Conditional Beliefs and Higher-Order Preferences

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CONDITIONAL BELIEFS AND HIGHER-ORDER PREFERENCES*

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Abstract

In this paper, we establish the Bayesian foundations of type structures in which beliefs are lexicographic probability systems (LPS's)—such as those used in Brandenburger *et al.* (2008)—rather than standard probability measures as in Mertens and Zamir (1985). This is a setting which the distinction between preferences hierarchies (Epstein and Wang, 1996) and beliefs hierarchies is meaningful and the former has conceptual advantages. Type structures in which beliefs are conditional probability systems (CPS's) are found to describe fewer hierarchies than LPS type structures can if a nonredundancy requirement is imposed. The two families of type structures are found to be capable of describing the same set of hierarchies in the absence of such a requirement. The existence of "largest"—a notion closely related to universality—LPS/CPS type structures is also shown. Finally, we find that some coherent hierarchies cannot be types but those hierarchies may be needed to express epistemic conditions for iterated elimination of weakly dominated strategies.

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1 INTRODUCTION

Blume *et al.* (1991b) showed that Nash equilibrium refinements such as admissible[1] equilibrium, perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978) could be more simply characterized using lexicographic probability systems (LPS's Blume *et al.*, 1991a) instead of infinite sequences of probability measures. Such equilibrium refinements on normal-form games have implications on rational behavior in extensive-form games. For example, in the words of Kohlberg and Mertens (1986), "given a game tree, a proper equilibrium of its normal form will give a sequential equilibrium in any variant of that tree obtained by applying any of the… inessential transformations."

Conditional probability systems (CPS's), as used by Myerson (1986) and Battigalli and Siniscalchi (1999) to describe beliefs in extensive-form games, can also be viewed as LPS's satisfying some restrictions on the set of events on which beliefs can be conditioned (cf. Hammond, 1994; Halpern, 2010). It is not unreasonable to hypothesize that normal-form analogs of various epistemic analyses of rationality in extensive-form games that use CPS's could be formulated by using LPS's.

While LPS's have the potential to be useful to analyses in the aforementioned settings, there was little progress on the use of LPS's to state conditions on beliefs about beliefs until Brandenburger *et al.* (2008) provided an epistemic characterization of iterated elimination of weakly dominated strategies. Their solution to the long-standing puzzle involved conditions that could not be expressed using the standard frameworks in which beliefs are represented by probability measures.

However, some nontrivial conceptual questions have been raised by the use of LPS type structures—i.e., type structures in which types are mapped to LPS beliefs about other types—in Brandenburger *et al.* (2008). In order to answer these questions, the foundations of LPS type structures and their connection to beliefs hierarchies must be established in the same way that papers like Mertens and Zamir (1985) and Brandenburger and Dekel (1993) did for standard probability type structures. The related issues are briefly enumerated and summarized below. The first and second issues are given more detailed treatment in Lee (2013).

Issue 1 Brandenburger *et al.* (2008) use type structures in which beliefs are lexicographic conditional probability systems (LCPS). That is, types are mapped to beliefs that are naturally interpreted as CPS's. When we refer to CPS's in the remainder of this paper, we will mean LCPS's. However, the finite-order beliefs implied by such type structures are not CPS's themselves.

Issue 2 The distinction between CPS's and LPS's disappears when the underlying space of uncertainty contains duplicate elements that represent the same descriptions of reality. The set of LPS's over any space of uncertainty, when interpreted

[1] i.e., equilibrium in undominated strategies

as representations of preferences over acts that are consistent with the axioms of Blume *et al.* (1991a), contain continuum-many duplicate representations of the same preferences. An immediate and stark consequence of this fact is that it there is that there is no difference—as preferences over preferences à la Epstein and Wang (1996)—between LPS beliefs about LPS beliefs and CPS beliefs about LPS beliefs unless such redundancies are eliminated.

Issue 3 Keisler and Lee (2012) shows that two CPS type structures that describe the exact same set of beliefs hierarchies can differ in whether they permit the existence of states that satisfy *rationality and common assumption of rationality (RCAR)*—Brandenburger *et al.* (2008)'s epistemic condition for iterated admissibility. It is unclear whether the distinction between LPS type structures and CPS type structures as far as this epistemic condition is concerned.

Issue 4 The epistemic condition for iterated admissibility formulated by Brandenburger *et al.* (2008) is RCAR in *complete* LPS type structures. Complete type structures are large type structures in the sense that the mappings from types to beliefs are surjective. Given that RCAR exists in some (cf. Keisler and Lee, 2012) but not all (cf. Brandenburger *et al.*, 2008) complete CPS type structures, the question of what the "largest" CPS type structure is becomes relevant.

In this paper, we construct the set of all coherent hierarchies of lexicographic expected utility (LEU) *preferences* so that no two hierarchies in the set represent the same preferences hierarchy. We then obtain the somewhat suprising result that the set of LEU preferences hierarchies does not coincide with the set of LPS beliefs hierarchies. In particular, there are some LEU preferences hierarchies that cannot be represented by types!

We also establish the existence of an LPS type structure that is "largest" in the sense of inducing all hierarchies that can be described by all other LPS type structures. An analogous result is proven for CPS type structures without redundant types.

We also formally establish the ways in which the descriptive powers of LPS type structures and CPS type structures differ. If redundant types are permitted, a hierarchy can be described by an LPS type structure if and only if it can be described by an CPS type structure. In the absence of redundant types, some hierarchies cannot be induced by CPS type structures. That said, if an epistemic condition can be stated using only finite-order beliefs, CPS type structures and LPS type structures are shown to be equally powerful.

Finally, we provide epistemic conditions for iterated admissibility that are formulated in an explicit model of preferences hierarchies as opposed to in an implicit model such as a type structure. As a byproduct, it is revealed that, regardless of how we topologize the spaces of finite-order beliefs, there are preferences hierarchies that satisfy this analog of Brandenburger *et al.* (2008)'s RCAR but cannot be types.

2 CONDITIONAL BELIEFS AND LEU PREFERENCES

2.1 PRELIMINARIES

Following Savage (1954), the beliefs considered in this paper are "personalistic" in the sense that they capture the degrees of confidence that a decision maker has in various statements. These degrees of confidence are reflected in her choices over menus of acts that promise utility prizes contingent on resolutions of uncertainty.

Definition 1. Let X be a topological space. X is a **Polish space** if its topology $\mathsf{Top}(X)$ is separable and completely metrizable.

Definition 2. Let X be a topological space. X is a **standard Borel space** if X is a Borel subset of some Polish space Y and $\mathsf{Top}(X)$ is the subspace topology with respect to Y. Equivalently, X is a standard Borel space if its Borel sets can be generated by a Polish topology.

For Definitions 3–9, let X be a standard Borel space.

Definition 3. An **act** defined on X is a bounded Borel map $f\colon X \to [0,1]$. The set of all acts defined on X is denoted by $\mathsf{Act}(X)$.

Definition 4. The set of all **probability Borel measures** defined on X is denoted by $\mathcal{P}(X)$, which is itself a standard Borel space.[2]

Definition 5. A **lexicographic probability system (LPS)** defined on X is a finite sequence of probability measures on X.

$$\forall n \in \mathbb{N} \quad \mathcal{L}_n(X) \equiv \prod_{j=1}^{n} \mathcal{P}(X) \equiv \text{set of all length-}n\text{ LPS's} \qquad \mathcal{L}(X) \equiv \biguplus_{n\in\mathbb{N}} \mathcal{L}(X) \equiv \text{set of all LPS's}$$

The sets $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are standard Borel spaces under appropriate topologies.[3]

Definition 6. Let $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(X)$ and $f \in \mathsf{Act}(X)$. The **lexicographic expected utility** of f under σ is the following n-tuple of expected utilities.

$$\mathrm{E}_\sigma f \equiv \left(\int f\, d\mu_1, \ldots, \int f\, d\mu_n \right) = (E_{\mu_1} f, \ldots, E_{\mu_n} f)$$

Definition 7. Each LPS $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(X)$ defines the preference relation $\succsim_\sigma$ on $\mathsf{Act}(X)$ as follows, where $\geq^{\mathcal{L}}$ is the lexicographic order.[4]

$$\forall f, g \in \mathsf{Act}(X) \quad f \succsim_\sigma g \iff \mathrm{E}_\sigma f \geq^{\mathcal{L}} \mathrm{E}_\sigma g$$

[2] If X is a Polish space, then the weak* topology on $\mathcal{P}(X)$ is Polish.
[3] $\mathcal{L}_n(X)$ has the product topology and $\mathcal{L}(X)$ is a topological union.
[4] ***

Definition 8. Probability measures $\mu, \nu \in \mathcal{P}(X)$ are **mutually singular** if there exist disjoint Borel sets $U, V \subseteq X$ such that $\mu(U) = \nu(V) = 1$. We write $\mu \perp \nu$ to denote the fact that μ and ν are mutually singular.[5]

Definition 9. An LPS $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}_n(X)$ is a **conditional probability system (CPS)** if its component measures $\mu_1, \ldots, \mu_n$ are pairwise mutually singular. Equivalently, σ is a CPS if there exists a family $\{U_1, \ldots, U_n\}$ of pairwise disjoint Borel sets such that $\mu_1(U_1) = \cdots = \mu_n(U_n) = 1$. $\mathcal{C}(X)$ denotes the set of all CPS's on X. For all $n \in \mathbb{N}$, $\mathcal{C}_n(X) \equiv \mathcal{C}(X) \cap \mathcal{L}_n(X)$.

2.2 REDUNDANCY AND CONDITIONAL BELIEFS ABOUT BELIEFS

Let X be a standard Borel space. Unlike the space $\mathcal{P}(X)$ of probability measures, the space $\mathcal{L}(X)$ contains distinct beliefs that represent the same preference relation on acts.

