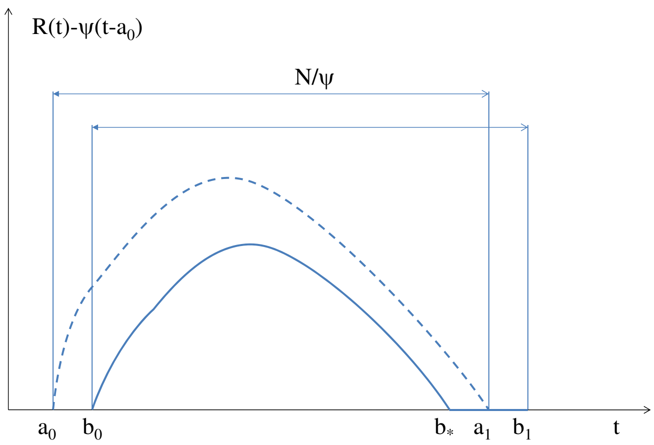

Figure 4: Evolution of the queue under no policy and under the optimal parking fee

optimal parking fee. Thus all drivers in this interval pay the same total amount for parking. The parking fee is concentrated on the interval $]b_*, b_1]$, where it ensures that there is no queue.

Figure 4 illustrates the evolution of queue length under no policy and under the optimal parking fee. The dashed line shows that under no policy the queue first builds and then dissipates between times a_0 and a_1 and that these times span a duration of N/ψ time units. Queueing begins later at time b_0 under the optimal parking fee and it also ends earlier at time b_*. Departures continue during $[b_*, b_1]$ at the capacity rate such that there is no queue during this interval. The latest arrival at time b_1 occurs later than it would under no policy.

7 Linear specification

This section specializes results to the case of so-called $\alpha - \beta - \gamma$ preferences (Vickrey, 1969; Arnott et al., 1993). Let $v(a) = \beta \cdot \min(a, 0) - \gamma \cdot \max(a, 0)$ and let utility be $u(t, a) - \tau = v(a) - \alpha \cdot (a - t) - \tau$. Then α is the value of time, the marginal cost of lateness is γ and the marginal cost of earliness is β. Let $0 < \beta < \alpha, 0 < \gamma$, as is typically assumed (Small, 1982). Then u satisfies the requirements stated in Section 2. The following proposition, proved in the Appendix, provides the optimal welfare gain in terms of the welfare function W, defined in (7). It thus states that the optimal welfare gain is obtained when the difference Δ in parking fee for the first and last drivers is equal to $\frac{\beta\gamma}{\beta+\gamma}\frac{N}{\psi}$. The proposition also evaluates the welfare gain in that case.

Proposition 1 *The optimal parking fee leads to a welfare gain of*

$$W\left(\frac{\beta\gamma}{\beta+\gamma}\frac{N}{\psi}\right) - W(0) = \frac{N^2}{\psi}\frac{\beta^2\gamma}{2(\beta+\gamma)^2}.$$

The interval without queuing has duration

$$b_1 - b_* = \frac{\beta}{\beta+\gamma}\frac{N}{\psi}.$$

Thus, a share $\frac{\beta}{\beta+\gamma}$ of drivers arrive during the later period when the parking fee removes queueing. The maximal welfare gain corresponds to a share of $\frac{\beta}{\beta+\gamma}$ of the maximal welfare gain that can be obtained by a time-varying toll at the bottleneck and the share is strictly less than $1/2$ when $\beta < \gamma$ as would commonly be assumed. It is also straightforward to verify that the revenue from the optimal parking fee corresponds to the same share of $\frac{\beta}{\beta+\gamma}$ of the revenue from the optimal time varying toll. The optimal coarse toll, i.e. a toll that has only two values, captures half the welfare gain that can be obtained by the optimal time-varying toll (Fosgerau, 2011) and so the optimal parking fee approaches this welfare gain when β is close to γ. These results are invariant under proportional changes in (β, α, γ) . A value of γ/β in the range $2-4$ is reasonable and leads to an optimal welfare gain in the range $[0.08, 0.11] \cdot N^2/\psi$, and this is between one fifth and one third of the gain that could be obtained by the optimal time varying toll or between two fifths and two thirds of the gain that could be obtained by the optimal coarse toll.

