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Abstract

This paper analyzes traffic bottleneck congestion when drivers randomly cause incidents that temporarily block the bottleneck. Drivers have general scheduling preferences for time spent at home and at work. They independently choose morning departure times from home to maximize expected utility without knowing whether an incident has occurred. The resulting departure time pattern may be compressed or dispersed according to whether or not the bottleneck is fully utilized throughout the departure period on days without incidents. For both the user equilibrium (UE) and the social optimum (SO) the departure pattern changes from compressed to dispersed when the probability of an incident becomes sufficiently high. The SO can be decentralized with a time-varying toll, but drivers are likely to be strictly worse off than in the UE unless they benefit from the toll revenues in some way. A numerical example is presented for illustration. Finally, the model is extended to encompass minor incidents in which the bottleneck retains some capacity during an incident.

JEL CLASSIFICATIONS: C61, D62, R41.

KEYWORDS: Departure-time decisions, bottleneck model, traffic incidents, congestion, scheduling utility, morning commute, evening commute.

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1 Introduction

Traffic congestion imposes a heavy burden in urban areas. The Texas Transportation Institute conducts an annual survey of traffic congestion in the US. According to its 2012 report, in 2011 congestion caused an estimated 5.5 billion hours of travel delay and 2.9 billion gallons of extra fuel consumption with a total cost of $121 billion (Schrank et al., 2012). The average cost per automobile commuter in the urban areas studied was $818. Nonrecurring traffic congestion due to accidents, bad weather, special events, and other shocks accounts for a large fraction of the total delays. According to Schrank et al. (2011, Appendix B, p. B-27) incident-related delays alone contribute 52 - 58 percent of total delay in US urban areas.[1]

Unanticipated travel delays upset peoples' travel plans, and may cause them to arrive late with serious consequences for commuting, business, and other types of trips. Travelers can sometimes adjust to the threat of delays by changing their transport mode or destination, or even cancelling trips, but a more common response is to adjust departure times. Researchers have long been interested in studying the adjustment process, and they have adopted various modeling approaches. In an early and insightful study, Gaver (1968) derived the optimal departure time for a driver faced with stochastic travel time who incurs costs from both travel time and schedule delay. The optimal policy, which Gaver called a headstart strategy, entails a probabilistic trade-off between arriving early and arriving late. Gaver assumed that travel time has a constant and exogenous variance, and he did not attempt to derive an endogenous travel time distribution as a dynamic equilibrium. His approach was adopted and extended by Knight (1974); Hall (1983); Noland and Small (1995), and Noland (1997).

All these studies use models with flow congestion. An alternative approach is to use the Vickrey (1969) bottleneck model in which congestion delay takes the form of queuing. A series of studies by Arnott et al. (1991, 1999) and Lindsey (1994, 1999) introduced stochasticity into the bottleneck model by assuming that capacity and/or demand fluctuate randomly from day to day, but are constant during the period of use on a given day. For want of a better term, we will call this the "daily-shocks" model.[2]

[1]The 2012 Report does not repeat this estimate. As Hall (1993) observes, the contribution of nonrecurrent congestion is difficult to determine because it depends on the magnitude and timing of recurrent congestion, and vice versa. Drivers may underestimate the prevalence of nonrecurrent congestion because incident-induced queues can persist long after the incidents are cleared away.

[2]Arnott et al. (1991, 1999) and Li et al. (2008) analyze user equilibrium in the daily-shocks model, whereas Lindsey (1994, 1999) focuses on the social optimum. Other recent papers have also studied random travel times using the bottleneck model. Xin and Levinson (2007) assume that travel times are exogenous and independently distributed over time, and their model does not feature incidents per se. Fosgerau (2010) shows how the dynamics of random congestion

Our paper differs from these earlier bottleneck-model studies in three ways. First, they adopted the traditional specification of trip-timing preferences used by Vickrey (1969) in which individuals have a preferred time to arrive at their destination and incur a schedule delay cost proportional to the amount of time they arrive earlier or later. Following Börjesson et al. (2012) we will call this the "step" model. Here we adopt a more general scheduling utility function approach that incorporates preferences for time spent at different activities. We apply the model to commuting trips by specifying preferences for time spent at home and at work.[3]

Second, and more fundamentally, we assume that capacity can fluctuate while trips are being made rather than being determined before travel begins. Third, we assume that capacity reductions are due to incidents caused by drivers during their trip. The timing of shocks is therefore endogenous to the model rather than exogenous as in earlier studies. Since drivers are responsible for most incidents, this within-day, endogenous specification of capacity fluctuations accounts for a significant portion of nonrecurring congestion that occurs. It also provides the basis for assessing tolling and other policies to reduce the costs of congestion by altering peoples' travel decisions. For most of the paper we assume that capacity is reduced to zero by an incident although in a final section we examine a variant of the model in which loss of capacity is partial.

Two unpublished studies cover part of the same ground as we do. Schrage (2006) derives the unregulated and socially optimal departure rates for a single road link when the accident rate is a function of the inflow rate and therefore endogenous. Her model differs from ours in three main respects. First, she uses the Henderson (1974) flow congestion model in which a driver's travel time is determined by the aggregate departure rate when he starts his trip. This model has no state variable analogous to queue length in the bottleneck model. Second, capacity is reduced only partially in an incident and it subsequently recovers slowly, and deterministically, rather than all at once. Third, drivers are assumed to know whether and when an accident has occurred before they depart. Schrage derives the optimal time-varying and state-dependent toll that decentralizes the social optimum, but she does not solve for the timing of departures in either the unregulated user equilibrium or the social optimum. In independent work, Peer et al. (2010)

induce characteristic loops in the relationship between the mean and the variance of travel time over different times of day. de Palma and Fosgerau (2011) analyze random queue sorting whereby travel time is random from the perspective of individual travellers, but capacity and demand are fixed.

[3] Jenelius et al. (2011) use a similar scheduling utility function approach to study the effects of unpredictable travel time shocks on trip-timing decisions. They apply the model to a full day of activity including morning and evening commutes. Their model differs in featuring shocks that are exogenous and independent of time of day. There is also no traffic congestion in their model.

use the bottleneck model to analyze incidents in which, like Schrage (2006), capacity loss is partial. They treat incident timing as exogenous and assume that an incident persists until all drivers have completed their trips. They also adopt the "step" model of trip-timing preferences. Finally, they limit attention to the unregulated user equilibrium and do not examine the social optimum or tolling.

In our paper we undertake a systematic analysis of both user (i.e., Nash) equilibrium and socially optimal trip-timing decisions when drivers do not not know whether an incident has occurred before they decide when to depart. We solve for the optimal time-varying (but state-independent) toll that decentralizes the social optimum. One of the questions we address is whether the bottleneck operates at capacity throughout the travel period on days when no incident occurs, or whether some capacity goes "unused". We show that for both the user equilibrium and social optimum, spare capacity does exist for part, or all, of the travel period if incidents are sufficiently probable.[4] In contrast to the daily-shocks model, departures can be more spread out in the user equilibrium than in the social optimum. Another difference is that the socially-optimal departure rate can decrease, rather than increase, over time.

The paper is organized as follows. Section 2 describes the model. Section 3 summarizes the main features of user equilibrium and social optimum for the deterministic variant of the model with no incidents. Section 4 derives properties of the user equilibrium with incidents. Section 5 conducts a parallel analysis of the social optimum. Section 6 presents a numerical example calibrated for morning commutes, and then considers a variant for evening commutes. Section 7 undertakes a partial analysis of an extension of the model in which the bottleneck retains some capacity during an incident. Finally, Section 8 concludes with a summary and ideas for extension.

2 The model

A continuum of N identical individuals drive alone from a common origin through a bottleneck to a common destination.[5] To be concrete, in most of the paper the trip is assumed to be a morning commute from home (H) to work (W). (However, an evening commute is also examined in the example section.) Departure time from home is denoted by t. Drivers[6] depart at a rate $\rho(t)$ during a set of times T;

[4] Holding spare capacity is broadly consistent with policies of reserving shoulder lanes for use during accidents and other disruptions.

[5] A notational glossary is provided at the end of the paper.

[6] Throughout the paper we will refer to "drivers" even though individuals are treated as a continuum in the model so that there are no discrete or atomic agents. Reference to "drivers", "users", "commuters" and so on is common in the bottleneck model literature, and it facilitates exposition.

cumulative departures are thus $R(t) = \int_{\{v \in T | v \leq t\}} \rho(v)\, dv$.[7] Free-flow travel time before and after reaching the bottleneck is normalized to zero. A driver departing at t encounters a queuing delay of $q(t)$ at the bottleneck and reaches work at time $a = t + q(t)$. Drivers have scheduling preferences[8] described by the utility function

$$u(t, a) = \int_{t_H}^{t} \beta(v)\, dv + \int_{a}^{t_W} \gamma(v)\, dv. \tag{1}$$

The limits of integration, t_H and t_W, are chosen such that all travel takes place within the interval $[t_H, t_W]$. Function $\beta(\cdot) > 0$ denotes the flow of utility from being at home, and function $\gamma(\cdot) > 0$ denotes utility from being at work. Functions $\beta(\cdot)$ and $\gamma(\cdot)$ are assumed to be continuously differentiable with derivatives $\beta' < 0$ and $\gamma' > 0$ and to intersect at time t^*.[9] Utility from time spent driving is normalized to zero. These assumptions ensure that, for any fixed trip duration, there is a unique departure time t, $t < t^*$, that maximizes scheduling utility. They also assure that $u(t, a)$ is strictly increasing in t, strictly decreasing in a, and globally strictly concave. Two final assumptions, $\underset{v \to t_H}{Lim}\, \beta(v) = \infty$ and $\underset{v \to t_W}{Lim}\, \gamma(v) = \infty$, will ensure existence of a Nash equilibrium in departure times.[10]

If no incident is in progress, the bottleneck has a flow capacity of s. Incidents are caused by a randomly determined driver and block the bottleneck for a deterministic period $\Delta > 0$. The incident occurs when the driver reaches the head of the queue (if any) and is about to cross the bottleneck.[11] At most one driver causes an incident on a given day. Let $\xi \in [0, N]$ be the random variable that indicates the position of the culpable driver in the departure schedule if an incident occurs. Variable ξ has a continuously differentiable density $f(\xi)$ and a cumulative function $F(\xi)$, where $F(0) = 0$ and $F(N) < 1$. Function $f(\cdot)$ will be called

[7] All statements about ρ in the paper will be "almost surely", since ρ can take arbitrary values on sets of Lebesgue measure zero without affecting aggregate behaviour or welfare. To ease exposition this detail will be ignored.

[8] This formulation of scheduling preferences originates from Vickrey (1969, 1973) and has been used by Tseng and Verhoef (2008), Fosgerau and Engelson (2011), Fosgerau and de Palma (2012), Jenelius et al. (2011), and Börjesson et al. (2012).

[9] The notation differs from that in the step model where β denotes the cost per minute of arriving before t^*, and γ denotes the cost per minute of arriving after t^*. The assumptions $\beta' < 0$ and $\gamma' > 0$ rule out the step model because the (implicit) $\beta(\cdot)$ and $\gamma(\cdot)$ functions in that model are constants except at t^* where $\gamma(\cdot)$ steps up. This is not particularly restrictive since the step-model preferences can be approximated arbitrarily closely by differentiable functions. Nevertheless, the assumptions could be generalized as in Fosgerau and Engelson (2011).

[10] These assumptions are relaxed in the example of Section 6 where $\beta(\cdot)$ and $\gamma(\cdot)$ are linear functions.

[11] The mechanics of the model are the same if an incident occurs anywhere between home and the exit point from the bottleneck. All drivers ahead of the culprit are unaffected by the incident.

"incident risk" and $F(N)$ "incident probability".[12] A baseline assumption is that incident risk is constant, and this will be assumed in the numerical example of Section 6. A day with an incident is called a "Bad day", and a day without an incident is called a "Good day". Any costs associated with incidents other than delay are ignored.[13] For the analysis of the social optimum, we require $f(\cdot)$ to be differentiable.

Drivers independently choose their departure times to maximize expected scheduling utility while taking the departure rate as given and without knowing whether an incident has occurred. The first driver departs at time t_0. If the bottleneck operates at capacity from t_0 on, and driver ξ causes an incident, the incident occurs at time $t_0 + \frac{R(\xi)}{s}$ and capacity is restored at $t_0 + \frac{R(\xi)}{s} + \Delta$. A driver who is delayed by an incident will be said to incur "queuing delay" even if the driver causes the incident and there is no queue of drivers ahead. The duration of an incident, Δ, is assumed to be long enough that the queue does not dissipate until after the last driver departs. For future reference this is called the "persistent-queue" assumption. If a queue develops on Good days, it may or may not dissipate before the last driver departs.

3 User equilibrium and system optimum without incidents

As a first step in analyzing the model, and also for later reference, we briefly describe the user equilibrium and social optimum for a setting in which incidents do not occur.

3.1 User equilibrium

Fosgerau and de Palma (2012) analyze user equilibrium (UE) in the model without incidents and their treatment is briefly summarized here. Let superscript e denote UE and 0 the setting without incidents. It is easy to show that departures take place during an interval $T^{e0} = [t_0^{e0}, t_N^{e0}]$. A queue exists throughout the interior of T^{e0}, but disappears at time t_N^{e0} so that $t_N^{e0} = t_0^{e0} + \frac{N}{s}$. de Palma and Fosgerau

[12]Incident risk depends on ξ, but not on the identity of the individual driver. The order in which drivers depart therefore does not affect aggregate variables of interest. Incident risk also does not depend on the rate at which drivers arrive at the bottleneck or on whether there is a queue. (However, the probability that an incident occurs within a short interval of time is proportional to the arrival rate at the bottleneck.) Relaxing these assumptions would complicate the analysis.

[13]Results of interest are unaffected if incidents create additional costs (e.g., related to emergency response, vehicle repair, filing of insurance claims, etc.) as long as the costs are independent of t and ξ.

(2011) refer to elimination of the queue at t_N^{e0} as the "no residual queue" property of UE.

Since a queue exists in the interior of T^{e0}, a driver departing at time t exits the bottleneck at $t_0^{e0} + \frac{R^{e0}(t)}{s}$. Scheduling utility is constant on T^{e0} and equal to

$$u\left(t, t_0^{e0} + \frac{R^{e0}(t)}{s}\right) = \int_{v=t_H}^{t} \beta(v)\, dv + \int_{v=t_0^{e0}+\frac{R^{e0}(t)}{s}}^{t_W} \gamma(v)\, dv.$$

The UE departure rate is derived by differentiating $u(\cdot)$ with respect to t, setting the derivative to zero, and rearranging terms:

$$\frac{\rho^{e0}(t)}{s} = \frac{\beta(t)}{\gamma\left(t_0^{e0} + \frac{R^{e0}(t)}{s}\right)}. \tag{2}$$

Since the first and last drivers receive the same expected utility, $u\left(t_0^{e0}, t_0^{e0}\right) = u\left(t_0^{e0} + \frac{N}{s}, t_0^{e0} + \frac{N}{s}\right)$. This condition can be written as

$$\int_{v=t_0^{e0}}^{t_0^{e0}+\frac{N}{s}} \left(\beta(v) - \gamma(v)\right) dv = 0. \tag{3}$$

Eqn. (3) gives an implicit formula for t_0^{e0}. It states that a driver who shifts from departing first to departing last gains additional utility at home that just offsets foregone utility at work.

A representative user equilibrium is shown in Figure 1. The first driver departing at t_0^{e0} has a scheduling utility equal to the area under curve $abgjdef$. This area is smaller by area bgd than the ideal of leaving home at t^*, arriving immediately at work, and gaining utility of $abcdef$. The last driver departing at t_N^{e0} has a utility equal to the area under the curve $abcdkef$ which is less than the ideal by area dke. Condition (3) assures that areas dke and bgd are equal. Now consider a driver departing at time t, and call this 'driver t'. From time t_0^{e0} to t, driver t gains area $bghc$ more utility than the first driver. From time t to $t_0^{e0} + \frac{R^{e0}(t)}{s}$, driver t is caught in the queue and gains less utility than the first driver by area $hlmj$. In equilibrium, queuing time is such that area $hlmj$ matches area $bghc$. This is equivalent to the condition that driver t's utility from time spent at home from t_0^{e0} to t equals the first driver's utility from time spent at work from t to $t_0^{e0} + \frac{R^{e0}(t)}{s}$.

