A Robust Turnpike Deduced by Economic Maturity

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Abstract

In the paper, a one-sector neoclassical model with stochastic growth has been constructed. The key concept of *economic maturity* is well-defined in the abstract model economy, and also a thorough characterization of the minimum time needed to economic maturity is supplied for the first time. Moreover, it is confirmed that the capital-labor ratio corresponding to the state of economic maturity indeed provides us with a robust turnpike of the optimal path of capital accumulation.

Keywords: Stochastic growth; Economic maturity; Asymptotic turnpike theorem; Neighborhood turnpike theorem; Robustness.

JEL classification: C60; E13; E22.

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1. INTRODUCTION

When concerning the problems of economic development for underdeveloped economies, one may notice that the principle of maximum speed is widely employed, for example, the Germany and Japan after World War II and China after 1978s (see, Song et al., 2011). That is to say, provided the existence of the maximum sustainable terminal path consumption per capita or von Neumann path consumption per capita, which would be regarded as the state of *economic maturity* in a certain sense, the major goal of the people and the government is to choose appropriate or optimal savings strategy and fiscal policies, respectively, such that the state of economic maturity can be reached as soon as possible. Indeed, the underlying motivation of the present exploration, in line with Kurz (1965), is to derive conditions under which the specified economy can reach the maximum terminal path in minimum time. In particular, we only analyze the economy before economic maturity in the model, i.e., we focus on underdeveloped economies, and we leave those types of economies having reached economic maturity to future research.

Although we focus on a one-sector neoclassical aggregate growth model (see, Solow, 1956; Cass, 1965), the present study extends Kurz's analyses in many ways, for instance, first, we consider the economy lying in a persistently non-stationary environment; second, the nature or social planner is naturally incorporated into the macroeconomic model, and it is asserted that the endogenous savings rate and the minimum time just form the sub-game perfect Nash equilibrium of the stochastic differential dynamic game between the nature or social planner and the representative agent; third, it is demonstrated that the minimum time needed to reach economic maturity is completely characterized by the maximum sustainable level of terminal path capital-labor ratio or the state corresponding to economic maturity, and also the terminal path of capital-labor ratio provides us with a robust turnpike; finally, the maximum sustainable level of terminal path consumption per capita or capital-labor ratio is endogenously determined in the present model rather than that of Kurz (1965), Samuelson (1965) and Cass (1966), where the terminal capital-labor ratio is exogenously given or prescribed.

The rest of the paper is organized as follows. In section 2, the basic model is constructed, some necessary assumptions and definitions, especially the definitions of economic maturity and the minimum time needed to economic maturity, are introduced. Section 3 will be the major part of the paper, where both Asymptotic Turnpike Theorem and Neighborhood Turnpike Theorem are established. Section 4 proves the robustness of the turnpike theorems demonstrated in section 3, i.e., we assert the existence of a robust turnpike deduced by economic maturity based upon section 3. There is a brief concluding section, where we have discussed about possible extensions of the basic framework. All proofs, unless otherwise noted in the text, appear in the Appendix.

2. THE ENVIRONMENT

Here, and throughout the paper, we consider a one-sector neoclassical model with stochastic growth. As usual, we employ the following neoclassical production function,

$$Y(t) = F\left(K(t), L(t)\right). \tag{1}$$

which is a strictly concave function, and also it exhibits constant returns to scale with $K(t)$ denoting the aggregate capital stock and $L(t)$ representing the labor force or population size. Thus, the following law of motion of capital accumulation is derived,

$$\frac{dK(t)}{dt} = F\left(K(t), L(t)\right) - \delta K(t) - C(t). \tag{2}$$

where δ , an exogenously given constant, denotes the depreciation factor and $C(t)$ stands for aggregate consumption in period t .

Now, suppose that $\left(B(t), 0 \le t \le T\right)$ stands for a standard Brownian motion defined on the following filtered probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t\right\}_{0\le t\le T}, \mathbb{P}\right)$ with $\mathbb{F} \triangleq \left\{\mathcal{F}_t\right\}_{0\le t\le T}$ the $\mathbb{P}-$augmented filtration generated by $\left(B(t), 0 \le t \le T\right)$ with $\mathcal{F} \triangleq \mathcal{F}_T$ for $\forall T > 0$, that is, the underlying stochastic basis satisfies the well-known *usual* *conditions*. Then, based upon the given probability space and in line with Merton (1975), we define,

$$dL(t) = nL(t)dt + \sigma L(t)dB(t) . \tag{3}$$

subject to $B(0) = 0$ a.s.- $\mathbb{P}$ and $\sigma \in \mathbb{R}_0 \triangleq \mathbb{R} - \{0\}$, a constant. Thus, combining (2) with (3) and applying Itô's rule leads us to,

$$dk(t) = \left[s\left(k(t)\right) f\left(k(t)\right) - \left(\delta + n - \sigma^2\right) k(t) \right] dt - \sigma k(t) dB(t) . \tag{4}$$

with $k(0) \equiv k_0 > 0$ and $k(t) \triangleq K(t)/L(t)$, $f\left(k(t)\right) \triangleq F\left(K(t), L(t)\right)/L(t) = F\left(\frac{K(t)}{L(t)}, 1\right)$, $s\left(k(t)\right) \triangleq 1 - \frac{c(t)}{f(k(t))}$ and $c(t) \triangleq C(t)/L(t)$ denoting the capital-labor ratio, per capita output, savings per unit output and per capita consumption, respectively, at time t. Specifically, for the SDE of capital-labor ratio given by (4), Chang and Malliaris (1987) proved the following theorem,

THEOREM 1: *If the production function* f *is strictly concave, continuously differentiable on* $[0, \infty)$, $f(0) = 0$, *and* $\lim_{k(t)\to\infty} f'\left(k(t)\right) \triangleq \lim_{k(t)\to\infty} \frac{df(k(t))}{dk(t)} = 0$, *then there exists a unique solution to* (4).

Thus, we directly give,

ASSUMPTION 1: *The assumptions or conditions given by Theorem* 1 *are assumed to be fulfilled throughout the current paper*.

2.1. *Economic Maturity*

It is assumed that the abstract economy consists of $L(t)$ identical individuals in period t, each of whom possesses perfect foresight as usual. Hence, we suppose there is a representative agent with the following objective function,

$$\mathbb{E}_{t_0}\left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1\left(\left(1 - s\left(k(t)\right)\right) f\left(k(t)\right)\right) dt + e^{-\rho(\tau - t_0)} U_2\left(f\left(k(\tau)\right)\right) \right] . \tag{5}$$

where $\mathbb{E}_{t_0}$ denotes the expectation operator depending on $\mathcal{F}_{t_0}$ with $t_0 \geq 0$, $0 < \rho < 1$ represents the discount factor, $\tau \triangleq \tau(\omega) \in \mathcal{T} \triangleq \{ \mathcal{F} -$ stopping times$\}$ for

$\omega \in \Omega$, and $U_1(\cdot)$, $U_2(\cdot)$ are strictly increasing, strictly concave instantaneous utility functions of per capita consumption and per capita output, respectively, with the well-known Inada conditions satisfied.

It is easily seen that the criterion defined by (5) is widely used in existing economic literatures, including the macroeconomic studies. Nevertheless, $\tau \triangleq \tau(\omega)$ is usually pre-specified and is deterministic, i.e., $\tau(\omega) \equiv T > 0$ for all $\omega \in \Omega$, for any exogenously given constant $0 < T \le \infty$, in most of the excellent macroeconomic literatures. Noting that τ truly implies interesting and also important economic implications in accordance to Kurz (1965), we will extend Kurz's work by introducing the nature or social planner into the present macroeconomic model. Rather, the nature or the social planner will choose an admissible value $\tau^* \triangleq \tau^*(\omega)$ so that (5) is maximized. Formally, we give,

DEFINITION 1: We define the *dynamic game* Γ between the nature and the representative agent according to the following order of action,

Step 1: The nature will choose a strategy $\tau^*(\omega) \in \mathcal{T}$ such that the criterion in (5) is maximized subject to the SDE of capital-labor ratio in (4).