Definition 10. Let $\rho, \sigma \in \mathcal{L}(X)$. We say that ρ, σ are **preference-equivalent** if $\succsim_\rho = \succsim_\sigma$ and write $\rho \cong \sigma$.

In fact, each LPS $\sigma \in \mathcal{L}(X)$ is preference-equivalent to an uncountable number of LPS's in $\mathcal{L}(X)$. This poses nontrivial conceptual challenges to meaningfully defining what is a conditional belief (i.e., CPS) about beliefs (i.e., LPS's).[6] The most obvious of these challenges is that the mutual singularity of probability measures—and therefore the distinction between CPS's and LPS's—loses its incisiveness in the presence of such redundancies.

For example, consider the space $\mathcal{C}(\mathcal{L}(X))$. If the subjectivist interpretation is taken seriously, it might be argued that the substantive content of an LPS $\sigma \in \mathcal{L}(X)$ is entirely captured by the associated preference relation $\succsim_\sigma$. As such, one might then argue that beliefs about beliefs have content only to the extent that they describe beliefs about preferences. The contents of $\mathcal{L}(\mathcal{L}(X))$ and $\mathcal{C}(\mathcal{L}(X))$ are essentially the same in that regard. As such, removing redundant beliefs from $\mathcal{L}(X)$ is a practical necessity if we want to make meaningful statements involving conditional beliefs about LEU preferences.

2.3 MARGINAL PREFERENCES

The concept of marginal beliefs is essential to the construction of beliefs about beliefs. Again, it may be argued that, under the personalistic interpretation, the substantive content of marginal beliefs should be marginal preferences. For Definitions 11–14,

[5] A probability measure can be viewed as an element of some linear space. A pair of mutually singular probability measures can be viewed as a pair of orthogonal vectors in this space.

[6] Lee (2013) contains a more detailed discussion of these issues.

Definition 11. Let X, Y be nonempty standard Borel spaces and $\succsim$ a preference relation on $\mathsf{Act}(X \times Y)$. Then $\succsim$ induces a **marginal preference relation** on $\mathsf{Act}(X)$, which is denoted by $\operatorname{marg}_X \succsim$. Formally, the statement $\succsim_X = \operatorname{marg}_X \succsim$ is equivalent to the following.[7]

$$\forall f, g \in \mathsf{Act}(X) \quad f \succsim_X g \iff f \succsim g.$$

Definition 12. Let X, Y be nonempty standard Borel spaces. The **marginal belief operator** (marg) on LPS's is defined as follows.

$$\forall \sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(X \times Y) \quad \operatorname{marg}_X \sigma \equiv (\operatorname{marg}_X \mu_1, \ldots, \operatorname{marg}_X \mu_n)$$

It is easy to see that $\operatorname{marg}_X \sigma$ represents the marginal preference relation $\operatorname{marg}_X \succsim_\sigma$ on $\mathsf{Act}(X)$. However, the marginal belief operator on LPS's displays an irregularity that the marginal belief operator on probability measures does not. There exist LPS's $\rho, \sigma \in \mathcal{L}(X \times Y)$ such that their marginal beliefs on X are preference-equivalent but not equal. Formally, this is stated as follows.

$$\begin{aligned} &\forall \mu, \nu \in \mathcal{P}(X \times Y) && \operatorname{marg}_X \mu \cong \operatorname{marg}_X \nu \iff \operatorname{marg}_X \mu = \operatorname{marg}_X \nu \\ &\exists \rho, \sigma \in \mathcal{L}(X \times Y) && \operatorname{marg}_X \rho \cong \operatorname{marg}_X \sigma \wedge \operatorname{marg}_X \rho \neq \operatorname{marg}_X \sigma \end{aligned}$$

Unfortunately, this behavior of the marginal operator on LPS's is not merely a technical artifact; it has conceptual implications. The marginal LPS $\operatorname{marg}_X \sigma$—of some $\sigma \in \mathcal{L}(X \times Y)$—describes more aspects of $\succsim_\sigma$ than $\operatorname{marg}_X \succsim_\sigma$ does. As was the case in the previous section, the redundancy of $\mathcal{L}(X)$ of plays a role in bringing about this irregularity. However, these issues can be sidestepped by appealing to the following result.

Proposition 1 (Lee (2013))**.** *Let X be a nonempty standard Borel space. There exists a Borel subset $\mathcal{U}(X) \subseteq \mathcal{L}(X)$ and a surjective Borel map $\mathfrak{s}_X \colon \mathcal{L}(X) \to \mathcal{U}(X)$ such that*

1. *$\mathfrak{s}_X(\sigma) \cong \mathfrak{s}_X(\sigma') \iff \sigma \cong \sigma'$ for all $\sigma, \sigma' \in \mathcal{L}(X)$; and*
2. *$\sigma \cong \mathfrak{s}_X(\sigma)$ for all $\sigma \in \mathcal{L}(X)$.*

Furthermore, $\mathcal{U}(X)$ only contains minimal-length representations of LEU preferences.

Definition 13. For each standard Borel space X, fix a set $\mathcal{U}(X)$ and a map $\mathfrak{s}_X$ that exist by Proposition 1. For all $n \in \mathbb{N}$, $\mathcal{U}_n(X) \equiv \mathcal{U}(X) \cap \mathcal{L}_n(X)$.

The sets $\mathcal{U}(X)$ and $\mathcal{L}(X)$ describe the same set of preferences, but $\mathcal{U}(X)$ does not suffer from the presence of redundant representations. It follows that $\mathcal{L}(\mathcal{U}(X))$ and $\mathcal{C}(\mathcal{U}(X))$ differ in meaningful ways. Furthermore, the map $\mathfrak{s}_X$ permits the definition of a well-behaved marginal operator that does not exhibit the conceptual problems discussed in this section.

[7] Note that any $f \in \mathsf{Act}(X)$ can be viewed as the map $(x, y) \mapsto f(x)$ that belongs to $\mathsf{Act}(X \times Y)$.

Definition 14. Let X, Y be nonempty standard Borel spaces. The **marginal preference operator** (margp) on LPS's is defined as follows.

$$\forall \sigma \in \mathcal{L}(X \times Y) \quad \operatorname{margp}_X \sigma \equiv \mathfrak{s}_X(\operatorname{marg}_X \sigma)$$

The mapping $\sigma \mapsto \operatorname{marg}_X \sigma$ is Borel since it is a composition of Borel maps. Furthermore, the following statement is readily verified.

$$\forall \rho, \sigma \in \mathcal{L}(X \times Y) \qquad \operatorname{margp}_X \rho \cong \operatorname{margp}_X \sigma \iff \operatorname{margp}_X \rho = \operatorname{margp}_X \sigma$$

3 INTERACTIVE UNCERTAINTY

3.1 BASIC ENVIRONMENT

Consider a game setting where $I = \{a, b\}$ denotes the set of all human players.[8] Players a and b are respectively called Ann and Bob.[9] Our definitions and results easily extend to environments with any finite number of players.

Each player has a basic (i.e., first-order) state, which is known to herself but not to her opponents. For all $i \in I$, the set of Player i's first-order states is denoted by X_i^1, which is assumed to be a standard Borel space. Player i is uncertain about the first-order state of her opponents. Her attitude toward this uncertainty is fully captured by her preferences over the set $\mathsf{Act}(X_{-i}^1)$. Naturally, these are called her first-order preferences.

3.2 HIGHER-ORDER PREFERENCES

For all $n \in \mathbb{N}$, the set of Player i's n^{th}-order states is denoted by X_i^n and the set of her n^{th}-order preferences is denoted by H_i^n. The sets X_i^n and H_i^n are defined in parallel by induction as follows:

$$\begin{aligned}
&\forall n \in \mathbb{N} && H_i^n \equiv \mathcal{U}(X_{-i}^n) \\
&\text{base case:} && X_i^2 \equiv X_i^1 \times H_i^1 \\
&\forall n \geq 2 && X_i^{n+1} \equiv \{(x_i^n, h_i^n) \in X_i^n \times H_i^n \colon \underbrace{\operatorname{proj}_{H_i^{n-1}} x_i^n = \operatorname{margp}_{X_{-i}^{n-1}} h_i^n}_{\text{coherency condition}}\}
\end{aligned}$$

It is apparent from the definition of X_i^{n+1} that each $(x_i^n, h_i^n) \in X_i^{n+1}$ describes Player i's $(n-1)^{\text{th}}$-order preferences in two places:

$$\operatorname{proj}_{H_i^{n-1}} x_i^n \qquad\qquad \operatorname{margp}_{X_{-i}^{n-1}} h_i^n$$

The coherency condition requires that these descriptions are mutually consistent. In other words, the diagram in Figure 1 commutes for all $n \geq 2$.

Lemma 1. *For all $n \in \mathbb{N}$, X_i^n and H_i^n are nonempty standard Borel spaces.*

[8] We adopt the usual convention that $-a \equiv b$ and $-b \equiv a$.

[9] We adopt the convention of using female pronouns when we refer to generic players whose gender is unknown (e.g., Player $i \in I$ knows *her* own preferences).

$$
\begin{tikzcd}
X_i^{n+1} \arrow[r, "\text{proj}"] \arrow[d, "\text{proj}"] & H_i^n \arrow[d, "\text{margp}_{X_{-i}^{n-1}}"] \\
X_i^n \arrow[r, "\text{proj}"] & H_i^{n-1}
\end{tikzcd}
$$

Figure 1: Coherency condition

3.3 COHERENT HIERARCHIES

Finally, we can define hierarchies of preferences, which are the objects of principal interest to us.