8 Private parking

We consider now a situation where some drivers cannot be charged for parking. This could be because they have private parking available that cannot be charged by the public authority. Let $N = N_c + N_u$, where N_c is the number of drivers that can be charged and N_u is the number of drivers that cannot be charged. Drivers are otherwise identical and they cannot affect whether they can be charged for parking or not. This assumption enables us to focus on the direct effects of parking fees without having to worry about selection into groups. Charged and uncharged drivers share the same queue at the bottleneck.

Let the departures of uncharged drivers take place during S^u with $Conv\left(S^u\right) = [b_0^u, b_1^u]$ and similarly let departures for charged drivers take place during S^c with $Conv\left(S^c\right) = [b_0^c, b_1^c]$. Let $b_0 = \min\left(b_0^u, b_0^c\right)$, and $b_1 = \max\left(b_1^u, b_1^c\right)$. The following theorem establishes some properties of Nash equilibrium.

Theorem 3 *Consider a parking fee satisfying the conditions (3-6) of Theorem 1. Then, in Nash equilibrium, capacity is fully utilized during* $[b_0, b_1]$ *and* $b_1 = b_0 + N/\psi$. *Uncharged drivers depart within the interval* $[b_0, b_*]$ *with* $b_* < b_1$.

The theorem shows that uncharged drivers depart within the period when there is congestion and schedule delay utility v is largest. Some of the charged drivers are induced to travel later and they all achieve lower utility.

Let N_u and N_c be given. We may then ask what is the optimal charge. Using Theorem 2 and the preceding discussion, the optimal charge that charges N_c drivers satisfies $v(b_0^u) = v\left(b_0^u + \frac{N_u}{\psi}\right)$ and $\pi(t) = -v'(t)$ for $t > b_0^u + \frac{N_u}{\psi} \equiv b_1^u \equiv b_0^c$. Departures of charged drivers take place from b_0^c to $b_1^c = b_0^c + N_c/\psi$. We have $\Delta = P(b_1^c) - P(b_0^c) = v(b_1^c) - v(b_0^c)$. In case Δ is larger than its optimal value from Theorem 2, then there can be an early period with zero charge for charged drivers such that the optimum outcome is obtained. If on the other hand, the number of drivers that can be charged is less than the optimal number, then the optimal charge under this restriction is the one just described.

9 Early bird specials

Early bird specials are common in cities around the world (Victoria Transport Policy Institute, 2012) and they are targeted at commuters. Early bird specials provide all day parking at a discounted price for all-day parkers who arrive at a parking lot by a certain time such as 8 am. This section presents an analysis of how early bird specials can be used to reduce traffic congestion and improve welfare. An early bird special is given by (N_{eb}, a_{eb}, P_{eb}), where the discounted price P_{eb} is available to the first N_{eb} drivers that arrive prior to a_{eb}. This definition does not require the constraints given by N_{eb} and a_{eb} to be binding and so it incorporates the cases where either N_{eb} and a_{eb} is large, such that it is only the number of early birds or the latest arrival time of early birds that is constrained. Denote by $[e_0, e_1]$ the interval during which the early birds travel.

Drivers who do not receive the early bird special, we label regular drivers and we carry forward all previous notation to them: regular drivers pay the regular parking fee π, they travel during $[b_0, b_1]$ and b_* is the time after t_* where $v(b_0) = v(b_*)$. The following theorem characterizes welfare optimum under a parking fee combined with an early bird special. The welfare measure is again the sum of driver utility and parking fee revenues, which is simply the scheduling utility achieved.

Theorem 4 *Under the socially optimal combination of a regular parking fee π with an early bird special (N_{eb}, a_{eb}, P_{eb}), capacity is fully utilized throughout a period of length N/ψ, where $b_0 = e_0 + N_{eb}/\psi$ and $b_1 = e_0 + N/\psi$. The time-varying parking fee is*

$$\pi(t) = \begin{cases} 0, & t \in [b_0, b_*] \\ -v'(t) & t \in]b_*, b_1]. \end{cases}$$

There is queueing during $[b_0, b_*]$ and no queue during $[b_*, b_1]$. Departures begin later than in unregulated equilibrium such that $v(e_0) > v(b_1)$. The early bird charge lies between the total parking fees paid by the first and last regular drivers $P(b_1) < P_{eb} < P(b_0)$.