Note that driver t's queuing time is shorter the higher his utility from work because the driver foregoes more utility while traveling rather than being at work. Thus, if utility functions $\beta(\cdot)$ and $\gamma(\cdot)$ were shifted upwards an equal amount, area $bghc$ would not change, but area $hlmj$ would become taller and narrower. This observation helps to explain a difference between the morning commute and

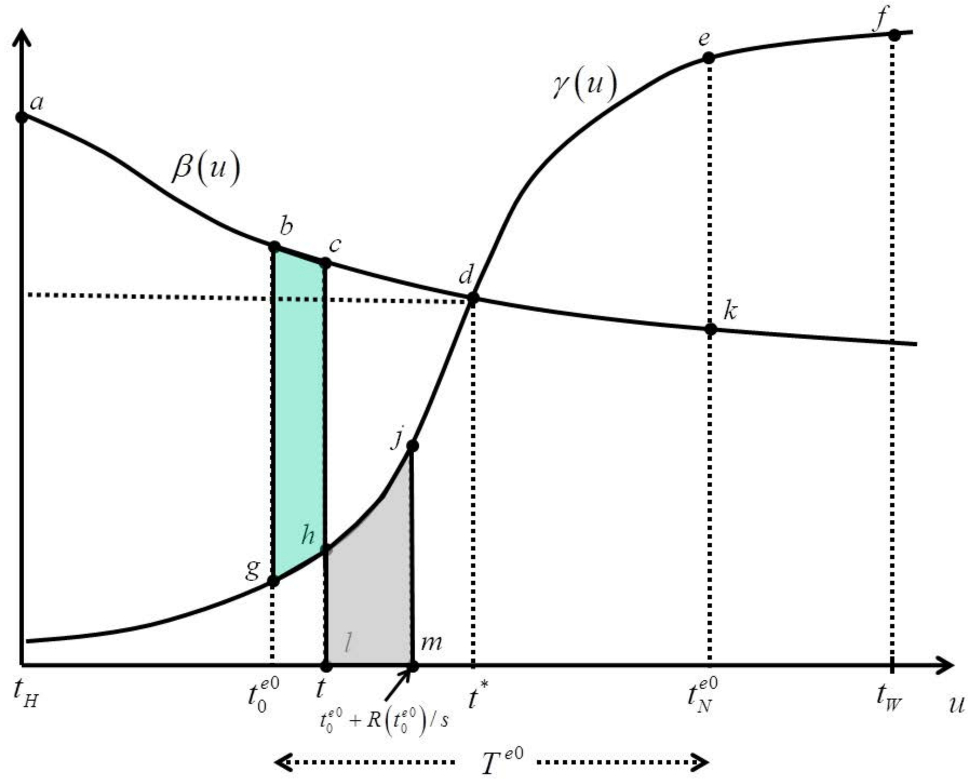

Figure 1: Scheduling utility and timing of departures

evening commute examples in Section 6.

3.2 The social optimum

The social optimum (SO), denoted by superscript w, is derived by choosing $\rho(t)$ to maximize aggregate scheduling utility:

$$U = \int_{t \in T^{w0}} \rho(t)\, u(t,a)\, dt,$$

where T^{w0} is the set of SO departure times. The departure rate is maintained at capacity throughout T^{w0} so that no queue is allowed to form, $a = t$ for each driver, and $t_N^{w0} = t_0^{w0} + \frac{N}{s}$. Aggregate scheduling utility is therefore

$$U = \int_{t=t_0^{w0}}^{t_0^{w0}+\frac{N}{s}} su(t,t)\, dt = s \int_{t=t_0^{w0}}^{t_0^{w0}+\frac{N}{s}} \left[\int_{v=t_H}^{t} \beta(v)\, dv + \int_{v=t}^{t_W} \gamma(v)\, dv \right] dt.$$

The first-order condition for t_0^{w0} is

$$\int_{v=t_0^{w0}}^{t_0^{w0}+\frac{N}{s}} (\beta(v) - \gamma(v))\, dv = 0. \quad (4)$$

Equation (4) for t_0^{w0} is identical to eqn. (3) for t_0^{e0}. Departures therefore occur over the same time interval in UE and SO: $T^{w0} = T^{e0}$.

It is straightforward to show that the SO can be decentralized by levying a time-varying toll, $\tau^{w0}(t)$, such that $u(t,t) - \tau^{w0}(t)$ is constant for $t \in T^{w0}$ and lower for $t \notin T^{w0}$. If demand were elastic, the toll must be zero at the beginning and end of the travel period (Arnott et al., 1993). Here the number of drivers is fixed, and a constant can be added to the toll schedule without affecting trip timing. Nevertheless, it is natural to set the toll to zero at the beginning and end of the departure period so that $\tau^{w0}(t_0^{w0}) = \tau^{w0}(t_N^{w0}) = 0$.

4 User equilibrium with incidents

Consider now the case of interest in which incidents can occur. We begin by establishing some general characteristics of UE. This is followed by separate analyses of the compressed departures and dispersed departures configurations.

4.1 General characteristics of user equilibrium

Lemmas 1 and 2 below summarize properties of a UE.

Lemma 1 *(a): The UE departure rate,* $\rho^e(t)$*, is strictly positive on an interval* $T^e = (t_0^e, t_N^e)$*. (b):* $t^* \in (t_0^e, t_N^e)$*. (c):* $R^e(t_N^e) = N \leq s(t_N^e - t_0^e)$*: on Good days the no-residual queue property holds.*

Proof. Part (a): $\rho^e(t)$ cannot drop to zero in the interior of T^e since otherwise some driver could increase expected utility by departing during the gap. Part (b): Clearly $t_0^e < t^*$; otherwise any driver departing after t^* could increase utility by departing at t^* instead. Suppose $t_N^e \leq t^*$ so that $\beta(t_N^e) \geq \gamma(t_N^e)$. If the last driver departed dt later, his expected utility would change by $dE(u|t_N^e) = (\beta(t_N^e) - (1 - F(N))\gamma(t_N^e))\,dt$. Given $\beta(t_N^e) \geq \gamma(t_N^e)$ and $F(N) > 0$, $dE(u|t_N^e) > 0$ and t_N^e cannot be an individually optimal departure time. Part (c): If $R^e(t_N^e) > s(t_N^e - t_0^e)$, there would be a queue at t_N^e. This would violate the no residual queue property, and the last driver could leave home later without arriving at work later. ■

Lemma 2 *In UE the last departure time,* t_N^e*, is such that*

$$F(N) \leq 1 - \frac{\beta(t_N^e)}{\gamma(t_N^e)}. \tag{5}$$

Proof. By Lemma 1, on Good days there is no residual queue at t_N^e. And by the persistent-queue assumption, if an incident occurs the queue persists until after t_N^e. A driver departing just after t_N^e therefore encounters a queue with probability $F(N)$, and the driver's expected utility changes at a rate

$$\frac{\partial E(u|t)}{\partial t} = \beta(t) - (1 - F(N))\gamma(t). \tag{6}$$

Expression (6) must be non-positive for $t \geq t_N^e$; otherwise the last driver could increase utility by departing later. Since (6) is largest for $t = t_N^e$, inequality (5) must hold.[14] ■

Expression (6) is readily interpreted. $\beta(t)$ is the marginal benefit at time t from staying longer at home, and $\gamma(t)$ is the marginal cost of delaying arrival at work. Departing later implies arriving later if no incident has occurred which is the case with probability $1 - F(N)$. If an incident has occurred, there is no cost of delaying departure since the driver merely spends less time queuing and reaches

[14] There is no additional equilibrium condition analogous to (5) that applies to t_0^e because an incident cannot occur before t_0^e. Given $t_0^e < t^*$, departing before t_0^e would clearly not be optimal.

work at the same time. If $F(N)$ is sufficiently large, and Δ is sufficiently small, the persistent queue assumption will be violated since any t_N^e that satisfies (5) will occur after the queue dissipates. The persistent queue assumption therefore imposes bounds on $F(N)$ and Δ.

User equilibrium follows one of two patterns. In one, queuing on Good days persists until the last driver has departed. Similar to UE with no incidents, all drivers complete their trips within the minimum feasible time interval of N/s. This pattern will be called "compressed" departures. In the second pattern, called "dispersed" departures, queuing ends on Good days before the last driver departs and departures extend for a time interval longer than N/s. The compressed-departures and dispersed-departures patterns are examined separately in the next two subsections.

4.2 User equilibrium with compressed departures

The main characteristics of a compressed-departures UE are summarized in the following theorem.

Theorem 1 *Assume departures are compressed. Then a unique Nash equilibrium exists. The equilibrium departure rate is*

$$\frac{\rho^e(t)}{s} = \frac{\beta(t)}{\left(\begin{array}{c}(1-F(R^e(t)))\gamma\left(t_0^e+\frac{R^e(t)}{s}\right)+F(R^e(t))\gamma\left(t_0^e+\frac{R^e(t)}{s}+\Delta\right)\\ +f(R^e(t))\int_{v=t_0^e+\frac{R^e(t)}{s}}^{t_0^e+\frac{R^e(t)}{s}+\Delta}\gamma(v)\,dv\end{array}\right)}. \tag{7}$$

The departure time set T^e is determined by the conditions $t_N^e = t_0^e + N/s$ and

$$\int_{v=t_0^e}^{t_N^e}(\beta(v)-\gamma(v))\,dv = F(N)\int_{v=t_N^e}^{t_N^e+\Delta}\gamma(v)\,dv. \tag{8}$$

Proof. Expected utility from departing at t is

$$
\begin{aligned}
E(u|t) &= (1-F(R^e(t)))\,u\left(t, t_0^e + \frac{R^e(t)}{s}\right) \\
&\quad + F(R^e(t))\,u\left(t, t_0^e + \frac{R^e(t)}{s} + \Delta\right) \\
&= \int_{v=t_H}^{t} \beta(v)\,dv + \int_{v=t_0^e+\frac{R^e(t)}{s}}^{t_W} \gamma(v)\,dv \\
&\quad - F(R^e(t)) \int_{v=t_0^e+\frac{R^e(t)}{s}}^{v=t_0^e+\frac{R^e(t)}{s}+\Delta} \gamma(v)\,dv,
\end{aligned} \tag{9}
$$

which is constant during T^e.[15] Differentiate and set to zero to obtain

$$
\begin{aligned}
&\beta(t) - \gamma\left(t_0^e + \frac{R^e(t)}{s}\right)\frac{\rho^e(t)}{s} - f(R^e(t))\,\rho^e(t) \int_{v=t_0^e+\frac{R^e(t)}{s}}^{t_0^e+\frac{R^e(t)}{s}+\Delta} \gamma(v)\,dv \\
&- F(R^e(t))\frac{\rho^e(t)}{s}\left(\gamma\left(t_0^e + \frac{R^e(t)}{s} + \Delta\right) - \gamma\left(t_0^e + \frac{R^e(t)}{s}\right)\right) = 0.
\end{aligned}
$$

Collecting terms in $\rho^e(t)$ yields (7) which simplifies to (2) if $f(n) = 0$, $\forall n$. Utility from departing at t_0^e is given by (9) with $t = t_0^e$:

$$
E(u|t_0^e) = u(t_0^e, t_0^e) = \int_{v=t_H}^{t_0^e} \beta(v)\,dv + \int_{v=t_0^e}^{t_W} \gamma(v)\,dv. \tag{10}
$$

Expected utility from departing at t_N^e is given by (9) with $t = t_N^e$:

$$
E(u|t_N^e) = \int_{v=t_H}^{t_N^e} \beta(v)\,dv + \int_{v=t_N^e}^{t_W} \gamma(v)\,dv - F(N) \int_{v=t_N^e}^{t_N^e+\Delta} \gamma(v)\,dv. \tag{11}
$$

Equating (10) and (11) yields condition (8). The left-hand side of (8) is decreasing in t_0^e, while (given $t_N^e = t_0^e + N/s$) the right-hand side is increasing. Any solution is thus unique. Existence is guaranteed by the assumptions $\underset{v \to t_H}{Lim}\, \beta(v) = \infty$ and $\underset{v \to t_W}{Lim}\, \gamma(v) = \infty$. ■

[15] The first expression for expected utility in (9) is explained as follows. The first term pertains to utility when there is no incident which occurs with probability $1-F(R^e(t))$. The driver departs at time t and arrives at $t_0^e + \frac{R^e(t)}{s}$ when the $R^e(t)$ preceding drivers have passed the bottleneck. The second term pertains to utility when an incident has occurred, with probability $F(R^e(t))$. Since the bottleneck is shut for a period Δ, the driver arrives at $t_0^e + \frac{R^e(t)}{s} + \Delta$.

The candidate compressed UE described in Theorem 1 can be tested by solving for t_N^e using (8), and substituting the result into Condition (5). If (5) is satisfied, the UE is indeed compressed. If (5) is violated, the UE is dispersed. Condition (5) is assumed to be satisfied in the balance of this subsection.

The right-hand side of (8) is an increasing function of $F(N)$ and Δ. Thus, the greater the incident probability and the longer an incident lasts, the earlier departures begin. This result is consistent with the Gaver (1968) analysis of headstart strategies, mentioned in the introduction.

Corollary 1 *Assume departures are compressed, and $f(n)$ is independent of n. Then ρ^e is decreasing so that R^e is concave.*

Proof. Concavity of R^e follows from equation (7) since $\beta(t)$ in the numerator is a decreasing function of t, whereas in the denominator: $\gamma\left(t_0^e + \frac{R^e(t)}{s}\right)$, $\gamma\left(t_0^e + \frac{R^e(t)}{s} + \Delta\right)$, $\int_{v=t_0^e+\frac{R^e(t)}{s}}^{t_0^e+\frac{R^e(t)}{s}+\Delta} \gamma(v)\, dv$ and $F(R^e(t))$ are all increasing; $\gamma\left(t_0^e + \frac{R^e(t)}{s} + \Delta\right) > \gamma\left(t_0^e + \frac{R^e(t)}{s}\right)$; and $f(R^e(t))$ is constant. ■

Concavity of the cumulative departure schedule implies that the departure rate decreases monotonically over time, and also that on good days queuing can occur only during one connected time period. Concavity is also a property of user equilibrium in the daily-shocks model (Arnott et al., 1999, Proposition 1).

From equation (7) the initial departure rate is

$$\frac{\rho^e(t_0^e)}{s} = \frac{\beta(t_0^e)}{\gamma(t_0^e) + f(0)\int_{v=t_0^e}^{t_0^e+\Delta} \gamma(v)\, dv}. \tag{12}$$

With no incidents, the initial departure rate is given by eqn. (2):

$$\frac{\rho^{e0}(t_0^{e0})}{s} = \frac{\beta(t_0^{e0})}{\gamma(t_0^{e0})}. \tag{13}$$

Compared to (13), eqn. (12) includes an additional term in the denominator, but since $t_0^e < t_0^{e0}$, $\beta(t_0^e) > \beta(t_0^{e0})$ and $\gamma(t_0^e) < \gamma(t_0^{e0})$. It is therefore unclear whether incident risk induces drivers to depart at a faster or slower initial rate. (In the example presented in Section 6 the departure rate is faster.) However, if there is no incident risk for the first drivers (i.e., $f(0) = 0$), then it follows from (13), (12), and $t_0^e < t_0^{e0}$ that the initial departure rate is higher when incidents can occur. This is also a property of the model with exogenous incidents in Peer et al. (2012).

4.3 User equilibrium with dispersed departures

If a candidate UE with compressed departures is computed using equation (8), and inequality (5) is violated, the UE is dispersed. Depending on the time path of $f(\cdot)$, the departure rate can be nonmonotonic, and on Good days queuing can occur in disjoint time intervals. This is conceivable if $f(\cdot)$ has a pronounced double peak, but it seems unlikely since the natural baseline is for incident risk to be constant. To keep the analysis tractable it is assumed that any queuing on Good days occurs during a single interval beginning at t_0^e. Thus, suppose that on Good days there is a queue for $t \in \left(t_0^e, \tilde{t}\right)$ and no queue for $t \in \left[\tilde{t}, t_N^e\right]$. For $t \in \left(t_0^e, \tilde{t}\right)$, the departure rate is given by (7). For $t \in \left[\tilde{t}, t_N^e\right]$, equation (9) does not apply and expected utility must be computed afresh. If an incident occurs at time $v \leq \tilde{t}$, a driver departing at t arrives at time $a = t_0^e + \frac{R^e(t)}{s} + \Delta$. If $v \geq \tilde{t}$, arrival time is defined by the condition $R^e(v) + s(a - v - \Delta) = R^e(t)$, or $a = v + \Delta + \frac{R^e(t) - R^e(v)}{s}$.[16] Expected utility is therefore[17]

$$\begin{aligned} E(u|t) &= (1 - F(R^e(t)))\, u(t,t) \\ &\quad + F\left(R^e\left(\tilde{t}\right)\right) u\left(t, t_0^e + \frac{R^e(t)}{s} + \Delta\right) \\ &\quad + \int_{v=\tilde{t}}^{t} \rho(v)\, f(R^e(v))\, u\left(t, v + \Delta + \frac{R^e(t) - R^e(v)}{s}\right) dv. \end{aligned} \tag{14}$$

Differentiating (14) with respect to t, and setting the derivative to zero, one

[16] This equation is explained as follows. When an incident occurs at time v, $R^e(v)$ drivers have passed the bottleneck. During the interval $[v, v+\Delta]$, no further drivers can pass. After $v+\Delta$, drivers pass the bottleneck again at rate s as long as a queue persists. Cumulative passages through the bottleneck by time $a > v + \Delta$ are therefore $R^e(v) + s(a - v - \Delta)$. A driver who departs at time t is preceded by $R^e(t)$ other drivers. This driver therefore arrives when the bottleneck has processed this number of drivers: $R^e(v) + s(a - v - \Delta) = R^e(t)$.