Definition 15. A **coherent preferences hierarchy** of Player i is a sequence $h_i = (h_i^1, h_i^2, \dots) \in \prod_{n\in\mathbb{N}} H_i^n = \prod_{n\in\mathbb{N}} \mathcal{U}(X_{-i}^n)$ such that the beliefs that form the sequence are mutually consistent. Formally, the set H_i of all coherent hierarchies is defined as follows:

$$H_i \equiv \left\{ (h_i^1, h_i^2, \dots) \in \prod_{n\in\mathbb{N}} H_i^n : \quad \forall n \in \mathbb{N} \quad \text{margp}_{X_{-i}^n} h_i^{n+1} = h_i^n \right\}$$

Definition 16. A **player state** of Player i is a sequence $(x_i^1, x_i^2, \dots) \in \prod_{n\in\mathbb{N}} X_i^n$ such that the higher-order states that form the sequence are mutually consistent. Formally, the set X_i of all player states is defined as follows.

$$X_i \equiv \varprojlim_n X_i^n = \left\{ (x_i^1, x_i^2, \dots) \in \prod_{n\in\mathbb{N}} X_i^n : \quad \forall n \in \mathbb{N} \quad \text{proj}_{X_i^n} x_i^{n+1} = x_i^n \right\}$$

There is an obvious and natural isomorphism between X_i and $X_i^1 \times H_i$.

$$(x_i^n)_{n\in\mathbb{N}} \mapsto (x_i^1, (\text{proj}_{H_i^n} x_i^{n+1})_{n\in\mathbb{N}})$$

It follows that we may use the following alternate definition of player states.

Definition 17. Elements of $X_i^1 \times H_i$ are called player states of i.

Lemma 2. *X_i and H_i are nonempty standard Borel spaces.*

3.4 FROM BELIEFS ABOUT STATES TO HIERARCHIES

Definition 18. The canonical map $\mathfrak{h}_i : \mathcal{L}(X_{-i}) \to H_i$ that extracts from each element of $\mathcal{L}(X_{-i})$ the preferences hierarchy it implies is defined as follows:

$$\forall n \in \mathbb{N} \quad \mathfrak{h}_i(b_i; n) \equiv \text{margp}_{X_{-i}^n} b_i \qquad\qquad \mathfrak{h}_i(b_i) \equiv (\mathfrak{h}_i(b_i; n))_{n\in\mathbb{N}}$$

Lemma 3. *The map $\mathfrak{h}_i$ is Borel. Furthermore, $\rho \cong \sigma$ if and only if $\mathfrak{h}_i(\rho) = \mathfrak{h}_i(\sigma)$ for all $\rho, \sigma \in \mathcal{L}(X_{-i})$.*

An immediate consequence of Lemma 3 is that $\mathfrak{h}_i|_{\mathcal{U}(X_{-i})}$ and $\mathfrak{h}_i|_{\mathcal{C}(X_{-i})}$ are one-to-one.

Corollary 1. *A hierarchy can be expressed as an LPS in $\mathcal{L}(X_{-i})$ if and only if it can be expressed as an LPS in $\mathcal{U}(X_{-i})$. In other words,*

$$\mathfrak{h}_i(\mathcal{L}(X_{-i})) = \mathfrak{h}_i(\mathcal{U}(X_{-i})).$$

Furthermore, the set of all such hierarchies—i.e., $\mathfrak{h}_i(\mathcal{L}(X_{-i}))$—is Borel in H_i.

Proof of Corollary 1. First, note that $\mathfrak{h}_i(\mathcal{L}(X_{-i})) \supseteq \mathfrak{h}_i(\mathcal{U}(X_{-i}))$ because $\mathcal{L}(X_{-i}) \supseteq \mathcal{U}(X_{-i})$. For all $\rho \in \mathcal{L}(X_{-i})$, there exists a $\sigma \in \mathcal{U}(X_{-i})$ such that $\rho \cong \sigma$. By Lemma 3, $\rho \cong \sigma$ implies that $\mathfrak{h}_i(\rho) = \mathfrak{h}_i(\sigma)$. It follows that $\mathfrak{h}_i(\mathcal{L}(X_{-i})) \subseteq \mathfrak{h}_i(\mathcal{U}(X_{-i}))$.

Since $\mathcal{U}(X_{-i})$ is free of redundant beliefs, the restriction of the Borel map $\mathfrak{h}_i$ to $\mathcal{U}(X_{-i})$ is one-to-one. This implies that the inverse mapping $(\mathfrak{h}_i|_{\mathcal{U}(X_{-i})})^{-1}$ is Borel as well. Therefore, $\mathfrak{h}_i(\mathcal{L}(X_{-i})) = \mathfrak{h}_i(\mathcal{U}(X_{-i}))$ is Borel. $\square$

Theorem 1. *Not every coherent hierarchy can be expressed as an LPS on player states. In other words, $\mathfrak{h}_i(\mathcal{L}(X_{-i})) \neq H_i$.*

Proof of Theorem 1. If X_i^1 is not a singleton for at least one $i \in I$, then the space X_j is uncountable for all $j \in I$. Without loss of generality, let X_i be uncountable for all $i \in I$.

Fix a $x_{-i} \in X_{-i}$ and let $a(1) \equiv x_{-i}$. Now, for each $n \in \mathbb{N}$, we can choose an $a(n+1) \in X_{-i}$ such that

$$\mathrm{proj}_{X_{-i}^n}\, a(n+1) = \mathrm{proj}_{X_{-i}^n}\, a(n) \ \wedge\ \mathrm{proj}_{X_{-i}^{n+1}}\, a(n+1) \neq \mathrm{proj}_{X_{-i}^{n+1}}\, a(n).$$

Due to the coherency condition imposed on X_{-i}, the following must then hold for all $m, n \in \mathbb{N}$.

$$\begin{aligned} m \leq m &\implies \mathrm{proj}_{X_{-i}^m}\, a(n) = \mathrm{proj}_{X_{-i}^m}\, a(m) \\ m > n &\implies \mathrm{proj}_{X_{-i}^m}\, a(n) \neq \mathrm{proj}_{X_{-i}^m}\, a(m) \end{aligned}$$

We can define a sequence $(\mu_n)_n$ of probability measures in $\mathcal{P}(X_{-i})$ such that $\mu_n(a(n)) = 1$ for all $n \in \mathbb{N}$. Define the hierarchy $(h_i^n)_n \in H_i$ as follows using these measures.

$$\forall n \in \mathbb{N} \quad h_i^n \equiv (\mathfrak{h}_i(\mu_n; n), \mathfrak{h}_i(\mu_{n-1}; n), \ldots, \mathfrak{h}_i(\mu_1; n))$$

Note that $\mathfrak{h}_i(\mu_m; n)(\mathrm{proj}_{X_{-i}^n}\, a(k)) = 1$ for all $m \leq n$. It follows that h_i^n is a CPS and has minimal length. Furthermore, h_i is a coherent hierarchy because the following holds for all $n \in \mathbb{N}$.

$$\begin{aligned}
&\operatorname{marg}_{X^n_{-i}} h_i^{n+1} \\
&= \overbrace{(\operatorname{marg}_{X^n_{-i}} \mathfrak{h}_i(\mu_{n+1}; n+1), \operatorname{marg}_{X^n_{-i}} \mathfrak{h}_i(\mu_n; n+1), \ldots, \operatorname{marg}_{X^n_{-i}} \mathfrak{h}_i(\mu_1; n+1))}^{n+1 \text{ measures}} \\
&= \overbrace{(\mathfrak{h}_i(\mu_{n+1}; n), \mathfrak{h}_i(\mu_n; n), \ldots, \mathfrak{h}_i(\mu_1; n))}^{n+1 \text{ measures}} \\
&\cong \overbrace{(\mathfrak{h}_i(\mu_n; n), \ldots, \mathfrak{h}_i(\mu_1; n))}^{n \text{ measures}} = h_i^n \qquad \because \mathfrak{h}_i(\mu_{n+1}; n) = \mathfrak{h}_i(\mu_n; n)
\end{aligned}$$

Suppose by way of contradiction that there exists an LPS $\sigma \in \mathcal{L}(X_{-i})$ that represents h_i. Its length must be some $N \in \mathbb{N}$. The $(N+1)^{\text{th}}$-order preferences implied by σ can then be represented by $\operatorname{marg}_{X_i^{N+1}} \sigma$, which must have length N. However, h_i^{N+1}, which has length $N+1$, cannot be represented by a shorter LPS. Since $h_i^{N+1} \cong \operatorname{marg}_{X_i^{N+1}} \sigma$, this yields a contradiction. □

4 TYPE STRUCTURES

4.1 FROM TYPES TO HIERARCHIES

Definition 19. An **$\mathcal{L}$-type structure** is a tuple $\langle T_i, \beta_i \rangle_{i \in I}$ such that the following holds for all $i \in I$:

1. T_i is a nonempty standard Borel space; and
2. $\beta_i \colon T_i \to \mathcal{L}(X^1_{-i} \times T_{-i})$ is a Borel map.

Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$. $\mathbf{T}$ is also called a **$\mathcal{P}$-type structure** if $\beta_i(T_i) \subseteq \mathcal{P}(X^1_{-i} \times T_{-i})$ for all i; and a **$\mathcal{C}$-type structure** if $\beta_i(T_i) \subseteq \mathcal{C}(X^1_{-i} \times T_{-i})$ for all i. The set T_i is called Player i's **type space** and elements of T_i are called **types**. The map β_i is called her **type-belief map**.

The familiar type structures of Mertens and Zamir (1985); Brandenburger and Dekel (1993); Tan and Werlang (1988) are $\mathcal{P}$-type structures. The lexicographic type structures in Brandenburger *et al.* (2008) are $\mathcal{C}$-type structures.