Figure 5 illustrates the social optimum for the general case. Evaluating the first-order conditions for social optimum for the combination of a time-varying

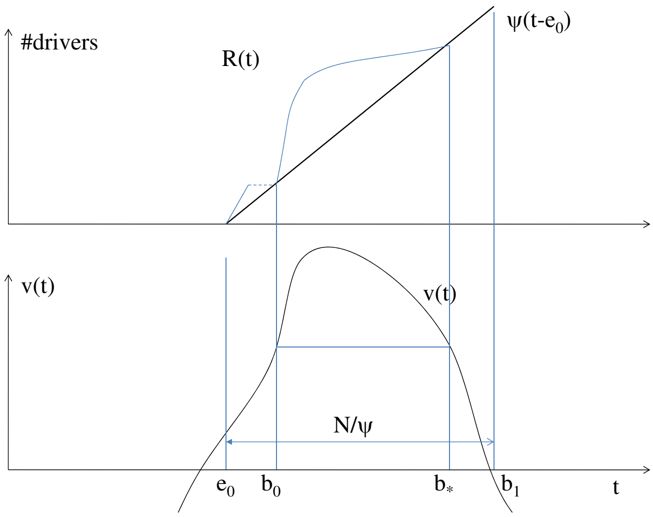

Figure 5: Social optimum with early bird special

parking fee with an early bird special in the case of linear scheduling preferences as discussed in section 7 leads to $e_0 = -\frac{\gamma}{2}\frac{\beta+2\gamma}{(\beta+\gamma)^2}\frac{N}{\psi}$ and $b_0 = \frac{\gamma}{\beta+2\gamma}e_0$, such that $b_0 - e_0 = \frac{1}{2}\frac{\gamma}{\beta+\gamma}\frac{N}{\psi}$ and the optimal share of early birds out of all drivers is $\frac{1}{2}\frac{\gamma}{\beta+\gamma}$. With γ/β in the range $[2, 4]$, this share lies in the range $[0.33, 0.40]$ and it is always smaller than $1/2$.

10 The evening commute

The analysis so far has concerned the morning commute, but with minor modifications it applies to the evening commute as well. This section will show that most conclusions carry more or less directly over from the morning to the evening commute.

Recall first that the analysis of the morning commute ignored any interaction with the evening commute, which could occur, e.g., through the duration of the period at work. This simplification greatly facilitates analysis and will be retained in the analysis of the evening commute.

Our general specification of scheduling preferences treats the departure time and the arrival time symmetrically, so it is not specific to the morning commute, and applies equally well to the evening. We may consider scheduling preferences that are specific to the evening commute with t_* now being the preferred time of instantaneous transfer from work to home.

The treatment of congestion can also be exactly the same in the two commutes. Hence the evening no-policy equilibrium and the optimal time-varying toll exactly parallel those of the morning.

The difference is in the effect of a parking fee paid at the origin of the trip rather than at the destination. The parking fee is charged at the work place. Hence it creates an incentive to reduce the time spent at the workplace. This is equally true in both commutes. In the morning, the parking fee decreases with later departure (from home), while in the evening the parking fee increases with later departure (from work). This reversal has the effect of reversing the order of the two distinct intervals under the socially optimal parking fee. Recall that in the morning social optimum, there is first an interval of queueing, where the parking fee rate is zero, this is followed by an interval where the parking fee rate is $-v'$ and where there is no queue. In the evening social optimum, the evening parking fee rate is first equal to v' during an interval and this maintains the departure rate from work at the bottleneck capacity such that a queue does not arise. Later, in the evening, the parking fee rate is zero and a queueing interval occurs.

Early birds or drivers with private parking are not affected, they have no incentive to depart early and will depart during the period when the parking fee rate is zero. Thus the conclusions for the morning commute regarding drivers with private parking carry over to these cases.

11 Morning and evening commutes integrated

This section considers the morning and the evening commutes simultaneously and shows that interaction between commutes can imply that a parking fee can be designed to remove queueing completely. The parking fee is still restricted to be positive at any time so a parking fee in the morning can only reduce queueing in the morning but not remove it; similarly a parking fee in the evening can only reduce queueing in the evening. But it is possible to exploit interaction between the two commutes that occurs through the length of the time spent at work. Then the morning parking fee affects not only the morning commute but also the evening commute through the length of the working day; similarly a parking fee during the evening commute will affect the morning commute. Somewhat surprisingly, queueing can then be removed in both commutes simultaneously.