[17] The first term in (14) covers instances in which no incident occurs. The probability of no incident is $1 - F(R^e(t))$. The driver encounters no queue, and therefore arrives immediately at t and receives a utility of $u(t,t)$. The second term covers instances in which an incident occurs before $\tilde{t}$ which happens with probability $F\left(R^e\left(\tilde{t}\right)\right)$. The number of drivers who have departed by t is $R^e(t)$. Were there no incident, the driver departing at t would pass the bottleneck at $t_0^e + \frac{R^e(t)}{s}$. With an incident, arrival is delayed by Δ and the driver passes the bottleneck at $t_0^e + \frac{R^e(t)}{s} + \Delta$. The last term covers instances in which an incident occurs after $\tilde{t}$ but before the driver departs at t. If an incident occurs at time v, the driver arrives at $v + \Delta + \frac{R^e(t) - R^e(v)}{s}$ as explained above eqn. (14). Since there is no queue before v, the driver responsible for an incident at time v leaves home at time v. The probability that this driver causes an accident is $f(R^e(v))$, and the rate at which drivers are departing at time v is $\rho(v)$. This explains the integrand of the last term.

obtains

$$\beta(t) - (1 - F(R^e(t)))\gamma(t) - f(R^e(t))\rho^e(t)\int_{v=t}^{t+\Delta}\gamma(v)\,dv$$
$$-\frac{\rho^e(t)}{s}F\left(R^e\left(\tilde{t}\right)\right)\gamma\left(t_0^e + \frac{R^e(t)}{s} + \Delta\right)$$
$$-\frac{\rho^e(t)}{s}\int_{v=\tilde{t}}^{t}\rho(v)\,f(R^e(v))\,\gamma\left(v + \Delta + \frac{R^e(t) - R^e(v)}{s}\right)dv = 0.$$

Collecting terms in $\rho^e(t)$ yields

(15)

$$\frac{\rho^e(t)}{s} = \frac{\beta(t) - (1 - F(R^e(t)))\gamma(t)}{\left[\begin{array}{c} sf(R^e(t))\int_{v=t}^{t+\Delta}\gamma(v)\,dv + F\left(R^e\left(\tilde{t}\right)\right)\gamma\left(t_0^e + \frac{R^e(t)}{s} + \Delta\right) \\ + \int_{v=\tilde{t}}^{t}\rho(v)\,f(R^e(v))\,\gamma\left(v + \Delta + \frac{R^e(t)-R^e(v)}{s}\right)dv \end{array}\right]}.$$

The denominator of (15) is strictly positive. The numerator must be positive until $t = t_N^e$, and nonpositive thereafter. Thus, the departure rate drops to zero at t_N^e which is defined by the condition

$$\beta\left(t_N^e\right) = (1 - F(N))\,\gamma\left(t_N^e\right). \tag{16}$$

Condition (5) therefore holds as an equality when departures are dispersed. The UE with dispersed departures can be solved numerically using the following iterative procedure:

1. Guess t_0^e.
2. Integrate eqn. (7) from $t = t_0^e$ to $t = \tilde{t}$, where $\tilde{t}$ is defined by the condition $R\left(\tilde{t}\right) = s\left(\tilde{t} - t_0^e\right)$.
3. Integrate eqn. (15) from $t = \tilde{t}$ to $t = t_N^e$ where t_N^e is defined by condition (16).
4. If $R^e\left(t_N^e\right)$ matches N within a tolerance limit, then stop. Otherwise return to step 1.

5 The social optimum with incidents

5.1 Preliminary results

The SO with incidents maximizes total expected utility:

$$E(U) = \int_{t \in T^w} \rho(t) E(u|t) \, dt, \tag{17}$$

where T^w is the set of SO departure times. The SO departure rate, $\rho^w(t)$, maximizes $E(U)$ subject to the feasibility constraints $\rho^w(t) \geq 0$ and $R^w(t_N^w) = N$. Before tackling this optimal control problem some general properties of $\rho^w(t)$ will be deduced.

Lemma 3 *(a): The SO departure rate never exceeds capacity: $\rho^w(t) \leq s$. (b): $t^* \in int(T^w)$.*

Lemma 3 is proved in Appendix B.1. Part (a) is obvious: exceeding capacity would cause queuing on Good days without giving any driver extra time at home or work. Part (b) is also intuitive: to maximize total scheduling utility at home and work, the first driver must depart when home time is more valuable, and the last driver must depart when work time has greater value.

Lemma 4 *In the SO, the last departure time, t_N^w, is such that*

$$F(N) \leq 1 - \frac{\beta(t_N^w)}{\gamma(t_N^w)}. \tag{18}$$

Proof. If the last driver is rescheduled to depart slightly after t_N^w, other drivers are unaffected and the change in total expected utility is limited to the last driver. The proof of Lemma 2 therefore applies to Lemma 4. ■

Condition (18) on t_N^w has the same functional form as condition (5) on t_N^e. This congruence will be used later to compare the timing of departures in the SO and UE.

Lemmas 3 and 4 establish some bounds on the rate and timing of departures in the SO, but a number of questions remain. Should $\rho^w(t)$ ever be reduced below capacity? If so, how does $\rho^w(t)$ vary over time thereafter? Is $\rho^w(t)$ a continuous function? Is it ever optimal to reduce $\rho^w(t)$ low enough and long enough to eliminate queuing for at least some incident states? Various behaviors are possible. For example, suppose an early incident is likely, the incident risk then declines, and Δ is large. The model is then similar to the daily-shocks model for which $\rho^w(t)$ is weakly increasing over time (Lindsey, 1994). If, alternatively, an early incident is likely, but Δ is small, it is optimal to hold $\rho^w(t)$ below capacity long enough to clear any queue from the probable, but short-lived incident. A third possibility is that incident risk is increasing with n. In this case it may be prudent to accelerate departures in order to induce incidents earlier, and allow them to be cleared away sooner so that most drivers do not arrive inordinately late.

Given the wide range of possible solutions, the SO is difficult to analyze in full generality. Technical obstacles also arise. These can be circumvented by reformulating the optimization problem (see below), but the solution is tedious. Consequently, attention will be focused on the optimal timing of departures and on whether the departure rate should ever be reduced below capacity.

5.2 Optimal control formulation

Three technical difficulties arise if optimal control theory[18] is applied to maximize (17) with respect to $\rho^w(t)$. First, the Hamiltonian depends on lagged values of ρ^w. This problem is partly overcome by using the index or position of drivers in the departure schedule, n, as the running variable rather than t. The control variable becomes the time headway between successive drivers rather than the departure rate. Second, the Hamiltonian depends on lagged values of the state variable R. This problem is circumvented by replacing R with a set of state-contingent queuing times.[19] Third, the equation of motion for queuing time $q(t)$ is not differentiable at $q = 0$. This problem is addressed by imposing a nonnegativity constraint on queuing time that binds during time intervals when $\rho^w(t) = s$.

The optimal solution of the reformulated problem is assumed to comprise two stages. In Stage 1, which includes drivers $n \in [0, \hat{n}]$, headway – denoted by h – is maintained at $h(n) = 1/s$. This is equivalent to holding the departure rate at capacity. In Stage 2, which encompasses the remaining drivers $n \in (\hat{n}, N]$, headway is increased above $1/s$. The optimal value of $\hat{n}$ is solved as described below. If $\hat{n} = N$, Stage 2 is degenerate and the departure rate is held at capacity throughout the travel period. On Good days, all drivers travel within a time span of N/s, and the departure schedule is compressed in the same sense as it is compressed in the UE. If $\hat{n} = 0$, Stage 1 is degenerate and the departure rate is held below capacity throughout.

Total expected utility for the two stages combined is:

$$E(U) = \int_{n=0}^{N} \left[\begin{array}{c} (1 - F(n)) U(t(n), t(n) + q_+(n)) \\ + \int_{\xi=0}^{n} f(\xi) U(t(n), t(n) + q_\xi(n)) d\xi \end{array} \right] dn, \tag{19}$$

where $t(n)$ is departure time for driver n, $q_+(n)$ is queuing time experienced by

[18] Optimal control methods are described in Kamien and Schwartz (1981, Part II) and Leonard and Van Long (1992, Chap. 6).

[19] Lindsey (1994) also uses this approach.

driver n if no incident has occurred, and $q_\xi(n)$ is queuing time experienced by driver n if driver $\xi \leq n$ has caused an incident. The equations of motion for $q_+(n)$ and $q_\xi(n)$ are:

$$\frac{dq_+(n)}{dn} = \left\{ \begin{array}{c} \frac{1}{s} - h(n) \text{ if } q_+(n) > 0, \text{ or } q_+(n) = 0 \text{ and } h(n) < \frac{1}{s} \\ 0 \quad \text{otherwise} \end{array} \right\},$$

$$\frac{dq_\xi(n)}{dn} = \left\{ \begin{array}{c} \frac{1}{s} - h(n) \text{ if } q_\xi(n) > 0, \text{ or } q_\xi(n) = 0 \text{ and } h(n) < \frac{1}{s} \\ 0 \quad \text{otherwise} \end{array} \right\}, \xi < n.$$

In Stage 1, the constraint $q_+(n) \geq 0$ is binding. Given the persistent-queue assumption, the nonnegativity constraint $q_\xi(n) \geq 0$ can be ignored in both stages.

5.2.1 Stage 1: Departure rate held at capacity

The equations of motion and constraints for Stage 1 are:

(20) $$\frac{dt(n)}{dn} = h(n) \quad \text{(costate variable } \mu_1(n) \leq 0\text{)},$$

(21) $$q_+(n) \geq 0 \quad \text{(multiplier } \Psi(n) \geq 0\text{)},$$

(22) $$\frac{dq_+(n)}{dn} = \frac{1}{s} - h(n) \quad \text{(costate variable } \lambda_+(n) \leq 0\text{)},$$

(23) $$\frac{dq_\xi(n)}{dn} = \frac{1}{s} - h(n), \ \xi < n \ \text{(costate variable } \lambda_\xi(n) \leq 0\text{)}.$$

Equation (20) stipulates that departure time, a state variable, increases at a rate equal to the headway between successive drivers. Costate variable μ_1 reflects the benefit of occupying or "using up" departure time slots. Because departure time slots are valuable in the interior of T^w, $\mu_1 < 0$. Initial conditions are:

(24) $$t(0) \quad \text{free},$$

(25) $$\mu_1(0) = 0,$$

$$q_{+}(0) = 0, \tag{26}$$

$$q_{\xi}(\xi) = \Delta, \xi \in [0, N]. \tag{27}$$

The departure time for the first driver is chosen freely as per Condition (24). Costate variable μ_1 is therefore zero for the first driver as per Condition (25). Queuing time on Good days is initially zero as per Condition (26). If an incident occurs, queuing time jumps from 0 to Δ as per Condition (27).

The Hamiltonian is[20]

$$\begin{aligned}\Omega &= (1 - F(n)) U(t(n), t(n) + q_{+}(n)) + \int_{\xi=0}^{n} f(\xi) U(t(n), t(n) + q_{\xi}(n)) d\xi \\ &+\mu_1(n) h(n) + (1 - F(n)) \Psi(n) q_{+}(n) + (1 - F(n)) \lambda_{+}(n) \left(\frac{1}{s} - h(n)\right) \\ &+ \int_{\xi=0}^{n} f(\xi) \lambda_{\xi}(n) d\xi \left(\frac{1}{s} - h(n)\right).\end{aligned} \tag{28}$$

Optimality conditions are[21]

$$\frac{\partial\Omega}{\partial h} = \underbrace{\mu_1(n)}_{(1)} - \underbrace{(1 - F(n)) \lambda_{+}(n)}_{(2)} - \underbrace{\int_{\xi=0}^{n} f(\xi) \lambda_{\xi}(n) d\xi}_{(3)} = 0, \tag{29}$$

$$\frac{\partial \mu_1(n)}{\partial n} = -\frac{\partial H}{\partial t(n)} = -\left\{\begin{array}{c} \underbrace{\beta(t(n))}_{(1)} - \underbrace{(1 - F(n)) \gamma(t(n) + q_{+}(n))}_{(2)} \\ -\underbrace{\int_{\xi=0}^{n} f(\xi) \gamma(t(n) + q_{\xi}(n)) d\xi}_{(3)} \end{array}\right\}, \tag{30}$$

$$\frac{\partial \lambda_{+}(n)}{\partial n} = -\frac{\partial H}{\partial q_{+}(n)} = \gamma(t(n) + q_{+}(n)) - \Psi(n), \tag{31}$$

[20] In (28) the multiplier $\Psi(n)$ and the costate variables $\lambda_{+}(n)$ and $\lambda_{\xi}(n)$ are multiplied by their respective probability and probability densities. This facilitates interpretation of the optimality conditions.

[21] Second-order conditions for an optimum are assumed to hold.

$$\frac{\partial \lambda_\xi(n)}{\partial n} = -\frac{\partial H}{\partial q_\xi(n)} = \gamma(t(n) + q_\xi(n)), \; \xi < n. \tag{32}$$

Equation (29) identifies the net benefit from marginally increasing the headway for driver n. Term (1) is the opportunity cost of allocating more departure time to driver n. Term (2) is the expected benefit from reducing queuing time when no incident has occurred. Similarly, Term (3) is the expected benefit from reducing queuing time when an incident has occurred. At the optimum, the opportunity cost matches the expected benefits. Equation (30) describes the evolution of $\mu_1(n)$, where $\mu_1(n) < 0$ corresponds to the disbenefit of using up departure time slots. The term in braces is the rate of change in driver n's expected utility as the driver's departure time increases. Term (1) is driver n's utility from staying longer at home. Term (2) is driver n's expected loss of work-time utility if no incident has occurred, and term (3) is the corresponding loss if an incident has occurred. Finally, equations (31) and (32) describe the evolution of the costate variables, $\lambda_+(n)$ and $\lambda_\xi(n)$, which specify the shadow benefit of queuing time, and thus are negative. Equation (31) governs the disbenefit of queuing time when there is no incident. This disbenefit declines over time as fewer drivers remain who will arrive late. Equation (32) is interpreted similarly.

5.2.2 Solution with compressed departures

Stage 1 can prevail for none, some, or all of the travel period. If it prevails for all of it, the SO is compressed. The departure rate is held at capacity, and first-order condition (29) is not needed to derive the solution. Total expected utility is given by (19) with $q_+(n) = 0$, $t(n) = t_0^w + n/s$, and $t(n) + q_\xi(n) = t(\xi) + \Delta + (n-\xi)/s = t_0^w + n/s + \Delta$. Using (1), this yields

$$E(U) = \int_{n=0}^{N} \left[\begin{array}{c} \int_{v=t_H}^{t_0^w+n/s} \beta(v)\,dv + \int_{v=t_0^w+n/s}^{t_W} \gamma(v)\,dv \\ -F(n) \int_{v=t_0^w+n/s}^{t_0^w+n/s+\Delta} \gamma(v)\,dv \end{array} \right] dn.$$

The first-order condition for t_0^w is

$$\frac{\partial E(U)}{\partial t_0^w} = \int_{n=0}^{N} \left[\begin{array}{c} \beta(t_0^w + n/s) - \gamma(t_0^w + n/s) \\ -F(n)\left(\gamma(t_0^w + n/s + \Delta) - \gamma(t_0^w + n/s)\right) \end{array} \right] dn = 0,$$

or

$$\int_{n=0}^{N}\left(\beta\left(t_0^w+n/s\right)-\gamma\left(t_0^w+n/s\right)\right)dn=$$
$$\int_{n=0}^{N}F\left(n\right)\left(\gamma\left(t_0^w+n/s+\Delta\right)-\gamma\left(t_0^w+n/s\right)\right)dn. \tag{33}$$

Using integration by parts, the right-hand side of (33) can be written

$$\begin{aligned}
& F\left(n\right)\int_{t=t_0^w+n/s}^{t_0^w+n/s+\Delta}\gamma\left(v\right)dv\Bigg|_{n=0}^{N}-\int_{n=0}^{N}f\left(n\right)\int_{t=t_0^w+n/s}^{t_0^w+n/s+\Delta}\gamma\left(v\right)dtdn \\
= \;& F\left(N\right)\int_{t=t_N^w}^{t_N^w+\Delta}\gamma\left(t\right)dt-\int_{n=0}^{N}f\left(n\right)\int_{t=t_0^w+n/s}^{t_0^w+n/s+\Delta}\gamma\left(t\right)dtdn.
\end{aligned}$$

Changing the variable of integration from n to $t=t_0^w+n/s$ yields, finally,

$$\int_{t=t_0^w}^{t_N^w}\left(\beta\left(t\right)-\gamma\left(t\right)\right)dt= \tag{34}$$
$$F\left(N\right)\int_{t=t_N^w}^{t_N^w+\Delta}\gamma\left(t\right)dt-s\int_{t=t_0^w}^{t_N^w}f\left(s\left(t-t_0^w\right)\right)\left(\int_{v=t}^{t+\Delta}\gamma\left(v\right)dv\right)dt.$$

Holding $F\left(N\right)$ fixed, the right-hand side of (34) is a decreasing function of $f\left(\cdot\right)$ on the interval $\left(0,N\right)$. Departures in the SO therefore begin later the higher the incident risk for any given incident probability. The right-hand side of (34) is an increasing function of Δ so that, similar to the UE, departures begin earlier the longer incidents last.

To determine whether the SO is indeed compressed, t_N^w can be solved with (34) and substituted into Condition (18).[22] If the SO and UE are both compressed it is possible to compare their trip timing and welfare as is done in the following theorem.

Theorem 2 *Assume departures are compressed in both UE and SO. Then (a): Departures begin later in the SO than in the UE, but earlier than in the model without incidents: $t_0^e<t_0^w<t_0^{e0}=t_0^{w0}$. (b): The SO can be implemented using a time-dependent toll. (c): If the toll is constrained to be non-negative, drivers are strictly worse off than in the UE if they do not benefit from the toll revenues.*

[22] Condition (18) can be derived using the optimal control formulation in this subsection by substituting $\mu_1\left(N\right)=0$, $\Psi\left(N\right)q_+\left(N\right)=0$, $h\left(N\right)=\frac{1}{s}$, and $t\left(N\right)+q_\xi\left(N\right)=t_0^w+N/s+\Delta$ into the Hamiltonian (28), differentiating it with respect to $t\left(N\right)$, and evaluating the derivative at $t\left(N\right)=t_N^w$.