It immediately apparent that the higher-order preferences, and therefore the coherent hierarchy, implied by a type can be recovered by repeatedly taking compositions of the type-belief maps. Before we do so, it is useful to first extend the notion of pushforward measures to LPS's.

Definition 20. Let X, Y be standard Borel spaces and $f : X \to Y$ a Borel map. Given a $\mu \in \mathcal{P}(Y)$, the pushforward belief $f\mu \in \mathcal{P}(X)$ is the probability measure defined as follows

$$\forall E \in \mathbf{B}(X) \quad f\mu(E) \equiv \mu(f^{-1}(E))$$

Definition 21. Let X, Y be standard Borel spaces and $f : X \to Y$ a Borel map. Given a $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(Y)$, the pushforward belief $f\sigma \in \mathcal{L}(X)$ is defined as follows.

$$f\sigma \equiv (f\mu_1, \ldots, f\mu_n)$$

Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure. We can inductively define a sequence of Borel maps $(\beta_i^n)_{n \in \mathbb{N}}$ such that $\beta_i^n \colon T_i \to H_i^n$ recovers the n^{th}-order preferences implied by each type.

$$\begin{aligned} & t_i \mapsto \beta_i^1(t_i) \equiv \operatorname{margp}_{X_{-i}^1} \beta_i(t_i) \\ \forall n \in \mathbb{N} \quad & t_i \mapsto \beta_i^{n+1}(t_i) \equiv \operatorname{margp}_{X_{-i}^{n+1}} \tilde{\beta}_{-i}^n \beta_i(t_i) \end{aligned}$$

The one-to-one Borel map $\tilde{\beta}_i^n \colon X_i^1 \times T_i \to X_i^{n+1} \times T_i$ is defined as follows.

$$(x_i^1, t_i) \mapsto \tilde{\beta}_i^n(x_i^1, t_i) \equiv (x_i^1, \beta_i^1(t_i), \ldots, \beta_i^n(t_i), t_i) \in \overbrace{X_i^1 \times H_i^1 \times \cdots \times H_i^n \times T_i}^{\supseteq X_i^n \times H_i^n \times T_i \supseteq X_i^{n+1} \times T_i}$$

The LPS $\tilde{\beta}_{-i}^n \beta_i(t_i)$—i.e., the pushforward of $\beta_i(t_i) \in \mathcal{L}(X_{-i}^1 \times T_{-i})$ under $\tilde{\beta}_{-i}^n$—is a belief about $X_{-i}^{n+1} \times T_{-i}$.

Definition 22. Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure and let the family of maps $\{\beta_i^n \colon (i, n) \in I \times \mathbb{N}\}$ be defined as above. For each $i \in I$, Player i's **type-hierarchy map** is the function $\beta_i^\infty \colon T_i \to H_i$, which is defined as follows.

$$t_i \xmapsto{\beta_i^\infty} (\beta_i^n(t_i))_{n \in \mathbb{N}}$$

Given a type-belief map β_i, the associated type-hierarchy map is indicated by adding the superscript ∞.

Lemma 4. *Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure. The type-hierarchy map $\beta_i^\infty \colon T_i \to H_i$ is Borel for each $i \in I$.*

Harsanyi (1967)'s insight that hierarchies, which are cumbersome objects, can be represented by types, which are comparatively simple objects, inspired numerous papers establishing the foundations of various type structures. A recurring theme in these investigations is whether there is a type structure that can describe "all higher-order beliefs"—a notion that varies according to the context in which the question is asked. Following several recent papers, we use *terminality* in this paper as an umbrella term to describe such properties of type structures.[10] The following common variants of the terminality question are considered.

[10]The word *universality* has also been used frequently.

Definition 23. Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure and $\langle \beta_i^\infty \rangle_{i \in I}$ its type-hierarchy maps.

1. **strongly terminal** if it describes every coherent preferences hierarchy, i.e.,

$$\forall i \in I \quad \beta_i^\infty(T_i) = H_i$$

2. **weakly terminal** in a given family $\mathcal{F}$ of $\mathcal{L}$-type structures if it describes every coherent preferences hierarchy that can be described by type structures in $\mathcal{F}$, i.e.,

$$\forall \langle Y_i, \gamma_i \rangle \in \mathcal{F} \quad \forall i \in I \quad \gamma_i^\infty(T_i) \subseteq \beta_i^\infty(T_i)$$

3. **finitely terminal** if it describes all finite-order preferences, i.e.,

$$\forall i \in I \quad \forall n \in \mathbb{N} \quad \operatorname{proj}_{H_i^n} \beta_i^\infty(T_i) = H_i^n$$

Corollary 2. *No strongly terminal $\mathcal{L}$-type structure exists.*

Proof of Corollary 2. Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure. For each $t_i \in T_i$, the belief in $\mathcal{L}(X_{-i}^1 \times H_{-i}) = \mathcal{L}(X_{-i})$ to which $\beta_i(t_i) \in \mathcal{L}(X_{-i}^1 \times T_{-i})$ corresponds can be obtained via the type-hierarchy maps. Therefore, hierarchies that can be described by $\mathbf{T}$ can also be described as beliefs about the opponents' player states—i.e., beliefs about X_{-i}. Since Theorem 1 states that some hierarchies cannot be described thusly, it follows that those hierarchies cannot be described by $\mathbf{T}$. $\square$

4.2 CANONICAL TYPE STRUCTURES

$$\begin{aligned}
U_i^1 &\equiv \mathfrak{h}_i(\mathcal{U}(X_{-i})) \subseteq H_i \\
\forall n \geq 2 \quad U_i^{n+1} &\equiv \mathfrak{h}_i(\{\sigma \in \mathcal{U}(X_{-i}) \colon \sigma(X_{-i}^1 \times U_{-i}^n) = \vec{1}\}) \subseteq U_i^n \\
T_i^{\mathcal{U}} &\equiv \bigcap_{n \in \mathbb{N}} U_i^n
\end{aligned}$$

Definition 24. The canonical LPS type structure is the tuple $\mathbf{T}^{\mathcal{U}} \equiv \langle T_i^{\mathcal{U}}, \beta_i^{\mathcal{U}} \rangle_{i \in I}$ such that $\beta_i^{\mathcal{U}} \colon T_i^{\mathcal{U}} \to \mathcal{U}(X_{-i}^1 \times T_{-i}^{\mathcal{U}})$ is the mapping $t_i \mapsto \mathfrak{h}_i^{-1}(t_i)$.

$$\begin{aligned}
C_i^1 &\equiv \mathfrak{h}_i(\mathcal{C}(X_{-i}^1 \times H_{-i})) \subseteq H_i \\
\forall n \geq 2 \quad C_i^{n+1} &\equiv \mathfrak{h}_i(\{\sigma \in \mathcal{C}(X_{-i}) \colon \sigma(X_{-i}^1 \times C_{-i}^n) = \vec{1}\}) \subseteq C_i^n \\
T_i^{\mathcal{C}} &\equiv \bigcap_{n \in \mathbb{N}} C_i^n
\end{aligned}$$

Definition 25. The canonical CPS type structure is the tuple $\mathbf{T}^{\mathcal{C}} \equiv \langle T_i^{\mathcal{C}}, \beta_i^{\mathcal{C}} \rangle_{i \in I}$ such that $\beta_i^{\mathcal{C}} : T_i^{\mathcal{C}} \to \mathcal{C}(X_{-i}^1 \times T_{-i}^{\mathcal{C}})$ is the mapping $t_i \mapsto \mathfrak{h}_i^{-1}(t_i)$.

Lemma 5. *$\mathbf{T}^{\mathcal{U}}$ and $\mathbf{T}^{\mathcal{C}}$ are $\mathcal{L}$-type structures and their type-belief maps are Borel isomorphisms.*

Proof of Lemma 5. By Lemma 3, $\mathfrak{h}_i$ is a one-to-one Borel map when its domain is restricted to a nonredundant set of LPS's. For any nonempty standard Borel space X, $\mathcal{C}(X)$ and $\mathcal{U}(X)$ are nonredundant Borel sets of LPS's. For all $(i, n) \in I \times \mathbb{N}$, U_i^n and C_i^n are Borel because they are images of nonredundant Borel sets of LPS's under one-to-one Borel maps. It follows that $T_i^{\mathcal{U}}$ and $T_i^{\mathcal{C}}$ are Borel sets for all $i \in I$.

For any nonempty standard Borel space X, $\mathcal{P}(X) \subseteq \mathcal{C}(X) \subseteq \mathcal{L}(X)$. It follows that both $\mathbf{T}^{\mathcal{U}}$ and $\mathbf{T}^{\mathcal{C}}$ contain the Brandenburger and Dekel (1993) type structure, which is nonempty.