Consider drivers who commute to and from work. In the morning they pass through a bottleneck with capacity $\psi_m,$ in the evening they pass through a bottleneck with capacity ψ_e and the two capacities may be different. The departure time from home in the morning is denoted t_m, departures begin at time c_m and cumulative departures in the morning are denoted R_m. Capacity will always be fully utilized during the commute such that $c_m + \frac{R_m(t_m)}{\psi_m}$ is the arrival time at work. The evening commute from home to work is denoted similarly with subscripts e.

We impose more structure on utility than we have before in this paper. In particular we assume that utility is separable in utility achieved at home at rate h_m

prior to departure, utility achieved at home at rate h_e after returning home in the evening and utility achieved associated with the duration at work Γ.[10] Define then the money-metric utility function

$$\begin{aligned} u(t_m, t_e) &= \int_{t_{min}}^{t_m} h_m(s)\, ds + \int_{c_e + \frac{R_e(t_e)}{\psi_e}}^{t_{max}} h_e(s)\, ds \\ &\quad + \Gamma\left(t_e - c_m - \frac{R_m(t_m)}{\psi_m}\right) - \int_{c_m + \frac{R_m(t_m)}{\psi_m}}^{t_e} \pi(s)\, ds. \end{aligned}$$

Utility accumulates at home from time t_{min} in the morning and after time t_{max} in the evening. The values of t_{min} and t_{max} are arbitrary as they have no effect on marginal utilities and hence have no effect on behavior. If it were the case that $\Gamma'' = 0$, then the utility function would be additively separable into a part depending only on t_m and another part depending only on t_e. In this case the morning and evening commutes could be analyzed separately and we would be back in the situation from previous sections. So we require that $\Gamma > 0, \Gamma' > 0, \Gamma'' < 0$. Moreover, utility rates h_m, h_e satisfy $h_m, h_e > 0, h_m' < 0 < h_e'$. We assume that the ranges of the functions involved and their derivatives are sufficient to ensure the existence of equilibrium. Parking is charged at the positive time-varying rate $\pi(\cdot)$ during the time spent at work.

It is clear from the previous analysis and for the same reasons as before that there are two commuting intervals in equilibrium, that capacity is fully utilized during these intervals if the parking fee is not too high, and that in each commute

[10] This assumes that workers can decide how much time to spend at work on any given day. An alternative would be to assume a fixed duration at work. This would however have the implication that the departure rate from work would be the same as the arrival rate at work, and this is at most a constant ψ_m. Then if $\psi_m < \psi_e$ there would never be queue in the evening or if $\psi_m > \psi_e$ there would be an increasing queue at all departure times from work where capacity ψ_m is utilized. Both implications seem strange.

the queue is exactly gone at the time of the last departure. We assume that utility is such that the commuting intervals do not overlap. The equilibrium departure rates can be found from the first-order conditions for utility maximization. The next lemma establishes that drivers pass the bottleneck in the same sequence in the two commutes. This insight is very convenient for the further analysis.

Lemma 1 *Drivers depart in the same sequence in the two commutes.*

The next theorem establishes that it is possible to construct a parking fee that implements equilibrium such that there is no queueing in either commute. The theorem statement defines a function $f()$, which relates departure times in the morning to departure times in the evening when departures in both commutes take place at the capacity rates and commuters depart in the same sequence in the two commutes.

Theorem 5 *Given times c_m and $c_e > c_m + N/\psi_m$, define the function*

$$f(t_m) = \frac{\psi_m}{\psi_e}(t_m - c_m) + c_e, t_m \in [c_m, c_m + N/\psi_m]. \tag{8}$$

Assume that c_m, c_e satisfy the following conditions:

$$\begin{aligned} h_m(c_m) &= \Gamma'(c_e - c_m) & (9)\\ h_e\left(c_e + \frac{N}{\psi_e}\right) &= \Gamma'\left(c_e + \frac{N}{\psi_e} - c_m - \frac{N}{\psi_m}\right) & (10)\\ \max\{h_m(t), h_e(f(t))\} &\leq \Gamma'(f(t) - t), t \in [c_m, c_m + N/\psi_m]. & (11)\end{aligned}$$

The following parking fee leads to an equilibrium with no queueing:

$$\pi(t) = \begin{cases} \Gamma'(f(t) - t) - h_m(t), & c_m \leq t < c_m + N/\psi_m \\ \Gamma'(t - f^{-1}(t)) - h_e(t), & c_e \leq t < c_e + N/\psi_e \\ 0, & otherwise. \end{cases}$$

Conditions (9) and (10) indicate the socially optimal equilibrium with no queueing.