Theorem 2 is proved in Appendix B.2. Part (a) of Theorem 2 indicates that departures begin and end later in the SO than UE if both are compressed.[23] To see why, note that in the UE the first and last drivers gain the same expected utility or, equivalently, incur the same expected private travel cost. The first driver is therefore indifferent between continuing to depart first, and switching to depart last. However, by switching to last the driver no longer imposes an accident risk on other drivers. Starting from the UE departure schedule this switch is therefore socially desirable at the margin, and the SO therefore has a later departure schedule than the user equilibrium. Looked at another way, the last driver does not impose an accident externality because there are no subsequent drivers to delay. However, the last driver is concerned about being delayed by an incident caused by one of the earlier drivers. By contrast, the first driver is not concerned about being delayed, but does impose an externality on all the other drivers. Individuals are therefore biased towards departing too early. In this respect, the UE is more sensitive than the SO to incident risks.

Part (b) of Theorem 2 asserts that, as in the model without incidents, the SO departure schedule can be decentralized with a time-varying toll. Part (c) states that the toll makes drivers worse off. This is because the toll must be higher at the beginning of the departure period than at the end so that drivers delay departure.[24] Since negative tolls are ruled out by assumption, the first driver has to pay a positive toll.

As noted above, Conditions (5) and (18) have the same functional form. Since $t_N^w > t_N^e$ if the UE and SO are both compressed, Condition (5) is more stringent than Condition (18). It is therefore possible for Condition (5) to fail for a candidate compressed UE so that the UE is dispersed, but for Condition (18) to hold so that the SO is compressed.[25] The results of Theorem 2 still apply in this case as formalized in the following theorem.

Theorem 3 *Assume departures are dispersed in the UE but compressed in the SO. Then (a): Departures begin and end later in the SO than in the UE, but earlier than in the model without incidents: $t_0^e < t_0^w < t_0^{e0} = t_0^{w0}$. (b): The SO can be implemented using a time-dependent toll. (c): If the toll is constrained to be non-negative, drivers are strictly worse off than in the UE if they do not benefit from the toll revenues.*

[23]This contrasts with the daily-shocks model in which the SO can begin earlier (Lindsey, 1994).

[24]This is unlike either the model without incidents or the daily-shocks model (see Lindsey (1994, Proposition 10)).

[25]This possibility contrasts with a property of (deterministic) flow-congestion models that departures are more spread out in the system optimum than user equilibrium as long as flow is not hypercongested. See Chu (1995), Small and Verhoef (2007, Section 4.1.2), and DePalma and Arnott (2012).

Theorem 3 is proved in Appendix B.3. The intuition underlying Theorem 3 is the same as for Theorem 2.

5.2.3 Stage 2: Departure rate held below capacity

The equations of motion and constraints for Stage 2 of the SO are similar to Stage 1:

$$\frac{dt\,(n)}{dn} = h\,(n) \qquad \text{(costate variable } \mu_2\,(n) \leq 0\text{)}, \tag{35}$$

$$\frac{dq_\xi\,(n)}{dn} = \frac{1}{s} - h\,(n)\,,\ \ \xi < n \ \ \text{(costate variable } \lambda_\xi\,(n) \leq 0\text{)}. \tag{36}$$

In Stage 2 the departure rate is held below capacity so that $q_+\,(n) = 0$. This variable and its associated multiplier, $\Psi\,(n)$, are thus omitted. Terminal conditions are:

$$\mu_2\,(N) = 0, \tag{37}$$

$$\lambda_\xi\,(N) = 0,\ \ \xi \in (0, N]\,. \tag{38}$$

Costate variable μ_2 for departure time is zero for the last driver, as per Condition (37), because departure time slots after $t\,(N)$ are available, but undesirable. The shadow value of queuing time is also zero for the last driver as per Condition (38) because there are no further drivers to be delayed by a queue.

The Hamiltonian for Stage 2 is

$$\begin{aligned} \Omega \;=\;& (1 - F\,(n))\,U\,(t\,(n)\,, t\,(n)) + \int_{\xi=0}^{n} f\,(\xi)\,U\,(t\,(n)\,, t\,(n) + q_\xi\,(n))\,d\xi \\ & + \mu_2\,(n)\,h\,(n) + \int_{\xi=0}^{n} f\,(\xi)\,\lambda_\xi\,(n)\,d\xi \left(\frac{1}{s} - h\,(n)\right). \end{aligned} \tag{39}$$

Optimality conditions are:

$$\frac{\partial\Omega}{\partial h} = \mu_2\,(n) - \int_{\xi=0}^{n} f\,(\xi)\,\lambda_\xi\,(n)\,d\xi = 0, \tag{40}$$

$$\frac{\partial \mu_2(n)}{\partial n} = -\frac{\partial H}{\partial t(n)} = -\left\{ \begin{array}{l} \beta(t(n)) - (1-F(n))\gamma(t(n)) \\ -\int_{\xi=0}^{n} f(\xi)\gamma(t(n)+q_\xi(n))\,d\xi \end{array} \right\}, \tag{41}$$

$$\frac{\partial \lambda_\xi(n)}{\partial n} = -\frac{\partial H}{\partial q_\xi(n)} = \gamma(t(n)+q_\xi(n)),\ \xi < n. \tag{42}$$

Equations (40) and (41) have similar interpretations to (29) and (30) for Stage 1. Integrating (42), and using terminal condition (38), gives an equation for $\lambda_\xi(n)$:

$$\lambda_\xi(n) = -\int_{v=n}^{N} \gamma\left(t(\xi)+\Delta+s^{-1}(v-\xi)\right)dv. \tag{43}$$

Differentiating (40) with respect to n, and matching the resulting expression for $\frac{\partial \mu_2(n)}{\partial n}$ with (41), one obtains:

$$\beta(t(n)) = (1-F(n))\gamma(t(n)) - f(n)\lambda_n(n),\ \ n\in[\hat{n},N). \tag{44}$$

Equation (44) characterizes the optimal departure time for driver n. Except for the last term, it has a similar interpretation to equation (6) for the UE. The left-hand side is the marginal benefit to driver n from delaying departure. The first term on the right-hand side is the expected marginal cost to driver n of delaying arrival at work when there is no incident. The second term on the right-hand side is the expected additional marginal external cost that driver n imposes on subsequent drivers. This cost arises because any incident will end later and hence cause subsequent drivers to arrive later and lose more time at work.

Given terminal condition (38), $\lambda_N(N) = 0$ and equation (44) simplifies with $n = N$ to

$$\beta(t^w(N)) = (1-F(N))\gamma(t^w(N)), \tag{45}$$

where superscript w is added to clarify comparison with the UE. Condition (18) therefore holds as an equality if departures are dispersed in the SO. Equation (45) for $t^w(N)$ is identical to equation (16) for t_N^e with dispersed departures in UE. Hence $t_N^w = t_N^e$: if departures are dispersed in both the SO and UE, then departures end at the same time. This contrasts with the case where both UE and SO are compressed in which SO departures begin and end later.

If driver ξ causes an incident, driver $n > \xi$ is delayed at the bottleneck until time

$$t(n) + q_\xi(n) = t(\xi) + \Delta + s^{-1}(n-\xi). \tag{46}$$

Using (46), and differentiating (44) with respect to n, one obtains (see Appendix B.4) an expression for the SO headway:

(47)
$$h(n) = \frac{f(n)\left(\begin{array}{c}\gamma(t(n)) + \gamma(t(n)+\Delta) \\ +s^{-1}\int_{v=n}^{N}\acute{\gamma}\left(t(n)+\Delta+s^{-1}(v-n)\right)dv\end{array}\right) + \lambda_n(n)\,\partial f(n)/\partial n}{\begin{array}{c}(1-F(n))\,\acute{\gamma}(t(n)) - \acute{\beta}(t(n)) \\ +f(n)\int_{v=n}^{N}\acute{\gamma}\left(t(n)+\Delta+s^{-1}(v-n)\right)dv\end{array}}.$$

The denominator of (47) is strictly positive. If $f(n) = \partial f(n)/\partial n = 0$, the numerator is zero which is inconsistent with the requirement that $h(n) > 1/s$ in Stage 2. Thus, Stage 2 can prevail for a given n only if incident risk $f(n)$ is sufficiently high and/or increasing. Optimal headway is an increasing function of $f(n)$ because queuing becomes more likely in the future, and increasing the headway reduces queuing time. In the last term of the numerator, $\lambda_n(n) < 0$ for $n < N$. Optimal headway is therefore smaller if incident risk is increasing. This is because accelerating departures induces incidents to occur and end sooner, thereby allowing later drivers to arrive with shorter delays.

As noted above, if departures are dispersed in the SO, then they are also dispersed in the UE. The SO and UE with dispersed departures are difficult to compare because the UE departure rate in (15) is very different in functional form from the departure rate implied by the SO headway in (47). It is easy to show that the SO can still be supported by a time-varying toll, but results analogous to those in parts (a) and (c) of Theorems 2 and 3 have eluded us.

5.2.4 Numerical solution method

The SO with dispersed departures cannot, in general, be solved analytically. To solve it numerically, eqn. (43) with $\xi = n$ can be substituted into (44) to obtain an equation for departure time during Stage 2, $t_2(n)$. The remaining unknowns, t_0^w and $\hat{n}$, are solved using two conditions. First, the departure time for driver $\hat{n}$ must be the same in Stage 1 and Stage 2:

$$t_1(\hat{n}) = t_0^w + \hat{n}/s = t_2(\hat{n}). \tag{48}$$

Second, the costate variables μ_1 and μ_2 must match at $\hat{n}$:

$$\mu_1(\hat{n}) = \mu_2(\hat{n}). \tag{49}$$

$\mu_1(\hat{n})$ is computed by integrating (30) forward with respect to n, starting at $n = 0$ with $\mu_1(0) = 0$. $\mu_2(\hat{n})$ is computed by integrating (41) backward with respect to n, starting at $n = N$ with $\mu_2(N) = 0$. (If Stage 2 is optimal throughout the departure period, then $\hat{n} = 0$ and eqns. (48) and (49) do not have an interior solution.) A notable feature of the solution with $\hat{n} > 0$ is that $h_2(\hat{n}) > 1/s$. Optimal headway is therefore discontinuous, and takes an upward jump at the transition point from Stage 1 to Stage 2.

6 A numerical example

An example is now presented to obtain further insights into the characteristics of the UE and SO, as well as to get a sense of the quantitative importance of incidents. Following Fosgerau and Engelson (2011), the scheduling utility functions for home and work are assumed to be linear: $\beta(t) = \beta_0 - \beta_1 t$ and $\gamma(t) = \gamma_0 + \gamma_1 t$, with β_0, β_1, γ_0 and γ_1 all strictly positive. Following Börjesson et al. (2012), this will be called the "slope" model.

As demonstrated in the previous two sections, both the UE and SO are sensitive to the shape of the incident risk function. A reasonable baseline assumption is that incident risk is constant and this is assumed for the example: $f(n) = f$, where f is a constant. Appendix B.5 gives partial analytical solutions for the UE and SO.

6.1 Calibration

Tseng and Verhoef (2008) derive non-parametric estimates of functions $\beta(t)$ and $\gamma(t)$ for morning commuting trips by car or public transport. Their estimates are approximated fairly well by the slope model. Linear regressions of $\beta(t)$ and $\gamma(t)$ were therefore performed using the seven observations for their mixed-logit estimates (see Table 3 of their paper). These produced estimates of β_1 = 8.86 €/hr^2 and γ_1= 25.42 €/hr^2. A complication with the estimates of β_0 and γ_0 is that time spent traveling is preferred to time spent at work for working more than about 10 min before t^*. Time spent traveling is also preferred to time spent at home shortly before t^*. Such preferences would induce very high or infinite (i.e., mass) departures in UE (see Arnott et al., 1990). To avoid this, parameters β_0 and γ_0 were set by trial and error to generate reasonable departure rates over the full travel period. The values chosen were $\beta_0 = \gamma_0 = 40$; with $\beta_0 = \gamma_0$ this implies $t^* = 0$. The parameterized utility functions used are therefore $\beta(t) = 40 - 8.86t$ and $\gamma(t) = 40 + 25.42t$.

There are relatively few studies of incident duration. For accidents that involve one lane closure, Golob et al. (1987) find a mean incident duration of about one hour.[26] Jones et al. (1991) obtain a similar mean of 55 min. Nam and Mannering (2000) obtain a much higher mean of 162.5 min for incidents in an Incident Response Team database that tend to be more severe than average. For the example here, a smaller value of 30 min was selected for parameter Δ. This compensates in a rough way for the assumption that incidents completely block the bottleneck which is often not the case for multi-lane highways. It is in line with Hall (1993) who assumes that durations are uniformly distributed between 5 and 60 min with a mean of 32.5 min. It is also consistent with Koster and Rietveld (2011) who assume that incidents last for 35 min and reduce capacity by 80 percent.

Parameters N and s were set to $N = 8,000$ and $s = 4,000$, and incident risk was set to $f = 0.2/N$ so that incident probability is $fN = 0.2$. This implies that an incident occurs once per work week. On average, half the commuters would pass the bottleneck before the incident occurred so that individual commuters would experience an incident-related delay on one day out of ten. The results of the example are reported in the next subsection. The following subsection examines a modified version of the example that is descriptive of an evening commute.

6.2 Results for the morning commute

In the example, departures are compressed in both the UE and SO.[27] The two solutions are compared in Table 1.

[26] For "rear-end and sideswipe collisions" the mean is 58 min, and for "hit-object, broadside, and 'other' types of collisions" the mean is 62 min (Table 7 of their paper).

[27] Condition (5) for the UE is satisfied for $fN < 0.4482$, and Condition (18) for the SO is satisfied for $fN < 0.4936$. The UE and SO are therefore both compressed unless incident probability is nearly 1/2.

		No incidents $fN = 0$		Incidents $fN = 0.2$		Effect of incident risk	
		UE	SO	UE	SO	UE	SO
1	First departure time [hr]	-1.00	-1.00	-1.101	-1.037	-0.101	-0.037
2	Last departure time [hr]	1.00	1.00	0.899	0.936	-0.101	-0.037
3	Expected average social trip cost [€]	17.14	5.71	20.78	8.43	3.64	2.72
4	Average social trip cost on Good days [€]	17.14	5.71	18.16	5.74	1.02	0.03
5	Average social trip cost on Bad days [€]	—	—	31.23	19.21	—	—
6	Row 5 - Row 4			13.07	13.47		

Table 1: Comparison of user equilibrium and social optimum (morning)

The UE begins about 6 min earlier than with no incident risk. The initial departure rate is 15,389 veh/hr: appreciably higher than the rate of 13,405 veh/hr without incidents.[28] Consistent with Theorem 2, the SO also begins earlier than without incidents but the time shift is much smaller than for the UE. Expected trip costs are measured by the loss of expected utility relative to an ideal in which bottleneck capacity is effectively infinite, incidents never occur, and all drivers can therefore travel from home to work simultaneously at t^*. Without incidents, the cost of a trip is €17.14 in the UE and €5.71 in the SO. SO cost is only one third as large as UE cost because, in the slope model, two thirds of trip costs in the UE are due to queuing time which is avoided in the SO. With incidents, expected trip cost increases by €3.64 to €20.78 in the UE, and by €2.72 to €8.43 in the SO. The proportional increase in expected cost is larger for the SO, and in this respect the SO is less effective than the UE at adapting to incidents. One reason is that departures begin later in the SO than the UE, so that incident-related delays impose a higher cost from late arrivals. Another is that the SO with compressed departures is designed to avoid queuing on Good days while maintaining full capacity utilization. The SO has no "margin of reserve", and it is therefore vulnerable to capacity breakdowns.

The increase in expected cost can be decomposed into an increase on Good days and an increase on Bad days. Row 4 of Table 1 shows that the cost increase on Good days is negligible for the SO (€0.03) but appreciable for the UE

[28]Introduction of a small incident risk invariably leads to an increase in the initial departure rate if $\beta_1 < 2\gamma_1$; see Appendix B.5.

(€1.02). This is because the shift toward earlier departures is more pronounced in the UE. On Bad days, expected trip costs are higher in the UE (€31.23) than the SO (€19.21). But the difference in costs between Good days and Bad days is actually slightly higher in the SO (€13.47) than the UE (€13.07). In the UE, drivers adjust their departure times in response to incident risk in order to reduce the costs of lateness on Bad days while sacrificing some utility on Good days. As explained above, the SO is less flexible because the rate of departures is fixed, and only the timing of the travel period is adjusted.

Figure 2 plots the total cost of an incident (measured relative to Good days) as a function of when the incident occurs.[29] The SO curve lies slightly above the UE curve at all times. Both curves decline monotonically because later incidents affect fewer drivers. However, both curves are concave because drivers who depart later are more adversely affected by incidents so that even late incidents impose a significant loss.