Step 2: Given Step 1 and $\tau = \tau^*(\omega) \in \mathcal{T}$, the representative agent chooses a savings strategy $s^*\left(k(t), \tau^* - t_0\right)$ such that the criterion defined in (5) is maximized subject to the SDE of capital-labor ratio in (4).

Then, following the classical Backward Induction Principle, we define,

PROBLEM 1: The representative agent will find a savings policy $s^*\left(k(t), \tau - t_0\right)$ so as to,

$$\max \mathbb{E}_{t_0}\left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1\left(\left(1 - s\left(k(t)\right)\right) f\left(k(t)\right)\right) dt + e^{-\rho(\tau - t_0)} U_2\left(f\left(k(\tau)\right)\right)\right].$$

subject to the SDE of capital-labor ratio in (4), for $\forall \tau \in \mathcal{T}$.

If Problem 1, the modified Ramsey (1928) problem, has a solution, we obtain the optimal path of capital-labor ratio as follows,

$$dk(t) = \left[s^*\left(k(t), \tau - t_0\right) f\left(k(t)\right) - \left(\delta + n - \sigma^2\right) k(t) \right] dt - \sigma k(t) dB(t) . \tag{6}$$

And we put,

PROBLEM 2: The optimization problem facing the nature or the social planner is to find a stopping rule $\tau^*(\omega) \in \mathcal{T}$, so as to,

$$\sup \mathbb{E}_{t_0} \left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1 \left(\left(1 - s^*\left(k(t), \tau - t_0\right)\right) f\left(k(t)\right) \right) dt + e^{-\rho(\tau - t_0)} U_2 \left(f\left(k(\tau)\right) \right) \right].$$

subject to the SDE of capital-labor ratio given by (6).

REMARK 2.1: (i) It is especially worth emphasizing that Problem 2 can also be modified by focusing entirely upon the final state as that of Radner (1961). That is, the criterion of preference facing the nature or social planner is given by,

$$\mathbb{E}_{t_0} \left[e^{-\rho(\tau - t_0)} U_2 \left(f\left(k(\tau)\right) \right) \right]$$

which of course will result in a new turnpike. Nevertheless, we argue that similar turnpike theorems can also be proved for the new turnpike.

(ii) Moreover, in particular, one may notice certain similarity of the present approach to those literatures studying endogenous lifetime or endogenous longevity in growth models (see, Chakraborty, 2004; de la Croix and Ponthiere, 2010, and among others), there exist obvious differences between the both especially when referring to economic intuitions and economic implications behind the formal models. Existing studies focus on OLG models and health-investment behaviors while the current exploration emphasizing issues of macroeconomic development, i.e., the characterization of economic maturity for underdeveloped economies and the corresponding characteristics of their optimal capital-accumulation paths.

(iii) It is easily seen that the maximum sustainable capital-labor ratio corresponding to the state of economic maturity is endogenously determined as well as the minimum time needed to economic maturity by using stochastic optimal stopping theory that is widely applied in mathematical finance. However, in Kurz's (1965) study, the targets or the maximum sustainable level of terminal path capital-labor ratios are exogenously specified, and the corresponding minimum time problem is expressed as: for any given initial capital-labor ratios, to chose strategies so that the prescribed targets can be reached as soon as possible. The major innovation of the present approach, therefore, is both endogenously determining the terminal path, the minimum time and taking the economic-welfare considerations of the representative agent into account in solving the minimum time problem.

(iv) It follows from the specification of Problem 2 that we focus on the period of the economy before reaching economic maturity as Kurz (1965), Samuelson (1965) and Cass (1966). That is, the present framework is suitable for the studies on underdeveloped economies and we leave the relative exploration of developed economies, i.e., economies having reached economic maturity, to future research.

Thus, if Problem 2 has a solution, we get the optimal stopping time $\tau^*(\omega) \in \mathcal{T}$. And also $\left(\tau^*(\omega), s^*\left(k(t), \tau^*(\omega) - t_0\right)\right)$ forms the sub-game perfect Nash equilibrium of the dynamic game Γ given by Definition 1. Moreover, we supply the following formal definition,

DEFINITION 2: Provided Definition 1 and if Problem 1 and Problem 2 are solvable, we then obtain the *minimum time needed to economic maturity* for the present abstract aggregate economy, and we denote it by $\tau^*(\omega) \in \mathcal{T}$.

REMARK 2.2: It is especially worth mentioning that we define the standard of economic maturity from the perspective of economic welfare, which is of course reasonable in the current model economy. Notice that the state of economic maturity for any given economy should imply a peak state that yields the highest level of economic welfare,[2] we argue that the minimum time needed to economic maturity given by Definition 2 is well-defined in some sense. In particular, we only consider the economy before the economic maturity and we leave the economy after economic maturity to future research, i.e., we focus on underdeveloped economy.

Finally, noting that we do not focus on the endogenous savings behavior of the representative agent and also the explicit expression of the minimum time needed to

[2] We of course admit that there are many other standards that can characterize the state of economic maturity. Nevertheless, we persuasively argue that *economic welfare* will always be the appropriate choice when noting that the major goal of economic growth and economic development is to improve the economic welfare of the people for any modern economies. And in order to make things easier and tractable, we focus on the highest level of economic welfare, and this is, however, without loss of any generality in the model economy.

economic maturity in the current paper, we directly put,

ASSUMPTION 2: *It is assumed that both Problem* 1 *and Problem* 2 *are solvable, i.e., there is an optimal savings policy* $s^*\left(k(t), \tau^*(\omega) - t_0\right)$ *and there is a minimum time needed to economic maturity* $\tau^*(\omega) \in \mathcal{T}$. *Moreover, suppose that there exists a constant* $0 < k^* < \infty$ *such that* $\tau^*(\omega) \triangleq \inf\left\{t \geq 0; k(t) = k^*\right\} < \infty$ *a.s.*-$\mathbb{P}$.

REMARK 2.3: (i) In fact, Problem 1 can be solved by employing stochastic dynamic programming, and Merton (1975) proved the existence of optimal savings policy in a quite similar case. On the other hand, Problem 2 can also be solved under certain conditions, and one can refer to Karatzas and Wang (2001), Jeanblanc et al. (2004), and Øksendal and Sulem (2005) for more details. And the major goal of the present exploration is to confirm that k^* defines a robust turnpike, which is certainly deduced by economic maturity based on the above constructions and assumptions.

(ii) Moreover, Assumption 2 ensures the existence of the turnpike from the viewpoint of pure mathematical techniques. We, however, emphasize here that the existence can be taken for granted in reality. In other words, for any developed economy today, it certainly has experienced the state of economic maturity in its history. Thus, the existence of the state of economic maturity is easily ensured in reality. And here we specifically express it via using mathematical formulas and meanwhile we equip these kinds of mathematical formulas with special economic intuitions.

3. TURNPIKE THEOREMS

Now, based on Assumption 2, we get,

$$dk(t) = \left[s^*\left(k(t), \tau^* - t_0\right) f\left(k(t)\right) - \left(\delta + n - \sigma^2\right) k(t)\right] dt - \sigma k(t) dB(t)$$

$$\triangleq \varphi\left(k(t)\right) dt + \psi\left(k(t)\right) dB(t). \tag{7}$$

subject to $k(0) \equiv k_0 > 0$, a deterministic constant. And,

$$\tau^*(\omega) \triangleq \inf\left\{t \geq 0; k(t) = k^*\right\} < \infty \ \text{ a.s.-}\mathbb{P} \tag{8}$$

for some endogenously given constant $0<k^*<\infty$. We are encouraged to show that k^* exhibits turnpike property provided the above assumptions. And this is the major goal of the present section.

THEOREM 2 (Asymptotic Turnpike Theorem)[3]: *Provided the SDE of capital-labor ratio defined in* (7) *and the minimum time needed to economic maturity given by* (8)*, then we always get that* $k(t)$ *converges in* $L^1(\mathbb{P})$ *and the corresponding limit belongs to* $L^1(\mathbb{P})$*, specifically, it uniformly converges to* k^* *a.s.-*$\mathbb{P}$*, or equivalently,*

$$\lim_{t'\to\infty}\mathbb{P}\left(\bigcup_{t=t'}^{\infty}\left[\left|k(t)-k^*\right|\geq\varepsilon\right]\right)=0.$$

for $\forall\varepsilon>0$ *, if we have* $\varphi\left(k(t)\right)=0$ *a.s.-*$\mathbb{P}$*, i.e.,* $s^*\left(k(t),\tau^*-t_0\right)f\left(k(t)\right)=(\delta+n-\sigma^2)k(t)$ *a.s.-*$\mathbb{P}$*, in* (7).