It can be verified that a solution to conditions (9) and (10) must be unique. These conditions are not generally restrictive, they merely require that the functions involved have ranges that are sufficiently large to allow a solution. Condition (11) may be restrictive if capacities are unequal, $\psi_m \neq \psi_e$, but it is redundant if they are equal.

The parking fee of the theorem implements a situation where the morning commute takes place during $[c_m, c_m + N/\psi_m]$ with departures at the capacity rate ψ_m. The definition (8) ensures that if drivers depart at the capacity rate ψ_m during the morning commute, then they depart at the capacity rate ψ_e during $[c_e, c_e + N/\psi_e]$ in the evening. Conditions (9-11) ensure that the parking fee rate is always positive and that $\pi(c_m) = \pi(c_e + N/\psi_e) = 0$.

Compared to a situation with no parking fee and first departures still at c_m and c_e, the welfare gain from the parking fee of Theorem 5 is the total parking fee payment during the two commutes. The parking fee during $[c_m, c_e + N/\psi_e]$ when all are at work is set to zero in the theorem but can be larger provided the equilibrium conditions are not affected. The parking fee revenue during this period does then not affect behavior (as we assume fixed demand) and does hence not contribute to any change in welfare.

12 Conclusion

This paper has analyzed the potential efficiency gains that may be realized through retiming of commuting trips due to a time-varying parking fee charged at a positive rate at the workplace. At the social optimum, the commute to work is divided into two distinct intervals by the optimal parking fee. During the first interval, parking is charged at a zero rate and there is queueing. During the second interval, parking is charged at a time-varying rate such that there is no queue while capacity remains fully utilized. The sequence of these two periods is reversed from the morning to the evening commute. Parking fees create an incentive to reduce the length of time spent at work.

With private parking, a group of drivers cannot be charged for parking. It turns out not to matter for equilibrium departure time outcomes for the optimal charge, provided the drivers who cannot be charged are few enough to fit within the congested part of the commute. It is thus possible to exempt a group of drivers from paying the parking fee without sacrificing the welfare gains that can be achieved. Early bird specials may be designed to increase efficiency even further.

The analysis up to this point treated the morning and evening commutes separately. During either commute, a parking fee can reduce congestion but not remove it. When there is interaction between the commutes through the duration of time spent at work then it is possible to affect the evening commute through a parking fee during the morning and vice versa. The paper has exhibited a case where it is then possible to utilize the interaction to remove congestion completely during both commutes through a parking fee.

It is an essential characteristic of parking fees considered in the paper that the total parking fee payment is decreasing as a function of the arrival time at work in the morning and increasing as a function of the departure time from work in the evening. This restriction leads to results that differ from the case of a time-varying toll. If it were possible to charge for parking at a negative rate, then any time-varying toll could be replicated and the well-known analysis of such a toll could be applied.

It is straightforward to extend the results of this paper to the case of elastic demand. A way to proceed is to let aggregate demand depend on the average utility obtained in equilibrium. The optimal toll can then be obtained by fixing $P(b_1)$, which amounts to adding a fixed component to the parking fee. If $P(b_1) = -Nv'(b_1)\frac{\partial b_1}{\partial N}$ then the marginal benefit of adding a car equals the marginal cost. In this way the model can be extended to deal with externalities including e.g. congestion cruising for a limited number of parking spaces and other congestion externalities.

The current analysis has focused on the interaction of a time-varying parking fee rate with congestion dynamics. We focus on the timing of parking and thus complement the earlier contributions discussed in the introduction that, simply put, consider where and for how long to park. Future research could seek to integrate these perspectives in a unified analysis. It would also be natural to seek to allow for heterogenous drivers, as has been done for the bottleneck model by Lindsey (2004) and recently van den Berg and Verhoef (2011).[11]

[11] With the dynamic bottleneck model, METROPOLIS, implemented for large networks, such complications could be envisaged. This will allow to test the robustness of our predictions for large scale networks (see de Palma et al., 1997).