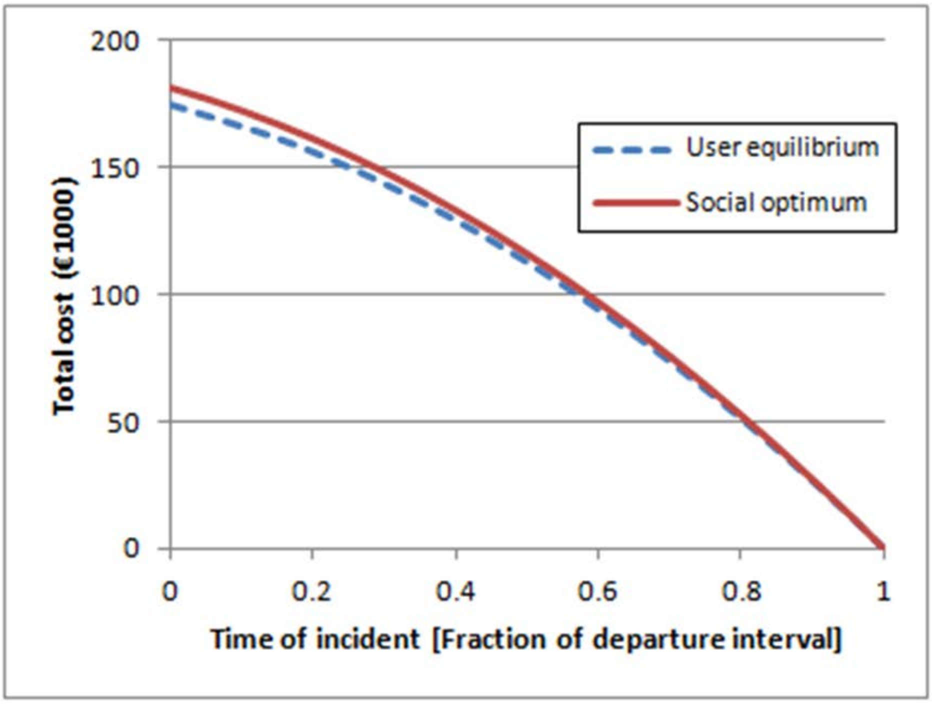

Figure 2: Total cost of incidents in user equilibrium and social optimum (morning)

[29] Since incidents occur as drivers pass the bottleneck, the probability density of incidents is uniform with respect to timing.

Figures 3 and 4 plot individual driver's trip costs in UE and SO (including toll) according to whether or not they encounter an incident. In each case the cost with no incident decreases with departure time while the cost with an incident increases. The increase in cost due to an incident grows from about €10 for the first driver to €35 for the last driver. Departing later is therefore riskier, and if drivers were risk-averse they would tend to depart in order of decreasing risk aversion.[30]

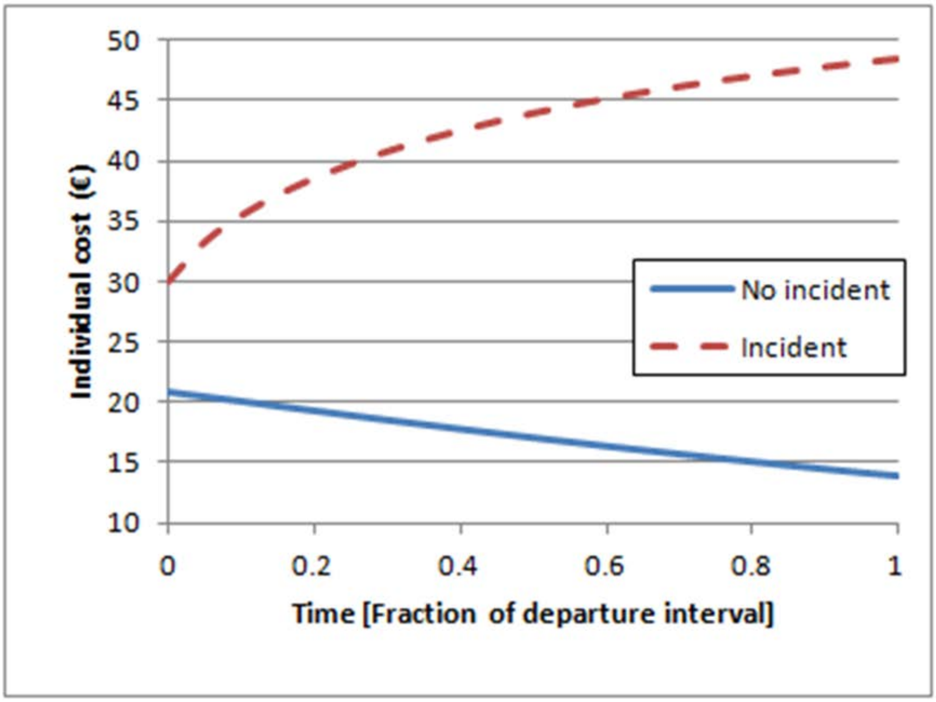

Figure 3: Individual cost of incident in user equilibrium (morning)

One way to reduce the costs of incidents is to reduce their frequency. Another is to reduce their duration. The effects of incident frequency and duration are easily determined in the model by varying parameters f and Δ. Figure 5 shows how incident probability affects expected costs per trip for the UE, the SO, and the SO including toll costs. In all three cases the relationship is nearly linear. The elasticities are respectively 1.009 for the UE, 0.9928 for the SO, and 0.9899 for the SO including toll. In the case of the SO the elasticity cannot exceed one; it would equal one if the departure schedule were held fixed independent of the incident

[30] de Palma et al. (2012) develop a model of route choice with drivers that differ in risk aversion, and show how the most risk-averse drivers select the safer route.

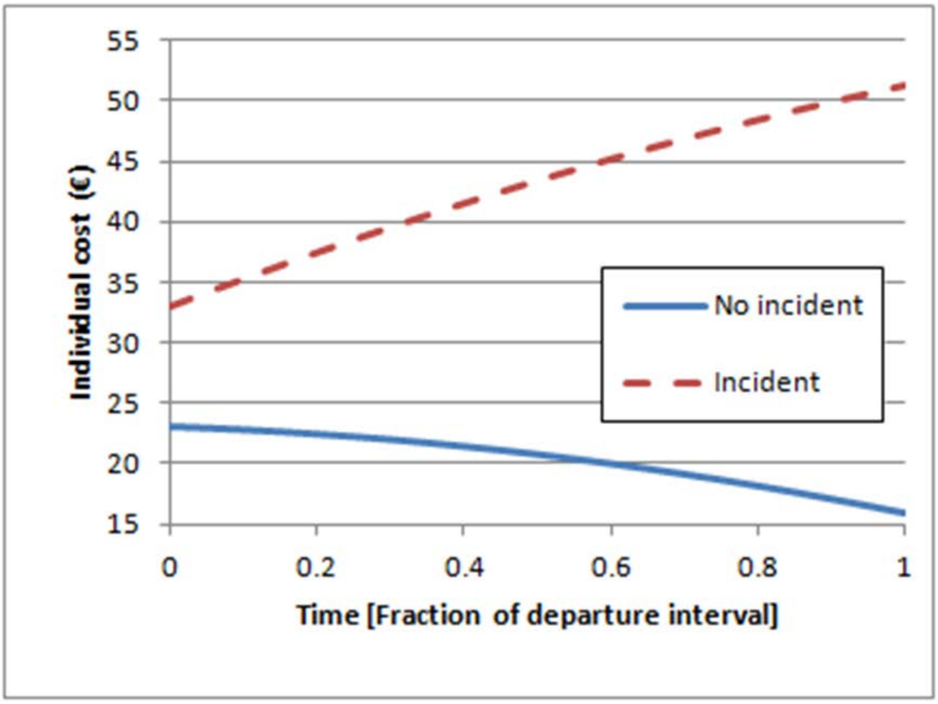

Figure 4: Individual cost of incident in social optimum with toll (morning)

probability, but this is generally not optimal. This reasoning does not apply to the UE, and the elasticity is slightly above one. Thus, drivers' uncoordinated response to incident risks is collectively inefficient.

Figure 6 presents analogous results for incident duration. In each case, expected cost increases more than in proportion to duration. The elasticities are 1.109 for the UE, 1.111 for the SO, and 1.100 for the SO including toll. This suggests that highest priority should be given to road links where major incidents are common. The duration of incidents that require assistance is equal to the sum of detection time, time required to dispatch emergency vehicles, travel time to the incident location, and service time (Hall, 2002). Incident duration can be reduced by expediting any of these stages. Carson et al. (1999) determine that even short reductions in incident response and clearance times can yield high benefit-to-cost ratios.

A notable feature of Figure 6 is how much the toll boosts the private costs of incidents borne by drivers in the SO. Figure 7 displays the toll schedule for three levels of incident duration as well as a case without incidents (effectively, incidents of zero duration). As incident duration increases, the toll schedule increases rapidly and it also advances slowly as departures begin earlier. With $\Delta = 1.5$ hr, the toll begins at €16.87 and rises to a maximum of €30.24 before declining smoothly to zero. Such a high toll is attributable to the combined effects of inelastic demand, relatively strong trip-timing preferences, and long-lasting incidents that shut capacity down completely. In practice, tolls are likely to be lower than this.

6.3 Results for the evening commute

To this point we have used the model to describe incidents that occur during the morning commute. Yet the model can be applied to other situations in which people wish to move from one location to another at the same time. An obvious instance is the evening commute. The evening commute has not been studied as extensively as the morning commute, either theoretically or empirically, but it is generally considered that work time constraints are a major – if not dominating – factor in determining when people leave work. In theoretical studies that have used the step model to describe the evening commute it is typically assumed that scheduling costs are determined by departure time (from work) rather than arrival time (at home), and that the unit cost of departing from work early is greater than the unit cost of departing late.[31] In the slope model the corresponding assumption is that $\beta_1 > \gamma_1$. This is what Börjesson et al. (2012) find using a stated preference

[31] See, for example, Fargier (1983), de Palma and Lindsey (2002), and Zhang et al. (2008).

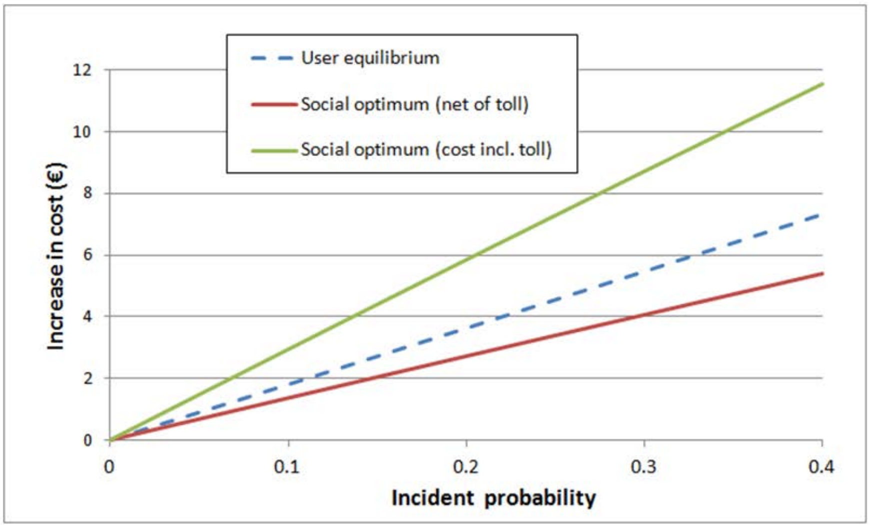

Figure 5: Effect of incident probability on incident costs (morning)

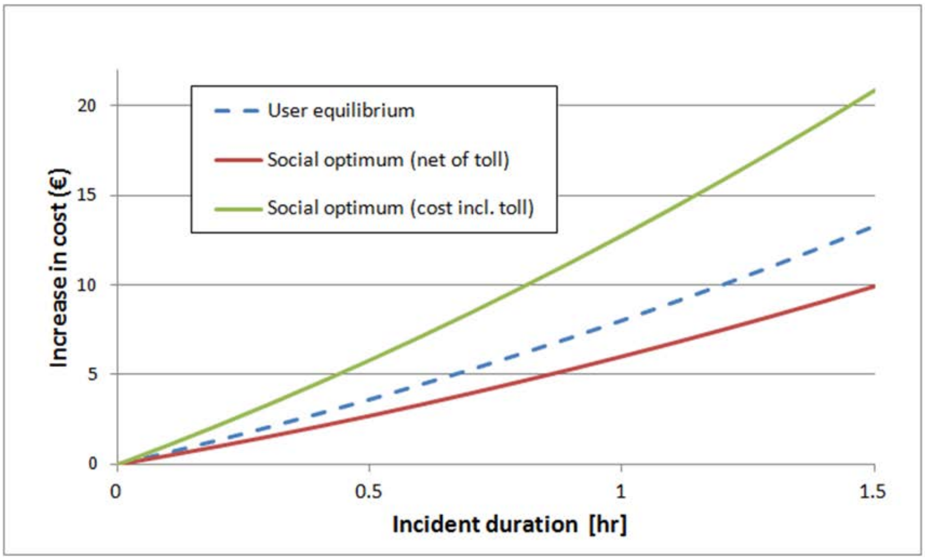

Figure 6: Effect of incident duration on incident costs (morning)

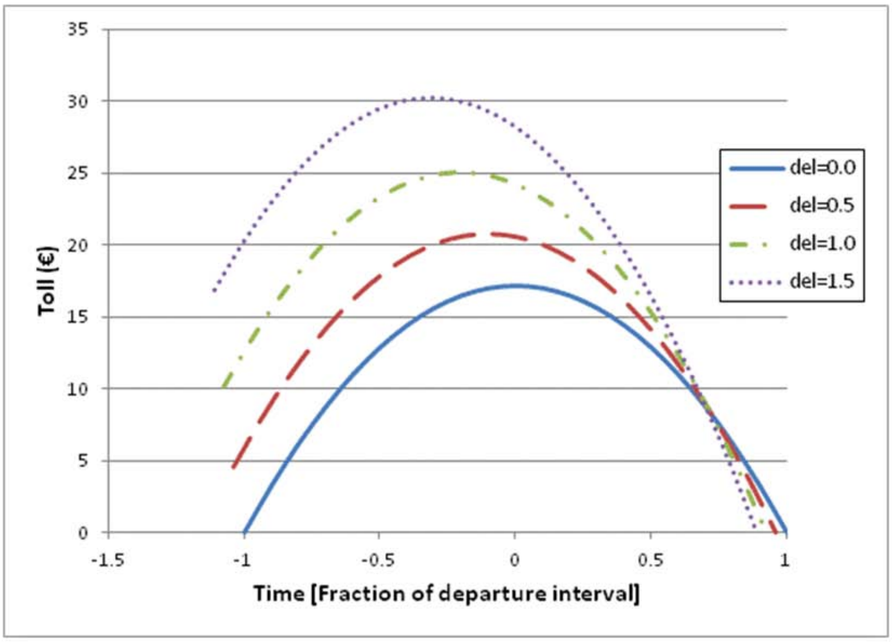

Figure 7: Effect of incident duration on socially optimal toll (morning)

data set for a sample of people traveling for a variety of purposes.[32] In contrast to Tseng and Verhoef (2008), their estimate of β_1 is more than three times as large as their estimate of γ_1 (see their Table 6).

We now briefly investigate how incident risk affects commuting behaviour in the evening commute, and compare it with the morning. To simplify the comparison, instead of adopting the Börjesson et al. (2012) estimates we simply interchange the estimates of β_1 and γ_1, and use $\beta_1 = 25.42$ €/hr^2 and $\gamma_1 = 8.86$ €/hr^2.[33] Other parameter values are kept the same as for the morning.

In the absence of incident risk, the evening commute is a mirror image of the morning commute for both UE and SO in terms of the departure period and trip costs. However, the initial departure rate is lower in the evening than the morning. Substituting parameter values into the formula $\rho\left(t_0^{e0}\right) = \frac{2\beta_0+\beta_1 N/s}{2\beta_0-\gamma_1 N/s}s$ one obtains an initial departure rate of 8,403 veh/hr for the evening compared to 13,405 veh/hr for the morning. Figure 1 offers an explanation for the difference. The scheduling utility functions drawn there depict a morning commute with $\gamma(\cdot)$ steeper than $\beta(\cdot)$ in the neighborhood of t^* (in the slope model the two functions are linear). For the evening commute, $\gamma(\cdot)$ is flatter than $\beta(\cdot)$. Area $hlmj$ is therefore taller than the area shown, and correspondingly narrower. Queuing time therefore grows more slowly. The intuition for this is that utility from time spent at home is relatively insensitive to time of day, and workers therefore have less to gain by delaying departure from work. Another difference from the morning is that introduction of a small incident risk leads to a reduction in the initial departure rate.[34]

Table 2 displays other properties of the evening commute corresponding to those for the morning commute shown in Table 1. Departures are again compressed in both the UE and SO.[35] Compared to the morning, departures in the UE and SO begin later. The UE begins about 4 min earlier than without incident risk compared to 6 min earlier for the morning. Incident risk has a smaller effect because, with a smaller value of γ_1, arriving late is less costly. For the same reason, average trip costs for UE and SO are lower on both Good days and Bad days than in the morning.

[32]Rather surprisingly, they find no significant differences in scheduling parameters for either morning and afternoon trips, or for different trip purposes.

[33] The ratio $\beta_1/\gamma_1 = 2.86$ is similar to the ratio of 3.26 that Börjesson et al. (2012) obtain.

[34]This happens if $2\beta_0\left(2\gamma_1 - \beta_1\right) + \beta_1\gamma_1\left(\frac{N}{s} - \Delta\right) < 0$, which is satisfied in the example; see Appendix B.5.

[35]Condition (5) for the UE is satisfied for $fN < 0.5778$, and Condition (18) for the SO is satisfied for $fN < 0.6763$.

		No incidents $fN = 0$		Incidents $fN = 0.2$		Effect of incident risk	
		UE	SO	UE	SO	UE	SO
1	First departure time [hr]	-1.00	-1.00	-1.074	-1.013	-0.074	-0.013
2	Last departure time [hr]	1.00	1.00	0.936	0.987	-0.074	-0.013
3	Expected average social trip cost [€]	17.14	5.71	19.75	7.97	2.61	2.26
4	Average social trip cost on Good days [€]	17.14	5.71	17.53	5.72	0.41	0.01
5	Average social trip cost on Bad days [€]	—	—	28.66	16.98	—	—
6	Row 5 - Row 4			11.13	11.26		

Table 2: Comparison of user equilibrium and social optimum (evening)

For the UE, expected trip cost increases by only €2.61 compare to €3.64 for the morning. For the SO, expected trip costs increases by €2.26 compared to €2.72 for the morning. Again, the proportional increase in expected cost is much larger for the SO than the UE for reasons similar to those for the morning.