PROOF: See Appendix A. ■

REMARK 3.1: It is interesting to notice that Joshi (1997) also studies the turnpike theory in a stochastic aggregate growth model, in which stochastic environments as independent variables are directly and exogenously incorporated into the production function, by applying supermartingale property to confirm the corresponding convergence. However, one may easily tell the difference between Joshi's method and our proof. Moreover, it is persuasively argued that the essential requirement in Theorem 2 can be easily met thanks to the stochastic volatility term represented by σ.

However, if $\varphi\left(k(t)\right)\neq 0$, we define a new process $\theta(t)$ by,

$$\varphi\left(k(t)\right)=\theta(t)\psi\left(k(t)\right).$$

for a.a. $(t,\omega)\in\left[0,T\right]\times\Omega$. Then we put,

$$Z(t)\triangleq\exp\left\{-\int_0^t\theta(s)dB(s)-\frac{1}{2}\int_0^t\theta^2(s)ds\right\}.$$

[3] This proof brings the idea from Dai (2012). And our turnpike theorems satisfy the classical characteristics, i.e., any optimal paths stay within a small neighborhood of the turnpike almost all the time and the turnpike is independent of initial conditions (see, McKenzie, 1976; Yano, 1984).

Define a new measure $\mathbb{Q}$ on $\mathcal{F}_T$ by,

$$d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega).$$

i.e., $Z(T)$ is the Radon-Nikodym derivative. Now, we need the following assumption,

ASSUMPTION 3: *Here, we suppose that at least one of the following two conditions holds,*

(i) *We have* $\mathbb{E}\left[Z(T)\right] = 1$;

(ii) *The following Novikov Condition holds, i.e.,*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta^2(t)dt\right)\right] < \infty \quad for \quad 0 \le T < \infty.$$

Thus, based upon Assumption 3 and according to the Girsanov Theorem, we get that $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$, $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $k(t)$ is a martingale w. r. t. $\mathbb{Q}$ on the stochastic basis $\left(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}\right)$. Using Girsanov Theorem again, we conclude that the following process,

$$\hat{B}(t) \triangleq \int_0^t \theta(s)ds + B(t), \quad \forall t \in \left[0, T\right].$$

is a Brownian motion w. r. t. $\mathbb{Q}$ with $\hat{B}(0) = B(0) = 0$ a.s., and expressed in terms of $\hat{B}(t)$ we can get,

$$dk(t) = \psi\left(k(t)\right)d\hat{B}(t). \tag{9}$$

subject to $k(0) \equiv k_0 > 0$, a deterministic constant. Now, based on (9) and similar to (8), we, by slightly modifying Assumption 2, give,

$$\hat{\tau}^*(\omega) \triangleq \inf\left\{t \ge 0; k(t) = \hat{k}^*\right\} < \infty \quad \text{a.s.-}\mathbb{Q} \tag{10}$$

for some endogenously determined constant $0 < \hat{k}^* < \infty$. Therefore, employing the same proof as that of Theorem 2, we establish,

THEOREM 3 (Asymptotic Turnpike Theorem): *Provided the SDE of capital-labor ratio defined in* (9) *and the minimum time needed to economic maturity given by* (10), *then we always get that* $k(t)$ *converges in* $L^1(\mathbb{Q})$ *and the corresponding limit belongs to* $L^1(\mathbb{Q})$*, specifically, it uniformly converges to* $\hat{k}^*$ *a.s.-*$\mathbb{Q}$*, or equivalently,*

$$\lim_{t'\to\infty}\mathbb{Q}\left(\bigcup_{t=t'}^{\infty}\left[\left|k(t)-\hat{k}^*\right|\geq\varepsilon\right]\right)=0.$$

for $\forall\varepsilon>0$.

In what follows, we proceed to prove the neighborhood turnpike theorem. We do this by first giving the following assumption,

ASSUMPTION 4: *Suppose that* $k(t)\in\mathbb{R}_{++}\triangleq[0,\infty]$ *, which is the one point compactification of* $\mathbb{R}$ *at infinity with the induced topology, for* $\forall t\geq 0$. *Moreover, it is assumed that there exists a unique invariant Borel probability measure* π *on* $\mathbb{R}_{++}$ *such that* $\pi\left[bd\left(\mathbb{R}_{++}\right)\right]\triangleq\pi\left[\{0\}\bigcup\{+\infty\}\right]=0$*, i.e.,* $bd\left(\mathbb{R}_{++}\right)$ *denotes the boundary of* $\mathbb{R}_{++}$. *And we denote the Borel probability measure corresponding to the SDE defined in* (9) *by* $\hat{\pi}$ *with the above requirements totally met.*

REMARK 3.2: Mirman (1972) constructs a one-sector growth model with uncertain technology, i.e., random variables, which are assumed to be independent and identically distributed, are directly introduced into the neoclassical production function, thereby resulting in a discrete-time Markov process of the capital stock. Specifically, Mirman defines the Borel probability measure on the Borel sets of the non-negative real line by using the corresponding probability transition function of the above Markov process. Moreover, Theorem 2.1 of Mirman confirms that there exists a stationary probability measure that has no mass at either zero or infinity. In contrast, the present paper constructs continuous time Markov process of capital-labor ratio. Nonetheless, one can still prove that there exists a unique invariant Borel probability measure satisfies the requirements of Assumption 4 under certain relatively weak conditions. For more details, one may refer to Theorem 2.1 of Imhof (2005), Theorem 3.1 of Benaïm et al. (2008) and Theorem 5 of Schreiber et al (2011). The present paper omits the corresponding proof is just for the sake of simplicity.

Thus, the following theorem is derived,

THEOREM 4 (Neighborhood Turnpike Theorem)[4]: *Based upon Theorem* 2 *and Assumption* 4, *we can get that there exists a constant* $\Sigma > 0$ *such that for* $\forall \alpha > 0$ *with* $\alpha > \Sigma$,

(i) $\mathbb{E}\left[\tau_{\overline{B}_\alpha(k^*)}(\omega)\right] \le \dfrac{dist\left(k_0, k^*\right)}{\alpha - \Sigma}$,

(ii) $\pi\left[\overline{B}_\alpha(k^*)\right] \ge 1 - \dfrac{\Sigma}{\alpha} \triangleq 1 - \varepsilon$.

where,

$$B_\alpha(k^*) \triangleq \left\{k(t) \in \mathbb{R}_{++}; \left|k(t) - k^*\right| < \alpha, t \ge 0\right\},$$

$$\tau_{\overline{B}_\alpha(k^*)}(\omega) \triangleq \inf\left\{t \ge 0; k(t) \in \overline{B}_\alpha(k^*) \triangleq clB_\alpha(k^*)\right\},$$

and,

$$dist\left(k_0, k^*\right) \triangleq k^* \log\left(k^*/k_0\right).$$

for $k_0 \triangleq k(0) > 0$.

PROOF: See Appendix B. ■

Similarly, we derive the following theorem,

THEOREM 5 (Neighborhood Turnpike Theorem): *Based upon Theorem* 3 *and Assumption* 4, *we can get that there exists a constant* $\hat{\Sigma} > 0$ *such that for* $\forall \hat{\alpha} > 0$ *with* $\hat{\alpha} > \hat{\Sigma}$,

(i) $\mathbb{E}^{\mathbb{Q}}\left[\hat{\tau}_{\overline{B}_{\hat{\alpha}}(\hat{k}^*)}(\omega)\right] \le \dfrac{dist\left(k_0, \hat{k}^*\right)}{\hat{\alpha} - \hat{\Sigma}}$,

(ii) $\hat{\pi}\left[\overline{B}_{\hat{\alpha}}(\hat{k}^*)\right] \ge 1 - \dfrac{\hat{\Sigma}}{\hat{\alpha}} \triangleq 1 - \hat{\varepsilon}$.

where,

$$B_{\hat{\alpha}}\left(\hat{k}^*\right) \triangleq \left\{k(t) \in \mathbb{R}_{++}; \left|k(t) - \hat{k}^*\right| < \hat{\alpha}, t \ge 0\right\},$$

$$\hat{\tau}_{\overline{B}_{\hat{\alpha}}(\hat{k}^*)}(\omega) \triangleq \inf\left\{t \ge 0; k(t) \in \overline{B}_{\hat{\alpha}}(\hat{k}^*) \triangleq clB_{\hat{\alpha}}\left(\hat{k}^*\right)\right\},$$

and,

[4] This proof brings the method employed by Imhof (2005) and Dai (2012).

$$dist\left(k_0,\hat{k}^*\right) \triangleq \hat{k}^* \log\left(\hat{k}^*/k_0\right).$$

for $k_0 \triangleq k(0) > 0$.

