Interest rate paradox

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Abstract

System's properties are not always determined by properties of its elements. In this paper was made an attempt to analyze securities not isolated, but with respect to environment, i.e. to adjacent participants' operations on a market. It was shown that risk-neutral probability measure depends on these operations. No arbitrage conditions were developed for this case. Paradoxical results were obtained by using them. It was shown that almost on every market it is possible to create such instruments that makes these conditions not holding. Arbitrage opportunities exist on such markets and they are inefficient.

Keywords: market efficiency, risk-neutral probability density, interest rate, arbitrage, efficiency conditions.

JEL classification codes: G10, G12

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1. Introduction

The theory of No Arbitrage plays a serious role in Mathematical Finance. Development of pricing mechanisms (Black and Scholes 1973 and Merton 1973), understanding of market efficiency, no arbitrage conditions (Harrison and Kreps 1979, Harrison and Pliska 1981) and many other important themes, which highly influence nowadays markets, are strongly connected to it. However, there are open questions, e.g. Fama (1997) concluded that existing anomalies require new theories of the stock market and we need to continue the search for better models of asset pricing.

In a modern world we use strategies and securities (e.g. CDOs) that become more and more complex. Often there are chains of operations between participant's account and elementary securities. However, most theories analyze elementary securities and extrapolate results to complex systems (markets). In this paper securities are analyzed by using the traditional approach of no arbitrage, but with respect to systems complexity.

We know that derivative's price is discounted expected value of future payoff under the risk-neutral measure Q (Cox and Ross 1976). Let at a future time T a derivative's payoff is H_T, a random variable on the probability space describing market. The discount factor from the moment when premium is being paid until expiration time T is $P(\mathrm{t}_0, \mathrm{T})$. Today's fair value of the derivative is

$$E_Q = P(t_0, T) \cdot E_Q(H_T) \tag{1}$$

If participants agree that premium is being paid at the moment of expiration then $P(\mathrm{T}, \mathrm{T}) = 1$.

Payoff can be transformed after expiration into a different asset, e.g. from dollars into euro, stock, bonds etc. Premium is being transformed from this asset into a numeraire.

Consequently,

$$H_0 \cdot E(X_0) = P'(t_0,T) \cdot E_Q(H_T \cdot E(X)) \tag{2}$$

where $E(X)$ is exchange function for payoff; X is some set of parameters, it may contain S; X_0 is value of X at the moment when premium is being paid.

This is not a numeraire change in a classical sense (Jamshidian 1989). Securities are not affected and remain the same. We just add a pair of adjacent operations as a part of a complex strategy but use standard securities.

For example, for EUR/USD call options at least two cases are possible:

$$\begin{gathered} E(X) = 1 \\ E(X_0) = 1 \\ H_T(S) = S - K \text{ if } S > K \end{gathered} \tag{3}$$

where K is a strike price and S is price of underlying asset at the moment of expiration. Also payoff, paid in dollars, could be transformed in euro after option exercising.

$$\begin{gathered} E(X) = \frac{1}{S} \\ E(X_0) = \frac{1}{S_0} \\ H_T(S) = S - K \text{ if } S > K \end{gathered} \tag{4}$$

where S_0 is price of underlying asset at the initial moment.

2. No arbitrage conditions

Let $H_T(S)$ be Dirac delta function $\delta(\mathrm{x-S})$. Then

$$H_0^{\delta}(S) = q(S) \cdot P(t_0,T) \cdot \frac{E(X)}{E(X_0)} \tag{5}$$

where $q(\mathrm{S})$ is probability density function. It is apparent that there are such functions $E_i(\mathrm{X})$, to which correspond different measures Q_i and $P_i(\mathrm{t_0,T})$.

Otherwise next equality is false.

$$q_i(S) \cdot P_i(t_0,T) \cdot \frac{E_i(X)}{E_i(X_0)} = q_j(S) \cdot P_j(t_0,T) \cdot \frac{E_j(X)}{E_j(X_0)} \tag{6}$$

If S is constant then $q_i(S)$ could be equal to each other. However, in general, they are not.

For every i:

$$E_{Q_i}(1) = 1 \tag{7}$$

If this is not true then Q_i is not a probability measure. Also from comparison of equations (2) and (7) follows that equation (7) describes derivative or group of derivatives, for which:

1. Premium is being paid at the moment of expiration.
2. Transformed payoff is equal to one.
3. Transformed premium is also equal to one.

If it is not true then arbitrage with risk-free profit is possible.

Using equation (6) this no arbitrage condition can be transformed in the next one:

$$E_{Q_i}\left(\frac{E_i(X)}{E_j(X)}\right) = \frac{P_j(t_0,T)}{P_i(t_0,T)} \cdot \frac{E_i(X_0)}{E_j(X_0)} \tag{8}$$

There have to be no such $E_i(X)$ and $E_j(X)$ that make equation (8) false. Otherwise arbitrage opportunities exist.