Figure 8 plots the total cost of an incident as a function of when the incident occurs. Unlike for the morning shown in Figure 2, the SO curve crosses the UE curve twice rather than lying wholly above it. Both curves are still concave, but less so than for the morning because late incidents are not as costly when the penalty for arriving late at home is less severe.

Figures 9 and 10 plot individual driver's trip costs in UE and SO according to whether or not they encounter an incident. The two sets of curves are much flatter than in Figures 3 and 4 for the morning. For both UE and SO, the cost incurred when encountering an incident grows from about €16 for the first driver to €25 for the last driver. Departing later is therefore riskier, but much less so than for the morning commute.

7 Incidents with partial reductions in bottleneck capacity

Most incidents on multi-lane highways do not reduce capacity to zero.[36] Suppose an incident reduces bottleneck capacity from s to $k \in [0, s)$. Call an incident

[36] See *Highway Capacity Manual 2000* (2000, Chapter 22, Table 22-6).

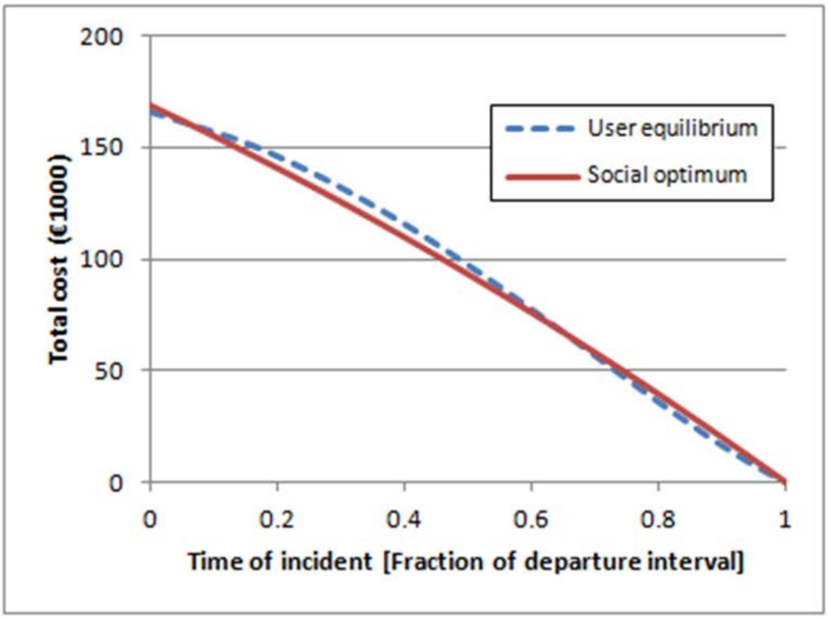

Figure 8: Total cost of incidents in user equilibrium and social optimum (evening)

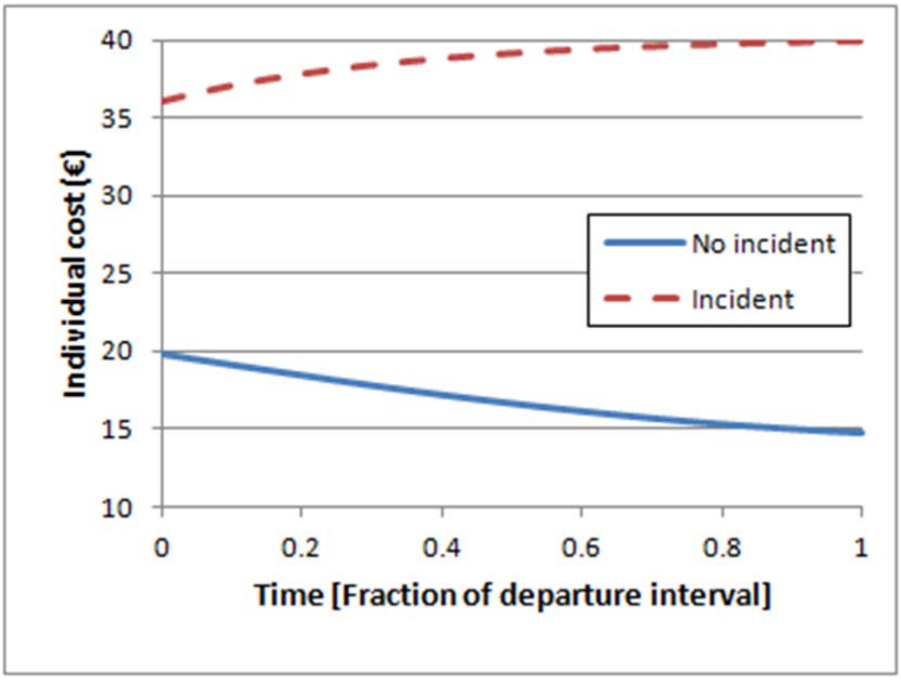

Figure 9: Individual cost of incident in user equilibrium (evening)

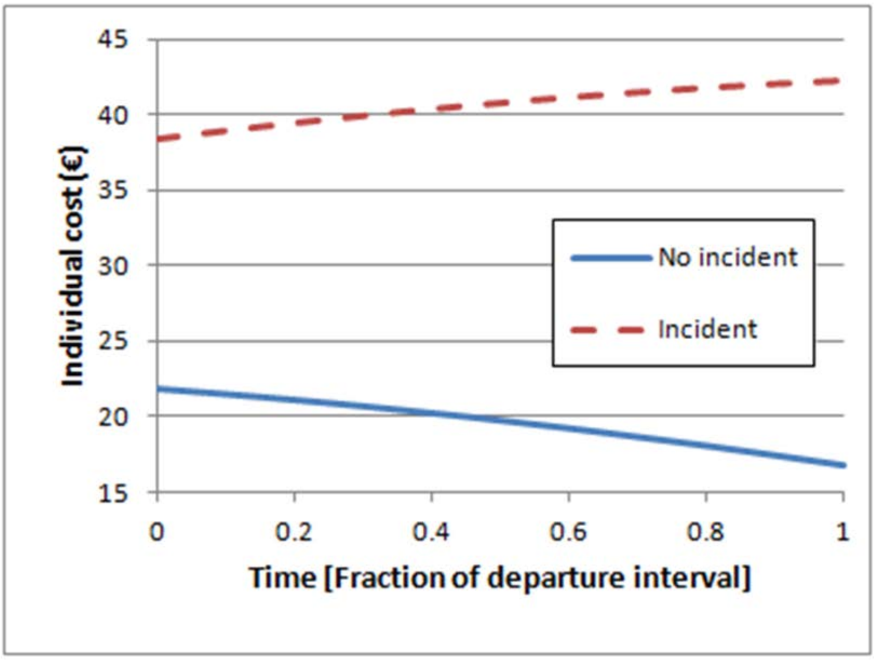

Figure 10: Individual cost of incident in social optimum with toll (evening)

"major" if $k = 0$ (as in the basic model), and "minor" if $k > 0$. Minor incidents are more complicated to analyze than major incidents, and some properties of major incidents do not extend to minor incidents. Consequently, the analysis of minor incidents here is brief.

7.1 User equilibrium

Lemmas 1 and 2 describing UE for major incidents also apply to minor incidents.[37] However, the delay due to a minor incident depends on when it occurs and it is necessary to consider the amount of time elapsed between an incident and when a driver departs. Figure 11 depicts the various possibilities. A driver departing in the interval $t \in [t_0, R^{-1}(k\Delta)]$ encounters one of two cases: (1) no incident has yet occurred, or (2) an incident has occurred that ends after the driver arrives at work. A driver departing in the interval $t \in [R^{-1}(k\Delta), t_0 + \Delta]$ can encounter a third case: (3) an incident has occurred that ends before the driver arrives at work. Finally, a driver departing after $t_0 + \Delta$ can encounter a fourth case: (4) an incident has ended before the driver departs. Case 2 can temporarily "disappear" either before or after $t_0 + \Delta$. (Figure 11 shows an example in which Case 2 disappears after $t_0 + \Delta$.) However, Case 2 necessarily "reappears" before the last driver departs at t_N. The delay due to an incident differs between the cases; formulas are derived in Appendix B.6.

The first departure in UE is determined (see Appendix B.6) by the condition

$$\text{(50)} \qquad \int_{v=t_0^e}^{t_N^e} (\beta(v) - \gamma(v))\, dv = F\left(R^e\left(t_N^e - \frac{k}{s}\Delta\right)\right) \int_{v=t_N^e}^{t_N^e + \frac{s-k}{s}\Delta} \gamma(v)\, dv$$
$$+ \int_{v=t_N^e - \frac{k}{s}\Delta}^{t_N^e} f(R^e(v))\, \rho^e(v) \int_{t=t_N^e}^{t_N^e + \frac{s-k}{s}(t_N^e - v)} \gamma(t)\, dt dv.$$

If $k = 0$, eqn. (50) simplifies to eqn. (8) for major incidents. Equation (50) involves lagged values of the departure rate, $\rho^e(v)$, for $v \in \left[t_N^e - \frac{k}{s}\Delta, t_N^e\right]$, and has to be solved numerically. The initial departure rate turns out to be as in eqn. (2): the same formula as without incidents. Since $t_0^e < t_0^{e0}$, $\rho^e(t_0^e) > \rho^{e0}(t_0^{e0})$. Thus, unlike for major incidents, the risk of minor incidents unambiguously raises the initial departure rate. This is because minor incidents do not eliminate capacity, and thus impose only a short delay on early drivers. The risk taken by postponing departure is therefore smaller, and the queue required to deter drivers from all departing later is correspondingly longer.

[37] The proof of Lemma 2 for major incidents is slightly modified; see Appendix B.6.

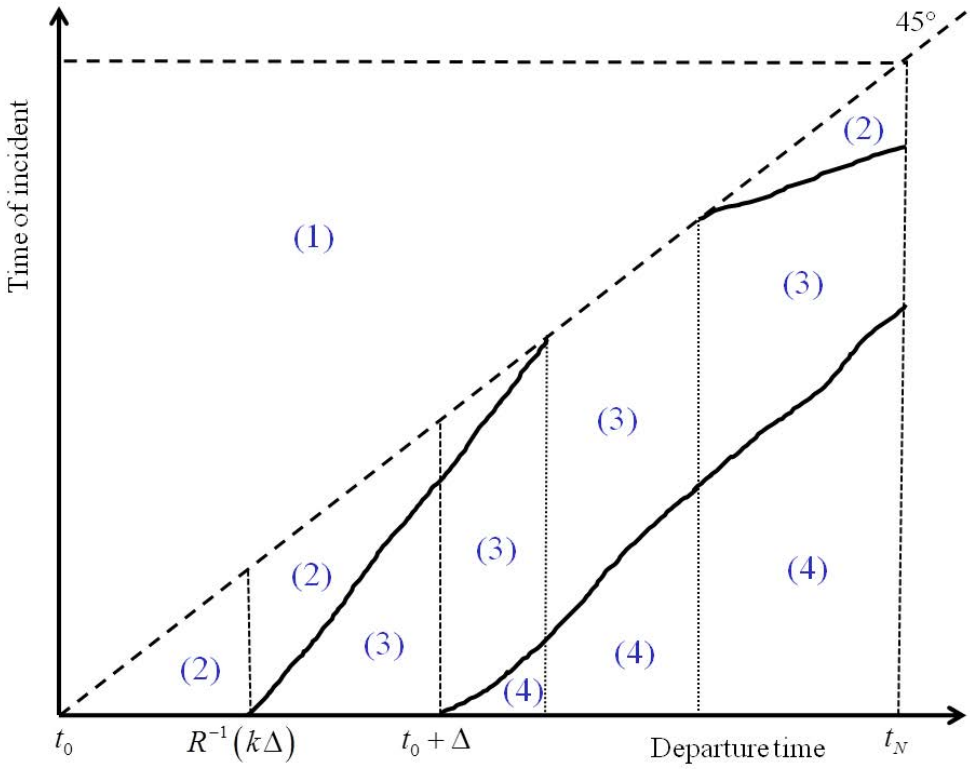

Figure 11: Timing of minor incidents (morning)

7.2 The social optimum

Lemma 3 for major incidents carries over to minor incidents and the proof is the same. However, the SO for minor incidents is tedious to derive since the SO departure rate depends on both lagged and leading values of itself. Attention is limited here to compressed departures. The SO departure period is derived in Appendix B.6 where it is shown that $\partial t_0^w / \partial k > 0$. The SO therefore begins earlier when incidents are more severe (i.e., k is smaller). We have been unable to rank t_0^w and t_0^e.

The UE and SO with compressed departures can be derived for the example in Section 6. Figure 12 plots for UE and SO of the morning commute the increase in expected trip cost due to incidents as the fraction of capacity lost in an incident (i.e., $1 - k/s$) increases from 0 to 1. The two curves rise smoothly, and at $k = 0$ they reach the values of €3.64 and €2.72 that apply for major incidents. Both curves are convex so that the expected cost of an incident is more than proportional to the capacity lost. This is broadly consistent with the finding in Section 6 that the cost of incidents is more than proportional to their duration.

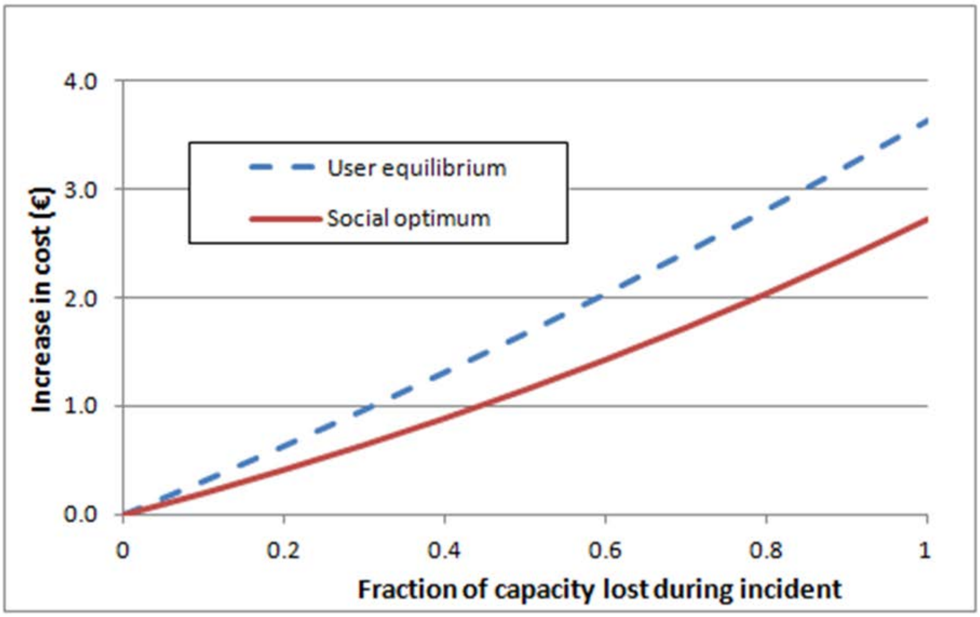

Figure 12: Increase in expected trip cost from minor incidents (morning)

8 Concluding remarks

Incidents are a major cause of traffic congestion in large urban areas. This paper uses the bottleneck model to analyze the effects of incidents on trip-timing decisions and trip costs. Incidents are assumed to be caused by individual drivers while they travel. Unlike in previous studies except for Schrage (2006), the timing of incidents is endogenous to traveler behavior. This enriches the realism and lessons derived from the model, but also adds complexity.

Several general results deserve highlighting. First, three biases are identified in the timing of departures in the UE relative to the SO. One is the familiar bias toward departing too quickly in the early stages of the travel period which leads to queuing on Good days in the UE, but not in the SO. The other two biases are driven by incident risk and they act in opposite directions. Drivers are biased toward departing too early because they do not want to be delayed by an incident and arrive seriously late. But they are also biased toward departing too late because they ignore the delays they will impose on subsequent drivers if they cause an incident. This last bias is missing from models in which incidents occur exogenously.

Second, if the probability of an incident is sufficiently high, then in both the UE and the SO bottleneck capacity is not fully used for the entire travel period on Good days when no incident occurs, and the departure pattern is "dispersed". For the SO this can be interpreted as a policy of maintaining reserve capacity in order to moderate the adverse effects of incidents on Bad days. A third result is that if departures are compressed in the SO, then drivers are worse off than in the UE if the SO is decentralized using a non-negative time-varying toll. This might aggravate resistance to congestion pricing and correspondingly strengthen arguments for using toll revenues in a way that benefits drivers.

Further lessons concerning how the probability, duration, and magnitude of incidents affect expected travel costs can be drawn from numerical examples. For the parameterized slope-model example we use, incidents that occur early in the travel period are more costly than later incidents because early incidents affect more travellers. The expected costs of incidents in both the UE and SO are approximately linear in the probability of incidents, but convex in their magnitude and duration. This suggests that, not surprisingly, priority for incident management should be given to shortening major and long-lived types of incidents.

There are many ways in which the paper could be extended. The analysis could be refined by relaxing the persistent-queue assumption. Heterogeneity in drivers' scheduling preferences could be introduced. The duration of incidents could be treated as a random variable. More than one incident per day could be allowed. The assumption that multiple incidents never occur is reasonable if incidents are rare. For the base case of the numerical example it is assumed that the probability of no incident is 0.8. Given the underlying hazard rate, more than one incident would occur with probability 0.0215, or about once every 45 days. Moreover, if a second incident occurs while the first is ongoing, the second incident has no effect until the first is cleared up. In the case of minor incidents, the effect of two concurrent incidents may be approximately equal to the impact of the one that is more severe (Hall, 2002).