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A Proofs

Proof of Theorem 2. The first part of the theorem has already been established. It remains to determine the welfare maximizing value of Δ. Compute the derivative of W as

$$W'(\Delta) = \psi(b'_* - b'_0) v(b_0) + \psi(b_* - b_0) v'(b_0) b'_0 + \psi(v(b_1) b'_1 - v(b_*) b'_*) .$$

Use that $v(b_0) = v(b_*)$, $b'_1 = b'_0$, and $\Delta = v(b_0) - v(b_1)$ to reduce this expression to

$$W'(\Delta) = [(b_* - b_0) v'(b_0) - \Delta] \psi b'_0,$$

and note that this is zero if and only if $\Delta = (b_* - b_0) v'(b_0)$. Next use that $1 = v'(b_0) b'_0 - v'(b_1) b'_1$ and $b'_1 = b'_0$ to find that $b'_0 = (v'(b_0) - v'(b_1))^{-1} > 0$. Note that

$$W'(0) = [(b_* - b_0) v'(b_0)] \psi b'_0 > 0$$

and that

$$W'(\Delta^*) = -\Delta^* \psi b'_0 < 0,$$

since $b_0 = b_*$ at $\Delta = \Delta^*$. Then there is at least one value of Δ between 0 and Δ^* with $W'(\Delta) = 0$. Evaluate next the second derivative of W at a point with $W'(\Delta) = 0$:

$$\begin{aligned} W''(\Delta) &= [(b'_* - b'_0) v'(b_0) + (b_* - b_0) v''(b_0) b'_0 - 1] \psi b'_0 \\ &+ [(b_* - b_0) v'(b_0) - \Delta] \psi b''_0 \\ &= [(b'_* - b'_0) v'(b_0) + (b_* - b_0) v''(b_0) b'_0 - 1] \psi b'_0 \end{aligned}$$

$$= \quad \left[\frac{v'(b_0) - v'(b_*)}{v'(b_*)} v'(b_0) b_0' + (b_* - b_0) v''(b_0) b_0' - 1 \right] \psi b_0',$$

where the last equality follows upon noting that $v'(b_0)b_0' = v'(b_*)b_*'$. This is negative if and only if

$$\frac{v'(b_0) - v'(b_*)}{v'(b_*)} v'(b_0) b_0' + (b_* - b_0) v''(b_0) b_0' < 1.$$

But this inequality holds since $v'(b_*) < 0$ and v is concave. Thus $W'(\Delta) = 0$ implies that $W''(\Delta) < 0$ and hence that W is quasiconcave on the interval $[0, \Delta^*]$ such that W has a unique maximum there. It is straightforward to verify that this maximum is global. ■

Proof of Theorem 3. Given the assumptions of Theorem 1, all departures will take place within the interval $[b_0, b_1]$ in Nash equilibrium. Now, $v(b_0) = u\left(b_0, \frac{R(b_0)}{\psi} + b_0\right) > u\left(b_1, \frac{R(b_1)}{\psi} + b_0\right) = v(b_1)$, where the inequality follows since the last driver pays a strictly smaller parking fee than the first but achieves the same utility. Moreover, $U^u \equiv u\left(t, \frac{R(t)}{\psi} + b_0\right), t \in S^u$ is constant, which requires that there is queue almost always during S^u. Equilibrium similarly requires that $U^c \equiv u\left(t, \frac{R(t)}{\psi} + b_0\right) - P(t), t \in S^c$ is constant. Thus $u\left(t, \frac{R(t)}{\psi} + b_0\right)$ is strictly decreasing on points of S^c where $\pi(t) > 0$. These conditions imply that all uncharged drivers obtain utility $v(b_0)$. Therefore they must all depart in the interval $[b_0, b_*]$, where b_* is defined by the equation $v(b_0) = v(b_*)$, which implies that $b_* < b_1$ by quasiconcavity of v. ■

Proof of Proposition 1. Given $\Delta = P(b_0) - P(b_1)$, with $0 < \Delta < v(t_*) -$