This condition may reveal market inefficiency if we are able find "wrong" asset. As an example assume that S is not expected to be constant and

$$\begin{aligned} &E_1(X) = S, E_1(X_0) = S_0 \\ &E_2(X) = 1, E_2(X_0) = 1 \\ &E_3(X) = \frac{1}{S}, E_3(X_0) = \frac{1}{S_0} \end{aligned} \tag{9}$$

Premiums are being paid at the moment of expiration. Consequently, $P(T,T) = 1$. Using equation (6) we can make next transformations:

$$\mathrm{E}_{Q_3}(1) = 1 \tag{10}$$

$$\mathrm{E}_{Q_3}(1) = \mathrm{E}_{Q_2}\left(\frac{\mathrm{S}}{S_0}\right) = \mathrm{E}_{Q_2}\left(\frac{\mathrm{S}_0}{S_0}\right) + \mathrm{E}_{Q_2}\left(\frac{\mathrm{S}-\mathrm{S}_0}{S_0}\right) = 1 \tag{11}$$

$$E_{Q_2}(\frac{S-S_0}{S_0})=0 \tag{12}$$

Analogous

$$E_{Q_1}(\frac{S-S_0}{S_0})=0 \tag{13}$$

But

$$E_{Q_2}(\frac{S-S_0}{S_0})=E_{Q_1}(\frac{S}{S_0}\cdot\frac{S-S_0}{S_0})=E_{Q_1}(\frac{S-S_0}{S_0})+E_{Q_1}(\frac{(S-S_0)^2}{S_0{}^2})=E_{Q_1}(\frac{(S-S_0)^2}{S_0{}^2})\neq 0 \tag{14}$$

Consequently, Q_1, Q_2 or Q_3 is not a probability measure. Probabilities of events (price in our case) S are certainly not real, because sum of probabilities more or less than one. This allows making risk-free profit.

3. Real example

Suppose that there are two securities: S_1 and S_2. S_2 is portfolio that at initial moment consists of $a(0)$ units of S_1.

At moment t manager sells some amount of S_1 and pays dividends in addition to those of S_1. Thus $a(t_1)\geq a(t_2), a(t_2)\geq 0$ for every $t_1<t_2$. Futures on S_2 cost

$$F_{S_2}(t)=a(t)\cdot F_{S_1}(t) \tag{15}$$

where $F_{S_1}(\mathrm{t})$ is price on futures on S_1.

$a(t)$ is a managed parameter. Consequently, $F_{S_2}(\mathrm{t})$ is also a managed parameter.

We can price futures on S_2 in futures on S_2 with another expiration time $t_1<t$. Price in this case:

$$F^t_{S_2}(t_1,t)=\frac{F_{S_2}(t)}{F_{S_2}(t_1)}=\frac{a(t)\cdot F_{S_1}(t)}{a(t_1)\cdot F_{S_1}(t_1)} \tag{16}$$

Information about $a(\mathrm{t})$ is open to all participants with some little piece of uncertainty. So the price of $F^{t}_{S_2}(t_1, t)$ is a random variable even if the price of S_1 is constant.

Manager can make next property by managing $a(t)$ if $F_{S_1}(t)$ is not equal to zero for period $[t_1, t_3], t_1 < t_2 < t_3$:

$$F^{t}_{S_2}(t_1, t_2) = F^{t}_{S_2}(t_2, t_3) \tag{17}$$

Suppose there is a security, priced in the way of equation (2), with next properties:

1. Underlying asset is futures on S_2 with expiration at t_3.
2. Numeraire is futures on S_2 with expiration at t_2.
3. Expiration time is $T < t_1$.
4. Premium is being paid at the moment of expiration.

There are three scenarios for payoff as stated in equation (9). It can be transformed into futures with expiration at t_3, t_1 or there is no transformation and payoff is in futures with expiration at t_2. Then

$$\begin{aligned} E_1(X) &= \frac{1}{F^{t}_{S_2}(t_2, t_1)} = F^{t}_{S_2}(t_1, t_2) \\ E_2(X) &= 1 \\ E_3(X) &= \frac{1}{F^{t}_{S_2}(t_2, t_3)} = \frac{1}{F^{t}_{S_2}(t_1, t_2)} \end{aligned} \tag{17}$$

Consequently, at least one of three "probability measures" is not a probability measure at all. It allows arbitrage and making risk-free profit in one of futures on S_2 with mentioned expiration times. If price of S_1 is positive in an interesting for participant numeraire then profit in this numeraire is also positive.

It should be noted that on a money market such situation may arise without creating special instruments like described above.

4. Conclusion

Well-known derivative's pricing formula was generalized to the case when participant transform payoff into another asset. It was shown that this operation affect fair premium. However, there may be many such assets, but derivative's price is one for all participants. By this reason risk-neutral measure changes when we change preferable asset, into which payoff is transformed.

No arbitrage conditions were obtained for this case. There have to be no such asset on a market that makes implied probability measures to be not probability measures.

First it was shown theoretical example of such group of assets. Then it was shown that such group of assets can be created almost on every market. The only thing that is needed is existence of some asset with positive price.

It sounds rather paradoxical and even absurdly. If it is true then it reveals very fundamental inefficiency.

References

Black, F., Scholes M. (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy 81 (3): 637–654.

Cox, J. and Ross S. (1976). The valuations of options for alternative stochastic processes. Journal of Financial Economics 3: 145-166.

Fama, E. (1998). Market Efficiency, Long-Term Returns and Behavioral Finance. Journal of Financial Economics 49: 283–306.

Harrison J.M. and Kreps D.M. (1979). Martingales and arbitrage in multiperiod arbitrage markets. Journal of Economic Theory 20: 381-408.

Harrison, J. M and Pliska, S. R. (1981). Martingales and Stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications 11 (3): 215–260.

Jamshidian, F. (1989). An Exact Bond Option Pricing Formula. The Journal of Finance 44: 205–209.

Merton, R. C. (1973). Theory of Rational Option Pricing. Bell Journal of Economics and Management Science 4 (1): 141–183.