Another extension is to treat travel demand as price sensitive. The number of trips taken would then differ for the user equilibrium and social optimum. It would also depend on the frequency, severity, and duration of incidents. By reducing the expected costs of travel, incident management schemes would stimulate more trips, and correspondingly more incidents, that would partially offset the policy benefits (Hall, 1993; Koster and Rietveld, 2011).

Another interesting extension is to assume that individuals are risk averse. Börjesson et al. (2012) find that scheduling models severely underestimate the disutility that individuals incur from travel time variability.[38] They offer two plausible explanations for this. One is that, contrary to what is assumed in scheduling models, people do not have a fixed preferred arrival time (t^* in the model here) because they can reschedule many activities given advance information about travel time durations. The other explanation is that people may dislike uncertainty itself because of the anxiety it creates, the costs incurred when decided when to depart, or the costs of formulating contingency plans. As Börjesson et al. (2012) put it, being 'delayed' can be worse than just being 'late'. Risk aversion could be introduced into the model in a crude way by adding to the expected utility function a term proportional to the standard deviation of travel time.

Incident risk could be assumed to depend on time of day or other circumstances. For instance, incident risk is higher at night (Varghese and Shankar, 2007) and in rain (FHWA Road Weather Management Program, 2009). A more radical step would be to reformulate the model using a flow congestion model as in Schrage (2006). One reason for doing so is that queuing congestion militates against maintaining spare capacity since it does not offer a smooth trade-off between higher capacity utilization on Good days and extra delay on Bad days. Such a trade-off does exist with flow congestion. Another reason is that both the frequency and severity of incidents per vehicle-km driven could depend on speed and/or the density of vehicles on the road (Shefer and Rietveld, 1997). The empirical evidence on this is limited and inconsistent.[39] A few studies have found that severe accidents are more common in light traffic. However, Wang et al. (2009) conclude that traffic congestion has no statistically significant effect on accident frequency on the M25 London orbital motorway. To the extent that incident rates

[38]This finding shows up in both the step model and the slope model they estimate.

[39]See, for example, Dickerson et al. (2000), Noland and Quddus (2005), and Small and Verhoef (2007, Section 3.4.6).

do vary with traffic conditions, the introduction of congestion tolls will affect not only congestion externalities but also incident/accident externalities (Dickerson et al., 2000).

A final extension worth noting is information about incidents. We have assumed that drivers know the risk function for incidents, $f(\cdot)$, but not whether an incident has occurred. It is true that pre-trip and en-route information about travel conditions are now available from various media. Nevertheless, incidents often occur after people have made their travel decisions and can alter them only with difficulty — if at all. Changes of route may be possible in urban areas with dense road networks, but drivers may be reluctant to switch route if alternatives are either circuitous or unfamiliar. For example, the shortest route from metro Vancouver to the US border is via the George Massey Tunnel on Highway 99. Long delays are frequently encountered at the tunnel. The closest alternative crossing is via the Alex Fraser Bridge on Highway 91 which is about 10 kilometres to the east. In settings such as these, travellers have little alternative to queuing once they are en route.

In other settings where travelers can respond to information, the effectiveness of the response will depend on a number of factors including the timeliness and precision of information, the fraction of travelers who have access to it, the proportion of them who choose to respond and how, whether alternative routes are congested, and so on. These and other design considerations for Advanced Traveler Information Systems have been extensively studied since the late 1980s.[40] The effects of information provision in an extended version of the current model are not easy to envisage.

[40] For reviews see Rietveld (2011) and Chorus and Timmermans (2011).

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A Notational Glossary

A.1 Latin characters

a : arrival time at work [clock time]
e: superscript denoting user equilibrium
$E\left(\right)$: expectations operator
f: probability density of an incident ("incident risk")
F: cumulative probability of an incident ("incident probability")
h: headway between successive departures [hr]
H: subscript denoting home
k : bottleneck capacity during incident ($k = 0$ for major incidents) [veh./hr]
n : index of drivers in order of departure
$\hat{n}$: value of n at which socially optimal departure rate drops below capacity
N : number of drivers
$q\left(t\right)$: queuing time [hr]
$q_{+}\left(n\right)$: queuing time in social optimum if no incident has occurred [hr]
$q_{\xi}\left(n\right)$: queuing time in social optimum if driver ξ causes an incident [hr]
R : cumulative departures [veh]
s : bottleneck capacity with no incident [veh./hr]
t : departure time from home [clock time]
t_0 : time when first driver departs from home [clock time]
t_H: time at which utility accounting begins (at home) [clock time]
t_N : time when last driver departs from home [clock time]
t_W: time at which utility accounting ends (at work) [clock time]
t^* : preferred time to transfer instantaneously from home to work [clock time]
T : set of departure times
u : scheduling utility
U : aggregate utility
v: clock time (used as variable for integration)
w: superscript denoting social optimum
W: subscript denoting work

A.2 Greek characters

β: rate of utility derived from time spent at home
β_0, β_1: parameters of linear $\beta\left(\right)$ function
γ: rate of utility derived from time spent at work
γ_0, γ_1: parameters of linear $\gamma\left(\right)$ function
Δ: duration of incident [hr]
$\lambda_{+}\left(n\right)$: costate variable for $q_{+}\left(n\right)$

$\lambda_\xi(n)$: costate variable for $q_\xi(n)$
μ_1: costate variable for departure time in Stage 1 of social optimum
μ_2: costate variable for departure time in Stage 2 of social optimum
ξ: index of driver who causes an incident
ρ: aggregate departure rate from origin [veh./hr]
τ: toll [€]
Ψ: multiplier on nonnegativity constraint $q_+(n) \geq 0$
Ω: Hamiltonian for social optimum

A.3 Other characters

0: superscript denoting model without incidents.

B Mathematical appendixes

B.1 Proof of Lemma 3

Part (a): Suppose $\rho^w(t) > s$ on some interval. Then $R^w(t)$ intersects a family of lines $s\left(t-\hat{t}\right)$ parameterized by $\hat{t}$. For any such $\hat{t}$, a queue exists for all $t \in \hat{T} \equiv \left\{t : R^w(t) > s\left(t-\hat{t}\right)\right\}$. Reducing $R^w(\cdot)$ to $s\left(t-\hat{t}\right)$ on $\hat{T}$ postpones departures from home for a cohort of drivers without delaying their arrivals at work. The cohort's utility increases strictly without affecting the utility of other drivers. Hence $\rho^w(t) \leq s$ on T^w.

Part (b), proof that $t_0^w < t^*$: Following the approach of Section 5.2, write total expected utility as:

$$E(U) = \int_{n=0}^{N} \left[\begin{array}{c} (1-F(n))\,U(t(n),t(n)) \\ +\int_{\xi=0}^{n} f(\xi)\,U(t(n),t(n)+q_\xi(n))\,d\xi \end{array} \right] dn,$$

where $t(n)$ is the departure time of driver n, and $q_\xi(n)$ is queuing time for driver n if driver $\xi < n$ causes an incident. Consider slightly advancing the departure times of all drivers by $dt > 0$. Total expected utility changes by

$$\begin{aligned} dE(U) &= dt \int_{n=0}^{N} \left[\begin{array}{c} (1-F(n))\left(\gamma(t(n)) - \beta(t(n))\right) \\ +\int_{\xi=0}^{n} f(\xi)\left(\gamma(t(n)+q_\xi(n)) - \beta(t(n))\right) d\xi \end{array} \right] dn \\ &= dt \int_{n=0}^{N} \left[\begin{array}{c} -\beta(t(n)) + (1-F(n))\,\gamma(t(n)) \\ +\int_{\xi=0}^{n} f(\xi)\,\gamma(t(n)+q_\xi(n))\,d\xi \end{array} \right] dn. \end{aligned}$$

If $t_0^w \geq t^*$, then $\beta(t(n)) \leq \gamma(t(n)) \leq \gamma(t(n)+q_\xi(n))$, $\forall n$, with $\beta(t(n)) < \gamma(t(n))$ for $n > 0$. Hence $dE(U) > 0$, and $t_0^w \geq t^*$ cannot be optimal.

Part (b), proof that $t_N^w > t^*$: Suppose $t_N^w \leq t^*$, and let the last driver depart dt later. The last driver's expected utility changes by $dE\left(u|\, t_N^w\right) = \left(\beta\left(t_N^w\right) - \left(1 - F\left(N\right)\right)\gamma\left(t_N^w\right)\right) dt$. Given $\beta\left(t_N^w\right) \geq \gamma\left(t_N^w\right)$ and $F\left(N\right) > 0$, $dE\left(u|\, t_N^w\right) > 0$, and $t_N^w \leq t^*$ cannot be optimal. ■

B.2 Proof of Theorem 2

Part (a): The right-hand side of (34) can be bounded above and below:

$$
\begin{aligned}
0 \quad &< \quad F\left(N\right)\int_{t=t_N^w}^{t_N^w+\Delta} \gamma\left(t\right) dt - s\int_{t=t_0^w}^{t_N^w} f\left(s\left(t - t_0^w\right)\right)\left(\int_{v=t}^{t+\Delta} \gamma\left(v\right) dv\right) dt \\
&< \quad F\left(N\right)\int_{t=t_N^w}^{t_N^w+\Delta} \gamma\left(t\right) dt.
\end{aligned}
$$

Comparison with equation (8) for t_0^e shows that $t_0^w \in \left(t_0^e, t_0^{e0}\right)$.

Part (b): If a toll implements the SO, expected utility net of the toll must be constant over the departure interval T^w and lower outside T^w. Such a toll has the form $\tau^w\left(t\right) = E\left(u|t\right) + c = \int_{v=t_H}^{t} \beta\left(v\right) dv + \int_{v=t}^{tw} \gamma\left(v\right) dv - F\left(n\right)\int_{v=t}^{t+\Delta} \gamma\left(v\right) dv + c$ where c is a constant.

Part (c): Expected utility when departing in the UE at t_N^e is given by (11): $E\left(u|t_N^e\right) = \int_{v=t_H}^{t_N^e} \beta\left(v\right) dv + \left(1 - F\left(N\right)\right)\int_{v=t_N^e}^{tw} \gamma\left(v\right) dv + F\left(N\right)\int_{v=t_N^e+\Delta}^{tw} \gamma\left(v\right) dv$. The same equation applies for $E\left(u|t_N^w\right)$ in the SO since, as in the UE, there is no queue on Good days, and the delay due to an incident does not depend on when it occurs. Therefore $E\left(u|t_N^w\right) - E\left(u|t_N^e\right) = \int_{v=t_N^e}^{t_N^w} \left(\beta\left(v\right) - \gamma\left(v\right)\right) dv + F\left(N\right)\left(\int_{v=t_N^e}^{t_N^w} \gamma\left(v\right) dv - \int_{v=t_N^e+\Delta}^{t_N^w+\Delta} \gamma\left(v\right) dv\right)$. The first term in this expression is negative since $t_N^w > t_N^e > t^*$. The second expression is also negative since $t_N^w > t_N^e$, $\Delta > 0$, and $\gamma\left(\cdot\right)$ is an increasing function. Expected utility gross of the toll is therefore lower in the SO than the UE. If the toll cannot be negative, drivers are therefore worse off in the SO if toll revenues are not used in a way that benefits them.

B.3 Proof of Theorem 3

Part (a): If UE departures are dispersed, then $\beta\left(t_N^e\right) = \left(1 - F\left(N\right)\right)\gamma\left(t_N^e\right)$ and $t_0^e < t_N^e - N/s$. If SO departures are compressed, then $\beta\left(t_N^w\right) < \left(1 - F\left(N\right)\right)\gamma\left(t_N^w\right)$ and $t_0^w = t_N^w - N/s$. Therefore, $t_0^w = t_N^w - N/s > t_N^e - N/s > t_0^e$. The proof that $t_0^w < t_0^{w0}$ is the same as for Theorem 2.

Part (b): The proof that the SO can be implemented using a time-dependent toll is the same as for Theorem 2.

Part (c): The proof is similar to the proof for Theorem 2. Expected utility when departing in the UE at t_N^e is given by (14):

$$
\begin{aligned}
E\left(u^e|t_N^e\right) &= \left(1-F\left(N\right)\right)u\left(t_N^e,t_N^e\right)\\
&+F\left(R^e\left(\tilde{t}\right)\right)u\left(t_N^e,t_0^e+\frac{N}{s}+\Delta\right)\\
&+\int_{v=\tilde{t}}^{t_N^e}\rho\left(v\right)f\left(R^e\left(v\right)\right)u\left(t_N^e,v+\Delta+\frac{N-R^e\left(v\right)}{s}\right)dv.
\end{aligned}
\tag{51}
$$

where a superscript e has been added to u since formulas for expected utility differ for the UE and SO. Integrating the last term of (51) by parts, and rearranging terms, one obtains

$$
\begin{aligned}
E\left(u^e|t_N^e\right) &= u\left(t_N^e,t_N^e\right)-F\left(N\right)\int_{v=t_N^e}^{t_N^e+\Delta}\gamma\left(v\right)dv\\
&+\int_{v=\tilde{t}}^{t_N^e}F\left(R^e\left(v\right)\right)\left(1-\frac{\rho^e\left(v\right)}{s}\right)\gamma\left(v+\Delta+\frac{N-R^e\left(v\right)}{s}\right)dv.
\end{aligned}
\tag{52}
$$

Expected utility when departing in the SO at t_N^w is

$$
E\left(u^w|t_N^w\right)=u\left(t_N^w,t_N^w\right)-F\left(N\right)\int_{v=t_N^w}^{t_N^w+\Delta}\gamma\left(v\right)dv. \tag{53}
$$

Subtracting (52) from (53):

$$
\begin{aligned}
E\left(u^w|t_N^w\right)-E\left(u^e|t_N^e\right) &= \int_{v=t_N^e}^{t_N^w}\left(\beta\left(v\right)-\gamma\left(v\right)\right)dv\\
&+F\left(N\right)\left(\int_{v=t_N^e}^{t_N^w}\gamma\left(v\right)dv-\int_{v=t_N^e+\Delta}^{t_N^w+\Delta}\gamma\left(v\right)dv\right)\\
&-\int_{v=\tilde{t}}^{t_N^e}F\left(R^e\left(v\right)\right)\left(1-\frac{\rho^e\left(v\right)}{s}\right)\gamma\left(v+\Delta+\frac{N-R^e\left(v\right)}{s}\right)dv.
\end{aligned}
$$

The first two terms in this expression are negative (see proof of Theorem 2). The third term is also negative since $\rho^e\left(v\right)<s$ for $v\in\left[\tilde{t},t_N^e\right]$. Expected utility net of the toll is therefore lower in the SO than UE. If the toll cannot be negative, and drivers do not benefit from toll revenues, drivers are worse off.

B.4 Optimal headway in Stage 2

Differentiating (44) totally with respect to n one obtains:

$$\begin{aligned} \acute{\beta}(t(n))\,h(n) &= (1-F(n))\,\acute{\gamma}(t(n))\,h(n) \\ &\quad -f(n)\,\gamma(t(n)) - f(n)\frac{\partial\lambda_n(n)}{\partial n} - \lambda_n(n)\frac{\partial f(n)}{\partial n}. \end{aligned} \tag{54}$$

Now

$$\frac{\partial\lambda_n(n)}{\partial n} = \left.\frac{\partial\lambda_\xi(n)}{\partial n}\right|_{\xi=n} + \left.\frac{\partial\lambda_\xi(n)}{\partial \xi}\right|_{\xi=n}.$$

Given (43),

$$\left.\frac{\partial\lambda_\xi(n)}{\partial n}\right|_{\xi=n} = \gamma(t(n)+\Delta), \tag{55}$$

and

$$\frac{\partial\lambda_\xi(n)}{\partial \xi} = -\int_{v=n}^{N}\acute{\gamma}\left(t(\xi)+\Delta+s^{-1}(v-\xi)\right)dv\left(h(\xi)-\frac{1}{s}\right),$$

so that

$$\begin{aligned} \left.\frac{\partial\lambda_\xi(n)}{\partial \xi}\right|_{\xi=n} &= -\int_{v=n}^{N}\acute{\gamma}\left(t(n)+\Delta+s^{-1}(v-n)\right)dv\left(h(n)-\frac{1}{s}\right) \\ &= \left(\frac{1}{s}-h(n)\right)\int_{v=n}^{N}\acute{\gamma}\left(t(n)+\Delta+s^{-1}(v-n)\right)dv. \end{aligned} \tag{56}$$

Adding (55) and (56) yields

$$\begin{aligned} \frac{\partial\lambda_n(n)}{\partial n} &= \gamma(t(n)+\Delta) \\ &\quad +\left(\frac{1}{s}-h(n)\right)\int_{v=n}^{N}\acute{\gamma}\left(t(n)+\Delta+s^{-1}(v-n)\right)dv. \end{aligned} \tag{57}$$

Substituting (57) into (54), and rearranging terms, yields equation (47) in the text for the optimal headway.