LEU preferences satisfy the limit closure property—i.e., if a sequence of Borel sets are 1-believed in the sense that their complements are Savage-null, then the intersection of those sets is also 1-believed in the same sense.[11] Therefore, the equalities below hold.

$$T_i^{\mathcal{U}} = \mathfrak{h}_i(\mathcal{U}(X_{-i}^1 \times T_{-i}^{\mathcal{U}})) \qquad\qquad T_i^{\mathcal{C}} = \mathfrak{h}_i(\mathcal{C}(X_{-i}^1 \times T_{-i}^{\mathcal{C}}))$$

It follows that the restriction of $\mathfrak{h}_i$ to $\mathcal{U}(X_{-i}^1 \times T_{-i}^{\mathcal{U}})$ is a Borel isomorphism from $\mathcal{U}(X_{-i}^1 \times T_{-i}^{\mathcal{U}})$ and $T_i^{\mathcal{U}}$; and the restriction of $\mathfrak{h}_i$ to $\mathcal{C}(X_{-i}^1 \times T_{-i}^{\mathcal{C}})$ is a Borel isomorphism from $\mathcal{C}(X_{-i}^1 \times T_{-i}^{\mathcal{C}})$ and $T_i^{\mathcal{C}}$. The type-belief maps in $\mathbf{T}^{\mathcal{U}}$ and $\mathbf{T}^{\mathcal{C}}$, being the functional inverses of those restrictions, are also Borel isomorphisms. □

Theorem 2. *$\mathbf{T}^{\mathcal{U}}$ is weakly terminal in the class of $\mathcal{L}$-type structures.*

Proof of Theorem 2. Let $\mathbf{T} = \langle T_i, \beta_i \rangle_i$ be an $\mathcal{L}$-type structure. We want to show that

$$\forall t_i \in T_i \quad \beta_i^\infty(t_i) \in T_i^{\mathcal{U}}.$$

Define $\tilde{\beta}_i^\infty$ as the following map.

$$(x_i^1, t_i) \mapsto (x_i^1, \beta_i^\infty(t_i), t_i) \qquad\qquad X_i^1 \times T_i \to X_i \times T_i$$

For each $t_i \in T_i$, the hierarchy $\beta_i^\infty(t_i)$ can also be represented by the belief

$$\operatorname{margp}_{X_{-i}} \tilde{\beta}_{-i}^\infty \beta_i(t_i) \in \mathcal{U}(X_{-i}) = \mathcal{U}(X_{-i}^1 \times H_{-i}).$$

It follows that $\beta_i^\infty(t_i) \in U_i^1 = \mathfrak{h}_i(\mathcal{U}(X_{-i}^1 \times H_{-i}))$. Furthermore, $\beta_i^\infty(t_i)$ belongs to U_i^2—i.e., the set of hierarchies that can be represented as beliefs that 1-believe

[11] In the literature, 1-belief, or simply belief, of an event E corresponds to belief with probability 1. Extending this notion to LPS's, an event E is 1-believed under LPS $(\mu_1, \ldots, \mu_n)$ if $\mu_1(E) = \cdots = \mu_n(E) = 1$.

$X^1_{-i} \times U^1_{-i}$—because $\beta^\infty_{-i}(t_{-i}) \in U^1_{-i}$ for all $t_{-i} \in T_{-i}$. By applying this line of argument inductively, it is shown that

$$\beta^\infty_i(t_i) \in \bigcap_{n \in \mathbb{N}} U^n_i = T^{\mathcal{U}}_i.$$

□

5 HIERARCHIES THAT ARE CONDITIONAL BELIEFS

5.1 REDUNDANT TYPES

Consider a $\mathcal{L}$-type structure $\langle T_i, \beta_i \rangle_{i \in I}$. Each $t_i \in T_i$ represents a belief $\beta_i(t_i)$ about $X^1_{-i} \times T_{-i}$. As briefly discussed in Section 2.2, the statement that $\beta_i(t_i)$ is a CPS loses its meaning when the space of uncertainty—i.e., $X^1_{-i} \times T_{-i}$—contains elements that are redundant with respect to the uncertainty that we wish to model. In the case of type structures, what we wish to model is uncertainty about hierarchies. As such, the set T_{-i} is descriptively useful only to the extent that it describes hierarchies in H_{-i}. We can define when a type structure is redundant in that regard.

Definition 26. Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure and $\langle \beta^\infty_i \rangle_{i \in I}$ its type-hierarchy maps. A type $t_i \in T_i$ is **redundant** if

$$\exists \hat{t}_i \in T_i \quad t_i \neq \hat{t}_i \wedge \beta^\infty_i(t_i) = \beta^\infty_i(\hat{t}_i).$$

We say that $\mathbf{T}$ is **redundant** if there is a Player i such that redundant types exist in T_i.

Any preferences hierarchy that can be described by a $\mathcal{L}$-type structure can also be described by a (possibly redundant) $\mathcal{C}$-type structure. This is an implication of the following result.

Theorem 3. *There exists a $\mathcal{C}$-type structure that is weakly terminal in the class of $\mathcal{L}$-type structures.*

Proof of Theorem 3. We wish to construct an $\mathcal{L}$-type structure $\mathbf{T} = \langle T_i, \beta_i \rangle_i$. Let $T_i = \mathbb{N} \times T^{\mathcal{U}}_i$. We will define the map $\beta_i \colon T_i \to \mathcal{C}(X^1_{-i} \times T_{-i})$ in a piecewise fashion.

First, for each $m \in \mathbb{N}$, the space $\{m\} \times T_i$ admits the following countable partition into Borel sets.

$$\Pi(m) = \{P_{mn} = \{(m, t_i) \in T_i \colon \beta^{\mathcal{U}}_i(t_i) \in \mathcal{L}_n(X^1_{-i} \times T_{-i})\} \colon n \in \mathbb{N}\}$$

For any $m, n \in \mathbb{N}$, the function β_i can be defined on the subdomain $P_{mn} \in \Pi(m)$ as follows:

$$\beta_i(m, t_i) \equiv (f_1\mu_1, \ldots, f_n\mu_n) \in \mathcal{C}(X^1_{-i} \times T_{-i}), \quad \text{where } \beta^{\mathcal{U}}_i(t_i) = (\mu_1, \ldots, \mu_n)$$

and $f_k\colon X^1_{-i} \times T^{\mathcal{U}}_{-i} \to X^1_{-i} \times \{k\} \times T^{\mathcal{U}}_{-i}$ is the Borel map $(x^1_{-i} t_{-i}) \mapsto (x^1_{-i}, k, t_{-i})$ for all $k \in \mathbb{N}$. The map β_i is clearly Borel on P_{mn} for all $m, n \in \mathbb{N}$. It follows that β_i is Borel on $\bigcup \Pi(m) = \bigcup_{n\in\mathbb{N}} P_{mn}$ because $\Pi(m)$ is countable. Therefore, β_i is Borel on each member of the countable partition

$$\left\{\bigcup \Pi(m)\colon m \in \mathbb{N}\right\}$$

of T_i. It follows that β_i is a Borel map.

Finally, $\mathbf{T}$ generates the same hierarchies as $\mathbf{T}^{\mathcal{U}}$ because the following equality holds for all $(m, t_i) \in T_i$ by construction.

$$\operatorname{marg}_{X^1_{-i} \times T^{\mathcal{U}}_{-i}} \beta_i(m, t_i) = \beta^{\mathcal{U}}_i(t_i)$$

Since $\mathbf{T}^{\mathcal{U}}$ is weakly terminal, so is $\mathbf{T}$. $\square$

In contrast, not every preferences hierarchy that can be described by a $\mathcal{L}$-type structure can also be described by a *nonredundant* $\mathcal{C}$-type structure. Furthermore, the canonical $\mathcal{C}$-type structure describes precisely the set of hierarchies that are described by nonredundant $\mathcal{C}$-type structures.

Theorem 4. *The canonical $\mathcal{C}$-type structure $\mathbf{T}^{\mathcal{C}}$ is*

1. *weakly terminal in the class of nonredundant $\mathcal{C}$-type structures; but*
2. *not weakly terminal in the class of $\mathcal{C}$-type structures.*[12]

Proof of Theorem 4. Let $\mathbf{T} = \langle T_i, \beta_i \rangle_i$ be a nonredundant $\mathcal{C}$-type structure. We want to show that

$$\forall t_i \in T_i \quad \beta^\infty_i(t_i) \in T^{\mathcal{C}}_i.$$

Define f_i as the following one-to-one Borel map. The map is one-to-one because $\mathbf{T}$ is nonredundant.

$$(x^1_i, t_i) \mapsto (x^1_i, \beta^\infty_i(t_i)) \qquad\qquad X^1_i \times T_i \to X^1_i \times H_i$$

For each $t_i \in T_i$, the hierarchy $\beta^\infty_i(t_i)$ can also be represented by the belief

$$f_{-i}\beta_i(t_i) \in \mathcal{C}(X_{-i}) = \mathcal{C}(X^1_{-i} \times H_{-i}).$$

Again, $f_{-i}\beta_i(t_i)$ is a CPS because $\beta_i(t_i)$ is a CPS and f_{-i} is one-to-one. It follows that $\beta^\infty_i(t_i) \in C^1_i = \mathfrak{h}_i(\mathcal{C}(X^1_{-i} \times H_{-i}))$. Furthermore, $\beta^\infty_i(t_i)$ belongs to C^2_i—i.e., the set of hierarchies that can be represented as beliefs that 1-believe $X^1_{-i} \times C^1_{-i}$—because $\beta^\infty_{-i}(t_{-i}) \in C^1_{-i}$ for all $t_{-i} \in T_{-i}$. By applying this line of argument inductively, it is shown that

$$\beta^\infty_i(t_i) \in \bigcap_{n\in\mathbb{N}} C^n_i = T^{\mathcal{C}}_i.$$

$\square$

[12]It is also therefore not weakly terminal in the class of $\mathcal{L}$-type structures.

In light of the preceding results, it makes sense to restrict our attention to nonredundant $\mathcal{C}$-type structures when we are interested in CPS beliefs about hierarchies.

5.2 "PROPER" CPS TYPES ARE ALMOST LPS TYPES

Although nonredundant $\mathcal{C}$-type structures describe a strict subset of the hierarchies that are described by $\mathcal{L}$-type structures, the two nevertheless have equal descriptive power in the following important way.