$v(a_0) = \frac{N}{\psi}\frac{\beta\gamma}{\beta+\gamma}$ and $P(b_1) = 0$, it is straightforward to find that

$$b_0 = \frac{\Delta - \gamma\frac{N}{\psi}}{\beta+\gamma}, b_* = -\frac{\beta}{\gamma}\frac{\Delta - \gamma\frac{N}{\psi}}{\beta+\gamma}, b_1 = \frac{\Delta + \beta\frac{N}{\psi}}{\beta+\gamma}.$$

Then the welfare given Δ is

$$\begin{aligned} W(\Delta) &= Nv(b_0) - \psi(b_1 - b_*)\Delta/2 \\ &= N\beta\frac{\Delta - \gamma\frac{N}{\psi}}{\beta+\gamma} - \psi\left(\frac{\Delta + \beta\frac{N}{\psi}}{\beta+\gamma} + \frac{\beta}{\gamma}\frac{\Delta - \gamma\frac{N}{\psi}}{\beta+\gamma}\right)\frac{\Delta}{2} \\ &= \frac{1}{\beta+\gamma}\left(-\beta\gamma\frac{N^2}{\psi} + \beta\Delta N - \psi\frac{\beta+\gamma}{\gamma}\frac{\Delta^2}{2}\right). \end{aligned}$$

This is maximal when

$$\Delta = \frac{\beta\gamma}{\beta+\gamma}\frac{N}{\psi}.$$

In this case

$$b_0 = \frac{-\gamma^2}{(\beta+\gamma)^2}\frac{N}{\psi}, b_* = \frac{\beta\gamma}{(\beta+\gamma)^2}\frac{N}{\psi}, b_1 = \frac{\beta^2 + 2\beta\gamma}{(\beta+\gamma)^2}\frac{N}{\psi}.$$

The optimal time-varying toll leads to a welfare gain of $\frac{N^2}{\psi}\frac{\beta\gamma}{2(\beta+\gamma)}$. ■

Proof of Theorem 4. Clearly, early birds depart before other drivers during $[e_0, e_1]$ where $e_1 < t_*$. They pay the same price for parking and will therefore queue, departing at the rate $\rho_{eb}(t) > \psi$,with $R_{eb}(e_1) = N_{eb}$ and the last arrival time being $e_0 + \frac{N_{eb}}{\psi}$. For other drivers, it is optimal that they are charged according to a fee as in section 6 where there is first an interval $[b_0, b_*]$ of arrival times where the parking fee rate is zero, there is queueing and $v(b_0) = v(b_*)$, next there is an interval $[b_*, b_1]$ of arrival times with no queueing and a parking fee rate that is

$\pi(t) = -v'(t)$. We recall that $b_0 \leq t_* \leq b_* < b_1$. It is also clear that capacity should be fully utilized during the commute. This requires that the last arrival time of the early birds is the same as the first arrival time of the ordinary drivers $e_0 + \frac{N_{eb}}{\psi} = b_0$. All drivers pass the bottleneck during $[e_0, b_1]$, so $b_1 = e_0 + N/\psi$. Thus the timing of departures is determined by e_0 and b_0. The difference between the parking fees P_{eb} and P is then also determined since all drivers achieve the same utility in equilibrium.

Welfare is

$$W = \psi \cdot (b_0 - e_0) v(e_0) + \psi \cdot (b_* - b_0) v(b_0) + \psi \int_{b_*}^{b_1} v(t) \, dt,$$

which is composed of $\psi \cdot (b_0 - e_0)$ early birds achieving scheduling utility $v(e_0)$, $\psi \cdot (b_* - b_0)$ ordinary drivers achieving scheduling utility $v(b_0)$ and the remaining $\psi \cdot (b_1 - b_*)$ achieving scheduling utility $v(t)$. The timing of departures is chosen through e_0 and b_0 to optimize welfare with first-order conditions (when $b_0 < t_* < b_*$)

$$\begin{aligned} v(e_0) &= (b_0 - e_0) v'(e_0) + v(b_1), \\ v(e_0) &= v(b_0) - (b_* - b_0) v'(b_0). \end{aligned}$$

Now $v'(e_0), v'(b_0) > 0$ such that $v(b_1) < v(e_0) < v(b_0)$. Utilities are equal in equilibrium so $P(b_1) < P_{eb} < P(b_0)$.