REMARK 3.3: Theorem 4 shows that the Borel probability measure π will place nearly all mass close to the turnpike k^*. And similarly, Theorem 5 reveals that the corresponding probability distribution $\hat{\pi}$ will place almost all mass close to the new turnpike $\hat{k}^*$. Indeed, Theorem 4 and 5 demonstrate the turnpike property from the viewpoints of both time dimension and space dimension, i.e., in the sense of Markov time and in the sense of invariant probability distribution, which of course will provide us with a much more complete characterization of the neighborhood turnpike property when compared with existing studies (see, McKenzie, 1976; Bewley, 1982; Yano, 1984, and among others).

4. ROBUSTNESS

It follows from (7) that,

$$\begin{aligned} dk(t) &= \varphi\left(k(t)\right)dt + \psi\left(k(t)\right)dB(t) \\ &\triangleq k(t)\varphi_0\left(k(t)\right)dt + k(t)\psi_0\left(k(t)\right)dB(t), \end{aligned} \tag{11}$$

Now, we introduce the following SDE,

$$\begin{aligned} d\tilde{k}(t) &= \tilde{\varphi}\left(\tilde{k}(t)\right)dt + \tilde{\psi}\left(\tilde{k}(t)\right)dB(t) \\ &\triangleq \tilde{k}(t)\tilde{\varphi}_0\left(\tilde{k}(t)\right)dt + \tilde{k}(t)\tilde{\psi}_0\left(\tilde{k}(t)\right)dB(t), \end{aligned} \tag{12}$$

where we have assumed that,

ASSUMPTION 5: *For any* $\xi > 0$, *we suppose that,*

$$\sup_{k,\tilde{k}>0}\left|\varphi_0(k) - \tilde{\varphi}_0(\tilde{k})\right| \vee \sup_{k,\tilde{k}>0}\left|\psi_0(k) - \tilde{\psi}_0(\tilde{k})\right| \le \xi.$$

That is to say, (12) *defines the* $\xi-$*perturbation of* (11).

Moreover, we need the following assumption for the sake of convenience,

ASSUMPTION 6: *We suppose that there exist constants* ϕ, $\tilde{\phi}$ *and* $\phi_0 < \infty$ *such* *that,*

$$\left|\varphi(k)k\right| \vee \left|\psi(k)\right|^2 \le \phi\left|k\right|^2, \quad \left|\tilde{\varphi}(\tilde{k})\tilde{k}\right| \vee \left|\tilde{\psi}(\tilde{k})\right|^2 \le \tilde{\phi}\left|\tilde{k}\right|^2,$$

and,

$$\sup_{k>0}\left|\varphi_0(k))\right|^2 \vee \sup_{k>0}\left|\psi_0(k)\right|^2 \le \phi_0 .$$

for $\forall k > 0$, $\forall \tilde{k} > 0$.

REMARK 4.1: One can easily find that Assumption 6 is truly reasonable thanks to Assumption 1. Assumption 6 is indeed without loss of any generality and is just for the sake of convenience in the following proofs.

LEMMA 1: *Provided the above assumptions, we find that there exist constants* $e(k_0, p, T) < \infty$ *and* $\tilde{e}(k_0, p, T) < \infty$ *such that,*

(i) $\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)\right|^p\right] \le e(k_0, p, T)$;

and,

(ii) $\mathbb{E}\left[\sup_{0\le t\le T}\left|\tilde{k}(t)\right|^p\right] \le \tilde{e}(k_0, p, T)$.

for $k(0) = \tilde{k}(0) = k_0 > 0$, $\forall T > 0$ *and* $\forall p \in \mathbb{N}$, $p \ge 2$.

PROOF: See Appendix C. ■

Specifically, if both $k(t)$ and $\tilde{k}(t)$ are martingales w. r. t. $\mathbb{P}$, then without the above assumptions we still get,

LEMMA 2: *If both* $k(t)$ *and* $\tilde{k}(t)$ *are martingales w. r. t.* $\mathbb{P}$, *then there exist constants* $\eta < \infty$ *and* $\tilde{\eta} < \infty$ *such that,*

(i) $\mathbb{E}\left[\lim_{T\to\infty}\sup_{0\le t\le T}\left|k(t)\right|^2\right] < \eta$;

And,

(ii) $\mathbb{E}\left[\lim_{T\to\infty}\sup_{0\le t\le T}\left|\tilde{k}(t)\right|^2\right] < \tilde{\eta}$.

for $k(0) = \tilde{k}(0) = k_0 > 0$.

PROOF: See Appendix D. ■

Now, we can derive the following proposition,

PROPOSITION 1: *Based on the above assumptions and Lemma* 1 *or Lemma* 2, *and suppose that* $k(0)=\tilde{k}(0)=k_0>0$, *then we get,*

$$\mathbb{E}\left[\lim_{T\to\infty}\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^2\right]\to 0 \quad as \quad \xi\to 0.$$

PROOF: See Appendix E. ■

REMARK 4.2: It should be pointed out here that in the proof of Proposition 1, we have implicitly used the following facts or assumptions, i.e., the speed of ξ approaching zero is much faster than that of time T approaching infinity and also $0\times\infty\equiv 0$. Moreover, we can get the same conclusion by taking the limit as $\xi\to 0$ first and then as $T\to\infty$.

Accordingly, the following theorem is established,

THEOREM 6 (Robust Turnpike): *Provided Theorem* 2 *and* 4, *we show that* k^* *is a robust turnpike.*

PROOF: To prove the robustness, one just need combine Theorem 2 with Proposition 1 or combine Theorem 4 with Proposition 1. And noting the following fact,

$$\left|\tilde{k}(t)-k^*\right|^2=\left|\tilde{k}(t)-k(t)+k(t)-k^*\right|^2\le 2\left[\left|\tilde{k}(t)-k(t)\right|^2+\left|k(t)-k^*\right|^2\right].$$

Thus, we leave the details to the interested reader. ■

Similarly, one can also assert,

THEOREM 7 (Robust Turnpike): *Provided Theorem* 3 *and* 5, *one can show that* $\hat{k}^*$ *is a robust turnpike.*

REMARK 4.3: Theorem 6 and 7 have confirmed the asymptotic stability of the turnpikes k^* and $\hat{k}^*$, respectively, under the above relatively weak assumptions. To summarize, by noticing that our theorems show that the optimal path of capital accumulation will *robustly* converge to the corresponding turnpike *in the sense of* *uniform topology*, we argue that the current study indeed extends existing turnpike theorems (see, Scheinkman, 1976; McKenzie, 1983; Yano, 1998) to much stronger cases. And this would be regarded as one innovation of the present paper.

5. CONCLUDING REMARKS

In the current exploration, we are encouraged to study the economic maturity of a given one-sector neoclassical model with stochastic growth. To the best of our knowledge, we, for the first time, supply a relatively complete characterization of the minimum time needed to economic maturity for any given abstract economy and further to show that the corresponding capital-labor ratio indeed exhibits both asymptotic turnpike property and neighborhood turnpike property under reasonable conditions. In other words, the optimal path of capital accumulation or the equilibrium path of capital accumulation will uniformly and robustly converge to the turnpike capital-labor ratio or will spend almost all the time staying in any given neighborhood of the turnpike capital-labor ratio, respectively, under relatively weak conditions and in a persistently non-stationary environment.