Theorem 5. *The canonical $\mathcal{C}$-type structure $\mathbf{T}^{\mathcal{C}}$ is finitely terminal in the class of $\mathcal{L}$-type structures.*

Proof of Theorem 5. $\mathbf{T}^{\mathcal{C}}$ is a belief-complete $\mathcal{C}$-type structure in the sense that $\beta_i^{\mathcal{C}}(T_i^{\mathcal{C}}) = \mathcal{C}(X_{-i}^1 \times T_{-i}^{\mathcal{C}})$. For all nonempty standard Borel spaces X, Y, U such that $U \subseteq X \times Y$ and $\operatorname{proj}_X U = X$, the set

$$\{\operatorname{marg}_X \sigma \colon \sigma \in \mathcal{L}(U)\}$$

is equal to $\mathcal{L}(X)$. The following statements can then be sequentially derived.

$$\begin{aligned}
&\operatorname{marg}_{X_{-i}^1} \beta_i^{\mathcal{C}}(T_i^{\mathcal{C}}) = \mathcal{L}(X_{-i}^1) \\
&\qquad\qquad\qquad\qquad \therefore \operatorname{margp}_{X_{-i}^1} \beta_i^{\mathcal{C}}(T_i^{\mathcal{C}}) = \mathcal{U}(X_{-i}^1) = H_i^1 \\
&\qquad\qquad\qquad\qquad \therefore \operatorname{proj}_{H_i^1} T_i^{\mathcal{C}} = H_i^1 \quad \text{and} \quad \operatorname{proj}_{X_i^2 = X_i^1 \times H_i^1} X_i^1 \times T_i^{\mathcal{C}} = X_i^2
\end{aligned}$$

Now assume the induction hypothesis—for induction on $n \in \mathbb{N}$—that $\operatorname{proj}_{X_i^m} X_i^1 \times T_i^{\mathcal{C}} = X_i^m$ for all $m \le n$.

$$\begin{aligned}
&\operatorname{marg}_{X_{-i}^n} \beta_i^{\mathcal{C}}(T_i^{\mathcal{C}}) = \mathcal{L}(X_{-i}^n) \\
&\qquad\qquad\qquad\qquad \therefore \operatorname{margp}_{X_{-i}^n} \beta_i^{\mathcal{C}}(T_i^{\mathcal{C}}) = \mathcal{U}(X_{-i}^n) = H_i^n \\
&\qquad\qquad\qquad\qquad \therefore \operatorname{proj}_{H_i^n} T_i^{\mathcal{C}} = H_i^n
\end{aligned}$$

Given that $T_i^{\mathcal{U}}$ is a set of *coherent* hierarchies, $\operatorname{proj}_{X_i^{n+1}} X_i^1 \times T_i^{\mathcal{C}} = X_i^{n+1}$ whenever $\operatorname{proj}_{H_i^n} T_i^{\mathcal{C}} = H_i^n$ $\square$

An important implication of Theorem 5 is that, for the purposes of analyzing epistemic conditions involving finite-order beliefs, there is effectively no difference between $\mathcal{L}$-type structures and nonredundant $\mathcal{C}$-type structures. To put it another way, every coherent preferences hierarchy can be approximated by a sequence of types in nonredundant $\mathcal{C}$-type structures.

6 COMMON ASSUMPTION OF RATIONALITY

6.1 ADMISSIBILITY

Definition 27. Let $G = \langle S_a, S_b, \pi_a, \pi_b \rangle$ be a finite game.

1. Strategy $s_i \in S_i$ is **1-admissible**—or simply **admissible**—if it is not weakly dominated in the game G.
2. Strategy $s_i \in S_i$ is m**-admissible**—for $m > 1$—if it is admissible in the reduced game obtained by removing all strategies of either player that are not $(m-1)$-admissible.
3. Strategy $s_i \in S_i$ is **iteratively admissible (IA)** if it is m-admissible for all $m \in \mathbb{N}$.
4. For all $m \in \mathbb{N}$, S_i^m denotes the set of Player i's m-admissible strategies. The set of Player i's IA strategies is denoted by $S_i^\infty \equiv \bigcup_{m \in \mathbb{N}} S_i^m$.

6.2 EPISTEMIC CONDITION FOR ITERATED ADMISSIBLITY

Consider a finite game $\langle S_a, S_b, \pi_a, \pi_b \rangle$ of complete information. The symbols S_i and π_i respectively denote Player i's strategy set and utility function. The fundamental uncertainty of each player concerns the strategy played by her opponent (i.e., $X_i^1 = S_i$).

Definition 28. Let X be a standard Borel space. An LPS $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(X)$ has **full-support** if

$$\bigcup_{j=1}^{n} \operatorname{supp} \mu_j = X.$$

The set of all full-support LPS's on X is denoted by $\mathcal{L}^+(X)$. The sets $\mathcal{C}^+(X)$ and $\mathcal{U}^+(X)$ are defined analogously:

$$\mathcal{C}^+(X) \equiv \mathcal{C}(X) \cap \mathcal{L}^+(X) \qquad \mathcal{U}^+(X) \equiv \mathcal{U}(X) \cap \mathcal{L}^+(X)$$

Definition 29. The hierarchy $h_i = (h_i^1, h_i^2, \ldots) \in H_i$ has **full-support** if each finite-order belief in the sequence has full-support, i.e.,

$$\forall n \in \mathbb{N} \quad h_i^n \in \mathcal{L}^+(X_{-i}^n).$$

Definition 30. Let X be a standard Borel space, $\sigma \in \mathcal{L}(X)$, and $E \subseteq X$ a Borel set. We say that event E is **assumed under** σ if there exists some $\rho = (\mu_1, \ldots, \mu_n) \in \mathcal{L}_n(E)$ such that

1. $\sigma \cong \rho$; and

2. there exists $m \leq n$ such that $\rho|m = (\mu_1, \ldots, \mu_m) \in \mathcal{L}^+(E)$ and $\mu_M(E) = 0$ for all $M > m$.

Definition 31. The player state $(s_i, h_i) \in S_i \times H_i$ is said to be **rational** if h_i has full-support and s_i maximizes LEU with respect to the first-order belief h_i^1. Let R_i^1 denote the set of all such pairs.

Definition 32. A player state satisfies **rationality and m^{th}-order assumption of rationality (RmAR)** if it belongs to the following set.

$$R_i^{m+1} \equiv R_i^m \cap \{(s_i, (h_i^n)_n) \in S_i \times H_i \colon h_i^{m+1} \text{ assumes } (\operatorname{proj}_{X_{-i}^{m+1}} R_{-i}^m)\}$$

Definition 33. A player state satisfies **rationality and common assumption of rationality (RCAR)** if it belongs to the following set.

$$R_i^\infty \equiv \bigcap_{n \in \mathbb{N}} R_i^m$$

Theorem 6. *For all $m \in \mathbb{N}$, $\operatorname{proj}_{S_i} R_i^m = S_i^m$.*

Proof of Theorem 6. By Lemma 7, $\operatorname{proj}_{S_i} R_i^k \setminus R_i^{k+1} = S_i^k$ for all k.

$$\begin{aligned}
R_i^m &= \bigcup_{k=m}^{\infty} (R_i^k \setminus R_i^{k+1}) \\
\operatorname{proj}_{S_i} R_i^m &= \bigcup_{k=m}^{\infty} \operatorname{proj}_{S_i} (R_i^k \setminus R_i^{k+1}) = \bigcup_{k=m}^{\infty} S_i^k = S_i^m \\
&\because S_i^1 \supseteq S_i^2 \supseteq S_i^3 \supseteq \ldots
\end{aligned}$$

□

Theorem 7. *R_i^∞ is nonempty and $\operatorname{proj}_{S_i} R_i^\infty = S_i^\infty$.*

Proof of Theorem 7. See Appendix. □

APPENDIX A TYPES TO HIERARCHIES

In order to recover hierarchies from types, it is useful to extend the notion of pushforward measures to LPS's.

Definition 34. Let X, Y be standard Borel spaces and $f : X \to Y$ a Borel map. Given a $\mu \in \mathcal{P}(Y)$, the pushforward belief $f\mu \in \mathcal{P}(X)$ is the probability measure defined as follows

$$\forall E \in \mathbf{B}(X) \quad f\mu(E) \equiv \mu(f^{-1}(E))$$

Definition 35. Let X, Y be standard Borel spaces and $f : X \to Y$ a Borel map. Given a $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(Y)$, the pushforward belief $f\sigma \in \mathcal{L}(X)$ is defined as follows.

$$f\sigma \equiv (f\mu_1, \ldots, f\mu_n)$$

Let $\mathbf{T} = \langle T_i, \beta_i \rangle_{i \in I}$ be an $\mathcal{L}$-type structure. We can inductively define a sequence of Borel maps $(\beta_i^n)_{n \in \mathbb{N}}$ such that $\beta_i^n \colon T_i \to H_i^n$ recovers the n^{th}-order preferences implied by each type.

$$\begin{aligned} & t_i \mapsto \beta_i^1(t_i) \equiv \operatorname{margp}_{X_{-i}^1} \beta_i(t_i) \\ \forall n \in \mathbb{N} \quad & t_i \mapsto \beta_i^{n+1}(t_i) \equiv \operatorname{margp}_{X_{-i}^{n+1}} \beta_{-i}^n \beta_i(t_i) \end{aligned}$$

The one-to-one Borel map $\tilde{\beta}_i^n \colon X_i^1 \times T_i \to X_i^{n+1} \times T_i$ is defined as follows.

$$(x_i^1, t_i) \mapsto \tilde{\beta}_i^n(x_i^1, t_i) \equiv (x_i^1, \beta_i^1(t_i), \ldots, \beta_i^n(t_i), t_i) \in \overbrace{X_i^{n+1} \times T_i}^{\subseteq X_i^1 \times (\prod_{k=1}^n H_i^k) \times T_i}$$

The LPS $\beta_{-i}^n \beta_i(t_i)$—i.e., the pushforward of $\beta_i(t_i) \in \mathcal{L}(X_{-i}^1 \times T_{-i})$ under β_{-i}^n—is a belief about $X_{-i}^{n+1} \times T_{-i}$

APPENDIX B HIGHER-ORDER PREFERENCES

Proof of Lemma 1. The proof is by induction.