B.5 Analytical solution for linear example

B.5.1 User equilibrium

Given (8), the first departure in UE is at time

$$t_0^e = \frac{\beta_0 - \gamma_0 - (\beta_1 + \gamma_1)\frac{N}{2s} - fs\Delta\left(\gamma_0 + \gamma_1\left(\frac{N}{s} + \frac{\Delta}{2}\right)\right)}{\beta_1 + \gamma_1 + f\gamma_1\Delta s}.$$

From (9), cumulative arrivals, $R^e(t)$, are given by the positive root of the quadratic equation

$$\frac{\gamma_1}{2s}(1 + 2fs\Delta)(R^e(t))^2 + \left((1 + fs\Delta)(\gamma_0 + \gamma_1 t_0^e) + \frac{fs\gamma_1}{2}\Delta^2\right)R^e(t)$$
$$-\beta_0 s(t - t_0^e) + \frac{\beta_1 s}{2}\left(t^2 - (t_0^e)^2\right) = 0.$$

Assuming without loss of generality that $\beta_0 = \gamma_0$, it can be shown that

$$Sgn\left(\left.\frac{\partial\rho(t_0^e)}{\partial f}\right|_{f=0}\right) = 2\beta_0(2\gamma_1 - \beta_1) + \beta_1\gamma_1\left(\frac{N}{s} - \Delta\right).$$

A small incident risk leads to an increase in the initial departure rate unless parameter γ_1 is small compared to β_1.

B.5.2 Social optimum

Stage 1 If the SO is compressed, eqn. (34) applies and the first departure occurs at time

$$t_0^w = \frac{\beta_0 - \gamma_0}{\beta_1 + \gamma_1} - \frac{N}{2s} - \frac{\gamma_1}{\beta_1 + \gamma_1}\frac{fN\Delta}{2}.$$

It is easy to check that $t_0^w > t_0^e$ if $f > 0$. Condition (18) is satisfied for the candidate compressed SO if the following quadratic equation in $Z \equiv fN\gamma_1$ is negative:

(58)

$$-\Delta Z^2 + \left(2\left(\beta_0 + \frac{\beta_1\gamma_0}{\gamma_1}\right) + (\beta_1 + \gamma_1)\left(\frac{N}{s} + \Delta\right)\right)Z - (\beta_1 + \gamma_1)^2\frac{N}{s} < 0.$$

Condition (58) is clearly satisfied for small values of f.

The costate variable for departure time during Stage 1 is a negative, convex, and quadratic function of n:

$$\mu_1(n) = \left(\gamma_0 - \beta_0 + (\beta_1 + \gamma_1)\, t_0^w\right) n + \left(f\Delta\gamma_1 + \frac{\beta_1 + \gamma_1}{s}\right)\frac{n^2}{2}, \quad n \in [0, \hat{n}].$$

Stage 2 Departure time in Stage 2 is given by eqn. (44) which has an analytical solution:

$$t_2(n) = \frac{\beta_0 - \gamma_0 - f\gamma_0 (N - 2n) - f\gamma_1 \left(\Delta + \frac{N-n}{2s}\right)(N - n)}{\beta_1 + \gamma_1 + f\gamma_1 (N - 2n)}, \quad n \in [\hat{n}, N].$$

The last driver departs at

$$t_N^w = t_2(N) = \frac{\beta_0 - \gamma_0 (1 - fN)}{\beta_1 + \gamma_1 (1 - fN)}.$$

If the SO entails only Stage 2, $\hat{n} = 0$ and the first driver departs at

$$t_0^w = t_2(0) = \frac{\beta_0 - \gamma_0 - fN\left(\gamma_0 + \gamma_1\left(\Delta + \frac{N}{2s}\right)\right)}{\beta_1 + \gamma_1 (1 + fN)}.$$

The costate variable for departure time in stage 2, $\mu_2(n)$, is solved numerically by integrating (41) backwards with respect to n from $n = N$.

B.6 Minor incidents

B.6.1 User equilibrium: Proof of Lemma 2 for minor incidents

A driver who deviates from a candidate compressed UE and departs at $t \in \left(t_N^e, t_N^e + \frac{s-k}{s}\Delta\right)$ has an expected utility

$$\begin{aligned} E(u|t) \;&=\; (1 - F(N))\, u(t,t) + \int_{v=t_N - \frac{k}{s}\Delta}^{t_N} f(R(v))\,\rho(v)\, u\left(t, \frac{s}{k} t_N - \frac{s-k}{k} v\right) dv \\ &\quad + F\left(R\left(t_N - \frac{k}{s}\Delta\right)\right) u\left(t, t_N + \frac{s-k}{s}\Delta\right). \end{aligned}$$

The derivative of expected utility is

$$\frac{\partial E(u|t)}{\partial t} = \beta(t) - (1 - F(N))\,\gamma(t),$$

which is the same as for major incidents. Hence condition (5) still applies. ■

B.6.2 Timing of departures in user equilibrium

As noted in the text, the delay imposed by an incident on a driver depends on when it occurs. Four cases are possible.

Case 1: No incident has occurred A driver departing at t arrives at $a = t_0 + R(t)/s$. The contribution to the driver's expected utility is:

$$\Delta u_1^e = (1 - F(R(t)))\, u(t, t_0 + R(t)/s). \tag{59}$$

Case 2: An incident has occurred that persists throughout the driver's trip If the incident occurs at time v, a driver departing at t arrives when the number of drivers who have passed the bottleneck catches up with cumulative departures at t: $s(v - t_0) + k(a - v) = R(t)$. Hence $a = \frac{R(t)+st_0}{k} - \frac{s-k}{k}v$. Since the driver must arrive before the incident ends, $a \leq v + \Delta$. This condition translates to $v \geq t_0 + \frac{R(t)-k\Delta}{s}$. The incident must also occur before the driver departs: $v \leq t$. Accounting for these conditions, the contribution to the driver's expected utility is:

(60)

$$\Delta u_2^e = \int_{v=Max\left[t_0, t_0 + \frac{R(t)-k\Delta}{s}\right]}^{t} f(R(v))\,\rho(v)\, u\left(t, \frac{R(t)+st_0}{k} - \frac{s-k}{k}v\right) dv.$$

Case 3: An incident has occurred that ends before the driver arrives Arrival time is determined by the condition $s(v - t_0) + k\Delta + s(a - v - \Delta) = R(t)$ which yields $a = t_0 + \frac{s-k}{k}\Delta + \frac{R(t)}{s}$. The incident cannot end before the driver departs: $v > t - \Delta$. But it must end before the driver arrives: $v \leq t_0 + \frac{R(t)-k\Delta}{s}$. The contribution to the driver's expected utility is:

$$\begin{aligned}
\Delta u_3^e &= \int_{v=Max[t_0, t-\Delta]}^{Min\left[t, t_0 + \frac{R(t)-k\Delta}{s}\right]} f(R(v))\,\rho(v)\, u\left(t, t_0 + \frac{s-k}{k}\Delta + \frac{R(t)}{s}\right) dv \\
&= \left(F\left(R\left(Min\left[t, t_0 + \frac{R(t)-k\Delta}{s}\right]\right)\right) - F(R(Max[t_0, t-\Delta]))\right) \\
&\quad u\left(t, t_0 + \frac{s-k}{k}\Delta + \frac{R(t)}{s}\right).
\end{aligned} \tag{61}$$

Case 4: An incident has ended before the driver departs Arrival time is determined by the same condition as for Case 3 so that $a = t_0 + \frac{s-k}{k}\Delta + \frac{R(t)}{s}$. The incident must end before the driver departs: $v < t - \Delta$. The contribution to the driver's expected utility is:

$$\Delta u_4^e = F\left(R\left(Max\left[t_0, t-\Delta\right]\right)\right) u\left(t, t_0 + \frac{s-k}{s}\Delta + \frac{R(t)}{s}\right). \tag{62}$$

For the last driver who departs at t_N, Cases 1-4 are all applicable. Using eqns. (59)-(62) and $R(t_N) = N$, the last driver's expected utility is

$$\begin{aligned} E(u|t_N) &= (1 - F(N))\, u(t_N, t_N) + \\ &\int\limits_{v=t_N-\frac{k}{s}\Delta}^{t_N} f(R(v))\,\rho(v)\, u\left(t_N, \frac{s}{k}t_N - \frac{s-k}{k}v\right) dv \\ &+F\left(R\left(t_N - \frac{k}{s}\Delta\right)\right) u\left(t_N, t_N + \frac{s-k}{s}\Delta\right). \end{aligned}$$

Equilibrium departure period The timing of the departure period $[t_0^e, t_N^e]$ is determined by the condition $t_N^e = t_0^e + N/s$ and the condition that $E(u|t_N) = u(t_0, t_0)$. Using eqn. (1), the latter condition works out to eqn. (50) in the text:

$$\begin{aligned} \int_{v=t_0^e}^{t_N^e} (\beta(v) - \gamma(v))\, dv &= F\left(R^e\left(t_N^e - \frac{k}{s}\Delta\right)\right) \int_{v=t_N^e}^{t_N^e+\frac{s-k}{s}\Delta} \gamma(v)\, dv \\ &+ \int_{v=t_N^e-\frac{k}{s}\Delta}^{t_N^e} f(R^e(v))\,\rho^e(v) \int_{t=t_N^e}^{t_N^e+\frac{s-k}{k}\left(t_N^e-v\right)} \gamma(t)\, dt dv. \end{aligned} \tag{63}$$

Equation (63) can be solved numerically using the following iterative procedure:

1. Guess t_0^e and set $t = t_0^e$.
2. Increment departure time by a small time step dt from t to $t + dt$.
3. Determine which of Cases 1-4 are applicable. Set the derivative of $E(u|t)$ to zero, and solve for $\rho^e(t)$.
4. Update $R^e(t + dt) = R^e(t) + \rho^e(t)\, dt$.
5. Repeat steps 2-4 until $t = t_0^e + N/s$.
6. Evaluate condition (63) using the values computed for $\rho^e(t), t \in \left[t_N^e - \frac{k}{s}\Delta, t_N^e\right]$. If condition (63) is satisfied to within a tolerance limit, then stop. Otherwise return to step 1.

Initial departure rate For the first few drivers, only Cases 1 and 2 apply. Using eqns. (59) and (60), expected utility is

$$\begin{aligned}&(1-F(R(t)))\,u(t,t_0+R(t)/s)\\+&\int\limits_{v=t_0}^{t} f(R(v))\,\rho(v)\,u\left(t,\frac{R(t)+st_0}{k}-\frac{s-k}{k}v\right)dv.\end{aligned}$$

Differentiating this expression with respect to t, and evaluating it at $t=t_0$, one obtains

$$\frac{\rho^e(t_0^e)}{s}=\frac{\beta(t_0^e)}{\gamma(t_0^e)}.$$

This is the same as eqn. (2) with $t=t_0^{e0}$ for the model without incidents. Since $t_0^e<t_0^{e0}$, $\rho^e(t_0^e)>\rho^{e0}(t_0^{e0})$.

B.6.3 Social optimum

Expected utility of drivers for the SO is derived in a similar way to UE, but using the formula that cumulative departures equal $R(t)=s(t-t_0)$.

Case 1: No incident has occurred A driver departing at t arrives at t. The contribution to the driver's expected utility is:

(64) $$\Delta u_1^w=(1-F(s(t-t_0)))\,u(t,t).$$

Case 2: An incident has occurred that persists throughout the driver's trip If the incident occurs at time v, a driver departing at t arrives when the number of drivers who have passed through the bottleneck catches up with cumulative departures at t: $s(v-t_0)+k(a-v)=s(t-t_0)$. The driver therefore arrives at $a=\frac{s}{k}t-\frac{s-k}{k}v$. Since the driver must arrive before the incident ends, $a\le v+\Delta$, which translates to $v\ge t-\frac{k}{s}\Delta$. The incident must also occur before the driver departs: $v\le t-\Delta$. Accounting for these conditions, the contribution to the driver's expected utility is:

(65) $$\Delta u_2^w=\int\limits_{v=Max\left[t_0,t-\frac{k}{s}\Delta\right]}^{t} f(s(v-t_0))\,su\left(t,\frac{s}{k}t-\frac{s-k}{k}v\right)dv.$$

Case 3: An incident has occurred that ends before the driver arrives Arrival time is determined by the condition $s(v-t_0)+k\Delta+(s-v-\Delta)=s(t-t_0)$

which yields $a = t + \frac{s-k}{s}\Delta$. Since the driver must arrive after the incident ends, $v \leq t - \frac{k}{s}\Delta$. The contribution to the driver's expected utility is:

$$\Delta u_3^w = \int\limits_{v=Max[t_0,t-\Delta]}^{t-\frac{k}{s}\Delta} f\left(s\left(v-t_0\right)\right) su\left(t, t+\frac{s-k}{s}\Delta\right) dv,$$

which can be broken out into three intervals:

(66)

$$\Delta u_3^w = \begin{cases} 0 \ \text{ if } \ t < t_0 + \frac{k}{s}\Delta \\ F\left(s\left(t-t_0\right)-k\Delta\right) u\left(t, t+\frac{s-k}{s}\Delta\right) \ \text{ if } \ t \in \left(t_0 + \frac{k}{s}\Delta, t_0+\Delta\right) \\ \left(F\left(s\left(t-t_0\right)-k\Delta\right) - F\left(s\left(t-\Delta-t_0\right)\right)\right) \\ u\left(t, t+\frac{s-k}{s}\Delta\right) \ \text{ if } \ t > t_0+\Delta. \end{cases}$$

Case 4: An incident has ended before the driver departs Arrival time is determined by the same condition as for Case 3 so that $a = t + \frac{s-k}{s}\Delta$. The contribution to expected utility is:

$$\Delta u_4^w = F\left(s\left(t-\Delta-t_0\right)\right) u\left(t, t+\frac{s-k}{s}\Delta\right). \tag{67}$$

Summing terms in (64)-(67) one obtains

$$E\left(u|t\right) = \begin{cases} \left(1-F\left(s\left(t-t_0\right)\right)\right) u\left(t,t\right) \\ +s \int\limits_{v=t_0}^{t} f\left(s\left(v-t_0\right)\right) u\left(t, \frac{s}{k}t - \frac{s-k}{k}v\right) dv \ \text{ if } \ t < t_0 + \frac{k}{s}\Delta \\ \left(1-F\left(s\left(t-t_0\right)\right)\right) u\left(t,t\right) + F\left(s\left(t-t_0\right)-k\Delta\right) u\left(t, t+\frac{s-k}{s}\Delta\right) \\ +s \int\limits_{v=t-\frac{k}{s}\Delta}^{t} f\left(s\left(v-t_0\right)\right) u\left(t, \frac{s}{k}t - \frac{s-k}{k}v\right) dv \ \text{ if } \ t > t_0 + \frac{k}{s}\Delta \end{cases}.$$

Changing the variable of integration from t to n, total expected utility becomes:

(68)

$$E(U) = \int_{n=0}^{k\Delta} \left[\begin{array}{c} (1-F(n)) \int_{v=t_H}^{t_0+n/s} \beta(v)\, dv + \int_{v=t_0+n/s}^{t_W} \gamma(v)\, dv \\ + \int_{\xi=0}^{n} f(\xi) \left(\int_{v=t_H}^{t_0+n/s} \beta(v)\, dv + \int_{v=t_0+\frac{n}{k}-\frac{s-k}{sk}\xi}^{t_W} \gamma(v)\, dv \right) dv \end{array} \right] dn$$

$$+ \int_{n=k\Delta}^{N} \left[\begin{array}{c} (1-F(n)) \int_{v=t_H}^{t_0+n/s} \beta(v)\, dv + \int_{v=t_0+n/s}^{t_W} \gamma(v)\, dv \\ +F(n-k\Delta) \left(\int_{v=t_H}^{t_0+n/s} \beta(v)\, dv + \int_{v=t_0+\frac{n}{s}+(s-k)\Delta}^{t_W} \gamma(v)\, dv \right) dv \\ + \int_{\xi=n-k\Delta}^{n} f(\xi) \left(\int_{v=t_H}^{t_0+n/s} \beta(v)\, dv + \int_{v=t_0+\frac{n}{k}-\frac{s-k}{sk}\xi}^{t_W} \gamma(v)\, dv \right) \end{array} \right] dn.$$

Differentiating (68) with respect to t_0, and collecting terms, the equation defining the optimal t_0^w can be written

(69)
$$\int_{n=0}^{N} (\beta(t(n)) - \gamma(t(n)))\, dn + \int_{n=0}^{N} F(n)\gamma(t(n))\, dn$$
$$- \int_{n=k\Delta}^{N} F(n-k\Delta)\gamma(t_\Delta(n))\, dn$$
$$- \int_{n=0}^{k\Delta} \int_{\xi=0}^{n} f(\xi)\gamma(t_\xi(n))\, d\xi dn - \int_{n=k\Delta}^{N} \int_{\xi=n-k\Delta}^{n} f(\xi)\gamma(t_\xi(n))\, d\xi dn = 0,$$

where $t(n) = t_0^w + n/s$, $t_\Delta(n) = t_0^w + n/s + \frac{s-k}{s}\Delta$, and $t_\xi(n) = t_0^w + n/s + \frac{s-k}{sk}(n-\xi)$. Condition (69) has the form

$$\int_{n=0}^{N} (\beta(t(n)) - \gamma(t(n)))\, dn + Z = 0.$$

Now

(70)
$$\begin{aligned} \frac{\partial Z}{\partial k} &= \frac{\Delta}{s} \int_{n=k\Delta}^{N} F(n-k\Delta)\acute{\gamma}(t_\Delta(n)) \\ &\quad + k^{-2} \int_{n=0}^{k\Delta} \int_{\xi=0}^{n} f(\xi)\acute{\gamma}(t_\xi(n))(n-\xi)\, d\xi dn \\ &\quad + k^{-2} \int_{n=k\Delta}^{N} \int_{\xi=n-k\Delta}^{n} f(\xi)\acute{\gamma}(t_\xi(n))(n-\xi)\, d\xi dn. \end{aligned}$$

All components of (70) are strictly positive. Given (69) and $t(n) = t_0^w + n/s$, this implies that $\partial t_0^w / \partial k > 0$.