Noting that we assume very general forms of preference for the representative agent and production technology for the firm, one can easily apply the present framework to study many different macroeconomic models with stochastic economic growth. Moreover, we argue that the present basic model can be naturally extended to other cases with different environments, including multi-sector models, heterogeneous -agent models or dynamic general equilibrium models (e.g., Bewley, 1982; Yano, 1984, and among others). Finally, one may easily notice that the present framework can be also extended to include multiple priors through applying the theory developed by Riedel (2009). And we leave these interesting and also important explorations to future research.

APPENDIX

A. Proof of Theorem 2

Put $\varphi\big(k(t)\big)=0$ in (7), then we find that $k(t)$ will be a martingale w. r. t. $\mathbb{P}$. Thus, by the Doob's Martingale Inequality, we obtain,

$$\mathbb{P}\left(\sup_{0\le t\le T}|k(t)|\ge\lambda\right)\le\frac{1}{\lambda}\mathbb{E}\big[|k(T)|\big]=\frac{k_0}{\lambda},\ \ \forall\lambda>0,\ \ \forall T>0. \tag{A.1}$$

Without loss of generality, we put $\lambda=2^m$ for $m\in\mathbb{N}$, then,

$$\mathbb{P}\left(\sup_{0\le t\le T}|k(t)|\ge 2^m\right)\le\frac{1}{2^m}k_0,\ \ \forall m\in\mathbb{N},\ \ \forall T>0.$$

Using the well-known Borel-Cantelli Lemma, we arrive at,

$$\mathbb{P}\left(\sup_{0\le t\le T}|k(t)|\ge 2^m\, i.m.m\right)=0,\ \ \forall T>0.$$

in which $i.m.m$ represents "infinitely many m". So for a.a. $\omega\in\Omega$, there exists $\bar{m}(\omega)\in\mathbb{N}$ such that,

$$\sup_{0\le t\le T}|k(t)|<2^m\ \text{ a.s. for } m\ge\bar{m}(\omega),\ \ \forall T>0.$$

i.e.,

$$\lim_{T\to\infty}\sup_{0\le t\le T}|k(t)|\le 2^m\ \text{ a.s. for } m\ge\bar{m}(\omega).$$

Consequently, $k(t)=k(t,\omega)$ is uniformly bounded for $t\in[0,T]$, $\forall T>0$ and for a.a. $\omega\in\Omega$. Thus, it is ensured that $k(t)=k(t,\omega)$ converges a.s.- $\mathbb{P}$ and the limit belongs to the space $L^1(\mathbb{P})$ thanks to the Doob's Martingale Convergence Theorem. Moreover, by Kolmogorov's or Chebyshev's Inequality, we get,

$$\mathbb{P}\left(\sup_{0\le t\le T}|k(t)|\ge\lambda\right)\le\frac{1}{\lambda^2}\operatorname{var}\big[|k(T)|\big],\ \ \forall 0<\lambda<\infty,\ \ \forall T>0.$$

It follows form (A.1) that,

$$\frac{1}{\lambda^2}\operatorname{var}\big[|k(T)|\big]\le\frac{k_0}{\lambda}\ \ \Leftrightarrow\ \ \operatorname{var}\big[|k(T)|\big]\le\lambda k_0,\ \ \forall T>0. \tag{A.2}$$

Noting that,

$$\operatorname{var}\big[|k(T)|\big]=\mathbb{E}\Big[|k(T)|^2\Big]-(k_0)^2,\ \ \forall T>0.$$

We get by (A.2),

$$\mathbb{E}\left[|k(T)|^2\right] \le (\lambda + k_0)k_0 < \infty\,,\ \ \forall 0 < \lambda < \infty\,,\ \ \forall T > 0\,.$$

which yields,

$$\sup_{T\ge 0}\mathbb{E}\left[|k(T)|^2\right] \le (\lambda + k_0)k_0 < \infty\,.$$

Hence, $k(t) = k(t,\omega)$ converges in $L^1(\mathbb{P})$ by applying the Doob's Martingale Convergence Theorem again.

Furthermore, it is easily seen that $k(t) - k^*$ is also a martingale w. r. t. $\mathbb{P}$. Thus, applying the Doob's Martingale Inequality again implies that,

$$\mathbb{P}\left(\sup_{0\le t\le T}\left|k(t) - k^*\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon}\mathbb{E}\left[\left|k(T) - k^*\right|\right],\ \ \forall \varepsilon > 0\,,\ \ \forall T > 0\,. \tag{A.3}$$

Provided that $\tau^*(\omega) \triangleq \inf\left\{t \ge 0; k(t) = k^*\right\} < \infty$ a.s.-$\mathbb{P}$ given by (8), we see that there exists $\beta > 0$ such that the above martingale inequality in (A.3) still holds for $\forall$ $\tau(\omega) \in B_\beta\left(\tau^*(\omega)\right) \triangleq \left\{\tau(\omega) \in \mathcal{T}; \left|\tau(\omega) - \tau^*(\omega)\right| \le \beta\right\}$ by using the Doob's Optional Sampling Theorem. Then, we get that $k(\tau) - k^*$ is uniformly bounded on the compact set $B_\beta\left(\tau^*(\omega)\right)$ by applying the Heine-Borel Theorem and Weierstrass Theorem. Therefore, we, without loss of any generality, set up $\beta = 2^{-m}$ for $\forall m \in \mathbb{N}$. Employing the continuity of martingale w. r. t. time t for any given $\omega \in \Omega$, for $\forall$ $\tau_m \in B_\beta\left(\tau^*(\omega)\right) \triangleq B_{2^{-m}}\left(\tau^*(\omega)\right)$, by using the Lebesgue Dominated Convergence Theorem, we are led to,

$$\limsup_{m\to\infty}\mathbb{P}\left(\sup_{0\le t\le \tau_m}\left|k(t) - k^*\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon}\limsup_{m\to\infty}\mathbb{E}\left[\left|k(\tau_m) - k^*\right|\right] = 0\,.$$

almost surely. And this implies that,

$$\limsup_{m\to\infty}\mathbb{P}\left(\sup_{0\le t\le \tau_m}\left|k(t) - k^*\right| < \varepsilon\right) \ge 1 \ \text{ a.s.-}\mathbb{P}$$

Letting $\varepsilon = 2^{-m_0}$, $\forall m_0 \in \mathbb{N}$, we get,

$$\limsup_{m\to\infty}\mathbb{P}\left(\sup_{0\le t\le \tau_m}\left|k(t) - k^*\right| < 2^{-m_0}\right) = 1 \ \text{ a.s.-}\mathbb{P}\,,\ \ \forall m_0 \in \mathbb{N}\,.$$

It follows from the well-known Fatou's Lemma that,

$$\mathbb{P}\left(\sup_{0\le t\le \tau^*(\omega)}\left|k(t)-k^*\right|<2^{-m_0}\right)=1 \text{ a.s.-}\mathbb{P},\ \forall m_0\in\mathbb{N}.$$

Then, applying the Borel-Cantelli Lemma again implies that,

$$\mathbb{P}\left(\sup_{0\le t\le \tau^*(\omega)}\left|k(t)-k^*\right|<2^{-m_0}\, i.m.m_0\right)=1.$$

where $i.m.m_0$ stands for "infinitely many m_0 ". So for a.a. $\omega\in\Omega$, there exists $\bar{m}_0(\omega)\in\mathbb{N}$ such that,

$$\sup_{0\le t\le \tau^*(\omega)}\left|k(t)-k^*\right|<2^{-m_0} \text{ a.s. for } \forall m_0\ge \bar{m}_0(\omega).$$

That is,

$$\limsup_{m_0\to\infty}\sup_{0\le t\le \tau^*(\omega)}\left|k(t)-k^*\right|\le 0, \text{ a.s.-}\mathbb{P}$$

which yields,

$$\limsup_{\tau^*(\omega)\to\infty}\sup_{0\le t\le \tau^*(\omega)}\left|k(t)-k^*\right|\le 0, \text{ a.s.-}\mathbb{P}$$

That is to say,

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty}\bigcap_{t'=0}^{\infty}\bigcup_{t=t'}^{\infty}\left[\left|k(t)-k^*\right|\ge\frac{1}{m}\right]\right)=0.$$