Base case ($n = 2$) It was assumed in the premise that X_i^1 is a standard Borel space for all $i \in I$. If X is a standard Borel space, then $\mathcal{U}(X)$ is a standard Borel space as well. It follows that $H_i^1 = \mathcal{U}(X_{-i}^1)$ is a standard Borel space for all $i \in I$.

For all $i \in I$, $X_i^2 = X_i^1 \times H_i^1$ is a product of standard Borel spaces and therefore itself a standard Borel space. For all $i \in I$, $H_i^2 = \mathcal{U}(X_{-i}^2)$ is a standard Borel space because X_{-i}^2 is a standard Borel space.

Inductive hypothesis Let $n \geq 2$. For all $m \leq n$, let X_i^m and H_i^m be standard Borel spaces.

Inductive step For all $i \in I$, $H_i^{n+1} = \mathcal{U}(X_{-i}^n)$ is a standard Borel space because X_{-i}^n is a standard Borel space.

X_i^{n+1} is a Borel subset of $X_i^n \times H_i^n$. Let $\kappa \colon X_i^n \times H_i^n \to H_i^{n-1} \times H_i^{n-1}$ be the map $(x_i^n, h_i^n) \mapsto (\operatorname{proj}_{H_i^{n-1}} x_i^n, \operatorname{margp}_{X_{-i}^{n-1}} h_i^n)$. The map is clearly Borel because it is Borel in each coordinate. X_i^{n+1} is the inverse image of the diagonal set in $H_i^{n-1} \times H_i^{n-1}$

under κ. Since the diagonal set is Borel, X_i^{n+1} is Borel.

$$\begin{aligned} D_i^{n-1} &\equiv \{(h_i^{n-1}, h_i^{n-1})\colon h_i^{n-1} \in H_i^{n-1}\} \\ X_i^{n+1} &= \{(x_i^n, h_i^n) \in X_i^n \times H_i^n\colon \operatorname{proj}_{H_i^{n-1}} x_i^n = \operatorname{margp}_{X_{-i}^{n-1}} h_i^n\} \\ &= \kappa^{-1}(D_i^{n-1}) \end{aligned}$$

□

Proof of Lemma 2. By definition, H_i is the inverse limit of the following inverse system of standard Borel spaces.

$$(H_i^n, f_i^n)_n \qquad f_i^n\colon H_i^{n+1} \to H_i^n \qquad = (h_i^{n+1} \mapsto \operatorname{margp}_{X_{-i}^n} h^{n+1})$$

The inverse limit H_i exists. Furthermore, H_i is a standard Borel space when endowed with the subspace Borel σ-algebra of the product $\prod_{n\in\mathbb{N}} H_i^n$. These results hold under the premise that f_i^n is a surjective Borel map for all $n \in \mathbb{N}$ (cf. 17.16 in Kechris, 1995). X_i is also a standard Borel space because it is isomorphic to the standard Borel space $X_i^1 \times H_i$. □

Proof of Lemma 3. It is obvious that the map $\mathfrak{h}_i$ is Borel is because the n^{th} coordinate of the output is defined by the Borel map $\mathfrak{h}_i(\cdot; n)$ for all $n \in \mathbb{N}$.

Let $\rho, \sigma \in \mathcal{L}(X_{-i})$. That $\rho \cong \sigma \implies \mathfrak{h}_i(\rho) = \mathfrak{h}_i(\sigma)$ follows naturally from the definitions of $\mathfrak{h}_i$ and the margp operator—if two preferences are identical, then so should the marginals of those preferences. We want to show the converse, i.e., $\mathfrak{h}_i(\rho) = \mathfrak{h}_i(\sigma) \implies \rho \cong \sigma$.

Without loss of generality, let $\rho = (\mu_1, \ldots, \mu_m)$ and $\sigma = (\nu_1, \ldots, \nu_m)$ be minimal-length LPS's, where $m \in \mathbb{N}$. Therefore, both ρ and σ are made up of a linearly independent components, i.e.,

$$\forall j \leq m \quad \mu_j \notin \operatorname{span}(\{\mu_k\colon j \neq k\}) \quad \wedge \quad \nu_j \notin \operatorname{span}(\{\nu_k\colon j \neq k\}).$$

It follows that there must exist $N \in \mathbb{N}$ such that $\operatorname{marg}_{X_{-i}^n} \rho$ and $\operatorname{marg}_{X_{-i}^n} \sigma$ are minimal-length LPS's in $\mathcal{L}_m(X_{-i}^n)$ for all $n \geq N$.

$$\begin{aligned} &\forall n \geq N \quad \operatorname{marg}_{X_{-i}^n} \mu_j \notin \operatorname{span}(\{\operatorname{marg}_{X_{-i}^n} \mu_k\colon j \neq k\}) \\ &\forall n \geq N \quad \operatorname{marg}_{X_{-i}^n} \nu_j \notin \operatorname{span}(\{\operatorname{marg}_{X_{-i}^n} \nu_k\colon j \neq k\}) \end{aligned}$$

It should also be true that $\operatorname{marg}_{X_{-i}^n} \rho \cong \operatorname{marg}_{X_{-i}^n} \sigma$ for all $n \geq N$ because $\mathfrak{h}_i(\rho; n) = \mathfrak{h}_i(\sigma; n)$ for all $n \in \mathbb{N}$. Preference-equivalence of length-m LPS's is characterized as follows.

$$\begin{aligned} &\operatorname{marg}_{X_{-i}^n} \rho \cong \operatorname{marg}_{X_{-i}^n} \sigma \\ &\qquad \iff \forall j \leq m \; \exists (\alpha_k^j)_{k=1}^j \in \mathbb{R}^j \quad \alpha_j^j > 0 \;\wedge\; \operatorname{marg}_{X_{-i}^n} \nu_j = \sum_{k=1}^j \operatorname{marg}_{X_{-i}^n} \alpha_k^j \mu_k \end{aligned}$$

The coefficients $\{\alpha_k^j \colon j \leq m \wedge k \leq j\}$ that exist by the statement above must be unique due to the linear independence of the set $\{\operatorname{marg}_{X_{-i}^n} \mu_j \colon j \leq m\}$. The coherency conditions that are built into the definitions of X_{-i} and X_{-i}^n imply that the following holds for all $n > 1$.

$$\begin{aligned} \operatorname{marg}_{X_{-i}^n} \nu_j &= \sum_{k=1}^{j} \operatorname{marg}_{X_{-i}^n} \alpha_k^j \mu_k \\ \implies \operatorname{marg}_{X_{-i}^{n-1}} \operatorname{marg}_{X_{-i}^n} \nu_j &= \sum_{k=1}^{j} \operatorname{marg}_{X_{-i}^{n-1}} \operatorname{marg}_{X_{-i}^n} \alpha_k^j \mu_k \\ \implies \operatorname{marg}_{X_{-i}^{n-1}} \nu_j &= \sum_{k=1}^{j} \operatorname{marg}_{X_{-i}^{n-1}} \alpha_k^j \mu_k \end{aligned}$$

In turn, this implies that the coefficients $\{\alpha_k^j \colon j \leq m \wedge k \leq j\}$ are the same no matter what the value of n is.[13]

$$\forall j \leq m \ \exists (\alpha_k^j)_{k=1}^j \in \mathbb{R}^j \ \forall n \in \mathbb{N} \quad \alpha_j^j > 0 \ \wedge \ \operatorname{marg}_{X_{-i}^n} \nu_j = \sum_{k=1}^{j} \operatorname{marg}_{X_{-i}^n} \alpha_k^j \mu_k$$

Furthermore, because the probability measure ν_j such that extends the measures $\{\operatorname{marg}_{X_{-i}^n} \nu_j \colon n \in \mathbb{N}\}$ to $X_{-i} = \varprojlim_n X_{-i}^n$ is unique, as is the measure μ_j such that extends the measures $\{\operatorname{marg}_{X_{-i}^n} \mu_j \colon n \in \mathbb{N}\}$ to X_{-i}, the following statement also holds.

$$\forall j \leq m \ \exists (\alpha_k^j)_{k=1}^j \in \mathbb{R}^j \ \forall n \in \mathbb{N} \quad \alpha_j^j > 0 \ \wedge \ \nu_j = \sum_{k=1}^{j} \alpha_k^j \mu_k$$

This is equivalent to saying that $\rho \cong \sigma$, which is the sought-after conclusion. $\square$

APPENDIX C ADMISSIBILITY

Lemma 6. *For all $(i, m) \in I \times \mathbb{N}$, R_i^m is Borel.*

Proof of Lemma 6. The proof is by induction.