A corner solution arises when $b_0 = t_* = b_*$. In that case only e_0 may vary and has first-order condition

$$v(e_0) = (t_* - e_0) v'(e_0) + v(b_1),$$

implying that again $v(b_1) < v(e_0) < v(b_0)$. ■

Proof of Lemma 1. The first-order condition for the choice of departure time in the morning, given the departure time in the evening, is

$$\begin{aligned} 0 &= \frac{\partial u(t_m, t_e)}{\partial t_m} \\ &= h_m(t_m) - \left(\Gamma' \left(t_e - c_m - \frac{R_m(t_m)}{\psi_m} \right) - \pi \left(c_m + \frac{R_m(t_m)}{\psi_m} \right) \right) \frac{\rho_m(t_m)}{\psi_m}. \end{aligned}$$

Observe that any t_m can only solve the first-order condition for one value of t_e. The function $t_e(t_m)$ thus defined then is single-valued. By the Berge maximum theorem (Aliprantis and Border, 2006), t_e has compact graph and hence t_e is continuous. We take for granted that it is continuously differentiable. The second-order condition requires that

$$\frac{\partial^2 u(t_m, t_e)}{\partial t_m^2} \leq 0.$$

Differentiating the first-order condition with respect to t_m leads to

$$0 = \frac{\partial^2 u(t_m, t_e)}{\partial t_m^2} - \Gamma'' \left(t_e - c_m - \frac{R_m(t_m)}{\psi_m} \right) \frac{\partial t_e}{\partial t_m}$$

and hence $\frac{\partial t_e}{\partial t_m} \geq 0$. It is possible to have $\frac{\partial t_e}{\partial t_m} = 0$ at points, but $\frac{\partial t_e}{\partial t_m} = 0$ cannot hold on any interval. If it did, then there would be a mass departure in the evening, which is ruled out in equilibrium (if a mass departure should occur, then it is always strictly utility increasing to postpone departure until immediately after the mass departure). This shows that $\frac{\partial t_e}{\partial t_m} > 0$ almost everywhere. ■

Proof of Theorem 5. Let $R_m(t_m) = \psi_m(t_m - c_m)$ in the first commute and $R_e(t_e) = \psi_e(t_e - c_e)$ in the second. Then there is no queueing while capacity is fully utilized. Utility for a driver with departure times t_m and t_e is then

$$u(t_m, t_e) = \int_{t_{min}}^{t_m} h_m(s)\,ds + \int_{t_e}^{t_{max}} h_e(s)\,ds + \Gamma(t_e - t_m) - \int_{t_m}^{t_e} \pi(s)\,ds.$$

Consider a driver departing from home at time $t_m \in [c_m, c_m + N/\psi_m]$. Then the first-order condition for the choice of the second departure time has only one solution, namely at $t_e = f(t_m)$ by the definition of π. Moreover, the second-order condition is satisfied,

$$\begin{aligned}
\left.\frac{\partial^2 u(t_m, t)}{\partial t^2}\right|_{t=t_e} &= -h'(t_e) + \Gamma''(t_e - t_m)\left(1 - \frac{1}{f'(t_m)}\right) - \pi'(t_e) \\
&= -h'(t_e) + \Gamma''(t_e - t_m) - \left(\Gamma''\left(t_e - f^{-1}(t_e)\right)\left(1 - \frac{1}{f'(t_m)}\right) - h'(t_e)\right) \\
&= \frac{\Gamma''(t_e - t_m)}{f'(t_m)} < 0.
\end{aligned}$$

With the optimal choice of departure time from work, $t_e = f(t_m)$, utility is constant over the interval $t_m \in [c_m, c_m + N/\psi_m]$, since

$$\frac{\partial u(t_m, f(t_m))}{\partial t_m} = h_m(t_m) - \Gamma'(t_e - t_m) + \pi(t_m) + (-h_e(t_e) + \Gamma'(t_e - t_m) - \pi(t_e))\, f'(t_m) = 0,$$

by the definition of π. Utility is smaller outside the interval by the conditions of the Theorem. Then the cumulative departure functions R_m, R_e defined above do in fact lead to equilibrium.

Condition (11) ensures that all drivers would benefit from increased duration at work. Due to conditions (9) and (10), this is not possible due to the restriction that the parking fee rate must be positive. Hence the equilibrium indicated by (9) and (10) is socially optimal. ■