Equivalently, for $\forall m\in\mathbb{N}$, we arrive at,

$$\mathbb{P}\left(\bigcap_{t'=0}^{\infty}\bigcup_{t=t'}^{\infty}\left[\left|k(t)-k^*\right|\ge\frac{1}{m}\right]\right)=0.$$

i.e., for $\forall\varepsilon>0$,

$$\lim_{t'\to\infty}\mathbb{P}\left(\bigcup_{t=t'}^{\infty}\left[\left|k(t)-k^*\right|\ge\varepsilon\right]\right)=0.$$

which gives the desired assertion. ■

B. Proof of Theorem 4

Given the SDE defined by (7), we can define the following characteristic operator of $k(t)$,

$$\mathcal{A}g\left(k_0\right)=\varphi\left(k_0\right)\frac{\partial g}{\partial k_0}\left(k_0\right)+\frac{1}{2}\psi^2\left(k_0\right)\frac{\partial^2 g}{\partial (k_0)^2}\left(k_0\right).$$

for any $k_0 \triangleq k(0)>0$. We now define Kullback-Leibler type distance (see, Bomze, 1991; Imhof, 2005) between k_0 and k^* as follows,

$$g\left(k_0\right)\triangleq dist\left(k_0,k^*\right)\triangleq k^*\log\left(\frac{k^*}{k_0}\right)\geq 0.$$

Then we get,

$$\mathcal{A}g\left(k_0\right)=\left[-\varphi\left(k_0\right)+\frac{1}{2k_0}\psi^2\left(k_0\right)\right]\frac{k^*}{k_0}.$$

By Theorem 2, we find that there exists $T_0<\infty$ such that,

$$\sup_{0\leq t\leq T}\left|k(t)-k^*\right|<\mu \quad \text{for} \quad \forall\mu>0,\ \ \forall T\geq T_0.$$

Thus, we have,

$$\mathcal{A}g\left(k_0\right)\leq\left[-\varphi\left(k_0\right)+\frac{1}{2k_0}\psi^2\left(k_0\right)\right]\frac{k^*}{k_0}+\mu-\left|k(t)-k^*\right|\triangleq\Sigma-\left|k(t)-k^*\right|. \tag{B.1}$$

Define,

$$B_\alpha(k^*)\triangleq\left\{k(t)\in\mathbb{R}_{++};\left|k(t)-k^*\right|<\alpha,t\geq 0\right\},$$

$$\tilde{\tau}(\omega)\triangleq\tau_{\overline{B}_\alpha(k^*)}(\omega)\triangleq\inf\left\{t\geq 0;k(t)\in\overline{B}_\alpha(k^*)\triangleq clB_\alpha(k^*)\right\}.$$

where $\overline{B}_\alpha(k^*)$ denotes the closure of $B_\alpha(k^*)$. Suppose that $\alpha>\Sigma$, for every $k(t)$ $\notin\overline{B}_\alpha(k^*)$, i.e., $k(t)\in\overline{B}_\alpha^C(k^*)$, we get,

$$\mathcal{A}g\left(k_0\right)\leq-\alpha+\Sigma.$$

by (B.1). Then by Dynkin's formula,

$$0\leq\mathbb{E}\left[g\left(k\left(t\wedge\tilde{\tau}\right)\right)\right]=g\left(k_0\right)+\mathbb{E}\left[\int_0^{t\wedge\tilde{\tau}}\mathcal{A}g\left(k(s)\right)ds\right]\leq g\left(k_0\right)+(\Sigma-\alpha)\mathbb{E}\left[t\wedge\tilde{\tau}(\omega)\right].$$

Since $t\wedge\tilde{\tau}\nearrow\tilde{\tau}$ as $t\to\infty$. Then by Lebesgue Monotone Convergence Theorem, we obtain,

$$0\leq g\left(k_0\right)+(\Sigma-\alpha)\mathbb{E}\left[\tilde{\tau}(\omega)\right].$$

which produces,

$$\mathbb{E}\left[\tau_{\bar{B}_\alpha(k^*)}(\omega)\right]=\mathbb{E}\left[\tilde{\tau}(\omega)\right]\le\frac{g\left(k_0\right)}{\alpha-\Sigma}=\frac{dist\left(k_0,k^*\right)}{\alpha-\Sigma}.$$

as required in (i). Moreover, for some constant $W>g\left(k_0\right)$, set up,

$$\tau_W=\tau_W(\omega)\triangleq\inf\left\{t\ge 0;g\left(k(t)\right)=W\right\}.$$

Thus, by Dynkin's formula and inequality (B.1),

$$0\le\mathbb{E}\left[g\left(k\left(t\wedge\tau_W\right)\right)\right]=g\left(k_0\right)+\mathbb{E}\left[\int_0^{t\wedge\tau_W}\mathcal{A}g\left(k(s)\right)ds\right]$$

$$\le g\left(k_0\right)-\mathbb{E}\left[\int_0^{t\wedge\tau_W}\left|k(s)-k^*\right|ds\right]+\Sigma\mathbb{E}\left[t\wedge\tau_W(\omega)\right].$$

If $W\to\infty$, we get $t\wedge\tau_W(\omega)\to t$, and by applying the well-known Lebesgue Bounded Convergence Theorem and Levi Lemma,

$$0\le g\left(k_0\right)-\mathbb{E}\left[\int_0^t\left|k(s)-k^*\right|ds\right]+\Sigma t.$$

which yields,

$$\mathbb{E}\left[\frac{1}{t}\int_0^t\left|k(s)-k^*\right|ds\right]\le\frac{g\left(k_0\right)}{t}+\Sigma.$$

Thus, we have,

$$\limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t\left|k(s)-k^*\right|ds\right]\le\Sigma. \tag{B.2}$$

If we let $\chi_{\bar{B}_\alpha^C(k^*)}\left(k(t)\right)$ denote the indicator function of set $\bar{B}_\alpha^C(k^*)$, then by (B.2) and Assumption 4, we arrive at,

$$\pi\left[\bar{B}_\alpha^C(k^*)\right]=\limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t\chi_{\bar{B}_\alpha^C(k^*)}\left(k(s)\right)ds\right]\le\limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t\frac{\left|k(s)-k^*\right|}{\alpha}ds\right]\le\frac{\Sigma}{\alpha}.$$

which implies that,

$$\pi\left[\bar{B}_\alpha(k^*)\right]\ge 1-\frac{\Sigma}{\alpha}\triangleq 1-\varepsilon.$$

which gives the desired assertion in (ii). ■

C. Proof of Lemma 1

Applying Itô's rule to (11) produces,

$$|k(t)|^2 = |k_0|^2 + 2\int_0^t \varphi\big(k(s)\big)k(s)ds + \int_0^t \big|\psi\big(k(s)\big)\big|^2 ds + 2\int_0^t \psi\big(k(s)\big)k(s)dB(s).$$

By using Assumption 6 we get that for $t_1 \in [0,T]$ and for some constant $e \triangleq e(p,T)$ $< \infty$, which may be different from line to line throughout the proof,

$$\sup_{0\le t\le t_1} |k(t)|^p \le e\left\{|k_0|^p + \left[\int_0^{t_1} \phi |k(s)|^2 ds\right]^{\frac{p}{2}} + \sup_{0\le t\le t_1}\left|\int_0^t k(s)\psi\left[k(s)\right]dB(s)\right|^{\frac{p}{2}}\right\}.$$

It follows from Cauchy-Schwarz Inequality that,

$$\sup_{0\le t\le t_1} |k(t)|^p \le e\left\{|k_0|^p + \int_0^{t_1} |k(s)|^p ds + \sup_{0\le t\le t_1}\left|\int_0^t k(s)\psi\left[k(s)\right]dB(s)\right|^{\frac{p}{2}}\right\}.$$

Taking expectations on both sides and applying the Burkholder-Davis-Gundy Inequality (see, Karatzas and Shreve, 1991, pp.166) shows that,

$$\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] \le e\left\{|k_0|^p + \int_0^{t_1} \mathbb{E}\left[|k(s)|^p\right]ds + \mathbb{E}\left[\int_0^{t_1} |k(s)|^2 \big|\psi\big(k(s)\big)\big|^2 ds\right]^{\frac{p}{4}}\right\}. \quad \text{(C.1)}$$