Base case: First, we show that R_i^1 is a nonempty Borel set such that $\operatorname{proj}_{S_i} R_i^1 = S_i^1$. Let H_i^+ be the nonempty Borel set of Player i's full-support hierarchies. In the proof of Lemma C.4 in Brandenburger *et al.* (2008), it is shown that the set of all $\sigma \in \mathcal{L}(S_{-i})$ such that a given s_i is optimal—in the sense of maximizing LEU—with respect to σ

[13] This is the n from $\operatorname{proj}_{X_{-i}^n}$.

is Borel. The following set is obtained by taking finite unions and finite intersections of Borel sets.

$$\tilde{R}_i^1 \equiv \{(s_i, \sigma) \in S_i \times \mathcal{L}(S_{-i})\colon s_i \text{ is optimal w.r.t. } \sigma\} \cap (S_i \times \mathcal{U}^+(S_{-i}))$$

It is therefore a Borel subset of $X_i^1 = S_i \times H_i^1 = S_i \times \mathcal{U}(S_{-i}) \subseteq S_i \times \mathcal{L}(S_{-i})$. Then R_i^1 can be rewritten as the following finite intersection of Borel sets.

$$R_i^1 = \left(\tilde{R}_i^1 \times \prod_{n \geq 2} H_i^n \right) \cap (S_i \times H_i)$$

Furthermore, a strategy s_i is admissible if and only it is optimal with respect to at least one full-support LPS over S_{-i}. Given that, for each $\rho \in \mathcal{L}(X)$, there exists a $\sigma \in \mathcal{U}(X)$ such that $\rho \cong \sigma$, $\operatorname{proj}_{S_i} R_i^1 = \operatorname{proj}_{S_i} \tilde{R}_i^1$ is equal to the following set.

$$\begin{aligned} &\{s_i \in S_i \times \mathcal{U}^+(S_{-i})\colon s_i \text{ is optimal w.r.t. some } \sigma\} \\ &= \{s_i \in S_i \times \mathcal{L}^+(S_{-i})\colon s_i \text{ is optimal w.r.t. some } \sigma\} \\ &= S_i^1 \end{aligned}$$

Inductive hypothesis Let $n \in \mathbb{N}$. Assume that $R_i^1, \ldots, R_i^n$ are nonempty Borel sets for all $i \in I$. Furthermore, let $\tilde{R}_i^1, \ldots, \tilde{R}_i^n$ be nonempty Borel sets for all $i \in I$, where $\tilde{R}_i^m$ is defined inductively by the following equation. The intersection $(\cdots \cap X_i^{m+2})$ at the end implies the consistency of $(h_i^1, \ldots, h_i^{m+1})$.

$$\tilde{R}_i^{m+1} \equiv \{(s_i, h_i^1, \ldots, h_i^{m+1}) \in \tilde{R}_i^m \times H_i^{m+1}\colon h_i^{m+i} \text{ assumes } \tilde{R}_{-i}^m\} \cap X_i^{m+2}$$

Inductive step We want to show that R_i^{n+1} is a nonempty Borel set.

$$\begin{aligned} R_i^{n+1} &= \{(s_i, h_i) \in R_i^n\colon (\operatorname{proj}_{H_i^{n+1}} h_i) \text{ assumes } (\operatorname{proj}_{X_{-i}^n} R_{-i}^n)\} \\ &= \{(s_i, h_i) \in R_i^n\colon (\operatorname{proj}_{H_i^{n+1}} h_i) \text{ assumes } \tilde{R}_{-i}^n)\} \\ &= \left(\tilde{R}_i^{n+1} \times \prod_{k \geq n+2} H_i^k \right) \cap (S_i \times H_i) \end{aligned}$$

The set of LPS's that assume a Borel set is Borel. It is then readily shown that $\tilde{R}_i^{n+1}$ and R_i^{n+1} are Borel because the set of all LPS's in H_i^{n+1} that assume a Borel set is Borel.[14]

From the definitions of R_{-i}^m it is easily shown that the following nesting of sets holds for all $m = 2, \ldots, n$.

$$E^m \equiv \operatorname{proj}_{X_{-i}^{n+1}} R_{-i}^m \subseteq E^{m-1} \equiv \operatorname{proj}_{X_{-i}^{n+1}} R_{-i}^{m-1}$$

[14] cf. Lee (2013)

These projections are also Borel sets.[15] The set of LPS's that assume a finite sequence of decreasing nonempty Borel sets is nonempty and Borel. Furthermore, if $(s_i, h_i) \in R_i^1$ and $(\text{proj}_{H_i^{n+1}} h_i)$ assumes $E^1, \ldots, E^n$, then (s_i, h_i) necessarily belongs to R_i^k for all $k \leq m$.[16] Therefore, R_i^{n+1} is nonempty. $\square$

Lemma 7. *Let $S_i^0 = S_i$ and $R_i^0 \equiv S_i^0 \times H_i$ for all $i \in I$. Then, the following holds for all $(i, m) \in I \times \mathbb{N}$.*

$$\text{proj}_{S_i}(R_i^{m-1} \setminus R_i^m) = S_i^{m-1}$$

Proof of Lemma 7. The proof is by induction.

Base case: That $\text{proj}_{S_i}(R_i^0 \setminus R_i^1) = S_i^0$ is trivial and immediate. There exists some $h_i^- \in H_i \setminus H_i^+$ and $S_i^0 \times \{h_i^-\} \subseteq R_i^0 \setminus R_i^1$.

Inductive hypothesis Let $n \in \mathbb{N}$. Let the following hold for all $(i, m) \in I$ such that $m = 1, \ldots, n$.

$$\text{proj}_{S_i}(R_i^{m-1} \setminus R_i^m) = S_i^{m-1}$$

Inductive step We want to show that $\text{proj}_{S_i}(R_i^n \setminus R_i^{n+1}) = S_i^n$. Let $E^1, \ldots, E^n$ be defined as below

$$\forall k \leq n \quad E^k \equiv \text{proj}_{X_{-i}^{n+1}} R_{-i}^k$$

A full-support LPS $\sigma = (\mu_1, \ldots, \mu_n) \in \mathcal{L}(X_{-i}^{n+1})$ can be constructed such that it assumes $E^1, \ldots, E^{n-1}$ and the following are true.

$$\begin{aligned} \exists e^* \in E^{n-1} \setminus E^n \quad \mu_1(\{e^*\}) &> 0 \\ \mu_1(E^{n-1}) &= 1 \\ \mu_2(E^{n-2} \setminus E^{n-1}) = \mu_3(E^{n-3} \setminus E^{n-2}) = \cdots = \mu_{n-1}(E^1 \setminus E^2) &= 1 \\ \mu_n(E^1 \cup \cdots \cup E^n) &= 0 \end{aligned}$$

Following the same logic as in the proof of Lemma 6, any $(s_i, h_i) \in S_i \times H_i$ such that $\text{proj}_{H_i^{n+1}} h_i \cong \sigma$ will then satisfy R$(n-1)$AR. However, μ_1 was constructed so that it cannot place probability 1 on both E^{n-1} and E^n simultaneously. It follows that E^n cannot be assumed by σ. By definition, $\text{proj}_{H_i^{n+1}} h_i$ assuming $E^n = \text{proj}_{X_{-i}^{n+1}} R_{-i}^n$ is equivalent to n^{th}-order assumption of rationality.

[15] It is not generally true that projections of Borel sets are not necessarily Borel. It is true for these particular projections of these particular Borel sets.

[16] If the $(n+1)^{\text{th}}$-order belief implied by h_i assumes E^k, then the $(k+1)^{\text{th}}$-order belief implied by h_i assumes $\tilde{R}_i^k$. The converse does not always hold.

Now, let Ξ be defined as follows.

$$\Xi \equiv \{(\nu_1, \dots, \nu_n) \in \mathcal{L}(S_{-i})\colon \operatorname{supp} \nu_j = S_{-i}^{n-j} \text{ for } j = 1, \dots n\}$$

Given any $\rho \in \Xi$, we can find another LPS $\sigma' \in \mathcal{L}(X_{-i}^{n+1})$ of equal length such that

1. The beliefs σ and σ' are mutually absolutely continuous—i.e., they have the same null sets; and
2. $\operatorname{marg}_{S_{-i}} \sigma' = \rho$.

Note that $\operatorname{marg}_{S_{-i}} \sigma$ also belongs to Ξ. A strategy is optimal with respect to a belief in Ξ if and only if it is n-admissible. It follows that, as long as we can find $(s_i, h_i) \in R_i^n \setminus R_i^{n+1}$ such that $\operatorname{proj}_{H_i^{n+1}} h_i = \sigma$, we can find, for each $s_i' \in S_i^n$, another hierarchy h_i' such that $(s_i', h_i') \in R_i^n \setminus R_i^{n+1}$. $\square$

Proof of Theorem 7. Let $S_i^0 = S_i$ and $R_i^0 \equiv S_i \times H_i$. First, note that $\operatorname{proj}_{S_i} R_i^\infty \subseteq S_i^\infty$. The desired conclusion is proven by showing that, for each $s_i \in S_i^\infty$, there exists some h_i such that $(s_i, h_i) \in R_i^\infty$. For any finite game, there must exist some $M > 0$ such that $S_i^M = S_i^\infty$ for all $i \in I$.

As an intermediate step, we want to show that, for all $k \geq M$, if $(s_i, \eta^k) \in R_i^k$, then there exists some $(s_i, \eta^{k+1}) \in R_i^{k+1}$ such that

$$\operatorname{proj}_{H_i^k} \eta^k = \operatorname{proj}_{H_i^k} \eta_i^{k+1}$$

[Additional details needed for previous step.]

For each $(s_i, \eta^M) \in R_i^M$, fix a sequence $(\eta^M, \eta^{M+1}, \dots)$ that is chosen as above. Let $\eta^\infty = (h_i^1, h_i^2, \dots) \in H_i$ be defined as follows.

$$\begin{aligned} &\forall k = 1, \dots, M && h_i^k = \operatorname{proj}_{H_i^k} \eta^M \\ &\forall k > M && h_i^k = \operatorname{proj}_{H_i^k} \eta^k \end{aligned}$$

Therefore, $(s_i, \eta^\infty) \in R_i^\infty$. Since such η^∞ can be found for each $(s_i, \eta^M) \in R_i^M$, it can be concluded that $\operatorname{proj}_{S_i} R_i^\infty = \operatorname{proj}_{S_i} R_i^M = S_i^M = S_i^\infty$. $\square$

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