Now, using the Young Inequality (see, Higham et al, 2003), Assumption 6, and Rogers -Hölder Inequality reveals that,

$$\mathbb{E}\left[\int_0^{t_1} |k(s)|^2 \big|\psi\big(k(s)\big)\big|^2 ds\right]^{\frac{p}{4}} \le \mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^{\frac{p}{2}} \left(\int_0^{t_1} \big|\psi\big(k(s)\big)\big|^2 ds\right)^{\frac{p}{4}}\right]$$

$$\le \frac{1}{2e}\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] + \frac{e}{2}\mathbb{E}\left[\int_0^{t_1} \big|\psi\big(k(s)\big)\big|^2 ds\right]^{\frac{p}{2}}$$

$$\le \frac{1}{2e}\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] + \frac{e}{2}\phi^{\frac{p}{2}}\mathbb{E}\left[\int_0^{t_1} |k(s)|^2 ds\right]^{\frac{p}{2}}$$

$$\le \frac{1}{2e}\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] + \frac{e}{2}\phi^{\frac{p}{2}}T^{\frac{p-2}{2}}\mathbb{E}\left[\int_0^{t_1} |k(s)|^p ds\right]$$

Substituting this into (C.1) yields,

$$\mathbb{E}\left[\sup_{0\le t\le T}|k(t)|^p\right]\le e\left\{|k_0|^p+\int_0^T\mathbb{E}\left[|k(t)|^p\right]dt\right\}.$$

Thus, by applying the following fact (see, Higham et al., 2003),

$$\mathbb{E}\left[|k(t)|^p\right]\le e\left(1+|k_0|^p\right).$$

We hence arrive at,

$$\mathbb{E}\left[\sup_{0\le t\le T}|k(t)|^p\right]\le e(k_0,p,T)<\infty.$$

which gives the desired result in (i). Noting that the proof of (ii) is quite similar to that of (i), we omit it. And this completes the whole proof. ■

D. Proof of Lemma 2

By the Doob's Martingale Inequality, we obtain,

$$\mathbb{P}\left(\sup_{0\le t\le T}|k(t)|\ge\lambda\right)\le\frac{1}{\lambda}\mathbb{E}\left[|k(T)|\right]=\frac{k_0}{\lambda},\quad\forall 0<\lambda<\infty,\quad\forall T>0. \tag{D.1}$$

Similarly, by Kolmogorov's or Chebyshev's Inequality, we get,

$$\mathbb{P}\left(\sup_{0\le t\le T}|k(t)|\ge\lambda\right)\le\frac{1}{\lambda^2}\operatorname{var}\left[|k(T)|\right],\quad\forall 0<\lambda<\infty,\quad\forall T>0.$$

It follows form (D.1) that,

$$\frac{1}{\lambda^2}\operatorname{var}\left[|k(T)|\right]\le\frac{k_0}{\lambda}\quad\Leftrightarrow\quad\operatorname{var}\left[|k(T)|\right]\le\lambda k_0,\quad\forall T>0. \tag{D.2}$$

Noting that,

$$\operatorname{var}\left[|k(T)|\right]=\mathbb{E}\left[|k(T)|^2\right]-(k_0)^2,\quad\forall T>0.$$

We get by (D.2),

$$\mathbb{E}\left[|k(T)|^2\right]\le(\lambda+k_0)k_0<\infty,\quad\forall 0<\lambda<\infty,\quad\forall T>0. \tag{D.3}$$

which implies that $k(t)$ is a square-integrable martingale. We define,

$$\zeta\triangleq|k(t)|,\quad\zeta^*\triangleq\|k(t)\|_\infty\triangleq\sup_{0\le s\le t}|k(s)|,\quad\|k(t)\|_2\triangleq\left\{\mathbb{E}\left[|k(t)|^2\right]\right\}^{\frac{1}{2}}.$$

Thus, applying Doob's Martingale Inequality and the well-known Fubini Theorem, we arrive at for some constant $N<\infty$,

$$\mathbb{E}\left[\left|\zeta^* \wedge N\right|^2\right] = 2\int_0^\infty x\mathbb{P}\left(\zeta^*(\omega) \wedge N \geq x\right)dx$$

$$\leq 2\int_0^\infty \int_{\{\zeta^*(\omega)\wedge N \geq x\}} \zeta(\omega)d\mathbb{P}(\omega)dx$$

$$= 2\int_0^\infty \int_\Omega \zeta(\omega)\chi_{\{\zeta^*(\omega)\wedge N\geq x\}}d\mathbb{P}(\omega)dx$$

$$= 2\int_\Omega \zeta(\omega) \int_0^{\zeta^*(\omega)\wedge N} dxd\mathbb{P}(\omega)$$

$$= 2\int_\Omega \zeta(\omega)\left(\zeta^*(\omega) \wedge N\right)d\mathbb{P}(\omega)$$

$$= 2\mathbb{E}\left[\zeta\left(\zeta^* \wedge N\right)\right]$$

It follows from Rogers-Hölder Inequality that,

$$\left\|\zeta^* \wedge N\right\|_2^2 = \mathbb{E}\left[\left|\zeta^* \wedge N\right|^2\right] \leq 2\left\|\zeta\right\|_2 \left\|\zeta^* \wedge N\right\|_2.$$

which produces,

$$\left\|\zeta^* \wedge N\right\|_2 \leq 2\left\|\zeta\right\|_2.$$

Noting that,

$$\mathbb{E}\left[\left|\zeta^* \wedge N\right|^2\right] \leq N^2 < \infty$$

And hence applying Lebesgue Dominated Convergence Theorem leads us to,

$$\left\|\zeta^*\right\|_2 \leq 2\left\|\zeta\right\|_2 \quad \Leftrightarrow \quad \left\|\zeta^*\right\|_2^2 \leq 4\left\|\zeta\right\|_2^2.$$

i.e.,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|k(s)\right|^2\right] \leq 4\mathbb{E}\left[\left|k(t)\right|^2\right] \leq 4(\lambda + k_0)k_0 < \infty, \quad \forall t \geq 0.$$

by using the inequality given by (D.3). Accordingly, a canonical application of Lebesgue Monotone Convergence Theorem or Levi Lemma gives the required assertion in (i). The proof of (ii) is similar to that of (i), we hence omit it. And therefore the whole proof is complete. ■

E. Proof of Proposition 1

Provided the SDEs defined in (11) and (12), and it follows from Lemma 1 that for $\forall$

$2 \le p < \infty$, $\forall T > 0$, there exists some constant $W < \infty$ such that,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)\right|^p\right] \vee \mathbb{E}\left[\sup_{0\le t\le T}\left|\tilde{k}(t)\right|^p\right] \le W \,. \tag{E.1}$$

where by Assumption 1,

$$k(t) = k_0 + \int_0^t k(s)\varphi_0\left(k(s)\right)ds + \int_0^t k(s)\psi_0\left(k(s)\right)dB(s)\,,$$

$$\tilde{k}(t) = k_0 + \int_0^t \tilde{k}(s)\tilde{\varphi}_0\left(\tilde{k}(s)\right)ds + \int_0^t \tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)dB(s)\,.$$

Moreover, we put $\left|k(t)\right| \vee \left|\tilde{k}(t)\right| \le \bar{W} < \infty$, for $\forall t \ge 0$, otherwise, we just consider $k(t) \wedge \bar{W}$ and $\tilde{k}(t) \wedge \bar{W}$ instead of $k(t)$ and $\tilde{k}(t)$, respectively, then we get the desired result by sending $\bar{W}$ to infinity thanks to the well-known Lebesgue Dominated Convergence Theorem. In what follows, we first define the following stopping times,

$$\tau_{\bar{W}} \triangleq \inf\left\{t \ge 0; \left|k(t)\right| \ge \bar{W}\right\},\quad \tilde{\tau}_{\bar{W}} \triangleq \inf\left\{t \ge 0; \left|\tilde{k}(t)\right| \ge \bar{W}\right\},\quad \tau^*_{\bar{W}} \triangleq \tau_{\bar{W}} \wedge \tilde{\tau}_{\bar{W}}\,.$$

By the Young Inequality (see, Higham et al, 2003) and for any $R > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^2\right] \\
&= \mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^2 \chi_{\{\tau_{\bar{W}}>T,\tilde{\tau}_{\bar{W}}>T\}}\right] + \mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^2 \chi_{\{\tau_{\bar{W}}\le T, or \tilde{\tau}_{\bar{W}}\le T\}}\right] \\
&\le \mathbb{E}\left[\sup_{0\le t\le T}\left|k\left(t\wedge\tau^*_{\bar{W}}\right)-\tilde{k}\left(t\wedge\tau^*_{\bar{W}}\right)\right|^2 \chi_{\{\tau^*_{\bar{W}}>T\}}\right] + \frac{2R}{p}\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^p\right] \\
&\quad + \frac{1-\frac{2}{p}}{R^{\frac{2}{p-2}}}\mathbb{P}\left(\tau_{\bar{W}}\le T, or \tilde{\tau}_{\bar{W}}\le T\right).
\end{aligned} \tag{E.2}$$

It follows from (E.1) that,

$$\mathbb{P}\left(\tau_{\bar{W}}\le T\right) = \mathbb{E}\left[\chi_{\{\tau_{\bar{W}}\le T\}}\frac{\left|k\left(\tau_{\bar{W}}\right)\right|^p}{\bar{W}^p}\right] \le \frac{1}{\bar{W}^p}\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)\right|^p\right] \le \frac{W}{\bar{W}^p}\,.$$

Similarly, one can get $\mathbb{P}\left(\tilde{\tau}_{\bar{W}}\le T\right) \le W/\bar{W}^p$. So,

$$\mathbb{P}\left(\tau_{\bar{W}} \le T, or \tilde{\tau}_{\bar{W}} \le T\right) \le \mathbb{P}\left(\tau_{\bar{W}} \le T\right) + \mathbb{P}\left(\tilde{\tau}_{\bar{W}} \le T\right) \le \frac{2W}{\bar{W}^{p}}.$$

Moreover, we obtain by (E.1),

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^{p}\right] \le 2^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left(\left|k(t)\right|^{p}+\left|\tilde{k}(t)\right|^{p}\right)\right] \le 2^{p}W .$$

Hence, (E.2) becomes,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^{2}\right]$$

$$\le \mathbb{E}\left[\sup_{0\le t\le T}\left|k\left(t\wedge\tau_{\bar{W}}^{*}\right)-\tilde{k}\left(t\wedge\tau_{\bar{W}}^{*}\right)\right|^{2}\right] + \frac{2^{p+1}RW}{p} + \frac{2(p-2)W}{pR^{\frac{2}{p-2}}\bar{W}^{p}}. \tag{E.3}$$

By the Cauchy-Bunyakovsky-Schwarz Inequality, we get,

$$\begin{aligned}
&\left|k\left(t\wedge\tau_{\bar{W}}^{*}\right)-\tilde{k}\left(t\wedge\tau_{\bar{W}}^{*}\right)\right|^{2}\\
&=\left|\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[k(s)\varphi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\varphi}_{0}\left(\tilde{k}(s)\right)\right]ds\right.\\
&\qquad\left.+\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[k(s)\psi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_{0}\left(\tilde{k}(s)\right)\right]dB(s)\right|^{2}\\
&\le 2\left\{T\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[\left|k(s)\varphi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\varphi}_{0}\left(\tilde{k}(s)\right)\right|^{2}\right]ds\right.\\
&\qquad\left.+\left|\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[k(s)\psi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_{0}\left(\tilde{k}(s)\right)\right]dB(s)\right|^{2}\right\}\\
&\le 4\left\{T\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[\left|k(s)\varphi_{0}\left(k(s)\right)-\tilde{k}(s)\varphi_{0}\left(k(s)\right)\right|^{2}\right]ds\right.\\
&\qquad+T\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[\left|\tilde{k}(s)\varphi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\varphi}_{0}\left(\tilde{k}(s)\right)\right|^{2}\right]ds\\
&\qquad\left.+\left|\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left[k(s)\psi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_{0}\left(\tilde{k}(s)\right)\right]dB(s)\right|^{2}\right\}
\end{aligned}$$

Taking expectations on both sides and using Itô's Isometry, we have for $\forall\,\tau \le T$,

$$\mathbb{E}\left[\sup_{0\le t\le \tau}\left|k\left(t\wedge\tau_{\bar{W}}^{*}\right)-\tilde{k}\left(t\wedge\tau_{\bar{W}}^{*}\right)\right|^{2}\right]$$

$$\le 4\left\{T\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)-\tilde{k}(s)\right|^{2}\left|\varphi_{0}\left(k(s)\right)\right|^{2}ds\right]\right.$$

$$+T\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|\tilde{k}(s)\right|^{2}\left|\varphi_{0}\left(k(s)\right)-\tilde{\varphi}_{0}\left(\tilde{k}(s)\right)\right|^{2}ds\right]$$

$$\left.+\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)\psi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_{0}\left(\tilde{k}(s)\right)\right|^{2}ds\right]\right\}$$

$$\le 8\left\{T\phi_{0}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)-\tilde{k}(s)\right|^{2}ds\right]+T\xi^{2}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|\tilde{k}(s)\right|^{2}ds\right]\right.$$

$$+\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)\psi_{0}\left(k(s)\right)-\tilde{k}(s)\psi_{0}\left(k(s)\right)\right|^{2}ds\right]$$

$$\left.+\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|\tilde{k}(s)\psi_{0}\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_{0}\left(\tilde{k}(s)\right)\right|^{2}ds\right]\right\}$$

$$\le 8\left\{T\phi_{0}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)-\tilde{k}(s)\right|^{2}ds\right]+T\xi^{2}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|\tilde{k}(s)\right|^{2}ds\right]\right.$$

$$\left.+\phi_{0}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)-\tilde{k}(s)\right|^{2}ds\right]+\xi^{2}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|\tilde{k}(s)\right|^{2}ds\right]\right\}$$

$$=8\left\{(T+1)\phi_{0}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|k(s)-\tilde{k}(s)\right|^{2}ds\right]+(T+1)\xi^{2}\mathbb{E}\left[\int_{0}^{t\wedge\tau_{\bar{W}}^{*}}\left|\tilde{k}(s)\right|^{2}ds\right]\right\}$$

$$\le 8\left\{(T+1)\phi_{0}\int_{0}^{T}\mathbb{E}\left[\sup_{0\le t_{0}\le s}\left|k\left(t_{0}\wedge\tau_{\bar{W}}^{*}\right)-\tilde{k}\left(t_{0}\wedge\tau_{\bar{W}}^{*}\right)\right|^{2}\right]ds+T(T+1)\bar{W}^{2}\xi^{2}\right\}$$

where we have used Assumption 5 and 6. Hence, applying Gronwall's Inequality (see, Higham et al, 2003) implies that,

$$\mathbb{E}\left[\sup_{0\le t\le \tau}\left|k\left(t\wedge\tau_{\bar{W}}^{*}\right)-\tilde{k}\left(t\wedge\tau_{\bar{W}}^{*}\right)\right|^{2}\right]\le 8T(T+1)\bar{W}^{2}\exp\left[8(T+1)\phi_{0}\right]\xi^{2}.$$

Inserting this into (E.3) leads us to,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|k(t)-\tilde{k}(t)\right|^{2}\right]\le 8T(T+1)\bar{W}^{2}\exp\left[8(T+1)\phi_{0}\right]\xi^{2}+\frac{2^{p+1}RW}{p}+\frac{2(p-2)W}{pR^{\frac{2}{p-2}}\bar{W}^{p}}.$$

Hence, for $\forall \varepsilon > 0$, we can choose R and $\bar{W}$ such that,

$$\frac{2^{p+1} RW}{p} \leq \frac{\varepsilon}{3} \text{ and } \frac{2(p-2)W}{pR^{\frac{2}{p-2}}\bar{W}^{p}} \leq \frac{\varepsilon}{3}.$$

And for any given $T > 0$, we put ξ such that,

$$8T(T+1)\bar{W}^2 \exp\left[8(T+1)\phi_0\right]\xi^2 \leq \frac{\varepsilon}{3}.$$

Thus, for $\forall \varepsilon > 0$, we obtain,

$$\mathbb{E}\left[\sup_{0 \le t \le T} \left|k(t) - \tilde{k}(t)\right|^2\right] \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Notice the arbitrariness of ε, and employ the Levi Lemma to give the desired result. And this proof is thus complete. ■

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